STUDIA MATHEMATICA 102 (3) (1992)

# On the uniform convergence and $L^1$-convergence of double Walsh–Fourier series

by

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**Abstract.** In 1970 C. W. Onneweer formulated a sufficient condition for a periodic $W$-continuous function to have a Walsh–Fourier series which converges uniformly to the function. In this paper we extend his results from single to double Walsh–Fourier series in a more general setting. We study the convergence of rectangular partial sums in $L^p$-norm for some $1 \le p \le \infty$ over the unit square $[0,1) \times [0,1)$. In case $p = \infty$, by $L^p$ we mean $C_W$, the collection of uniformly $W$-continuous functions $f(x,y)$, endowed with the supremum norm. As special cases, we obtain the extensions of the Dini–Lipschitz test and the Dirichlet–Jordan test for double Walsh–Fourier series.

**1. Introduction.** We consider the Walsh orthonormal system $\{w_j(x) : j \ge 0\}$ defined on the unit interval $I := [0,1)$ in the Paley enumeration (see [8]). To be more specific, let

$$r_0(x) := \begin{cases} 1 & \text{if } x \in [0, 2^{-1}), \\ -1 & \text{if } x \in [2^{-1}, 1), \end{cases} \qquad r_0(x+1) := r_0(x)\,,$$
$$r_j(x) := r_0(2^j x), \qquad j \ge 1 \text{ and } x \in I\,,$$

be the well-known Rademacher functions. For $j = 0$ set $w_0(x) = 1$, and if

$$j := \sum_{i=0}^{\infty} j_i 2^i, \qquad j_i = 0 \text{ or } 1\,,$$

1991 *Mathematics Subject Classification*: Primary 41A50; Secondary 42C10.

*Key words and phrases*: Walsh–Paley system, $W$-continuity, moduli of continuity and smoothness, bounded variation in the sense of Hardy and Krause, generalized bounded variation, complementary functions in the sense of W. H. Young, rectangular partial sum, Dirichlet kernel, convergence in $L^p$-norm, uniform convergence, Salem's test, Dini–Lipschitz test, Dirichlet–Jordan test.

This research was partially supported by the Hungarian National Foundation for Scientific Research under Grant #234.

is the dyadic representation of an integer $j \ge 1$, then set

$$w_j(x) := \prod_{i=0}^{\infty} [r_i(x)]^{j_i}.$$

Given $m \ge 0$ and $0 \le j < 2^m$, we set

$$I_m(j) := [j2^{-m}, (j+1)2^{-m}).$$

It is plain that $w_M(x)$ is constant on $I_m(j)$ for $2^m \le M < 2^{m+1}$.

We consider the double system $\{w_j(x)w_k(y) : j,k \ge 0\}$ on the unit square $I^2 := [0,1) \times [0,1)$. Given a function $f \in L^1(I^2)$, we form its double Walsh–Fourier series (abbreviated as WFS)

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk} w_j(x) w_k(y) \tag{1.1}$$

with

$$a_{jk} := \int_0^1 \int_0^1 f(u,v) w_j(u) w_k(v)\, du\, dv.$$

The rectangular partial sums of series (1.1) are defined by

$$S_{MN}(f;x,y) := \sum_{j=0}^{M-1}\sum_{k=0}^{N-1} a_{jk} w_j(x) w_k(y), \qquad M,N \ge 1.$$

As is well-known,

$$S_{MN}(f;x,y) = \int_0^1 \int_0^1 f(x\dot{+}u, y\dot{+}v) D_M(u) D_N(v)\, du\, dv \tag{1.2}$$

where

$$D_M(u) := \sum_{j=0}^{M-1} w_j(u)$$

is the Dirichlet kernel. Here $\dot{+}$ denotes dyadic addition. For this and further notations, definitions, and properties of WFS we refer to [10].

We will study approximation by $S_{MN}(f) := S_{MN}(f;x,y)$ to functions $f \in L^p := L^p(I^2), 1 \le p < \infty$, and $C_W := C_W(I^2)$ in the norm of $L^p$ and $C_W$, respectively. We remind the reader that $C_W(I^2)$ is the collection of functions $f : I^2 \to \mathbb{R}$ that are uniformly continuous from the dyadic topology of $I^2$ to the usual topology of $\mathbb{R}$, or for short: uniformly $W$-continuous. It is known that if the periodic extension of $f$ from $I^2$ to $\mathbb{R}^2$ with period 1 in each variable is classically continuous, then $f$ is also uniformly $W$-continuous on $I^2$. But the converse is not true in general (cf. [10, pp. 9–11]).



For brevity of notation, we write $L^\infty$ instead of $C_W$ and set

$$\|f\|_p := \left\{ \int_0^1 \int_0^1 |f(x,y)|^p\, dx\, dy \right\}^{1/p}, \qquad 1 \le p < \infty,$$

$$\|f\|_\infty := \sup\{|f(x,y)| : x,y \in I\}.$$

From the results of [10, pp. 142 and 156–158] it follows that $L^p$ is the closure of the double Walsh polynomials (i.e., the finite linear combinations of the Walsh functions $w_j(x)w_k(y)$ with $j,k \ge 0$) under the norm $\|\cdot\|_p$, $1 \le p \le \infty$. In particular, $C_W$ is the uniform closure of the double Walsh polynomials.

## 2. Preliminaries.

We remind the reader that the (*total*) *modulus of continuity* of a function $f \in L^p$ in $L^p$-norm, $1 \le p \le \infty$, is defined by

$$\omega_1(f;\delta_1,\delta_2)_p := \sup\{\|f(x\dot{+}u, y\dot{+}v) - f(x,y)\|_p : 0 \le u < \delta_1 \text{ and } 0 \le v < \delta_2\},$$

while the *partial moduli of continuity* are defined by

$$\omega_{1,x}(f;\delta_1)_p := \omega_1(f;\delta_1,0)_p \quad \text{and} \quad \omega_{1,y}(f;\delta_2)_p := \omega_1(f;0,\delta_2)_p$$

for $\delta_1, \delta_2 \ge 0$. By the Banach–Steinhaus theorem, for any $f \in L^p$ we have

$$\lim_{\delta_1,\delta_2 \to 0} \omega_1(f;\delta_1,\delta_2)_p = 0, \qquad 1 \le p \le \infty. \tag{2.1}$$

We also use the notion of the (*total*) *modulus of smoothness* of a function $f \in L^p$ in $L^p$-norm, $1 \le p \le \infty$, defined by

$$\omega_2(f;\delta_1,\delta_2)_p := \sup\{\|f(x\dot{+}u, y\dot{+}v) - f(x\dot{+}u, y) - f(x, y\dot{+}v) + f(x,y)\|_p : 0 \le u < \delta_1 \text{ and } 0 \le v < \delta_2\}.$$

Obviously, these moduli are nondecreasing functions in $\delta_1$ and $\delta_2$, respectively, and

$$\max\{\omega_{1,x}(f;\delta_1)_p, \omega_{1,y}(f;\delta_2)_p\} \le \omega_1(f;\delta_1,\delta_2)_p \le \omega_{1,x}(f;\delta_1)_p + \omega_{1,y}(f;\delta_2)_p,$$

$$\omega_2(f;\delta_1,\delta_2)_p \le \omega_{1,x}(f;\delta_1)_p + \omega_{1,y}(f;\delta_2)_p.$$

We need the notion of bounded variation in the sense of Hardy [3] and Krause. (See the discussion in [5, §254].) To go into details, given two partitions

$$\mathcal{D}_1 : 0 = x_0 < x_1 < \ldots < x_m = 1, \qquad \mathcal{D}_2 : 0 = y_0 < y_1 < \ldots < y_n = 1, \tag{2.2}$$

we form a rectangular grid $\mathcal{D} := \mathcal{D}_1 \times \mathcal{D}_2$ on $I^2$ and set

$$\mathcal{D}(f) := \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} |f(x_j, y_k) - f(x_{j+1}, y_k) - f(x_j, y_{k+1}) + f(x_{j+1}, y_{k+1})|$$

where $f : I^2 \to \mathbb{R}$ is an arbitrary function. We define the (*total*) *variation* of $f$ on $I^2$ by

$$\operatorname{var}(f; I^2) := \sup\{\mathcal{D}(f) : \mathcal{D} \text{ is any rectangular grid on } I^2\}$$

and say that $f$ is *of bounded variation* (according to Hardy and Krause) if each of the numbers

$$\operatorname{var}(f; I^2), \quad \operatorname{var}(f(\cdot, 0); I), \quad \operatorname{var}(f(0, \cdot); I)$$

is finite. Here the last two quantities are the ordinary variations of the single variable functions $f(x, 0)$ and $f(0, y)$, respectively. For instance,

$$\operatorname{var}(f(\cdot, 0); I) := \sup\{\mathcal{D}_1(f(\cdot, 0)) : \mathcal{D}_1 \text{ is any partition of } I\},$$

$$\mathcal{D}_1(f(\cdot, 0)) := \sum_{j=0}^{m-1} |f(x_j, 0) - f(x_{j+1}, 0)|,$$

and $\operatorname{var}(f(0, \cdot); I)$ is defined analogously.

We denote by $\mathrm{BV}(I^2)$ the collection of all functions $f : I^2 \to \mathbb{R}$ of bounded variation. It is readily verified that, with the norm given by

$$|||f||| := |f(0,0)| + \operatorname{var}(f(\cdot, 0); I) + \operatorname{var}(f(0, \cdot); I) + \operatorname{var}(f; I^2),$$

$\mathrm{BV}(I^2)$ is a Banach space.

A few remarks about the above definition are in order. Let $f \in \mathrm{BV}(I^2)$. Then it is easily checked that $f$ is bounded on $I^2$, and satisfies $\|f\|_\infty \le |||f|||$. Also, for each fixed $x, y \in I$, the marginal functions $f(\cdot, y)$ and $f(x, \cdot)$ are of bounded variation on $I$ with

$$\operatorname{var}(f(\cdot, y); I) \le |||f||| \quad \text{and} \quad \operatorname{var}(f(x, \cdot); I) \le |||f|||.$$

Finally, we remind the reader that Minkowski's inequality in the generalized form says that if $f \in L^p([a, b] \times [c, d])$ for some $1 \le p < \infty$, then

$$\Big\{ \int_a^b \Big| \int_c^d f(x, y)\, dy \Big|^p dx \Big\}^{1/p} \le \int_c^d \Big\{ \int_a^b |f(x, y)|^p\, dx \Big\}^{1/p} dy$$

(see, e.g., [4, p. 179]). We will also use the multivariate version, i.e., when the single integrals $\int_a^b$ and $\int_c^d$ are replaced by the double ones $\int_{a_1}^{b_1}\int_{a_2}^{b_2}$ and $\int_{c_1}^{d_1}\int_{c_2}^{d_2}$, respectively.

## 3. Main result.

First, we introduce a few notations. Given a function $f(x, y)$, periodic in both variables with period 1, for $0 \le j < 2^m$ and



$0 \le k < 2^n$ and integers $m, n \ge 0$ we set

$$_1\Delta_j^m f(x, y) := f(x \dot{+} 2j2^{-m-1}, y) - f(x \dot{+} (2j+1)2^{-m-1}, y),$$
$$_2\Delta_k^n f(x, y) := f(x, y \dot{+} 2k2^{-n-1}) - f(x, y \dot{+} (2k+1)2^{-n-1}),$$
$$\begin{aligned}\Delta_{jk}^{mn} f(x, y) &:= {_1\Delta_j^m}({_2\Delta_k^n} f(x, y)) = {_2\Delta_k^n}({_1\Delta_j^m} f(x, y))\\ &= f(x \dot{+} 2j2^{-m-1}, y \dot{+} 2k2^{-n-1})\\ &\quad - f(x \dot{+} (2j+1)2^{-m-1}, y \dot{+} 2k2^{-n-1})\\ &\quad - f(x \dot{+} 2j2^{-m-1}, y \dot{+} (2k+1)2^{-n-1})\\ &\quad + f(x \dot{+} (2j+1)2^{-m-1}, y \dot{+} (2k+1)2^{-n-1}).\end{aligned}$$

Furthermore, set $\lambda_0 := 1$ and $\lambda_j := j^{-1}$ for $j \ge 1$, and

$$V_m^{(1)}(f; x, y) := \sum_{j=0}^{2^m-1} \lambda_j \,|\, {_1\Delta_j^m} f(x, y)|,$$
$$V_n^{(2)}(f; x, y) := \sum_{k=0}^{2^n-1} \lambda_k \,|\, {_2\Delta_k^n} f(x, y)|,$$
$$V_{mn}(f; x, y) := \sum_{j=0}^{2^m-1}\sum_{k=0}^{2^n-1} \lambda_j\lambda_k |\Delta_{jk}^{mn} f(x, y)|.$$

We will prove the following

THEOREM. *Let $M, N$ be positive integers such that*

(3.1) $$M = 2^m + i,\ 1 \le i \le 2^m \quad \text{and} \quad N = 2^n + l,\ 1 \le l \le 2^n,$$

*for some integers $m, n \ge 0$. If $f \in L^p(I^2)$ for some $1 \le p \le \infty$, then*

(3.2) $$\|S_{MN}(f) - f\|_p \le \omega_1(f; 2^{-m}, 2^{-n})_p + \|V_m^{(1)}(f)\|_p + \|V_n^{(2)}(f)\|_p + \|V_{mn}(f)\|_p.$$

This is an extension of a result by Onneweer [7] from single to double WFS.

Proof. We will prove (3.2) in the case when $1 \le p < \infty$. The proof for $p = \infty$ is similar.

We start with the familiar representations

$$D_M(u) = D_{2^m}(u) + r_m(u)D_i(u),$$
$$D_N(v) = D_{2^n}(v) + r_n(v)D_l(v)$$

(cf. (3.1)). By (1.2) and Minkowski's inequality in the usual form, (3.3) $$\|S_{MN}(f)-f\|_p$$

$$= \Big\{ \iint_{I^2} \Big| \iint_{I^2} D_M(u)D_N(v)[f(x\dot{+}u, y\dot{+}v) - f(x,y)]\,du\,dv\Big|^p\,dx\,dy\Big\}^{1/p}$$

$$\le \Big\{ \iint_{I^2} \Big| \iint_{I^2} D_{2^m}(u)D_{2^n}(v)[f(x\dot{+}u, y\dot{+}v) - f(x,y)]\,du\,dv\Big|^p\,dx\,dy\Big\}^{1/p}$$

$$+ \Big\{ \iint_{I^2} \Big| \iint_{I^2} r_m(u)D_i(u)D_{2^n}(v) \times [f(x\dot{+}u, y\dot{+}v) - f(x,y)]\,du\,dv\Big|^p\,dx\,dy\Big\}^{1/p}$$

$$+ \Big\{ \iint_{I^2} \Big| \iint_{I^2} D_{2^m}(u)r_n(v)D_l(v) \times [f(x\dot{+}u, y\dot{+}v) - f(x,y)]\,du\,dv\Big|^p\,dx\,dy\Big\}^{1/p}$$

$$+ \Big\{ \iint_{I^2} \Big| \iint_{I^2} r_m(u)D_i(u)r_n(v)D_l(v) \times [f(x\dot{+}u, y\dot{+}v) - f(x,y)]\,du\,dv\Big|^p\,dx\,dy\Big\}^{1/p}$$

$$=: A^{(1)}_{MN} + A^{(2)}_{MN} + A^{(3)}_{MN} + A^{(4)}_{MN}, \quad \text{say.}$$

Since

(3.4) $$D_{2^m}(u) = \begin{cases} 2^m & \text{if } u\in[0,2^{-m}),\\ 0 & \text{if } u \in [2^{-m},1)\end{cases}$$

(see, e.g., [10, p. 7]), by Minkowski's inequality in the generalized form, we find that

(3.5) $$A^{(1)}_{MN} \le \iint_{I^2} D_{2^m}(u)D_{2^n}(v)\Big\{\iint_{I^2} |f(x\dot{+}u, y\dot{+}v) -f(x,y)|^p\,dx\,dy\Big\}^{1/p} du\,dv \le \omega_1(f;2^{-m},2^{-n})_p\,.$$

Next we will estimate $A^{(2)}_{MN}$. To this end, we keep in mind that

(a) $D_i(u)$ takes on a constant value on each dyadic interval $I_m(j)$, where $0\le j<2^m$ and $1\le i\le 2^m$;

(b) $I_m(j) = I_{m+1}(2j)\cup I_{m+1}(2j+1)$;

(c) $r_m(u) = \begin{cases} 1 & \text{if } u\in I_{m+1}(2j),\\ -1 & \text{if } u\in I_{m+1}(2j+1);\end{cases}$

(d) $t := u\dot{+}2^{-m-1}$ is a one-to-one mapping of $I_{m+1}(2j)$ onto $I_{m+1}(2j+1)$.



Thus, by (3.4) and (a)–(d),

(3.6) $$A^{(2)}_{MN}$$

$$= \Big\{\iint_{I^2} \Big| \sum_{j=0}^{2^m-1} D_i(j2^{-m})2^n\Big\{ \int_{I_{m+1}(2j)}\int_{I_n(0)} [f(x\dot{+}u,y\dot{+}v)-f(x,y)]\,du\,dv - \int_{I_{m+1}(2j+1)}\int_{I_n(0)} [f(x\dot{+}u,y\dot{+}v)-f(x,y)]\,du\,dv\Big\}\Big|^p\,dx\,dy\Big\}^{1/p}$$

$$= \Big\{\iint_{I^2} \Big| \sum_{j=0}^{2^m-1} D_i(j2^{-m})2^n \int_{I_{m+1}(2j)}\int_{I_n(0)} [f(x\dot{+}u,y\dot{+}v) - f(x\dot{+}u\dot{+}2^{-m-1},y\dot{+}v)]\,du\,dv\Big|^p\,dx\,dy\Big\}^{1/p}$$

$$= \Big\{\iint_{I^2} \Big| \sum_{j=0}^{2^m-1} D_i(j2^{-m})2^n \times \int_{I_{m+1}(0)}\int_{I_n(0)} {}_1\Delta^m_j f(x\dot{+}u,y\dot{+}v)\,du\,dv\Big|^p\,dx\,dy\Big\}^{1/p}.$$

We recall (see [1]) that

(3.7) $$|D_i(u)| \le \min(i, 2u^{-1}), \qquad u\in I\,.$$

Thus, applying the generalized Minkowski inequality, from (3.6) it follows that

$$A^{(2)}_{MN} \le \Big\{\iint_{I^2}\Big[i2^n \int_{I_{m+1}(0)}\int_{I_n(0)} |{}_1\Delta^m_0 f(x\dot{+}u,y\dot{+}v)|\,du\,dv + \sum_{j=1}^{2^m-1} 2^{m+1}j^{-1}2^n \int_{I_{m+1}(0)}\int_{I_n(0)} |{}_1\Delta^m_j f(x\dot{+}u,y\dot{+}v)|\,du\,dv\Big]^p\,dx\,dy\Big\}^{1/p}$$

$$\le 2^{m+n+1}\Big\{\iint_{I^2}\Big[\int_{I_{m+1}(0)}\int_{I_n(0)} V^{(1)}_m(f;x\dot{+}u,y\dot{+}v)\,du\,dv\Big]^p\,dx\,dy\Big\}^{1/p}$$

$$\le 2^{m+n+1}\int_{I_{m+1}(0)}\int_{I_n(0)}\Big\{\iint_{I^2} [V^{(1)}_m(f;x\dot{+}u,y\dot{+}v)]^p\,dx\,dy\Big\}^{1/p} du\,dv\,.$$

Since the norm $\|\cdot\|_p$ is translation invariant, hence we get

(3.8) $$A^{(2)}_{MN} \le \|V^{(1)}_m(f)\|_p\,.$$

Analogously,

(3.9) $$A^{(3)}_{MN} \le \|V^{(2)}_n(f)\|_p\,.$$

Finally, we deal with $A_{MN}^{(4)}$. Following a similar pattern to the case of $A_{MN}^{(2)}$, by (a)–(d) we obtain

$$A_{MN}^{(4)} = \Big\{ \iint_{I^2} \Big| \sum_{j=0}^{2^m-1} \sum_{k=0}^{2^n-1} D_i(j2^{-m}) D_l(k2^{-n}) \Big\{ \Big( \int_{I_{m+1}(2j)} \int_{I_{n+1}(2k)} - \int_{I_{m+1}(2j+1)} \int_{I_{n+1}(2k)} - \int_{I_{m+1}(2j)} \int_{I_{n+1}(2k+1)} + \int_{I_{m+1}(2j+1)} \int_{I_{n+1}(2k+1)} \Big) [f(x\dot{+}u, y\dot{+}v) - f(x,y)]\, du\, dv \Big\} \Big|^p dx\, dy \Big\}^{1/p}$$

$$= \Big\{ \iint_{I^2} \Big| \sum_{j=0}^{2^m-1} \sum_{k=0}^{2^n-1} D_i(j2^{-m}) D_l(k2^{-n}) \times \int_{I_{m+1}(2j)} \int_{I_{n+1}(2k)} [f(x\dot{+}u, y\dot{+}v) - f(x\dot{+}u\dot{+}2^{-m-1}, y\dot{+}v) - f(x\dot{+}u, y\dot{+}v\dot{+}2^{-n-1}) + f(x\dot{+}u\dot{+}2^{-m-1}, y\dot{+}v\dot{+}2^{-n-1})]\, du\, dv \Big|^p dx\, dy \Big\}^{1/p}$$

$$= \Big\{ \iint_{I^2} \Big| \sum_{j=0}^{2^m-1} \sum_{k=0}^{2^n-1} D_i(j2^{-m}) D_l(k2^{-n}) \times \int_{I_{m+1}(0)} \int_{I_{n+1}(0)} \Delta_{jk}^{mn} f(x\dot{+}u, y\dot{+}v)\, du\, dv \Big|^p dx\, dy \Big\}^{1/p}.$$

By (3.7) and the generalized Minkowski inequality, we conclude that

$$A_{MN}^{(4)} \le \Big\{ \iint_{I^2} \Big[ il \int_{I_{m+1}(0)} \int_{I_{n+1}(0)} |\Delta_{00}^{mn} f(x\dot{+}u, y\dot{+}v)|\, du\, dv + \sum_{j=1}^{2^m-1} 2^{m+1} j^{-1} l \int_{I_{m+1}(0)} \int_{I_{n+1}(0)} |\Delta_{j0}^{mn} f(x\dot{+}u, y\dot{+}v)|\, du\, dv + \sum_{k=1}^{2^n-1} 2^{n+1} i k^{-1} \int_{I_{m+1}(0)} \int_{I_{n+1}(0)} |\Delta_{0k}^{mn} f(x\dot{+}u, y\dot{+}v)|\, du\, dv + \sum_{j=1}^{2^m-1} \sum_{k=1}^{2^n-1} 2^{m+1} j^{-1} 2^{n+1} k^{-1} \times \int_{I_{m+1}(0)} \int_{I_{n+1}(0)} |\Delta_{jk}^{mn} f(x\dot{+}u, y\dot{+}v)|\, du\, dv \Big]^p dx\, dy \Big\}^{1/p}$$



$$\le 2^{m+n+2} \Big\{ \iint_{I^2} \Big[ \int_{I_{m+1}(0)} \int_{I_{n+1}(0)} V_{mn}(f; x\dot{+}u, y\dot{+}v)\, du\, dv \Big]^p dx\, dy \Big\}^{1/p}$$

$$\le 2^{m+n+2} \int_{I_{m+1}(0)} \int_{I_{n+1}(0)} \Big\{ \iint_{I^2} [V_{mn}(f; x\dot{+}u, y\dot{+}v)]^p\, dx\, dy \Big\}^{1/p} du\, dv\,.$$

Since $\|\cdot\|_p$ is translation invariant, hence

(3.10) $$A_{MN}^{(4)} \le \|V_{mn}(f)\|_p\,.$$

Combining (3.3), (3.5), (3.8)–(3.10) yields (3.2).

**4. Corollaries.** In this section we will show that our theorem implies the extension of two classical results from single to double WFS. The first of them is the Dini–Lipschitz test proved in [1, Theorem 13] for single WFS.

COROLLARY 1. *If $f \in L^p(I^2)$ for some $1 \le p \le \infty$ and*

(4.1) $$\omega_2(f; \delta_1, \delta_2)_p = o(\ln \delta_1^{-1} \ln \delta_2^{-1})^{-1} \quad \textit{as } \delta_1, \delta_2 \to 0\,,$$

(4.2) $$\omega_{1,x}(f; \delta)_p = o(\ln \delta^{-1})^{-1} \quad \textit{as } \delta \to 0\,,$$

(4.3) $$\omega_{1,y}(f; \delta)_p = o(\ln \delta^{-1})^{-1} \quad \textit{as } \delta \to 0\,,$$

*then the double WFS of $f$ converges to $f$ in $L^p$-norm.*

In particular, if $\omega_{1,x}(f;\delta)_p = o(\ln \delta^{-1})^{-2}$ and $\omega_{1,y}(f;\delta)_p = o(\ln \delta^{-1})^{-2}$, then the conclusion of Corollary 1 holds true.

We note that Corollary 1 in the particular case when $f \in C_W(I^2)$ ($p = \infty$) was stated in [2] without any proof.

Furthermore, Corollary 1 can be essentially improved in the cases when $1 < p < \infty$. Namely, for every such $p$ there exists a constant $K_p$ depending only on $p$ such that for any $f \in L^p(I^2)$ we have

(4.4) $$\|S_{mn}(f)\|_p \le K_p \|f\|_p, \qquad m, n \ge 1\,.$$

This is ultimately a consequence of the corresponding univariate inequality of Paley [8]. (See more details in [6].) On the other hand, from (4.4) and the Banach–Steinhaus theorem it follows that the double WFS of a function $f \in L^p(I^2)$, for some $1 < p < \infty$, converges to $f$ in $L^p$-norm.

PROBLEM 1. Nevertheless, we conjecture that Corollary 1 is sharp in the cases when $p = 1$ and $p = \infty$. That is, if "$o$" is replaced by "$O$" in any one of the conditions (4.1)–(4.3), then the conclusion of Corollary 1 is no longer true. But we are unable to present counterexamples.

Proof of Corollary 1. We see immediately that

$$\|{}_1\Delta_j^m f\|_p \le \omega_{1,x}(f; 2^{-m-1})_p\,,$$

whence

$$\|V_m^{(1)}(f)\|_p \le \Big(1+\sum_{j=1}^{2^m-1} j^{-1}\Big)\omega_{1,x}(f;2^{-m-1})_p \le \omega_{1,x}(f;2^{-m-1})_p \ln 2^{m+1}.$$

Analogously,

$$\|V_n^{(2)}(f)\|_p \le \omega_{1,y}(f;2^{-n-1})_p \ln 2^{n+1}$$

and

$$\|V_{mn}(f)\|_p \le \omega_2(f;2^{-m-1},2^{-n-1})_p \ln 2^{m+1}\ln 2^{n+1}.$$

It remains to apply (3.2), (2.1) and (4.1)–(4.3).

The next corollary is the Dirichlet–Jordan test for double WFS, whose univariate version was first proved in [11, Theorem 4].

COROLLARY 2. *If $f \in C_W(I^2)\cap \mathrm{BV}(I^2)$, then the double WFS of $f$ converges to $f$ uniformly on $I^2$.*

Proof. We can find two nondecreasing sequences $\{i(m): m\ge 0\}$ and $\{l(n): n\ge 0\}$ of positive integers such that

(i) $i(m)\le 2^m-1$ and $l(n)\le 2^n-1$ for all $m$ and $n$, respectively;
(ii) $i(m)\to\infty$ and $l(n)\to\infty$ as $m\to\infty$ and $n\to\infty$, respectively;
(iii)

$$\omega_{1,x}(f;2^{-m-1})\ln i(m)\to 0 \quad \text{as } m\to\infty,$$
$$\omega_{1,y}(f;2^{-n-1})\ln l(n)\to 0 \quad \text{as } n\to\infty,$$
$$\omega_2(f;2^{-m-1},2^{-n-1})\ln i(m)\ln l(n)\to 0 \quad \text{as } m,n\to\infty.$$

Here and in the sequel, we drop the subscript $p=\infty$. Then

$$\|V_M^{(1)}(f)\| \le \Big(1+\sum_{j=1}^{i(m)-1} j^{-1}\Big)\omega_{1,x}(f;2^{-m-1}) + \Big\|\sum_{j=i(m)}^{2^m-1} j^{-1}|_1\Delta_j^m f|\Big\|$$
$$\le \omega_{1,x}(f;2^{-m-1})\ln 2i(m) + [i(m)]^{-1}|||f|||.$$

Analogously,

$$\|V_n^{(2)}(f)\| \le \omega_{1,y}(f;2^{-n-1})\ln 2l(n) + [l(n)]^{-1}|||f|||.$$

Finally,

$$\text{(4.5)}\quad \|V_{mn}(f)\| \le \omega_2(f;2^{-m-1},2^{-n-1})\sum_{j=0}^{i(m)-1}\sum_{k=0}^{l(n)-1}\lambda_j\lambda_k$$
$$+\Big\|\Big\{\sum_{j=i(m)}^{2^m-1}\sum_{k=0}^{2^n-1}+\sum_{j=0}^{i(m)-1}\sum_{k=l(n)}^{2^n-1}\Big\}\lambda_j\lambda_k|\Delta_{jk}^{mn}f|\Big\|$$
$$\le \omega_2(f;2^{-m-1},2^{-n-1})\ln 2i(m)\ln 2l(n) + \max\{i^{-1}(m),l^{-1}(n)\}|||f|||.$$



Now, it is enough to apply (3.2), (2.1), (ii), and (iii).

On closing, we will extend Corollary 2 to certain collections of $W$-continuous functions of generalized bounded variation. To this end, we recall the definition of bounded $\Phi$-variation.

Let $\varphi(t)$ be a (classically) continuous, strictly increasing function defined for $t\ge 0$ such that $\varphi(0)=0$ and $\varphi(t)\to\infty$ as $t\to\infty$. Let $\psi$ be the inverse of $\varphi$. Next, let

$$\Phi(u) := \int_0^u \varphi(t)\,dt \quad\text{and}\quad \Psi(u) := \int_0^u \psi(t)\,dt.$$

Such functions $\Phi$ and $\Psi$ are called *complementary* in the sense of W. H. Young, and they satisfy the following inequality:

$$\text{(4.6)}\qquad ab\le \Phi(a)+\Psi(b),\qquad a,b\ge 0.$$

(See, e.g., [13, p. 16].)

Now, a function $f: I^2\to\mathbb{R}$ is said to be *of bounded $\Phi$-variation* if there exists a constant $K$ such that for any partitions $\mathcal{D}_1$ and $\mathcal{D}_2$ of $I$ (see (2.2)) we have

$$\text{(4.7)}\qquad \sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\Phi(|f(x_j,y_k)-f(x_{j+1},y_k)$$
$$-f(x_j,y_{k+1})+f(x_{j+1},y_{k+1})|)\le K,$$

furthermore, for any fixed $y\in I$,

$$\text{(4.8)}\qquad \sum_{j=0}^{m-1}\Phi(|f(x_j,y)-f(x_{j+1},y)|)\le K;$$

and for any fixed $x\in I$,

$$\text{(4.9)}\qquad \sum_{k=0}^{n-1}\Phi(|f(x,y_k)-f(x,y_{k+1})|)\le K.$$

We note that if $\Phi$ does not increase too fast in the sense that

$$\text{(4.10)}\qquad K_1 := \sup_{u>0}\Phi(2u)/\Phi(u) < \infty,$$

then it is enough to require the fulfillment of (4.8) and (4.9) for $y=0$ and $x=0$, respectively. In fact, it follows from (4.10) that for all $0\le u_1\le u_2$ we have

$$\Phi(u_1+u_2)\le\Phi(2u_2)\le K_1\Phi(u_2)\le K_1\{\Phi(u_1)+\Phi(u_2)\}.$$

Thus, assuming (4.7) and (4.8) for $y=0$, we obtain (4.8) for any $y\in I$ with $2K_1K$ instead of $K$ on the right-hand side.

COROLLARY 3. *Let $\Phi$ and $\Psi$ be complementary functions such that*

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\Psi(j^{-1}k^{-1}) < \infty. \tag{4.11}$$

*If $f \in C_W(I^2)$ is a function of bounded $\Phi$-variation, then the double WFS of $f$ converges to $f$ uniformly on $I^2$.*

This corollary is analogous to results obtained by L. C. Young [12] and Salem [9] for trigonometric Fourier series and by Onneweer [7] for WFS in the univariate case.

Proof. It is plain that from (4.11) it follows that $\sum_{j=1}^{\infty}\Psi(j^{-1}) < \infty$. Thus, we can find a sequence $\{\varepsilon(i) : i \geq 0\}$ of positive numbers decreasing to 0 as $i \to \infty$ and such that

$$K_2 := \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Psi(\lambda_j\lambda_k\varepsilon^{-1}(\max(j,k))) < \infty. \tag{4.12}$$

According to (4.6), we may write

$$|\Delta_{jk}^{mn}f(x,y)|\lambda_j\lambda_k\varepsilon^{-1}(\max(j,k)) \leq \Phi(|\Delta_{jk}^{mn}f(x,y)|) + \Psi(\lambda_j\lambda_k\varepsilon^{-1}(\max(j,k))).$$

By (4.7) and (4.12), for any $0 \leq i < 2^m$ and $0 \leq l < 2^n$ we get

$$\Big\{\sum_{j=i}^{2^m-1}\sum_{k=0}^{2^n-1} + \sum_{j=0}^{i}\sum_{k=l}^{2^n-1}\Big\}|\Delta_{jk}^{mn}f(x,y)|\lambda_j\lambda_k\varepsilon^{-1}(\max(j,k)) \leq K + K_2,$$

whence

$$\Big\{\sum_{j=i}^{2^m-1}\sum_{k=0}^{2^n-1} + \sum_{j=0}^{i}\sum_{k=l}^{2^n-1}\Big\}\lambda_j\lambda_k|\Delta_{jk}^{mn}f(x,y)| \leq (K+K_2)\varepsilon(\max(i,l)).$$

Now, we choose $\{i(m)\}$ and $\{l(n)\}$ as in the proof of Corollary 2 (see (i)–(iii) there). Analogously to (4.5), we conclude that

$$|V_{mn}(f;x,y)| \leq \omega_2(f;2^{-m-1},2^{-n-1})\ln 2i(m)\ln 2l(n) + (K+K_2)\varepsilon(\max(i(m),l(n))),$$

which tends to zero as $m,n \to \infty$, uniformly in $(x,y)$.

Similarly to the above reasoning, we find that

$$|V_m^{(1)}(f;x,y)| \leq \omega_{1,x}(f;2^{-m-1})\ln 2i(m) + (K+K_2)\varepsilon(i(m)),$$
$$|V_n^{(2)}(f;x,y)| \leq \omega_{1,y}(f;2^{-n-1})\ln 2l(n) + (K+K_2)\varepsilon(l(n)).$$

These also tend to zero as $m \to \infty$ and $n \to \infty$, respectively, and the convergence is uniform in $(x,y)$.



PROBLEM 2. It may be of some interest to construct counterexamples showing that conditions (4.10) and (4.11) cannot be weakened in general.

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*Received July 31, 1991* (2826)