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# Reiteration and a Wolff theorem for interpolation methods defined by means of polygons

by

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**Abstract.** We prove a reiteration theorem for interpolation methods defined by means of polygons, and a Wolff theorem for the case when the polygon has 3 or 4 vertices. In particular, we establish a Wolff theorem for Fernandez' spaces, which settles a problem left over in [5].

**0. Introduction.** Interpolation methods defined by means of polygons were introduced by Peetre and one of the present authors in [6]. These methods are close to the classical real interpolation method but work on $N$-tuples instead of couples and require two parameters instead of only one. The $N$ spaces should be thought of as sitting on the vertices of a convex polygon $\Pi$ in the plane $\mathbb{R}^2$.

Previously Sparr [12], Fernandez [8] and some other authors (see the monograph [3], 4.7.1, for precise references) have studied generalizations of the classical real method to $N$-tuples of Banach spaces. In fact, the first example of interpolation of more than two spaces was given by Foiaş and Lions [10] already in 1961. It turned out that several important results of the classical theory fail for $N$-tuples. Nevertheless, these generalizations have found interesting applications in Functional Analysis.

The methods defined by means of polygons include (the first nontrivial case of) spaces introduced by Sparr [12] and by Fernandez [8]. The former correspond to the case when $\Pi$ is equal to the simplex, while the latter appear for $\Pi$ equal to the unit square. Using this new geometrical approach, Cobos and Peetre described, among other results, the relationship between Sparr and Fernandez spaces and obtained new estimates for the norms of interpolated operators in the case of Fernandez spaces. This investigation was continued by Cobos, Kühn and Schonbek in [4] where, among other things, they studied the behaviour of operators acting from a $J$-space into a $K$-space.

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In the present paper we give a reiteration theorem for the methods defined by means of polygons, and a Wolff theorem for the case when the polygon has 3 or 4 vertices.

As is well known, Wolff's theorem [13] can be considered as a kind of converse of the reiteration theorem and has many interesting applications in Interpolation Theory. Wolff's theorem has been extended to Sparr spaces by Cobos and Peetre in [5], where they left as an open problem the question of extending Wolff's theorem to Fernandez' case. We shall see later that our Wolff theorem settles this question.

The reiteration theorem is given in Section 2, while Section 3 contains the Wolff theorem. Our approach highlights the geometrical aspects of these two classical results. When specializing our theorems to the case of the simplex we recover a stability result of Sparr [12] and the extended version of Wolff theorem established by Cobos and Peetre [5] (we always refer to the case of three spaces). Writing down our theorems for the case of the unit square we get new information on Fernandez spaces: a reiteration theorem which improves the one known before (see [8] and [9]), and a Wolff theorem.

**1. Preliminaries.** Let $\overline{A}=\{A_1,\ldots,A_N\}$ be a Banach $N$-tuple, that is to say, $N$ Banach spaces $A_j$ which are continuously embedded in a common Hausdorff topological vector space.

A Banach space $A$ is said to be an *intermediate space* with respect to $\overline{A}$ if $\Delta(\overline{A})=A_1\cap\ldots\cap A_N\hookrightarrow A\hookrightarrow A_1+\ldots+A_N=\Sigma(\overline{A})$ (continuous inclusions). Given any other $N$-tuple $\overline{B}=\{B_1,\ldots,B_N\}$, we write $T\in\mathcal{L}(\overline{A},\overline{B})$ to mean that $T$ is a linear operator from $\Sigma(\overline{A})$ into $\Sigma(\overline{B})$ whose restriction to each $A_j$ gives a bounded operator from $A_j$ to $B_j$ $(j=1,\ldots,N)$. We put

$$\|T\|_{\overline{A},\overline{B}}=\max\{\|T\|_{A_1,B_1},\ldots,\|T\|_{A_N,B_N}\}.$$

In what follows, $\Pi=\overline{P_1\ldots P_N}$ stands for a convex polygon in the affine plane $\mathbb{R}^2$, with vertices $P_j=(x_j,y_j)$. We imagine each space $A_j$ from our $N$-tuple $\overline{A}$ as sitting on the vertex $P_j$ $(j=1,\ldots,N)$.

Given any two positive numbers $t,s$, we define the $K$- and the $J$-functionals (with respect to the polygon $\Pi$) by

$$K(t,s;a)=\inf\Big\{\sum_{j=1}^{N}t^{x_j}s^{y_j}\|a_j\|_{A_j}:a=\sum_{j=1}^{N}a_j,\ a_j\in A_j\Big\},$$

$$J(t,s;a)=\max_{1\le j\le N}\{t^{x_j}s^{y_j}\|a\|_{A_j}\}.$$

Observe that $\{K(t,s;\cdot):t,s>0\}$ [resp. $\{J(t,s;\cdot):t,s>0\}$] is a family of norms on $\Sigma(\overline{A})$ [resp. $\Delta(\overline{A})$], any two of them being equivalent.



Let now $(\theta,\mu)$ be an interior point of $\Pi$ $[(\theta,\mu)\in\operatorname{Int}\Pi]$, and let $1\le q\le\infty$. The space $\overline{A}_{(\theta,\mu),q;K}$ consists of all $a\in\Sigma(\overline{A})$ for which the norm

$$\|a\|_{(\theta,\mu),q;K}=\Big(\sum_{(m,n)\in\mathbb{Z}^2}(2^{-m\theta-n\mu}K(2^m,2^n;a))^q\Big)^{1/q}$$

is finite.

The space $\overline{A}_{(\theta,\mu),q;J}$ is defined as the set of all $a\in\Sigma(\overline{A})$ which can be represented as

$$a=\sum_{(m,n)\in\mathbb{Z}^2}u_{m,n}$$

(convergence in $\Sigma(\overline{A})$) with $(u_{m,n})\subset\Delta(\overline{A})$ and

$$\Big(\sum_{(m,n)\in\mathbb{Z}^2}(2^{-m\theta-n\mu}J(2^m,2^n;u_{m,n}))^q\Big)^{1/q}<\infty.$$

The norm on $\overline{A}_{(\theta,\mu),q;J}$ is

$$\|a\|_{(\theta,\mu),q;J}=\inf\Big\{\Big(\sum_{(m,n)\in\mathbb{Z}^2}(2^{-m\theta-n\mu}J(2^m,2^n;u_{m,n}))^q\Big)^{1/q}\Big\}$$

where the infimum is taken over all representations $(u_{m,n})$ of $a$ as above.

It will also be convenient to give a meaning to $\overline{A}_{(\theta,\mu),q;K}$ and $\overline{A}_{(\theta,\mu),q;J}$ in the case when $(\theta,\mu)$ is a vertex $P_j$ of $\Pi$. We then define for all $1\le q\le\infty$

$$\overline{A}_{P_j,q;K}=\overline{A}_{P_j,q;J}=A_j.$$

Next we show some examples.

EXAMPLE 1.1. Let $\mathcal{H}$ be the collection of all 3-tuples $\overline{\theta}=(\theta_1,\theta_2,\theta_3)$ of numbers in the interval $(0,1)$ such that $\sum_{j=1}^{3}\theta_j=1$. If we take $\Pi$ equal to the simplex $\{(0,0),(1,0),(0,1)\}$, then, for any $\overline{\theta}\in\mathcal{H}$, $\overline{A}_{(\theta_2,\theta_3),q;K}$ and $\overline{A}_{(\theta_2,\theta_3),q;J}$ coincide with Sparr's spaces $\overline{A}^S_{(\theta_1,\theta_2,\theta_3),q;K}$ and $\overline{A}^S_{(\theta_1,\theta_2,\theta_3),q;J}$, respectively (see [12]).

EXAMPLE 1.2. Write $\mathcal{D}$ for the set of all $(\theta,\mu)\in\mathbb{R}^2$ such that $0<\theta,\mu<1$. If $\Pi$ is the unit square $\{(0,0),(1,0),(0,1),(1,1)\}$ and $(\theta,\mu)\in\mathcal{D}$ then we recover Fernandez' spaces $\overline{A}^F_{(\theta,\mu),q;K}$ and $\overline{A}^F_{(\theta,\mu),q;J}$ (see [8] and [9]). Observe that the role of the polygon justifies the restriction on parameters in Fernandez' case.

In the case of the classical real method, the $K$- and the $J$-spaces coincide, with equivalent norms (see [2]). But in our multidimensional context, they do not agree in general. Counterexamples can be found in [12], [7] and [6].

We only have now the continuous inclusion

$$\overline{A}_{(\theta,\mu),q;J} \hookrightarrow \overline{A}_{(\theta,\mu),q;K}$$

(see [6], Thm. 1.3.).

Take now a sequence of Banach spaces $(\mathcal{F}_{m,n})_{(m,n)\in\mathbb{Z}^2}$. For $j = 1,\ldots,N$, put

$$\mathcal{F}^j_{m,n} = (\mathcal{F}_{m,n}, 2^{-mx_j-ny_j}\|\ \|_{\mathcal{F}_{m,n}})$$

and let $\ell_q(\mathcal{F}^j_{m,n})$ be the vector-valued $\ell_q$-space, that is to say,

$$\ell_q(\mathcal{F}^j_{m,n}) = \{(x_{m,n}) : x_{m,n} \in \mathcal{F}_{m,n} \quad \text{and} \quad \|(x_{m,n})\|_{\ell_q(\mathcal{F}^j_{m,n})} < \infty\}$$

where

$$\|(x_{m,n})\|_{\ell_q(\mathcal{F}^j_{m,n})} = \Big(\sum_{m,n}(2^{-mx_j-ny_j}\|x_{m,n}\|_{\mathcal{F}_{m,n}})^q\Big)^{1/q}.$$

When all $\mathcal{F}_{m,n}$ are equal to the scalar field $\mathbb{K}$, the resulting $\ell_q$-space is denoted by $\ell_q(2^{-mx_j-ny_j})$. Note that $\ell_q(2^{-mx_j-ny_j})$ is just the scalar sequence space $\ell_q$ with weight $2^{-mx_j-ny_j}$ on the $(m,n)$th coordinate.

The following interpolation formulae were established in [6], Thm. 3.1, and will be useful in our later considerations.

LEMMA 1.3. *Let $(\theta,\mu) \in \operatorname{Int}\Pi$ and let $1 \le q_1,\ldots,q_N, q \le \infty$. Then we have with equivalence of norms*

$$(\ell_{q_j}(\mathcal{F}^j_{m,n}))_{(\theta,\mu),q;K} = (\ell_{q_j}(\mathcal{F}^j_{m,n}))_{(\theta,\mu),q;J} = \ell_q(2^{-m\theta-n\mu}\mathcal{F}_{m,n}).$$

Next we consider a class of affine transformations associated with the polygon $\Pi$. A mapping $R$ defined by

$$R\begin{pmatrix}u\\v\end{pmatrix} = Q + U\begin{pmatrix}u\\v\end{pmatrix}, \quad (u,v) \in \mathbb{R}^2,$$

where $Q \in \mathbb{R}^2$ and $U$ is an isomorphism of $\mathbb{R}^2$, is said to be of *type* $(\Pi)$ if for each $j = 1,\ldots,N$ either $RP_j \in \operatorname{Int}\Pi$ or $RP_j = P_k$ for some $k = 1,\ldots,N$.

Notice that if $R$ is of type $(\Pi)$ then $R$ transforms $\Pi$ into another convex polygon $R(\Pi) = \overline{RP_1\ldots RP_N}$ contained in $\Pi$. Moreover, $R(\operatorname{Int}\Pi) = \operatorname{Int}R(\Pi)$.

The following lemma shows the relationship between this kind of mappings and interpolation methods. When the $K$- or the $J$-spaces are defined by means of a polygon other than $\Pi$, we write the polygon as a superscript.

LEMMA 1.4. *Let $W \in \operatorname{Int}\Pi$, $1 \le q \le \infty$ and let $R$ be a mapping of type $(\Pi)$. Then the $K$- and the $J$-spaces defined by means of $\Pi$ and $W$ coincide (with equivalence of norms) with those defined by means of $R(\Pi)$ and $RW$, i.e.*

$$\overline{A}_{W,q;K} = \overline{A}^{R(\Pi)}_{RW,q;K} \quad \text{and} \quad \overline{A}_{W,q;J} = \overline{A}^{R(\Pi)}_{RW,q;J}.$$

The proof is an easy change of variables (see [4], Remark 4.1).



**2. Reiteration.** The reiteration theorem for the classical real method says that

$$(*) \qquad (\overline{A}_{\theta_0,q_0}, \overline{A}_{\theta_1,q_1})_{\eta,q} = \overline{A}_{\theta,q}$$

where $0 < \theta_0 \ne \theta_1 < 1$, $0 < \eta < 1$, $\theta = (1-\eta)\theta_0 + \eta\theta_1$ and $1 \le q_0, q_1, q \le \infty$ (see [2]).

Equality $(*)$ can be reformulated as

$$(**) \qquad (\overline{A}_{R0,q_0}, \overline{A}_{R1,q_1})_{\eta,q} = \overline{A}_{R\eta,q}$$

where we denote by $R$ the mapping in $\mathbb{R}$ given by $Rt = \theta_0 + (\theta_1 - \theta_0)t$. Clearly $R$ is an affine mapping in $\mathbb{R}$ of type $([0,1])$.

Next we show that formula $(**)$ extends to the $K$- and the $J$-methods defined by means of polygons.

THEOREM 2.1. *Let $\Pi = \overline{P_1\ldots P_N}$ be a convex polygon with vertices $P_j = (x_j,y_j)$, let $(\theta,\mu) \in \operatorname{Int}\Pi$ and let $R$ be a mapping of type $(\Pi)$. Put*

$$RP_j = (\alpha_j,\beta_j) \quad (j = 1,\ldots,N), \qquad R\begin{pmatrix}\theta\\\mu\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}.$$

*Let $\overline{A} = \{A_1,\ldots,A_N\}$ be an $N$-tuple and assume that $\overline{X} = \{X_1,\ldots,X_N\}$, $\overline{Y} = \{Y_1,\ldots,Y_N\}$ are $N$-tuples formed by intermediate spaces with respect to $\overline{A}$. If*

$$(1) \qquad X_j \hookrightarrow \overline{A}_{(\alpha_j,\beta_j),\infty;K}, \quad j = 1,\ldots,N$$

$$(2) \qquad [\textit{resp. } \overline{A}_{(\alpha_j,\beta_j),1;J} \hookrightarrow Y_j, \quad j = 1,\ldots,N]$$

*then*

$$\overline{X}_{(\theta,\mu),q;K} \hookrightarrow \overline{A}_{(\alpha,\beta),q;K} \quad [\textit{resp. } \overline{A}_{(\alpha,\beta),q;J} \hookrightarrow \overline{Y}_{(\theta,\mu),q;J}]$$

*where $1 \le q \le \infty$.*

Proof. For $(m,n) \in \mathbb{Z}^2$ let

$$\mathcal{F}_{m,n} = (\Sigma(\overline{A}), K(2^m,2^n;\cdot))$$

and for $j = 1,\ldots,N$ write

$$\mathcal{F}^j_{m,n} = (\Sigma(\overline{A}), 2^{-m\alpha_j-n\beta_j}K(2^m,2^n;\cdot)).$$

By (1), the operator

$$\nu : X_j \to \ell_\infty(\mathcal{F}^j_{m,n}), \quad a \to \nu a = (\ldots,a,a,a,\ldots),$$

is bounded for $j = 1,\ldots,N$. Interpolating by the $K$-method, we see that

$$\nu : \overline{X}_{(\theta,\mu),q;K} \to (\ell_\infty(\mathcal{F}^j_{m,n}))_{(\theta,\mu),q;K}$$

is also bounded. Let us identify the last space. Lemma 1.4 shows that

$$(\ell_\infty(\mathcal{F}^1_{m,n}),\ldots,\ell_\infty(\mathcal{F}^N_{m,n}))_{(\theta,\mu),q;K}$$

coincides with the $K$-space defined by means of the polygon $R(\Pi)$ and the point $(\alpha,\beta)=R(\theta,\mu)$,

$$(\ell_\infty(\mathcal{F}^1_{m,n}),\dots,\ell_\infty(\mathcal{F}^N_{m,n}))^{R(\Pi)}_{(\alpha,\beta),q;K},$$

and this space, according to Lemma 1.3, is equal to $\ell_q(2^{-m\alpha-n\beta}\mathcal{F}_{m,n})$. Hence there is a constant $M>0$ such that for all $a\in\overline{X}_{(\theta,\mu),q;K}$,

$$\Big(\sum_{(m,n)\in\mathbb{Z}^2}(2^{-m\alpha-n\beta}K(2^m,2^n;a))^q\Big)^{1/q}\le M\|a\|_{\overline{X}_{(\theta,\mu),q;K}}.$$

This establishes the inclusion $\overline{X}_{(\theta,\mu),q;K}\hookrightarrow\overline{A}_{(\alpha,\beta),q;K}$.

To check the case of the $J$-method, we shall work with the Banach spaces

$$G_{m,n}=(\Delta(\overline{A}),J(2^m,2^n;\cdot)),\qquad G^j_{m,n}=(\Delta(\overline{A}),2^{-m\alpha_j-n\beta_j}J(2^m,2^n;\cdot)).$$

Combining (2) with the definition of the $J$-method shows that the operator

$$\pi:\ell_1(G^j_{m,n})\to Y_j,\qquad (u_{m,n})\to\pi(u_{m,n})=\sum_{(m,n)\in\mathbb{Z}^2}u_{m,n},$$

is bounded for $j=1,\dots,N$. By interpolation,

$$\pi:(\ell_1(G^j_{m,n}))_{(\theta,\mu),q;J}\to\overline{Y}_{(\theta,\mu),q;J}$$

is bounded as well. The first space can be identified using again Lemmata 1.4 and 1.3. We have

$$(\ell_1(G^j_{m,n}))_{(\theta,\mu),q;J}=(\ell_1(G^j_{m,n}))^{R(\Pi)}_{(\alpha,\beta),q;J}=\ell_q(2^{-m\alpha-n\beta}G_{m,n}).$$

Thus $\pi:\ell_q(2^{-m\alpha-n\beta}G_{m,n})\to\overline{Y}_{(\theta,\mu),q;J}$ is bounded, and the inclusion $\overline{A}_{(\alpha,\beta),q;J}\hookrightarrow\overline{Y}_{(\theta,\mu),q;J}$ follows. ■

As we pointed out in the preliminaries, we always have $\overline{A}_{(\alpha,\beta),q;J}\hookrightarrow\overline{A}_{(\alpha,\beta),q;K}$ but in general $\overline{A}_{(\alpha,\beta),q;K}\not\hookrightarrow\overline{A}_{(\alpha,\beta),q;J}$.

A sufficient condition for equality between $K$- and $J$-spaces can be formulated in terms of the $N$-tuple $\overline{A}$ and the polygon $\Pi$:

CONDITION $\mathcal{E}(\overline{A},\Pi)$. There is a constant $\gamma$ (depending only on $\overline{A}$ and on $\Pi$) such that for every $a\in\Sigma(\overline{A})$ for which

$$\sum_{(m,n)\in\mathbb{Z}^2}\min_{1\le j\le N}\{2^{-mx_j-ny_j}\}K(2^m,2^n;a)<\infty \tag{3}$$

there exists a representation

$$a=\sum_{(m,n)\in\mathbb{Z}^2}u_{m,n}$$

(convergence in $\Sigma(\overline{A})$) with $(u_{m,n})\subset\Delta(\overline{A})$ and

$$J(2^m,2^n;u_{m,n})\le\gamma K(2^m,2^n;a),\qquad (m,n)\in\mathbb{Z}^2.$$



(Compare with the fundamental lemma of interpolation theory [2], Lemma 3.3.2. Notice also that if $\Pi$ is the simplex, then $\mathcal{E}(\overline{A},\Pi)$ is the condition $\mathcal{F}(\overline{A})$ of Sparr [12].)

Let $(\alpha,\beta)\in\operatorname{Int}\Pi$ and $1\le q\le\infty$. One can check that (3) holds for every $a\in\overline{A}_{(\alpha,\beta),q;K}$. Hence, if $\mathcal{E}(\overline{A},\Pi)$ is satisfied, we have

$$\overline{A}_{(\alpha,\beta),q;J}=\overline{A}_{(\alpha,\beta),q;K}\quad\text{(equivalent norms).}$$

When the $K$- and $J$-spaces coincide, we write simply $\overline{A}_{(\alpha,\beta),q}$ to denote either of them.

The following reiteration result is a direct consequence of Theorem 2.1.

THEOREM 2.2. *Let $\Pi=\overline{P_1\dots P_N}$ be a convex polygon with vertices $P_j=(x_j,y_j)$, let $(\theta,\mu)\in\operatorname{Int}\Pi$ and let $R$ be a mapping of type $(\Pi)$. Put*

$$RP_j=(\alpha_j,\beta_j)\quad(j=1,\dots,N),\qquad R\begin{pmatrix}\theta\\ \mu\end{pmatrix}=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}.$$

*Let $1\le q_1,\dots,q_N,q\le\infty$, and let $\overline{A}$ be an $N$-tuple such that*

$$\overline{A}_{(\alpha_j,\beta_j),q_j;J}=\overline{A}_{(\alpha_j,\beta_j),q_j;K},\qquad j=1,\dots,N,$$

*and*

$$\overline{A}_{(\alpha,\beta),q;J}=\overline{A}_{(\alpha,\beta),q;K}$$

*(in particular, these inequalities hold if $\mathcal{E}(\overline{A},\Pi)$ is satisfied). Then*

$$(\overline{A}_{(\alpha_1,\beta_1),q_1},\dots,\overline{A}_{(\alpha_N,\beta_N),q_N})_{(\theta,\mu),q}=\overline{A}_{(\alpha,\beta),q}$$

*(with equivalence of norms).*

The figure below illustrates the theorem. Here we have written $X_j=\overline{A}_{(\alpha_j,\beta_j),q_j}$.

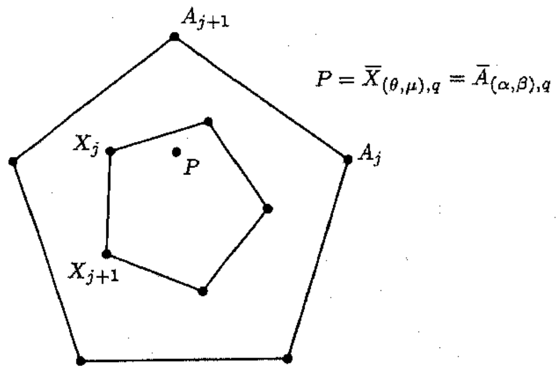


Fig. 2.1

As an application of Theorem 2.2, we next derive a stability result due to Sparr [12]; $\mathcal{H}$ is as defined in Example 1.1.

COROLLARY 2.3. *Let* $1 \le q_1, q_2, q_3, q \le \infty$, *let* $\overline{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \mathcal{H}$ *and let* $\overline{\theta}_j = (\theta_{j1}, \theta_{j2}, \theta_{j3}) \in \mathcal{H}$ *for* $j = 1, 2, 3$. *Assume that* $\mathbb{R}^3$ *is spanned by* $\{\overline{\theta}_j\}_{j=1}^3$ *and write* $\overline{\theta} = \sum_{j=1}^3 \lambda_j \overline{\theta}_j$. *If* $\overline{A}$ *is a 3-tuple such that* $\overline{A}^S_{\overline{\theta}_j, q_j; K} = \overline{A}^S_{\overline{\theta}_j, q_j; J}$, $j = 1, 2, 3$, *and* $\overline{A}^S_{\overline{\theta}, q; K} = \overline{A}^S_{\overline{\theta}, q; J}$ *then*

$$(\overline{A}^S_{\overline{\theta}_1, q_1}, \overline{A}^S_{\overline{\theta}_2, q_2}, \overline{A}^S_{\overline{\theta}_3, q_3})^S_{\overline{\lambda}, q} = \overline{A}^S_{\overline{\theta}, q}.$$

Proof. Recall that $\overline{A}^S_{\overline{\theta}, q} = \overline{A}_{(\theta_2, \theta_3), q}$ (Example 1.1) where $\Pi$ is the simplex $\{(0,0), (1,0), (0,1)\}$.

Let

$$R\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \theta_{12} \\ \theta_{13} \end{pmatrix} + \begin{pmatrix} \theta_{22} - \theta_{12} & \theta_{32} - \theta_{12} \\ \theta_{23} - \theta_{13} & \theta_{33} - \theta_{13} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Since $\mathbb{R}^3$ is spanned by $\{\overline{\theta}_j\}_{j=1}^3$ and $\{\overline{\theta}_j\}_{j=1}^3 \subset \mathcal{H}$, it follows that

$$\begin{vmatrix} \theta_{22} - \theta_{12} & \theta_{32} - \theta_{12} \\ \theta_{23} - \theta_{13} & \theta_{33} - \theta_{13} \end{vmatrix} = \begin{vmatrix} \theta_{11} & \theta_{21} & \theta_{31} \\ \theta_{12} & \theta_{22} & \theta_{32} \\ \theta_{13} & \theta_{23} & \theta_{33} \end{vmatrix} \neq 0.$$

Moreover,

$$R\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \theta_{12} \\ \theta_{13} \end{pmatrix}, \quad R\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \theta_{22} \\ \theta_{23} \end{pmatrix}, \quad R\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \theta_{32} \\ \theta_{33} \end{pmatrix},$$

and

$$\begin{aligned} R\begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix} &= (1 - \lambda_2 - \lambda_3)\begin{pmatrix} \theta_{12} \\ \theta_{13} \end{pmatrix} + \lambda_2 \begin{pmatrix} \theta_{22} \\ \theta_{23} \end{pmatrix} + \lambda_3 \begin{pmatrix} \theta_{32} \\ \theta_{33} \end{pmatrix} \\ &= \lambda_1 \begin{pmatrix} \theta_{12} \\ \theta_{13} \end{pmatrix} + \lambda_2 \begin{pmatrix} \theta_{22} \\ \theta_{23} \end{pmatrix} + \lambda_3 \begin{pmatrix} \theta_{32} \\ \theta_{33} \end{pmatrix} = \begin{pmatrix} \theta_2 \\ \theta_3 \end{pmatrix}. \end{aligned}$$

Consequently, Theorem 2.2 implies that

$$\begin{aligned} (\overline{A}^S_{\overline{\theta}_1, q_1}, \overline{A}^S_{\overline{\theta}_2, q_2}, \overline{A}^S_{\overline{\theta}_3, q_3})^S_{\overline{\lambda}, q} &= (\overline{A}_{(\theta_{12}, \theta_{13}), q_1}, \overline{A}_{(\theta_{22}, \theta_{23}), q_2}, \overline{A}_{(\theta_{32}, \theta_{33}), q_3})_{(\lambda_2, \lambda_3), q} \\ &= \overline{A}_{(\theta_2, \theta_3), q} = \overline{A}^S_{\overline{\theta}, q}. \ \blacksquare \end{aligned}$$

Our next result refers to Fernandez spaces; $\mathcal{D}$ is as in Example 1.2.

COROLLARY 2.4. *Let* $1 \le q_1, \ldots, q_4, q \le \infty$, *let* $\overline{\theta}_j = (\theta_{j1}, \theta_{j2}) \in \mathcal{D}$ $(1 \le j \le 3)$ *such that* $\mathbb{R}^2$ *is spanned by* $\{\overline{\theta}_2 - \overline{\theta}_1, \overline{\theta}_3 - \overline{\theta}_1\}$ *and assume that* $\overline{\theta}_4 = \overline{\theta}_2 + \overline{\theta}_3 - \overline{\theta}_1$ *belongs to* $\mathcal{D}$. *Let* $\overline{\lambda} = (\lambda_1, \lambda_2) \in \mathcal{D}$ *and put*

$$\overline{\theta} = \overline{\theta}_1 + \lambda_1(\overline{\theta}_2 - \overline{\theta}_1) + \lambda_2(\overline{\theta}_3 - \overline{\theta}_1).$$

*If* $\overline{A}$ *is a 4-tuple such that* $\overline{A}^F_{\overline{\theta}_j, q_j; K} = \overline{A}^F_{\overline{\theta}_j, q_j; J}$, $j = 1, \ldots, 4$, *and* $\overline{A}^F_{\overline{\theta}, q; K} = \overline{A}^F_{\overline{\theta}, q; J}$ *then* $(\overline{A}^F_{\overline{\theta}_1, q_1}, \overline{A}^F_{\overline{\theta}_2, q_2}, \overline{A}^F_{\overline{\theta}_3, q_3}, \overline{A}^F_{\overline{\theta}_4, q_4})^F_{\overline{\lambda}, q} = \overline{A}^F_{\overline{\theta}, q}$.



Proof. As pointed out in Example 1.2, $\overline{A}^F_{\overline{\theta}, q} = \overline{A}_{\overline{\theta}, q}$ for $\Pi$ equal to the square $\{(0,0), (1,0), (0,1), (1,1)\}$. Put

$$R\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \theta_{11} \\ \theta_{12} \end{pmatrix} + \begin{pmatrix} \theta_{21} - \theta_{11} & \theta_{31} - \theta_{11} \\ \theta_{22} - \theta_{12} & \theta_{32} - \theta_{12} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

One can easily check that $R$ is a mapping of type $\Pi$ and that $RP_j = \overline{\theta}_j$ for $j = 1, \ldots, 4$. In addition,

$$R\overline{\lambda} = \overline{\theta}_1 + \lambda_1(\overline{\theta}_2 - \overline{\theta}_1) + \lambda_2(\overline{\theta}_3 - \overline{\theta}_1) = \overline{\theta}.$$

Hence, applying Theorem 2.2 we obtain the result. ∎

Corollary 2.4 improves the reiteration theorem known before for Fernandez spaces (see [8], Thm. 4.5, and [9], Prop. 3.3).

**3. Wolff theorem.** Let us start by analyzing the Wolff theorem for the classical real method (see [13], [11], and [3], 4.5/C).

Assume that $A_0, A_1, X_0, X_1$ are four Banach spaces continuously embedded in a common Hausdorff topological vector space. Let $1 \le p, q \le \infty$, $0 < \theta < \eta < 1$ and $0 < \lambda$, $\mu < 1$ with $\theta = \lambda\eta$ and $\eta = (1 - \mu)\theta + \mu$. Wolff proved that if

$$X_0 = (A_0, X_1)_{\lambda, p} \quad \text{and} \quad X_1 = (X_0, A_1)_{\mu, q}$$

then

$$X_0 = (A_0, A_1)_{\theta, p} \quad \text{and} \quad X_1 = (A_0, A_1)_{\eta, q}.$$

The figure below illustrates the result:

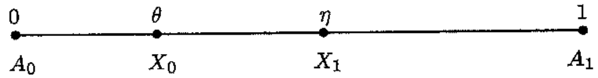


Fig. 3.0

Wolff's theorem can also be formulated in terms of mappings of type $([0,1])$. Indeed, let us associate with $X_0 = (A_0, X_1)_{\lambda, p}$ the mapping $Rt = \eta t$ which satisfies

$$R0 = 0, \quad R1 = \eta, \quad R\lambda = \theta,$$

while with $X_1 = (X_0, A_1)_{\mu, q}$ we associate the transformation $St = \theta + (1 - \theta)t$ satisfying

$$S0 = \theta, \quad S1 = 1, \quad S\mu = \eta.$$

Then the conclusion of the theorem reads

$$X_0 = (A_0, A_1)_{R\lambda, p} \quad \text{and} \quad X_1 = (A_0, A_1)_{S\mu, q}.$$

Next we extend this result to the methods defined by means of polygons. In what follows, we take $N$ equal to 3 or 4, and $I_1$, $I_2$ stand for two sets of positive integers such that $I_1 \cup I_2 = \{1, \ldots, N\}$ and $I_1 \cap I_2 = \emptyset$. Moreover, if $\overline{A}$ is an $N$-tuple and $X_1, \ldots, X_N$ are intermediate spaces with respect to $\overline{A}$, we write $\overline{A\text{-}s\text{-}X}$ for the $N$-tuple $\{Z_j\}$ where

$$Z_j = \begin{cases} A_j & \text{if } j \in I_s, \\ X_j & \text{otherwise.} \end{cases}$$

The assumption we shall require on mappings $R_1, R_2$ will explain the restriction $N = 3, 4$.

THEOREM 3.1. *Let $\Pi = \overline{P_1 \ldots P_N}$ be a convex polygon with $P_j = (x_j, y_j)$. Assume that $\{Q_j\}_{j=1}^N$ and $\{W_j\}_{j=1}^N$ are two sets of $N$ interior points of $\Pi$, such that there are two mappings $R_1$ and $R_2$ of type $(\Pi)$ satisfying*

$$R_s P_j = \begin{cases} P_j & \text{for } j \in I_s, \\ Q_j & \text{otherwise,} \end{cases}$$

$$Q_j = R_s W_j \quad \text{for } j \in I_s \ (s = 1, 2).$$

*Let $\overline{A} = \{A_1, \ldots, A_N\}$ be an $N$-tuple, let $X_1, \ldots, X_N$ be $N$ intermediate spaces with respect to $\overline{A}$ and assume that $\mathcal{E}(\overline{A}, \Pi)$, $\mathcal{E}(\overline{A\text{-}1\text{-}X}, \Pi)$ and $\mathcal{E}(\overline{A\text{-}2\text{-}X}, \Pi)$ are satisfied. If for $1 \le q_1, \ldots, q_N \le \infty$,*

(i) $$\overline{A\text{-}s\text{-}X}_{W_j, q_j} = X_j \quad \text{for } j \in I_s \ (s = 1, 2)$$

*then $\overline{A}_{Q_j, q_j} = X_j$ for $j = 1, \ldots, N$.*

Proof. Suppose $j \in I_s$. In order to prove that $\overline{A}_{Q_j, q_j} = X_j$ we take the equality $\overline{A\text{-}s\text{-}X}_{W_j, q_j} = X_j$ and apply the reiteration theorem. Indeed, if we show that for $j = 1, \ldots, N$ we have

(4) $$\overline{A}_{Q_j, 1} \hookrightarrow X_j$$

and

(5) $$X_j \hookrightarrow \overline{A}_{Q_j, \infty},$$

then according to Theorem 2.1 and the fact that $\mathcal{E}(\overline{A\text{-}s\text{-}X}, \Pi)$ is satisfied, we get

$$X_j = \overline{A\text{-}s\text{-}X}_{W_j, q_j} = \overline{A}_{R_s W_j, q_j} = \overline{A}_{Q_j, q_j}.$$

To establish (4) and (5), we shall use the techniques developed in [11] and [5]. In particular, we shall need the Aronszajn–Gagliardo description [1] of the $K$- and the $J$-methods.

Let $\overline{\ell}_1 = \{\ell_1(2^{-mx_j - ny_j})\}_{j=1}^N$, let $(\alpha, \beta) \in \operatorname{Int} \Pi$ and $1 \le q \le \infty$. Given any $N$-tuple $\overline{B}$, the space $\overline{B}_{(\alpha,\beta),q;J}$ can be characterized as the collection of all $b \in \Sigma(\overline{B})$ such that

$$b = \sum_{k=1}^{\infty} T_k x_k$$



where $T_k \in \mathcal{L}(\overline{\ell}_1, \overline{B})$, $x_k \in \ell_q(2^{-m\alpha - n\beta})$ and

$$\sum_{k=1}^{\infty} \|T_k\|_{\overline{\ell}_1, \overline{B}} \|x_k\|_{\ell_q(2^{-m\alpha-n\beta})} < \infty.$$

Moreover,

$$\|b\|_G = \inf\Big\{ \sum_{k=1}^{\infty} \|T_k\|_{\overline{\ell}_1, \overline{B}} \|x_k\|_{\ell_q(2^{-m\alpha-n\beta})} : b = \sum_{k=1}^{\infty} T_k x_k \Big\} = \|b\|_{(\alpha,\beta),q;J}.$$

That is to say, $\overline{B}_{(\alpha,\beta),q;J}$ coincides with the space obtained by applying to $\overline{B}$ the $N$-dimensional version of the Aronszajn–Gagliardo minimal functor defined by the $N$-tuple $\overline{\ell}_1$ and the intermediate space $\ell_q(2^{-m\alpha-n\beta})$,

$$G[\overline{\ell}_1; \ell_q(2^{-m\alpha-n\beta})](\overline{B}) = \overline{B}_{(\alpha,\beta),q;J}$$

(see [6], Thm. 3.3).

Let us introduce the inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^2$ and write $-mx_j - ny_j$ as $-\langle P_j, (m,n) \rangle$. We also put

$$\overline{\ell}_{1,s} = \{\ell_1(2^{-\langle R_s P_j, (m,n) \rangle})\} \quad (s = 1, 2).$$

Since $\overline{B}_{(\alpha,\beta),1;J} \hookrightarrow \overline{B}_{(\alpha,\beta),q;J}$, it follows from assumption (i) that

(i′) $$G[\overline{\ell}_{1,s}; \ell_1(2^{-\langle Q_j, (m,n)\rangle})](\overline{A\text{-}s\text{-}X}) \hookrightarrow X_j \quad \text{for } j \in I_s \ (s = 1, 2).$$

Next we show that (i′) implies condition (4), which can be equivalently written as

(4′) $$G[\overline{\ell}_1; \ell_1(2^{-\langle Q_j, (m,n)\rangle})](\overline{A}) \hookrightarrow X_j \quad \text{for } j = 1, \ldots, N.$$

If $j \in I_s$ we see by Lemmata 1.3 and 1.4 that $\ell_1(2^{-\langle Q_j, (m,n)\rangle})$ is obtained from $\overline{\ell}_{1,s}$ by applying the $J$-method with parameters $W_j$, 1. On the other hand, since $\mathcal{E}(\overline{A\text{-}s\text{-}X}, \Pi)$ is satisfied, the $K$-method with parameters $W_j$, 1 acting on $\overline{A\text{-}s\text{-}X}$ yields

$$\overline{A\text{-}s\text{-}X}_{W_j, 1; K} = \overline{A\text{-}s\text{-}X}_{W_j, 1} \hookrightarrow \overline{A\text{-}s\text{-}X}_{W_j, q_j} = X_j.$$

Hence, according to [4], Thm. 4.3, we deduce that there are two constants $\gamma_j > 0$ and $0 < \tau_j < 1$ (depending only on $\Pi$ and $W_j$) such that for any operator $T \in \mathcal{L}(\overline{\ell}_{1,s}, \overline{A\text{-}s\text{-}X})$ we have

(6) $$\|T\|_{\ell_1(2^{-\langle Q_j, (m,n)\rangle}), X_j} \le \gamma_j m^{\tau_j} M^{1-\tau_j}$$

where $m = \min\{\delta_j : j = 1, \ldots, N\}$, $M = \max\{\delta_j : j = 1, \ldots, N\}$ and

$$\delta_j = \begin{cases} \|T\|_{\ell_1(2^{-\langle P_j, (m,n)\rangle}), A_j} & \text{for } j \in I_s, \\ \|T\|_{\ell_1(2^{-\langle Q_j, (m,n)\rangle}), X_j} & \text{otherwise.} \end{cases}$$

For $j=1,\dots,N$ consider the function $\Re_j(\xi_1,\dots,\xi_N)$ of $N$ nonnegative real arguments defined by

$$\Re_j(\xi_1,\dots,\xi_N)=\gamma_j\big(\min_{1\le k\le N}\{\xi_k\}\big)^{\tau_j}\big(\max_{1\le k\le N}\{\xi_k\}\big)^{1-\tau_j}.$$

In the terminology of [5], the inequality (6) says that for $j\in I_s$ the spaces $\ell_1(2^{-\langle Q_j,(m,n)\rangle})$ and $X_j$ are $\Re_j$-interpolation spaces with respect to the $N$-tuples $\overline{\ell}_{1,s}$ and $\overline{A\text{-}s\text{-}X}$ $(s=1,2)$. The functions $\Re_j$ satisfy conditions (a) to (d) of [5]. Moreover, given any positive real number $t$, if we put

$$\overline{t}_s=(\xi_j)\quad\text{where}\quad \xi_j=\begin{cases}t & \text{if } j\in I_s,\\ 1 & \text{otherwise,}\end{cases}$$

we have

(e′) $$\lim_{t\to0}\Re_j(\overline{t}_s)=0\quad\text{for } j\in I_s\ (s=1,2),$$

(compare with condition (e) in [5]). Hence the argument in [5], Lemma 1.3, is still valid and therefore there exists a constant $C<\infty$ such that for every $T\in\mathcal{L}(\Sigma(\overline{\ell}_1),\Delta(\overline{A}))$,

(7) $$\max_{1\le j\le N}\{\|T\|_{\ell_1(2^{-\langle Q_j,(m,n)\rangle}),X_j}\}\le C\|T\|_{\overline{\ell}_1,\overline{A}}.$$

Now we are ready to establish (4′). We first show that if $T\in\mathcal{L}(\overline{\ell}_1,\overline{A})$ and $x\in\ell_1(2^{-\langle Q_j,(m,n)\rangle})$, then $Tx\in X_j$.

For $k$ a positive integer, let $V=V_k$ be the operator associating with each scalar sequence $\xi=(\xi_{m,n})$ the sequence $V\xi=(\lambda_{m,n})$ where

$$\lambda_{m,n}=\begin{cases}\xi_{m,n} & \text{if } \max\{|m|,|n|\}\le k,\\ 0 & \text{otherwise.}\end{cases}$$

Given $x\in\ell_1(2^{-\langle Q_j,(m,n)\rangle})$, choose $V$ such that

$$\|x-Vx\|_{\ell_1(2^{-\langle Q_j,(m,n)\rangle})}\le\tfrac12\|x\|_{\ell_1(2^{-\langle Q_j,(m,n)\rangle})}.$$

Clearly $V\in\mathcal{L}(\Sigma(\overline{\ell}_1),\Delta(\overline{\ell}_1))$ and $\|V\|_{\overline{\ell}_1,\overline{\ell}_1}=1$. Hence it follows from (7) that

$$\|TV\|_{\ell_1(2^{-\langle Q_j,(m,n)\rangle}),X_j}\le C\|T\|_{\overline{\ell}_1,\overline{A}}.$$

Put $x_1=Vx$ and $x_1'=x-Vx$; then $x=x_1+x_1'$ with

$$\|x_1'\|_{\ell_1(2^{-\langle Q_j,(m,n)\rangle})}\le\tfrac12\|x\|_{\ell_1(2^{-\langle Q_j,(m,n)\rangle})},\quad\text{and}$$
$$\|Tx_1\|_{X_j}\le C\|T\|_{\overline{\ell}_1,\overline{A}}\|x\|_{\ell_1(2^{-\langle Q_j,(m,n)\rangle})}.$$

Next we repeat the same procedure with $x_1'$ instead of $x$. Proceeding inductively we find a sequence $(x_k)\subset\Delta(\overline{\ell}_1)$ such that $x=\sum_{k=1}^\infty x_k$ (in $\ell_1(2^{-\langle Q_j,(m,n)\rangle})$) and

$$\|Tx_k\|_{X_j}\le\frac{1}{2^{k-1}}C\|T\|_{\overline{\ell}_1,\overline{A}}\|x\|_{\ell_1(2^{-\langle Q_j,(m,n)\rangle})}.$$



Thus $Tx\in X_j$ and

(9) $$\|Tx\|_{X_j}\le\sum_{k=1}^\infty\|Tx_k\|_{X_j}\le 2C\|T\|_{\overline{\ell}_1,\overline{A}}\|x\|_{\ell_1(2^{-\langle Q_j,(m,n)\rangle})}.$$

Take now any $a\in G[\overline{\ell}_1;\ell_1(2^{-\langle Q_j,(m,n)\rangle})](\overline{A})$. Given $\varepsilon>0$ arbitrarily, find a representation

$$a=\sum_{k=1}^\infty T_kx_k\quad\text{with}\quad T_k\in\mathcal{L}(\overline{\ell}_1,\overline{A}),\ x_k\in\ell_1(2^{-\langle Q_j,(m,n)\rangle})$$

and

$$\sum_{k=1}^\infty\|T_k\|_{\overline{\ell}_1,\overline{A}}\|x_k\|_{\ell_1(2^{-\langle Q_j,(m,n)\rangle})}\le\|a\|_G+\varepsilon.$$

By (9) we have $a\in X_j$ with

$$\|a\|_{X_j}\le\sum_{k=1}^\infty\|T_kx_k\|_{X_j}$$
$$\le 2C\sum_{k=1}^\infty\|T_k\|_{\overline{\ell}_1,\overline{A}}\|x_k\|_{\ell_1(2^{-\langle Q_j,(m,n)\rangle})}\le 2C(\|a\|_G+\varepsilon).$$

Passing to the limit as $\varepsilon\to0$, we obtain (4′).

It remains to prove (5). Let us start by recalling that given any $N$-tuple $\overline{B}$, the space $\overline{B}_{(\alpha,\beta),q;K}$ can be described as the set of all elements $b\in\Sigma(\overline{B})$ such that $Tb\in\ell_q(2^{-m\alpha-n\beta})$ for all operators $T\in\mathcal{L}(\overline{B},\overline{\ell}_\infty)$, where $\overline{\ell}_\infty=\{\ell_\infty(2^{-\langle P_j,(m,n)\rangle})\}$. Moreover,

$$\|b\|_H=\sup\{\|Tb\|_{\ell_q(2^{-m\alpha-n\beta})}:\|T\|_{\overline{B},\overline{\ell}_\infty}\le1\}$$

coincides with $\|b\|_{(\alpha,\beta),q;K}$. In other words, the $K$-method defined by the polygon $\Pi$ and the parameters $(\alpha,\beta)\in\operatorname{Int}\Pi$, $q\in[1,\infty]$ agrees with the $N$-dimensional version of the Aronszajn–Gagliardo maximal functor defined by means of the $N$-tuple $\overline{\ell}_\infty$ and the intermediate space $\ell_q(2^{-m\alpha-n\beta})$,

$$H[\overline{\ell}_\infty;\ell_q(2^{-m\alpha-n\beta})](\overline{B})=\overline{B}_{(\alpha,\beta),q;K}\quad\text{(see [6], Thm. 3.3).}$$

Put

$$\overline{\ell}_{\infty,s}=\{\ell_\infty(2^{-\langle R_sP_j,(m,n)\rangle})\}\quad(s=1,2).$$

Assumption (i) implies that

(i″) $$X_j\hookrightarrow H[\overline{\ell}_{\infty,s};\ell_\infty(2^{-\langle Q_j,(m,n)\rangle})](\overline{A\text{-}s\text{-}X})\quad\text{for } j\in I_s'\ (s=1,2).$$

Our task is to show that (5) follows from (i″). Observe that (5) written in terms of maximal functors reads

$$(5') \qquad X_j \hookrightarrow H[\overline{\ell}_\infty; \ell_\infty(2^{-\langle Q_j,(m,n)\rangle})](\overline{A}).$$

Reasoning as we did before for deriving (6), one can check that if $j \in I_s$ then $X_j$ and $\ell_\infty(2^{-\langle Q_j,(m,n)\rangle})$ are $\Re_j$-interpolation spaces with respect to the $N$-tuples $\overline{A\text{-}s\text{-}X}$ and $\overline{\ell}_{\infty,s}$. From this it follows that there is a constant $C < \infty$ such that for any $T \in \mathcal{L}(\Sigma(\overline{A}), \Delta(\overline{\ell}_\infty))$ we have

$$(7') \qquad \max_{1\le j\le N}\{\|T\|_{X_j,\ell_\infty(2^{-\langle Q_j,(m,n)\rangle})}\} \le C\|T\|_{\overline{A},\overline{\ell}_\infty}.$$

Next we establish (5′). Take any $b \in X_j$ and any $T \in \mathcal{L}(\overline{A}, \overline{\ell}_\infty)$. We must show that $Tb \in \ell_\infty(2^{-\langle Q_j,(m,n)\rangle})$, which is equivalent to checking that

$$\sup_{k\in\mathbb{N}}\{\|V_k Tb\|_{\ell_\infty(2^{-\langle Q_j,(m,n)\rangle})}\} < \infty,$$

where $V_k$ is the operator defined in (8).

Observe that $V_k \in \mathcal{L}(\Sigma(\overline{\ell}_\infty), \Delta(\overline{\ell}_\infty))$ and that $\|V_k\|_{\overline{\ell}_\infty,\overline{\ell}_\infty} = 1$. So it follows from (7′) that

$$\sup_{k\in\mathbb{N}}\{\|V_k Tb\|_{\ell_\infty(2^{-\langle Q_j,(m,n)\rangle})}\} \le C\|T\|_{\overline{A},\overline{\ell}_\infty}\|b\|_{X_j}.$$

Consequently, $Tb \in \ell_\infty(2^{-\langle Q_j,(m,n)\rangle})$ and

$$\|b\|_H = \sup\{\|Tb\|_{\ell_\infty(2^{-\langle Q_j,(m,n)\rangle})} : \|T\|_{\overline{A},\overline{\ell}_\infty} \le 1\} \le C\|b\|_{X_j}.$$

The proof is complete. ■

Writing down Theorem 3.1 for Sparr spaces we recover (the case of three spaces of) [5], Cor. 3.5. We denote by $\overline{e}_k$ the vector in $\mathbb{R}^3$ with all coordinates zero but the $k$th which is one.

COROLLARY 3.2. *Let* $\overline{\lambda}_j = (\lambda_{jr})$ *and* $\overline{\theta}_j = (\theta_{jr})$ *belong to* $\mathcal{H}$ *for* $j = 1, 2, 3$, *and assume that there is some positive integer* $k$ $(1 \le k \le 3)$ *such that*

$$(10) \qquad \theta_{jr} = \lambda_{jr} + \sum_{s=k+1}^{3} \lambda_{js}\theta_{sr} \qquad (1 \le j \le k,\ 1 \le r \le k),$$

$$(11) \qquad \theta_{jr} = \sum_{s=k+1}^{3} \lambda_{js}\theta_{sr} \qquad (1 \le j \le k,\ k+1 \le r \le 3),$$

$$(12) \qquad \theta_{jr} = \sum_{s=1}^{k} \lambda_{js}\theta_{sr} \qquad (k+1 \le j \le 3,\ 1 \le r \le k),$$

$$(13) \qquad \theta_{jr} = \lambda_{jr} + \sum_{s=1}^{k} \lambda_{js}\theta_{sr} \qquad (k+1 \le j \le 3,\ k+1 \le r \le 3),$$



*and*

$$(14) \qquad \mathbb{R}^3 \textit{ is spanned by } \{\overline{e}_1, \ldots, \overline{e}_k, \overline{\theta}_{k+1}, \ldots, \overline{\theta}_3\} \textit{ and also by } \{\overline{\theta}_1, \ldots, \overline{\theta}_k, \overline{e}_{k+1}, \ldots, \overline{e}_3\}.$$

*Suppose further that* $X_1, X_2, X_3$ *are intermediate spaces with respect to the 3-tuple* $\overline{A}$, *and that conditions* $\mathcal{F}(\overline{A})$, $\mathcal{F}(\overline{A\text{-}1\text{-}X})$ *and* $\mathcal{F}(\overline{A\text{-}2\text{-}X})$ *are satisfied, where* $I_1 = \{1, \ldots, k\}$ *and* $I_2 = \{k+1, \ldots, 3\}$. *If*

$$\overline{A\text{-}s\text{-}X}^{\mathcal{S}}_{\overline{\lambda}_j,q_j} = X_j \quad \textit{for } j \in I_s \ (s = 1, 2)$$

*where* $1 \le q_1, q_2, q_3 \le \infty$, *then* $\overline{A}^{\mathcal{S}}_{\overline{\theta}_j,q_j} = X_j$ *for* $j = 1, 2, 3$.

Proof. Suppose $k = 2$. The situation is illustrated by the following figure:

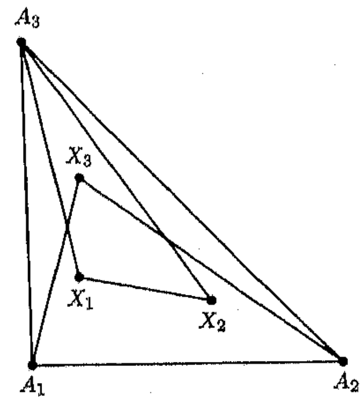


Fig. 3.1

Recall that $\overline{A}^{\mathcal{S}}_{\overline{\theta},q} = \overline{A}_{(\theta_2,\theta_3),q}$ where $\Pi$ is the simplex $\{(0,0), (1,0), (0,1)\}$. Define mappings $R_1$ and $R_2$ by

$$R_1\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & \theta_{32} \\ 0 & \theta_{33} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

$$R_2\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \theta_{12} \\ \theta_{13} \end{pmatrix} + \begin{pmatrix} \theta_{22}-\theta_{12} & -\theta_{12} \\ \theta_{23}-\theta_{13} & 1-\theta_{13} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

Assumption (14) implies that $R_1$ and $R_2$ are of type $(\Pi)$. Clearly

$$R_1\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad R_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad R_1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \theta_{32} \\ \theta_{33} \end{pmatrix}.$$

Moreover, by (10) and (11), we have for $j = 1, 2$

$$R_1\begin{pmatrix} \lambda_{j2} \\ \lambda_{j3} \end{pmatrix} = \begin{pmatrix} \lambda_{j2} + \lambda_{j3}\theta_{32} \\ \lambda_{j3}\theta_{33} \end{pmatrix} = \begin{pmatrix} \theta_{j2} \\ \theta_{j3} \end{pmatrix}.$$

For the map $R_2$ we get, using (12) and (13),

$$R_2\begin{pmatrix}\lambda_{32}\\ \lambda_{33}\end{pmatrix}=\begin{pmatrix}\theta_{12}+\lambda_{32}(\theta_{22}-\theta_{12})-\lambda_{33}\theta_{12}\\ \theta_{13}+\lambda_{32}(\theta_{23}-\theta_{13})+\lambda_{33}(1-\theta_{13})\end{pmatrix}$$
$$=\begin{pmatrix}\lambda_{31}\theta_{12}+\lambda_{32}\theta_{22}\\ \lambda_{33}+\lambda_{31}\theta_{13}+\lambda_{32}\theta_{22}\end{pmatrix}=\begin{pmatrix}\theta_{32}\\ \theta_{33}\end{pmatrix}.$$

Moreover,

$$R_2\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}\theta_{12}\\ \theta_{13}\end{pmatrix},\quad R_2\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\theta_{22}\\ \theta_{23}\end{pmatrix},\quad R_2\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}.$$

Hence, applying Theorem 3.1 with $Q_j=(\theta_{j2},\theta_{j3})$ and $W_j=(\lambda_{j2},\lambda_{j3})$, $j=1,2,3$, we conclude that $X_j=\overline{A}_{Q_j,q_j}=\overline{A}^S_{\overline{\theta}_j,q_j}$ $(1\le j\le 3)$.

If $k=1$ the proof can be carried out in the same way. Finally, the case $k=3$ is trivial. ■

We close the paper with another application, this time to Fernandez spaces. We write $\overline{1}=(1,1)$.

COROLLARY 3.3. *Let $\overline{\theta}_j=(\theta_{jr})$ and $\overline{\lambda}_j=(\lambda_{jr})$ belong to $\mathcal{D}$ for $j=1,2$, and assume that*

$$\theta_{11}+\theta_{12}\neq 1, \tag{15}$$
$$\theta_{21}\neq\theta_{22}, \tag{16}$$
$$\overline{\theta}_1=\lambda_{11}\overline{\theta}_2+\lambda_{12}(\overline{1}-\overline{\theta}_2), \tag{17}$$
$$\overline{\theta}_2=\overline{\lambda}_2+(1-\lambda_{21}-\lambda_{22})\overline{\theta}_1. \tag{18}$$

*Put*

$$\overline{\theta}_3=\overline{1}-\overline{\theta}_2,\quad \overline{\theta}_4=\overline{1}-\overline{\theta}_1,\quad \overline{\lambda}_3=\overline{1}-\overline{\lambda}_2,\quad \overline{\lambda}_4=\overline{1}-\overline{\lambda}_1.$$

*Assume further that $X_1,\ldots,X_4$ are four intermediate spaces with respect to the 4-tuple $\overline{A}$, and that $\mathcal{E}(\overline{A},\Pi)$, $\mathcal{E}(\{A_1,X_2,X_3,A_4\},\Pi)$ and $\mathcal{E}(\{X_1,A_2,A_3,X_4\},\Pi)$ are satisfied where $\Pi=\{(0,0),(1,0),(0,1),(1,1)\}$. If*

$$\{A_1,X_2,X_3,A_4\}^F_{\overline{\lambda}_j,q_j}=X_j\quad \textit{for } j=1,4,$$
$$\{X_1,A_2,A_3,X_4\}^F_{\overline{\lambda}_j,q_j}=X_j\quad \textit{for } j=2,3$$

*where $1\le q_1,\ldots,q_4\le\infty$, then $\overline{A}^F_{\overline{\theta}_j,q_j}=X_j$ for $j=1,\ldots,4$.*

Proof. Consider the mappings

$$R_1\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}\theta_{21}&1-\theta_{21}\\ \theta_{22}&1-\theta_{22}\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix},$$
$$R_2\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}\theta_{11}\\ \theta_{12}\end{pmatrix}+\begin{pmatrix}1-\theta_{11}&-\theta_{11}\\ -\theta_{12}&1-\theta_{12}\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}.$$



It follows from (15) and (16) that $R_1$ and $R_2$ are of type ($\Pi$). Clearly

$$R_1\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},\quad R_1\begin{pmatrix}1\\0\end{pmatrix}=\overline{\theta}_2,\quad R_1\begin{pmatrix}0\\1\end{pmatrix}=\overline{\theta}_3,\quad R_1\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}.$$

Moreover, (17) implies that $R_1\overline{\lambda}_j=\overline{\theta}_j$ for $j=1,4$.

For $R_2$ we have

$$R_2\begin{pmatrix}0\\0\end{pmatrix}=\overline{\theta}_1,\quad R_2\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix},\quad R_2\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix},\quad R_2\begin{pmatrix}1\\1\end{pmatrix}=\overline{\theta}_4.$$

In addition, by (18), $R_2\overline{\lambda}_j=\overline{\theta}_j$ for $j=2,3$. Consequently, using Theorem 3.1 with $I_1=\{1,4\}$, $I_2=\{2,3\}$, $Q_j=\overline{\theta}_j$ and $W_j=\overline{\lambda}_j$, we conclude that $\overline{A}^F_{\overline{\theta}_j,q_j}=X_j$ for $j=1,\ldots,4$. ■

The following figure illustrates the corollary:

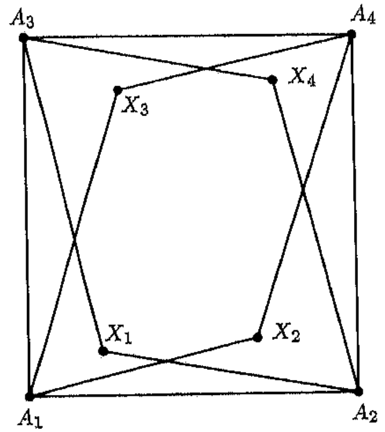


Fig. 3.2

Corollary 3.3 settles a question left open in [5].

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# Fréchet spaces of continuous vector-valued functions: Complementability in dual Fréchet spaces and injectivity

by

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**Abstract.** Fréchet spaces of strongly, weakly and weak*-continuous Fréchet space valued functions are considered. Complete solutions are given to the problems of their injectivity or embeddability as complemented subspaces in dual Fréchet spaces.

## 1. Introduction.

There is a famous conjecture that every injective Banach (or Fréchet) space is isomorphic to the space of scalar-valued continuous functions over an extremally disconnected topological space [19, p. 269]. It might seem that there is a chance to find some essentially new examples of injective Fréchet spaces by considering the spaces of vector-valued continuous functions. Unfortunately, as should be clear from the results of the present paper, this is not so, at least in the case of spaces of strongly or weakly continuous functions. (The situation is not so clear, however, for the spaces of weak*-continuous functions.) For Banach spaces of strongly continuous vector functions this was observed earlier by Cembranos [6].

We now briefly describe the contents of our paper.

Let $X$ be a Hausdorff topological space that is locally compact and *hemicompact* (i.e., has a fundamental sequence of compact sets); we will call such spaces *LCH-spaces* for short. Let $E$ be a Fréchet space or, when

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