Selections and representations of multifunctions
in paracompact spaces

by

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Abstract. Let \((X, T)\) be a paracompact space, \(Y\) a complete metric space, \(F : X \to 2^Y\) a lower semicontinuous multifunction with nonempty closed values. We prove that if \(T^+\) is a (stronger than \(T\)) topology on \(X\) satisfying a compatibility property, then \(F\) admits a \(T^+\)-continuous selection. If \(Y\) is separable, then there exists a sequence \((f_n)\) of \(T^+\)-continuous selections such that \(F(x) = \{f_n(x); n \geq 1\}\) for all \(x \in X\). Given a Banach space \(E\), the above result is then used to construct directionally continuous selections on arbitrary subsets of \(R \times E\).

1. Introduction. In the study of differential inclusions, it is often desirable to reduce the multivalued problem to an ordinary differential equation in the same space, constructing a continuous selection of the right hand side. Among the earliest selection theorems, the following results of Michael are well known:

[7, Thm. 1] If \(X\) is a paracompact topological space, every lower semicontinuous multifunction \(F\) from \(X\) into the nonempty, closed and convex subsets of a Banach space \(Y\) admits a continuous selection.

[7, Thm. 2] If \(X\) is paracompact and zero-dimensional, every lower semicontinuous multifunction \(F\) from \(X\) into the nonempty closed subsets of a complete metric space \(Y\) admits a continuous selection.

We recall that a normal topological space is zero-dimensional if and only if every locally finite open covering of it admits a disjoint open refinement \([7, \text{Prop. 2}].\)

In cases where these results do not apply, one can introduce a finer topology \(T^+\) on \(X\) and ask for selections of \(F\) which are \(T^+\)-continuous. A result in this direction is:

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[6, Thm. 1] Let \( X \) be a locally compact subset of a metric space and let \( F \) be a lower semicontinuous multifunction from \( X \) into the nonempty, closed subsets of a complete metric space \( Y \). Let \( T^+ \) be a second topology on \( X \) with the property:

\[(P_0) \quad \text{every } x \in X \text{ has a basis of neighborhoods (in the metric topology) consisting of } T^+\text{-closed-open sets.}\]

Then \( F \) admits a \( T^+ \)-continuous selection.

Given a Banach space \( E \) and a number \( M > 0 \), the family of all conical neighborhoods

\[(1.1) \quad \Gamma^M(t_0, x_0, \delta) = \{ (t, x) \in \mathbb{R} \times E : \|x - x_0\| \leq M(t - t_0), t_0 \leq t \leq t_0 + \delta \}\]

for all \( \delta > 0 \), \( (t_0, x_0) \in \mathbb{R} \times E \), generates a topology \( T^+ \) which satisfies the assumption \((P_0)\). Functions which are continuous w.r.t. this stronger topology will be called \( \Gamma^M \)-continuous, or simply directionally continuous. By constructing suitable directionally continuous selections, various results on the qualitative theory of differential inclusions were recently proved [3, 4, 5]. Until now, however, all applications have been confined to problems in locally compact spaces, mainly finite-dimensional. The aim of this paper is to remove the restriction on the domain of \( F \) in [6, Thm. 1], letting \( X \) be any paracompact topological space, as in Michael's theorems. This will allow the use of our selection technique in connection with infinite-dimensional differential inclusions, in full generality.

Instead of \((P_0)\), we consider the stronger property:

\[(P) \quad \text{For every pair of sets } A \subset B, \text{ with } A \text{ closed and } B \text{ open (in the original topology } T \text{), there exists a set } C, \text{ closed-open w.r.t. } T^+, \text{ such that } A \subset C \subset B.\]

The following results will be proved.

**Theorem 1.** Let \( (X, T) \) be a paracompact space, \( (Y, d) \) a complete metric space and \( F : X \to 2^Y \) a lower semicontinuous multifunction with nonempty closed values. If \( T^+ \) is a topology on \( X \) with the property \((P)\), then \( F \) admits a \( T^+ \)-continuous selection.

**Theorem 2.** Let \( E \) be a Banach space, \( M > 0 \), \( \Omega \) a subset of \( \mathbb{R} \times E \). Any lower semicontinuous multifunction \( F : \Omega \to 2^E \) with nonempty closed values admits a \( \Gamma^M \)-continuous selection.

**Theorem 3.** Let \( X \) be a metric space, \( Y \) a separable complete metric space and \( F : X \to 2^Y \) a lower semicontinuous multifunction with nonempty closed values. If \( T^+ \) is a topology on \( X \) with the property \((P)\), then there exists a sequence \( f_n \) of \( T^+ \)-continuous selections from \( F \) such that, for every \( x \in X \), the closure of the set \( \{ f_n(x) : n \geq 1 \} \) coincides with \( F(x) \).

For the basic theory of multifunctions and differential inclusions we refer to [1].

**2. Proof of Theorem 1.** Following a well-established argument due to Michael, the selection is obtained as the limit of a uniformly converging sequence of \( T^+ \)-continuous approximations. By induction, we shall construct functions \( (f_n)_{n \geq 1} \) with the properties:

\[(i)_n \quad \text{there exists a } T^+\text{-open and disjoint covering } \mathcal{O}^n = \{ \Omega^n_\alpha \}_{\alpha \in A^n} \text{ of } X; \]

\[(ii)_n \quad f_n \text{ is constant on } \Omega^n_\alpha, \text{ say, } f_n(x) = y^n_\alpha \text{ for all } x \in \Omega^n_\alpha; \]

\[(iii)_n \quad d(y^n_\alpha, F(x)) < 2^{-n} \forall x \in cl(\Omega^n_\alpha), \forall \alpha \in A^n; \]

\[(iv)_n \quad d(f_n(x), f_{n-1}(x)) < 2^{-n+1} \forall x \in X (n \geq 2).\]

To construct \( f_1 \), using the lower semicontinuity of \( F \), for every \( x \in X \) choose a point \( y^1_\alpha \in F(x) \) and a neighborhood \( U^1 \) of \( x \) such that

\[(2.1) \quad d(y^1_\alpha, F(x')) < 2^{-1} \forall x' \in U^1.\]

Let \( (V^1_\alpha)_{\alpha \in A^1} \) be a locally finite open refinement of \( (U^1_\alpha)_{\alpha \in A^1} \), say with \( V^1_\alpha \subset U^1_\alpha \), and let \( (W^1_\alpha)_{\alpha \in A^1} \) be another open refinement such that \( cl(W^1_\alpha) \subset V^1_\alpha \) for all \( \alpha \in A^1 \). Here and throughout the paper, \( cl(W) \) and \( int(W) \) will denote the closure and the interior of a set \( W \) in the original topology \( T \).

By the property \((P)\), for each \( \alpha \) one can choose a set \( Z^1_\alpha \), closed-open w.r.t. \( T^+ \), such that

\[(2.2) \quad cl(W^1_\alpha) \subset int(Z^1_\alpha) \subset cl(Z^1_\alpha) \subset V^1_\alpha.\]

Then \( (Z^1_\alpha) \) is a locally finite \( T^+\)-closed-open covering of \( X \). Choose a well-ordering \( \preceq \) of the set \( A^1 \) and define, for each \( \alpha \in A^1 \),

\[\Omega^1_\alpha = Z^1_\alpha \setminus \bigcup_{\beta \prec \alpha} Z^1_\beta.\]

Set \( \mathcal{O}^1 = (\Omega^1_\alpha)_{\alpha \in A^1} \). By the well-ordering, every \( x \in X \) belongs to exactly one set \( \Omega^1_\alpha \), where \( \overline{\alpha} = \min\{ \alpha \in A^1 : x \in Z^1_\alpha \} \). Hence, \( \mathcal{O}^1 \) is a partition of \( X \). Moreover, since \( (Z^1_\alpha) \) is locally finite (w.r.t. \( T \) and therefore w.r.t. \( T^+ \)), the sets \( \bigcup_{\beta \leq \alpha} Z^1_\beta \) are \( T^+\)-closed-open. Hence \( \mathcal{O}^1 \) is a \( T^+\)-closed-open, disjoint covering of \( X \) such that, by (2.2), \( cl(\Omega^1_\alpha) \) refines \( (V^1_\alpha) \). By setting \( y^n_\alpha = y^n_{\overline{\alpha}} \) and

\[f_1(x) = y^n_\alpha \forall x \in \Omega^n_\alpha, \forall \alpha \in A^n,\]

we obtain a \( T^+\)-continuous function which, by (2.1), satisfies \((i)_1, (ii)_1\).

Suppose now that functions \( f_k \) have been constructed satisfying the properties \((i)_k, (ii)_k, (iii)_k\), for \( 1 \leq k < n \). Since \( \mathcal{O}^{n-1} \) is a disjoint \( T^+\)-open covering, the map \( f_n \) can be defined separately on each set \( \Omega^{n-1}_\alpha \). Fix \( \alpha \in A^{n-1} \).
For every \( x \in \text{cl}(\Omega^{n-1}_\alpha) \), by (ii) \((ii)_{n-1}\) and lower semicontinuity there exist \( y_x \in F(x) \) and a neighborhood \( U_x \) of \( x \) such that
\[
\begin{align*}
    d(y_x, F(x')) < 2^{-n} & \quad \forall x' \in U_x, \\
    d(y_x, y^{n-1}_x) < 2^{-n+1}.
\end{align*}
\]

Since \( \text{cl}(\Omega^{n-1}_\alpha) \) is paracompact, the same argument used in the first induction step provides two locally finite open refinements \((V'_\beta), (W_\beta), \beta \in A^n_\alpha\) of \((U_x)_{x \in \text{cl}(\Omega^{n-1}_\alpha)}\) such that \( \text{cl}(W_\beta) \subset U_{y^\beta} \subset U_x \) for some \( x^\beta \in \text{cl}(\Omega^{n-1}_\alpha) \), for all \( \beta \in A^n_\alpha \). Using the property (P), construct a family \((Z_\beta)_{\beta \in A^n_{\alpha}}\) of \( T^+\)-closed-open subsets of \( \text{cl}(\Omega^{n-1}_\alpha) \) such that
\[
\text{cl}(W_\beta) \subset \text{int}(Z_\beta) \subset \text{cl}(Z_\beta) \subset U_{y^\beta} \quad \forall \beta \in A^n_\alpha,
\]
and let \( \preceq \) be a well-ordering on \( A^n_\alpha \). As in the first step, by setting
\[
\Omega_{\alpha, \beta}^n = Z_\beta \setminus \left( \bigcup_{\gamma \prec \beta} Z_\gamma \right)
\]
we get a disjoint \( T^+\)-closed-open covering of \( \text{cl}(\Omega^{n-1}_\alpha) \) such that \( \text{cl}(\Omega_{\alpha, \beta}^n) \) refines \((V'_\beta)\). Set now \( y^n_{\alpha, \beta} = y_{x^\beta} \) and define the \( T^+\)-continuous map \( f_{\alpha} : \Omega_{\alpha}^n \to Y \) as
\[
f_{\alpha}(x) = y^n_{\alpha, \beta} \quad \forall x \in \Omega_{\alpha, \beta}^n.
\]
Repeat the above construction for all \( \alpha \in A^{n-1} \) and define
\[
A^n = \bigcup_{\alpha \in A^{n-1}} (\{\alpha\} \times A^n_\alpha), \quad \Omega^n = (\Omega_{\alpha, \beta}^n)_{(\alpha, \beta) \in A^n_\alpha}.
\]
By (2.3), (2.4), the properties (i)–(iii) are satisfied.

The completeness of \( Y \) and (iii) now imply that the sequence \( (f_{n}) \) converges uniformly to a \( T^+\)-continuous function \( f \). Since the values of \( F \) are closed, by (i) and (ii) it follows that \( f \) is a selection from \( F \).

3. Proof of Theorem 2. Theorem 2 is an immediate consequence of Theorem 1 and of the following Lemma, showing that the property (P) holds for the topology \( T^+ \) generated by the family of all conical neighborhoods \( T^+ \) defined in (1.1).

**Lemma.** Let \( E \) be a Banach space and let \( A, B \) be disjoint and closed subsets of \( \mathbb{R} \times E \). Then there exists \( C \subseteq \mathbb{R} \times E \) such that
\[
\begin{align*}
    (i) A \subset \text{cl}(C) \cap B = \emptyset; \\
    (ii) C \text{ is } T^+\text{-closed-open.}
\end{align*}
\]

**Proof.** For every \((t, x) \in \mathbb{R} \times E, n \geq 1\), consider the open neighborhood
\[
L_n(t, x) = \{(s, y) : \|y - x\| < 2^{-n} - M|t - s|\}.
\]

Observe that, for every \( n \),
\[
\bigcup_{k \in \mathbb{Z}, x \in E} L_n \left( \frac{k}{M \cdot 2^n}, x \right) = \mathbb{R} \times E.
\]

In the following, on \( \mathbb{R} \times E \) we use the distance \( d((t, x), (s, y)) = \max\{\|t - s\|, \|x - y\|\} \). Define the closed set \( A^* = \{(t, x) \in \mathbb{R} \times E : d((t, x), A) \leq d((t, x), B)\} \). For every \((t, x) \in A \) choose \( k = k(t, x) \in \mathbb{Z} \) and \( n = n(t, x) \geq 1 \) such that,
\[
L_{t, x} = L_n(t, x) \left( \frac{k(t, x)}{M \cdot 2^{n(t, x)}}, x \right).
\]

One has
\[
(t, x) \in A^* \quad \text{and} \quad L_{t, x} \subset A^*.
\]

The sets \( L_{t, x} \) can also be written as
\[
L_{t, x} = \left\{(s, y) : \psi_{t, x}(y) < s < \psi_{t, x}(y) \right\},
\]
where
\[
\begin{align*}
    \psi_{t, x}(y) &= \frac{k(t, x)}{M \cdot 2^{n(t, x)}} - \frac{2^{-n(t, x)} - \|y - x\|}{M}, \\
    \psi_{t, x}(y) &= \frac{k(t, x)}{M \cdot 2^{n(t, x)}} + \frac{2^{-n(t, x)} - \|y - x\|}{M}.
\end{align*}
\]

Define
\[
\varphi_{n, k}(y) = \inf \left\{ \psi_{t, x}(y) : (t, x) \in A, \ n(t, x) = n, \ k(t, x) = k \right\},
\]
\[
\psi_{n, k}(y) = \sup \left\{ \psi_{t, x}(y) : (t, x) \in A, \ n(t, x) = n, \ k(t, x) = k \right\}
\]
and
\[
W_{n, k} = \{(s, y) \in \mathbb{R} \times E : \varphi_{n, k}(y) \leq s < \psi_{n, k}(y) \}
\]

Since all maps \( \varphi_{t, x}, \psi_{t, x} \) are Lipschitz continuous with constant \( 1/M \), the same is true for every \( \varphi_{n, k}, \psi_{n, k} \). We claim that each \( W_{n, k} \) is closed-open in the stronger topology \( T^+ \). Indeed, if \((t, x) \in W_{n, k}\), then \( t = \psi_{n, k}(x) - \delta \) for some \( \delta > 0 \). By Lipschitz continuity we thus have
\[
\{(s, y) : \|y - x\| \leq M|s - t|, \ t \leq s < t + \delta/2\} \subset W_{n, k},
\]
showing that \((t, x)\) is an interior point. On the other hand, if \((t, x) \notin W_{n, k}\), then either \( t \geq \psi_{n, k}(x) \) or \( t = \varphi_{n, k}(x) - \delta \) for some \( \delta > 0 \) and
\[
\{(s, y) : \|y - x\| \leq M|s - t|, \ t \leq s < t + \delta/2\} \cap W_{n, k} = \emptyset.
\]

In both cases, \((t, x)\) does not belong to the closure of \( W_{n, k}\).
We now show that the requirements of the Lemma are satisfied by the set
\[ C = \bigcup_{n \geq 1} W_{n,k}. \]

By (3.3), every \((t,x) \in A\) belongs to \(\{(s,y) : \psi_{t,s}(y) < s < \psi_{t,x}(y)\}\), and therefore to the interior of some \(W_{n,k}\). Moreover, \(C \subseteq cl(\bigcup_{(t,x) \in A} L_{t,x}) \subseteq A^*\); hence \(cl(C) \cap B = \emptyset\). This proves (i).

Concerning (ii), it is clear that \(C\) is open in the topology \(T^+\), being a union of open sets. To prove that \(C\) is closed, assume \((\bar{t}, \bar{x}) \notin C\). Then \(r = d((\bar{t}, \bar{x}), A) > 0\). Observe that, if \(d((s,y), A) > r/2\) for some \((s,y) \in L_{t,x}\), then \(diam(L_{t,x}) = \max \{2^{1-n(x,s)}, 2^{1-n(\bar{t}, \bar{y})}/M\} > r/2\), i.e.
\[ n(t, x) < N_r = -\log_2(\max\{r/4, M/r/4\}). \]

As a consequence, the ball \(B((\bar{t}, \bar{x}), r/2)\) intersects only finitely many sets \(W_{n,k}\). Indeed, if \(1 \leq n < N_r\), the sets
\[ L_n(k/(M \cdot 2^n), x) \cap B((\bar{t}, \bar{x}), r/2) \]

can be nonempty only for the finitely many \(k \in \mathbb{Z}\) such that \(|kM^{-1}2^{-n} - \bar{t}| < r/2 + M^{-1}2^{-n}\). We now have
\[ (\bar{t}, \bar{x}) \notin B((\bar{t}, \bar{x}), r/2) \setminus \bigcup_{n,k} W_{n,k}, \]
the union being taken over the finitely many indices \((n, k)\) for which \(W_{n,k} \cap B((\bar{t}, \bar{x}), r/2) \neq \emptyset\). Clearly the set in (3.6) is open w.r.t. \(T^+\) and does not intersect \(C\). This proves that \(C\) is closed in the topology \(T^+\), establishing (ii).

Remark. For the topology \(T^+\) generated by the family \(\{1, t\}\), each conical neighborhoods \(\Gamma^M(t, x, \epsilon, \delta)\) is actually a \(T^+\)-closed-open set. However, our Theorem 2 does not follow from [7, Thm. 2], because \((\mathbb{R} \times E, T^+)\) is never paracompact, provided \(E \neq \{0\}\). Indeed, consider the \(T^+\)-open covering of \(R \times E\)
\[ \mathcal{O} = \{ \Gamma^M(k, x, 1) \mid k \in \mathbb{Z}, x \in E \}. \]
Any point of the form \((0, x)\) belongs to exactly one set of \(\mathcal{O}\). Hence every \(T^+\)-open refinement of \(\mathcal{O}\) must contain a conical neighborhood \(\Gamma^M(0, x, \epsilon, \delta)\) with \(\epsilon = \epsilon(x) > 0\), for each \(x \in E\). For any fixed \(\bar{z}\) we have
\[ \bigcup_{\nu \geq 1} \{ \lambda \in [0, 1] : \lambda \bar{z} > \nu^{-1} \} = \bigcup_{\nu \geq 1} \Sigma_\nu = [0, 1]. \]
Therefore, some \(\Sigma_{\nu}\) is infinite (actually, uncountable), and has a cluster point, say \(\bar{\lambda}\). Any \(T^+\)-neighborhood of \(\bar{\lambda}\bar{z}\) then intersects infinitely many cones \(\Gamma^M(0, x, \epsilon(x))\), showing that \(\mathcal{O}\) is not locally finite.

4. Proof of Theorem 3. The following proof is an adaptation of the arguments used in [8, Lemma 5.2].
Let \((y_i)_{i \geq 1}\) be a countable, dense subset of \(Y\) and set, for each \(i, j \geq 1\),
\[ U_{i,j} = \{ x \in X : F(x) \cap B(y_i, 2^{-j}) \neq \emptyset \}. \]
By the lower semicontinuity of \(F\), each \(U_{i,j}\) is open, and therefore it is a countable union of closed subsets of \(X\), say
\[ U_{i,j} = \bigcup_{k \geq 1} C_{i,j,k}. \]
Define
\[ F_{i,j,k}(x) = \begin{cases} F(x) & \text{if } x \notin C_{i,j,k}, \\ cl(F(x) \cap B(y_i, 2^{-j})) & \text{if } x \in C_{i,j,k}. \end{cases} \]
Then each \(F_{i,j,k}\) is lower semicontinuous with closed values. By Theorem 1, for every \(i, j, k\), there exists a \(T^+\)-continuous selection \(f_{i,j,k}\) from \(F_{i,j,k}\).
To conclude the proof of Theorem 3, it suffices to check that the countable set \(\{f_{i,j,k}(x) : i, j, k \geq 1\}\) is dense in \(F(x)\) for every \(x \in X\). To see this, fix any \(\epsilon > 0\), \(x \in X\) and \(y \in F(x)\). Let \(i, j\) be such that \(2^{-j} < \epsilon\) and \(d(y, y_i) < 2^{-j}\). Then \(x \in U_{i,j}\) and therefore there exists \(k\) such that \(x \in C_{i,j,k}\). Hence
\[ d(f_{i,j,k}(x), y) \leq d(f_{i,j,k}(x), y_i) + d(y_i, y) \leq 2^{-j} + 2^{j} < \epsilon. \]

Remark. Theorem 3 still holds under the weaker assumption that \(X\) be a perfectly normal space, i.e. that any open subset of \(X\) be the union of countably many closed sets. The proof relies on the same arguments used for Proposition 5.2 and Theorem 3.1 in [8].

References

Representing and absolutely representing systems

by

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Abstract. We introduce various classes of representing systems in linear topological spaces and investigate their connections in spaces with different topological properties. Let us cite a typical result of the paper. If $H$ is a weakly separated sequentially separable linear topological space then there is a representing system in $H$ which is not absolutely representing.

A sequence $X = (x_k)_{k=1}^{\infty}$ of elements of a space (everywhere below the word "space" means "linear topological space") $H$ over a field $\Phi$ is called a basis in $H$ (see e.g. [6]) if for each $x$ in $H$ there exists a uniquely determined sequence $\{\eta_k\}_{k=1}^{\infty}$ of scalars from $\Phi$ such that the series $\sum_{k=1}^{\infty} \eta_k x_k$ converges to $x$ (everywhere below $\Phi = \mathbb{C}$ or $\mathbb{R}$). A basis $X$ in a locally convex space $H$ is said to be absolute if for each $x$ in $H$ the corresponding series $\sum_{k=1}^{\infty} \eta_k x_k$ converges absolutely in $H$ (to $x$). As is well known, there exist bases in Banach spaces which are not absolute. On the other hand, according to the Dynin–Mityagin theorem [6], each basis in a nuclear Fréchet space is absolute. A. A. Talalyan [7] introduced representing systems in a complete metrizable space as a natural generalization of bases. A sequence $X = (x_k)_{k=1}^{\infty}$ of elements of a space $H$ is called a representing system (r.s.) if each $x$ in $H$ can be represented in the form of a series

$$x = \sum_{k=1}^{\infty} \alpha_k x_k$$

converging in $H$. The class of spaces having at least one r.s. is much wider than the class of spaces with basis. According to [1], every nuclear Fréchet space not isomorphic to $\omega$ has a quotient space without a basis. As for r.s., we can give a criterion for a space to have an r.s. We say that a space $H$ is sequentially separable if there exists a "universal" sequence $V = (v_k)_{k=1}^{\infty}$ in $H$ such that for each $x$ in $H$ one can find a subsequence $(v_{k_n})_{k=1}^{\infty}$ tending to $x$ in $H$. For example, every separable space with a countable defining