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Selections and representations of multifunctions in paracompact spaces

by

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Abstract. Let $(X, \mathcal{T})$ be a paracompact space, Y a complete metric space, $F : X \to 2^Y$ a lower semicontinuous multifunction with nonempty closed values. We prove that if $\mathcal{T}^+$ is a (stronger than $\mathcal{T}$) topology on X satisfying a compatibility property, then F admits a $\mathcal{T}^+$-continuous selection. If Y is separable, then there exists a sequence (f_n) of $\mathcal{T}^+$-continuous selections such that $F(x) = \overline{\{f_n(x); n \geq 1\}}$ for all $x \in X$. Given a Banach space E, the above result is then used to construct directionally continuous selections on arbitrary subsets of $\mathbb{R} \times E$.

1. Introduction. In the study of differential inclusions, it is often desirable to reduce the multivalued problem to an ordinary differential equation in the same space, constructing a continuous selection of the right hand side. Among the earliest selection theorems, the following results of Michael are well known:

[7, Thm. 1] *If X is a paracompact topological space, every lower semicontinuous multifunction F from X into the nonempty, closed and convex subsets of a Banach space Y admits a continuous selection.*

[7, Thm. 2] *If X is paracompact and zero-dimensional, every lower semicontinuous multifunction F from X into the nonempty closed subsets of a complete metric space Y admits a continuous selection.*

We recall that a normal topological space is zero-dimensional if and only if every locally finite open covering of it admits a disjoint open refinement [7, Prop. 2].

In cases where these results do not apply, one can introduce a finer topology $\mathcal{T}^+$ on X and ask for selections of F which are $\mathcal{T}^+$-continuous. A result in this direction is:

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[6, Thm. 1] *Let X be a locally compact subset of a metric space and let F be a lower semicontinuous multifunction from X into the nonempty, closed subsets of a complete metric space Y. Let $\mathcal{T}^+$ be a second topology on X with the property*:

(P_0) *every $x \in X$ has a basis of neighborhoods (in the metric topology) consisting of $\mathcal{T}^+$-closed-open sets.*

Then F admits a $\mathcal{T}^+$-continuous selection.

Given a Banach space E and a number $M > 0$, the family of all conical neighborhoods

$$(1.1) \quad \Gamma^M(t_0, x_0, \delta) \doteq \{(t, x) \in \mathbb{R} \times E : \|x - x_0\| \le M(t - t_0),\ t_0 \le t < t_0 + \delta\}$$

for all $\delta > 0$, $(t_0, x_0) \in \mathbb{R} \times E$, generates a topology $\mathcal{T}^+$ which satisfies the assumption (P_0). Functions which are continuous w.r.t. this stronger topology will be called *Γ^M-continuous*, or simply *directionally continuous*. By constructing suitable directionally continuous selections, various results on the qualitative theory of differential inclusions were recently proved [3, 4, 5]. Until now, however, all applications have been confined to problems in locally compact spaces, mainly finite-dimensional. The aim of this paper is to remove the restriction on the domain of F in [6, Thm. 1], letting X be any paracompact topological space, as in Michael's theorems. This will allow the use of our selection technique in connection with infinite-dimensional differential inclusions, in full generality.

Instead of (P_0), we consider the stronger property:

(P) *For every pair of sets $A \subset B$, with A closed and B open (in the original topology $\mathcal{T}$), there exists a set C, closed-open w.r.t. $\mathcal{T}^+$, such that $A \subset C \subset B$.*

The following results will be proved.

THEOREM 1. *Let $(X, \mathcal{T})$ be a paracompact space, (Y, d) a complete metric space and $F : X \to 2^Y$ a lower semicontinuous multifunction with nonempty closed values. If $\mathcal{T}^+$ is a topology on X with the property* (P), *then F admits a $\mathcal{T}^+$-continuous selection.*

THEOREM 2. *Let E be a Banach space, $M > 0$, Ω a subset of $\mathbb{R} \times E$. Any lower semicontinuous multifunction $F : \Omega \to 2^E$ with nonempty closed values admits a Γ^M-continuous selection.*

THEOREM 3. *Let X be a metric space, Y a separable complete metric space and $F : X \to 2^Y$ a lower semicontinuous multifunction with nonempty closed values. If $\mathcal{T}^+$ is a topology on X with the property* (P), *then there exists a sequence f_n of $\mathcal{T}^+$-continuous selections from F such that, for every $x \in X$, the closure of the set $\{f_n(x) : n \ge 1\}$ coincides with $F(x)$.*

For the basic theory of multifunctions and differential inclusions we refer to [1].

2. Proof of Theorem 1. Following a well-established argument due to Michael, the selection is obtained as the limit of a uniformly converging sequence of $\mathcal{T}^+$-continuous approximations. By induction, we shall construct functions $(f_n)_{n \ge 1}$ with the properties:

$(\mathrm{i})_n$ there exists a $\mathcal{T}^+$-open and disjoint covering $\mathcal{O}^n = (\Omega^n_\alpha)_{\alpha \in \mathcal{A}^n}$ of X; for every α, f_n is constant on Ω^n_α, say, $f_n(x) = y^n_\alpha$ for all $x \in \Omega^n_\alpha$;

$(\mathrm{ii})_n$ $d(y^n_\alpha, F(x)) < 2^{-n} \quad \forall x \in \mathrm{cl}(\Omega^n_\alpha),\ \forall \alpha \in \mathcal{A}^n$;

$(\mathrm{iii})_n$ $d(f_n(x), f_{n-1}(x)) < 2^{-n+1} \quad \forall x \in X \ (n \ge 2)$.

To construct f_1, using the lower semicontinuity of F, for every $x \in X$ choose a point $y_x \in F(x)$ and a neighborhood U_x of x such that

$$(2.1) \qquad d(y_x, F(x')) < 2^{-1} \quad \forall x' \in U_x .$$

Let $(V_\alpha)_{\alpha \in \mathcal{A}^1}$ be a locally finite open refinement of $(U_x)_{x \in X}$, say with $V_\alpha \subset U_{x_\alpha}$, and let $(W_\alpha)_{\alpha \in \mathcal{A}^1}$ be another open refinement such that $\mathrm{cl}(W_\alpha) \subset V_\alpha$ for all $\alpha \in \mathcal{A}^1$. Here and throughout the paper, $\mathrm{cl}(W)$ and $\mathrm{int}(W)$ will denote the closure and the interior of a set W in the original topology $\mathcal{T}$. By the property (P), for each α one can choose a set Z_α, closed-open w.r.t. $\mathcal{T}^+$, such that

$$(2.2) \qquad \mathrm{cl}(W_\alpha) \subset \mathrm{int}(Z_\alpha) \subset \mathrm{cl}(Z_\alpha) \subset V_\alpha .$$

Then $(Z_\alpha)_\alpha$ is a locally finite $\mathcal{T}^+$-closed-open covering of X. Choose a well-ordering $\preceq$ of the set $\mathcal{A}^1$ and define, for each $\alpha \in \mathcal{A}^1$,

$$\Omega^1_\alpha = Z_\alpha \setminus \Big(\bigcup_{\beta \prec \alpha} Z_\beta \Big) .$$

Set $\mathcal{O}^1 = (\Omega^1_\alpha)_{\alpha \in \mathcal{A}^1}$. By the well-ordering, every $x \in X$ belongs to exactly one set $\Omega^1_{\overline{\alpha}}$, where $\overline{\alpha} = \min\{\alpha \in \mathcal{A}^1 : x \in Z_\alpha\}$. Hence, $\mathcal{O}^1$ is a partition of X. Moreover, since (Z_α) is locally finite (w.r.t. $\mathcal{T}$ and therefore w.r.t. $\mathcal{T}^+$), the sets $\bigcup_{\beta \prec \alpha} Z_\beta$ are $\mathcal{T}^+$-closed-open. Hence $\mathcal{O}^1$ is a $\mathcal{T}^+$-closed-open, disjoint covering of X such that, by (2.2), $\{\mathrm{cl}(\Omega^1_\alpha)\}$ refines $(V_\alpha)_\alpha$. By setting $y^1_\alpha = y_{x_\alpha}$ and

$$f_1(x) = y^1_\alpha \quad \forall x \in \Omega^1_\alpha,\ \forall \alpha \in \mathcal{A}^1 ,$$

we obtain a $\mathcal{T}^+$-continuous function which, by (2.1), satisfies $(\mathrm{i})_1$, $(\mathrm{ii})_1$.

Suppose now that functions f_k have been constructed satisfying the properties $(\mathrm{i})_k$–$(\mathrm{iii})_k$, for $1 \le k < n$. Since $\mathcal{O}^{n-1}$ is a disjoint $\mathcal{T}^+$-open covering, the map f_n can be defined separately on each set Ω^{n-1}_α. Fix $\alpha \in \mathcal{A}^{n-1}$.

For every $x \in \mathrm{cl}(\Omega_\alpha^{n-1})$, by $(\mathrm{ii})_{n-1}$ and lower semicontinuity there exist $y_x \in F(x)$ and a neighborhood U_x of x such that

$$d(y_x, F(x')) < 2^{-n} \quad \forall x' \in U_x\,, \tag{2.3}$$

$$d(y_x, y_\alpha^{n-1}) < 2^{-n+1}\,. \tag{2.4}$$

Since $\mathrm{cl}(\Omega_\alpha^{n-1})$ is paracompact, the same argument used in the first induction step provides two locally finite open refinements (V_β), (W_β), $\beta \in \mathcal{A}_\alpha^n$, of $(U_x)_{x\in \mathrm{cl}(\Omega_\alpha^{n-1})}$ such that $\mathrm{cl}(W_\beta) \subset V_\beta \subset U_{x_\beta}$ for some $x_\beta \in \mathrm{cl}(\Omega_\alpha^{n-1})$, for all $\beta \in \mathcal{A}_\alpha^n$. Using the property (P), construct a family $(Z_\beta)_{\beta\in\mathcal{A}_\alpha^n}$ of $\mathcal{T}^+$-closed-open subsets of $\mathrm{cl}(\Omega_\alpha^{n-1})$ such that

$$\mathrm{cl}(W_\beta) \subset \mathrm{int}(Z_\beta) \subset \mathrm{cl}(Z_\beta) \subset V_\beta \quad \forall \beta \in \mathcal{A}_\alpha^n\,,$$

and let $\preceq$ be a well-ordering on $\mathcal{A}_\alpha^n$. As in the first step, by setting

$$\Omega_{\alpha,\beta}^n = Z_\beta \setminus \Big(\bigcup_{\gamma\prec\beta} Z_\gamma\Big)$$

we get a disjoint $\mathcal{T}^+$-closed-open covering of $\mathrm{cl}(\Omega_\alpha^{n-1})$ such that $(\mathrm{cl}(\Omega_{\alpha,\beta}^n))_\beta$ refines $(V_\beta)_\beta$. Set now $y_{\alpha,\beta}^n = y_{x_\beta}$ and define the $\mathcal{T}^+$-continuous map $f_n : \Omega_\alpha^{n-1} \to Y$ as

$$f_n(x) = y_{\alpha,\beta} \quad \forall x \in \Omega_{\alpha,\beta}^n\,.$$

Repeat the above construction for all $\alpha \in \mathcal{A}^{n-1}$ and define

$$\mathcal{A}^n = \bigcup_{\alpha\in\mathcal{A}^{n-1}} (\{\alpha\}\times \mathcal{A}_\alpha^n)\,, \qquad \mathcal{O}^n = (\Omega_{\alpha,\beta}^n)_{(\alpha,\beta)\in\mathcal{A}^n}\,.$$

By (2.3), (2.4), the properties $(\mathrm{i})_n$–$(\mathrm{iii})_n$ are satisfied.

The completeness of Y and $(\mathrm{iii})_n$ now imply that the sequence (f_n) converges uniformly to a $\mathcal{T}^+$-continuous function f. Since the values of F are closed, by $(\mathrm{i})_n$ and $(\mathrm{ii})_n$ it follows that f is a selection from F. ∎

3. Proof of Theorem 2. Theorem 2 is an immediate consequence of Theorem 1 and of the following Lemma, showing that the property (P) holds for the topology $\mathcal{T}^+$ generated by the family of all conical neighborhoods $\Gamma^M(t_0, x_0, \delta)$ defined in (1.1).

LEMMA. *Let E be a Banach space and let A, B be disjoint and closed subsets of $\mathbb{R}\times E$. Then there exists $C \subseteq \mathbb{R}\times E$ such that*

(i) $A \subset \mathrm{int}(C)$ *and* $\mathrm{cl}(C) \cap B = \emptyset$;
(ii) C *is $\mathcal{T}^+$-closed-open.*

Proof. For every $(t,x) \in \mathbb{R}\times E$, $n \ge 1$, consider the open neighborhood

$$L_n(t,x) = \{(s,y) : \|y-x\| < 2^{-n} - M|t-s|\}. \tag{3.1}$$

Observe that, for every n,

$$\bigcup_{k\in\mathbb{Z},\,x\in E} L_n\Big(\frac{k}{M\cdot 2^n}, x\Big) = \mathbb{R}\times E\,.$$

In the following, on $\mathbb{R}\times E$ we use the distance $d((t,x),(s,y)) = \max\{|t-s|, \|x-y\|\}$. Define the closed set $A^* = \{(t,x) \in \mathbb{R}\times E : d((t,x),A) \le d((t,x),B)\}$. For every $(t,x)\in A$ choose $k = k(t,x) \in \mathbb{Z}$ and $n = n(t,x) \ge 1$ such that, setting

$$L_{t,x} = L_{n(t,x)}\Big(\frac{k(t,x)}{M\cdot 2^{n(t,x)}}, x\Big)\,, \tag{3.2}$$

one has

$$(t,x) \in L_{t,x} \subset A^*\,. \tag{3.3}$$

The sets $L_{t,x}$ can also be written as

$$L_{t,x} = \{(s,y) : \varphi_{t,x}(y) < s < \psi_{t,x}(y)\}\,,$$

where

$$\begin{aligned} \varphi_{t,x}(y) &= \frac{k(t,x)}{M\cdot 2^{n(t,x)}} - \frac{2^{-n(t,x)} - \|y-x\|}{M}\,,\\ \psi_{t,x}(y) &= \frac{k(t,x)}{M\cdot 2^{n(t,x)}} + \frac{2^{-n(t,x)} - \|y-x\|}{M}\,. \end{aligned} \tag{3.4}$$

Define

$$\varphi_{n,k}(y) = \inf\{\varphi_{t,x}(y) : (t,x)\in A,\ n(t,x) = n,\ k(t,x) = k\}\,,$$
$$\psi_{n,k}(y) = \sup\{\psi_{t,x}(y) : (t,x)\in A,\ n(t,x) = n,\ k(t,x) = k\}$$

and

$$W_{n,k} = \{(s,y) \in \mathbb{R}\times E : \varphi_{n,k}(y) \le s < \psi_{n,k}(y)\}\,.$$

Since all maps $\varphi_{t,x}$, $\psi_{t,x}$ are Lipschitz continuous with constant $1/M$, the same is true for every $\varphi_{n,k}$ and $\psi_{n,k}$. We claim that each $W_{n,k}$ is closed-open in the stronger topology $\mathcal{T}^+$. Indeed, if $(t,x) \in W_{n,k}$, then $t = \psi_{n,k}(x) - \delta$ for some $\delta > 0$. By Lipschitz continuity we thus have

$$\{(s,y) : \|y-x\| \le M(s-t),\ t \le s < t+\delta/2\} \subset W_{n,k}\,,$$

showing that (t,x) is an interior point. On the other hand, if $(t,x) \notin W_{n,k}$, then either $t \ge \psi_{n,k}(x)$ and

$$\{(s,y) : \|y-x\| \le M(s-t)\} \cap W_{n,k} = \emptyset\,,$$

or $t = \varphi_{n,k}(x) - \delta$ for some $\delta > 0$ and

$$\{(s,y) : \|y-x\| \le M(s-t),\ t \le s < t+\delta/2\} \cap W_{n,k} = \emptyset\,.$$

In both cases, (t,x) does not belong to the closure of $W_{n,k}$.

We now show that the requirements of the Lemma are satisfied by the set

$$C \doteq \bigcup_{n\geq 1, k\in\mathbb{Z}} W_{n,k}\,. \tag{3.5}$$

By (3.3), every $(t,x)\in A$ belongs to $\{(s,y):\varphi_{t,x}(y)<s<\psi_{t,x}(y)\}$, and therefore to the interior of some $W_{n,k}$. Moreover, $C\subset \mathrm{cl}(\bigcup_{(t,x)\in A} L_{t,x})\subset A^*$; hence $\mathrm{cl}(C)\cap B=\emptyset$. This proves (i).

Concerning (ii), it is clear that C is open in the topology $\mathcal{T}^+$, being a union of open sets. To prove that C is closed, assume $(\overline{t},\overline{x})\notin C$. Then $r\doteq d((\overline{t},\overline{x}),A)>0$. Observe that, if $d((s,y),A)>r/2$ and $(s,y)\in L_{t,x}$ for some $(t,x)\in A$, then $\mathrm{diam}(L_{t,x})=\max\{2^{1-n(t,x)},2^{1-n(t,x)}/M\}>r/2$, i.e.

$$n(t,x)<N_r\doteq -\log_2(\max\{r/4, Mr/4\})\,.$$

As a consequence, the ball $B((\overline{t},\overline{x}),r/2)$ intersects only finitely many sets $W_{n,k}$. Indeed, if $1\leq n<N_r$ the sets

$$L_n(k/(M\cdot 2^n),x)\cap B((\overline{t},\overline{x}),r/2)$$

can be nonempty only for the finitely many $k\in\mathbb{Z}$ such that $|kM^{-1}2^{-n}-\overline{t}|<r/2+M^{-1}2^{-n}$. We now have

$$(\overline{t},\overline{x})\in B((\overline{t},\overline{x}),r/2)\setminus\bigcup W_{n,k}, \tag{3.6}$$

the union being taken over the finitely many indices (n,k) for which $W_{n,k}\cap B((\overline{t},\overline{x}),r/2)\neq\emptyset$. Clearly the set in (3.6) is open w.r.t. $\mathcal{T}^+$ and does not intersect C. This proves that C is closed in the topology $\mathcal{T}^+$, establishing (ii). ■

Remark. For the topology $\mathcal{T}^+$ generated by the family (1.1), each conical neighborhoods $\Gamma^M(t_0,x_0,\delta)$ is actually a $\mathcal{T}^+$-closed-open set. However, our Theorem 2 does not follow from [7, Thm. 2], because $(\mathbb{R}\times E,\mathcal{T}^+)$ is never paracompact, provided $E\neq\{0\}$. Indeed, consider the $\mathcal{T}^+$-open covering of $\mathbb{R}\times E$

$$\mathcal{O}=(\Gamma^M(k,x,1))_{k\in\mathbb{Z},x\in E}\,.$$

Any point of the form $(0,x)$ belongs to exactly one set of $\mathcal{O}$. Hence every $\mathcal{T}^+$-open refinement of $\mathcal{O}$ must contain a conical neighborhood $\Gamma^M(0,x,\varepsilon)$ with $\varepsilon=\varepsilon(x)>0$, for each $x\in E$. For any fixed $\widehat{x}$ we have

$$\bigcup_{\nu\geq 1}\{\lambda\in[0,1]:\varepsilon(\lambda\widehat{x})>\nu^{-1}\}\doteq\bigcup_{\nu\geq 1}S_\nu=[0,1]\,.$$

Therefore, some S_ν is infinite (actually, uncountable), and has a cluster point, say $\widehat{\lambda}$. Any $\mathcal{T}^+$-neighborhood of $\widehat{\lambda}\widehat{x}$ then intersects infinitely many cones $\Gamma^M(0,x,\varepsilon(x))$, showing that $\mathcal{O}$ is not locally finite.

4. Proof of Theorem 3. The following proof is an adaptation of the arguments used in [8, Lemma 5.2].

Let $(y_i)_{i\geq 1}$ be a countable, dense subset of Y and set, for each $i,j\geq 1$,

$$U_{i,j}=\{x\in X: F(x)\cap B(y_i,2^{-j})\neq\emptyset\}\,.$$

By the lower semicontinuity of F, each $U_{i,j}$ is open, and therefore it is a countable union of closed subsets of X, say

$$U_{i,j}=\bigcup_{k\geq 1} C_{i,j,k}\,.$$

Define

$$F_{i,j,k}(x)=\begin{cases} F(x) & \text{if } x\notin C_{i,j,k}\,,\\ \mathrm{cl}(F(x)\cap B(y_i,2^{-j})) & \text{if } x\in C_{i,j,k}\,.\end{cases}$$

Then each $F_{i,j,k}$ is lower semicontinuous with closed values. By Theorem 1, for every i,j,k, there exists a $\mathcal{T}^+$-continuous selection $f_{i,j,k}$ from $F_{i,j,k}$.

To conclude the proof of Theorem 3, it suffices to check that the countable set $\{f_{i,j,k}(x): i,j,k\geq 1\}$ is dense in $F(x)$ for every $x\in X$. To see this, fix any $\varepsilon>0$, $x\in X$ and $y\in F(x)$. Let i, j be such that $2^{1-j}<\varepsilon$ and $d(y,y_i)<2^{-j}$. Then $x\in U_{i,j}$ and therefore there exists k such that $x\in C_{i,j,k}$. Hence

$$d(f_{i,j,k}(x),y)\leq d(f_{i,j,k}(x),y_i)+d(y_i,y)\leq 2^{-j}+2^{-j}<\varepsilon\,. \ ■$$

Remark. Theorem 3 still holds under the weaker assumption that X be a perfectly normal space, i.e. that any open subset of X be the union of countably many closed sets. The proof relies on the same arguments used for Proposition 5.2 and Theorem 3.1 in [8].

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Representing and absolutely representing systems

by

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Abstract. We introduce various classes of representing systems in linear topological spaces and investigate their connections in spaces with different topological properties. Let us cite a typical result of the paper. If H is a weakly separated sequentially separable linear topological space then there is a representing system in H which is not absolutely representing.

A sequence $X = (x_k)_{k=1}^{\infty}$ of elements of a space (everywhere below the word "space" means "linear topological space") H over a field Φ is called a *basis* in H (see e.g. [6]) if for each x in H there exists a uniquely determined sequence $\{\eta_k\}_{k=1}^{\infty}$ of scalars from Φ such that the series $\sum_{k=1}^{\infty} \eta_k x_k$ converges to x (everywhere below $\Phi = \mathbb{C}$ or $\mathbb{R}$). A basis X in a locally convex space H is said to be *absolute* if for each x in H the corresponding series $\sum_{k=1}^{\infty} \eta_k x_k$ converges absolutely in H (to x). As is well known, there exist bases in Banach spaces which are not absolute. On the other hand, according to the Dynin–Mityagin theorem [6], each basis in a nuclear Fréchet space is absolute. A. A. Talalyan [7] introduced representing systems in a complete metrizable space as a natural generalization of bases. A sequence $X = (x_k)_{k=1}^{\infty}$ of elements of a space H is called a *representing system* (r.s.) if each x in H can be represented in the form of a series

$$x = \sum_{k=1}^{\infty} \alpha_k x_k \tag{1}$$

converging in H. The class of spaces having at least one r.s. is much wider than the class of spaces with basis. According to [1], every nuclear Fréchet space not isomorphic to ω has a quotient space without a basis. As for r.s., we can give a criterion for a space to have an r.s. We say that a space H is *sequentially separable* if there exists a "universal" sequence $V = \{v_k\}_{k=1}^{\infty}$ in H such that for each x in H one can find a subsequence $(v_{n_k})_{k=1}^{\infty}$ tending to x in H. For example, every separable space with a countable defining

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