Contents of Volume 102, Number 3

STUDIA MATHEMATICA

Managing Editors: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

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Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-293997

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Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TEX at the Institute
Printed and bound by
drukarnia
herman & herman
02-240 WARSZAWA, ul. Jakobinów 23

PRINTED IN POLAND

ISBN 83-85116-49-4 ISSN 0039-3223

STUDIA MATHEMATICA 102 (3) (1992)

Uniqueness of unconditional bases of $c_0(l_p)$, $0 < p < 1$

by

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Abstract. We prove that if $0 < p < 1$ then a normalized unconditional basis of a complemented subspace of $c_0(l_p)$ must be equivalent to a permutation of a subset of the canonical unit vector basis of $c_0(l_p)$. In particular, $c_0(l_p)$ has unique unconditional basis up to permutation. Bourgain, Casazza, Lindenstrauss, and Tzafriri have previously proved the same result for $c_0(l_1)$.

1. Introduction. If X is a quasi-Banach space with unconditional basis we say that X has *unique normalized unconditional basis* (*up to permutation*) if whenever $(e_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ are two normalized unconditional bases of X, $(x_n)_{n\in\mathbb{N}}$ is equivalent to (a permutation of) $(e_n)_{n\in\mathbb{N}}$.

It is well known that the only Banach spaces with unique normalized unconditional basis are c_0, l_1, and l_2 ([8], [10]). In the wider class of quasi-Banach spaces, however, we find many other spaces with that property, including l_p $(0 < p < 1)$ (see [3], [5]). Bourgain, Casazza, Lindenstrauss and Tzafriri considered in [2] the uniqueness up to permutation of the normalized unconditional basis of direct sums of the spaces c_0, l_1, and l_2. They proved the following theorem:

THEOREM 1.1 (Theorem 4.1 of [2]). *Let Q be a bounded linear projection from $c_0(l_1)$ onto a subspace Z which has a normalized K-unconditional basis $(z_n)_{n=1}^{\eta}$. Then there exist a constant D, depending only on K and $\|Q\|$, and a partition $(B_j)_{j=1}^{J}$ of the integers $\{1,\ldots,\eta\}$ into mutually disjoint subsets so that*

$$(1.1)\qquad D^{-1}\max_{1\le j\le J}\sum_{n\in B_j}|a_n| \le \Big\|\sum_{n=1}^{\eta}a_nz_n\Big\| \le D\max_{1\le j\le J}\sum_{n\in B_j}|a_n|$$

for any choice of scalars $a_1,\ldots,a_\eta$. In particular, $c_0(l_1)$ has unique normalized unconditional basis up to permutation.

1991 *Mathematics Subject Classification*: 46A35, 46A16, 46B15.
The research of the author was partially supported by NSF-grant DMS 8901636.

Motivated by this result and by the fact that l_p $(0<p<1)$ has unique normalized unconditional basis, we will show in Section 2 the analogue of Theorem 1.1 for $c_0(l_p)$ $(0<p<1)$.

We recall that if X is a quasi-Banach space whose dual separates points and $0<q\le 1$ then the gauge functional of the q-convex hull of the closed unit ball of X is a q-norm on X that we will denote by $\|\cdot\|_q$. The q-Banach space $\widehat{X}_q$ resulting from the completion of $(X,\|\cdot\|_q)$ is called the *q-Banach envelope* of X (see [4] and [6]). The q-Banach envelope has the property that every continuous linear operator from X into a q-Banach space extends to $\widehat{X}_q$ with preservation of norm. In particular, the dual of $\widehat{X}_q$ is X^*. The 1-Banach envelope is a Banach space and will be called simply the *Banach envelope*. If $(e_n)_{n\in\mathbb{N}}$ is a K-unconditional basis of X then it is also a K-unconditional basis of $\widehat{X}_q$, and

$$K^{-1}\|e_n\|\le\|e_n\|_q\le\|e_n\| \quad \text{for all } n\in\mathbb{N}. \tag{1.2}$$

Since the Banach envelope of $c_0(l_p)$ $(0<p<1)$ is $c_0(l_1)$, the result in Theorem 1.1 will be essential to our arguments. However, the techniques in [2] do not extend to the non-locally convex case. Instead, we will follow the approach used for proving uniqueness results in quasi-Banach spaces in [3] and [5].

We recall that a quasi-Banach lattice X is said to be *p-convex*, where $0<p<\infty$, if for some constant C and for all $x_1,\dots,x_n\in X$ we have

$$\Big\|\Big(\sum_{i=1}^n|x_i|^p\Big)^{1/p}\Big\|\le C\Big(\sum_{i=1}^n\|x_i\|^p\Big)^{1/p}.$$

The procedure to define the element $(\sum_{i=1}^n|x_i|^p)^{1/p}\in X$ is described in [9], pp. 40–41.

THEOREM 1.2. *Let X be a p-convex quasi-Banach lattice $(0<p\le 1)$.*

(i) (Proposition 2.2 of [5]) *There is a constant A_1 such that*

$$\Big\|\Big(\sum_{i=1}^n\sum_{j=1}^n|x_{ij}|^2\Big)^{1/2}\Big\|\le A_1\int_0^1\int_0^1\Big\|\sum_{i=1}^n\sum_{j=1}^n x_{ij}r_i(t)r_j(s)\Big\|\,dt\,ds$$

for any X-valued matrix $(x_{ij})_{i,j=1}^n$, where $(r_i)_{i\in\mathbb{N}}$ denotes the sequence of Rademacher functions on [0,1].

(ii) (Proposition 2.1 of [5]) *If $(x_n)_{n\in\mathbb{N}}$ is an unconditonal basis of X then there is a constant A_2, depending only on X and the unconditional basis constant, so that*

$$A_2^{-1}\Big\|\Big(\sum_{i=1}^n|a_ix_i|^2\Big)^{1/2}\Big\|\le\Big\|\sum_{i=1}^n a_ix_i\Big\|\le A_2\Big\|\Big(\sum_{i=1}^n|a_ix_i|^2\Big)^{1/2}\Big\|$$

for any scalars $a_1,\dots,a_n$.

(i) is a vector version of Bonami's extension of Khinchin's inequality ([1]), and (ii) is a generalization of the Maurey–Khinchin inequalities ([9], [11]).

THEOREM 1.3 (cf. Lemma 6.3 of [3] and Theorem 2.3 of [5]). *Let X be a p-convex quasi-Banach lattice $(0<p<1)$ with a normalized unconditional basis $(e_n)_{n\in\mathbb{N}}$; and let Q be a bounded linear projection from X onto a subspace Z with a normalized unconditional basis $(x_n)_{n\in S}$ $(S\subseteq\mathbb{N})$. Let $(e_n^*)_{n\in\mathbb{N}}$ and $(x_n^*)_{n\in S}$ be the sequences of biorthogonal linear functionals associated with $(e_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in S}$ respectively, i.e.*

$$x=\sum_{n=1}^\infty e_n^*(x)e_n \quad\text{and}\quad Q(x)=\sum_{n\in S}x_n^*(x)x_n$$

for all $x\in X$. Suppose that there is a constant $\beta>0$ and an injective map $\sigma:S\to\mathbb{N}$ so that

$$|e_{\sigma(n)}^*(x_n)|\ge\beta \quad\text{and}\quad |x_n^*(e_{\sigma(n)})|\ge\beta$$

for all $n\in S$. Then the unconditional basic sequences $(x_n)_{n\in S}$ and $(e_{\sigma(n)})_{n\in S}$ are equivalent.

2. Uniqueness of unconditional bases of $c_0(l_p)$. Let $0<p<1$ be fixed. Throughout this section $\|\cdot\|$ will denote without confusion both the quasi-norm in $c_0(l_p)$ and the norm in the dual $l_1(l_\infty)$; and if $0<p<q\le 1$, $\|\cdot\|_q$ will denote the (quasi-)norm in the q-Banach envelope $c_0(l_q)$. The canonical 1-unconditional basis of unit vectors of $c_0(l_p)$ will be denoted by $(e_{lk})_{l,k=1}^\infty$. Here if $b=\sum_{l,k=1}^\infty b_{lk}e_{lk}\in c_0(l_p)$ then

$$\|b\|=\sup_l\Big(\sum_{k=1}^\infty|b_{lk}|^p\Big)^{1/p}.$$

The lattice structure induced by the canonical basis in $c_0(l_p)$ is clearly p-convex. We will devote this section to the proof of the following theorem:

THEOREM 2.1. *Let Q be a bounded linear projection from $c_0(l_p)$ onto a subspace X with a normalized K-unconditional basis $(x_n)_{n=1}^\eta$. Then there exist a constant Δ and a partition of the integers $\{1,\dots,\eta\}$ into mutually disjoint subsets $(L_i)_{i=1}^I$ so that*

$$\Delta^{-1}\max_{1\le i\le I}\Big(\sum_{n\in L_i}|a_n|^p\Big)^{1/p}\le\Big\|\sum_{n=1}^\eta a_nx_n\Big\|\le\Delta\max_{1\le i\le I}\Big(\sum_{n\in L_i}|a_n|^p\Big)^{1/p}$$

for any scalars $a_1,\dots,a_\eta$.

Before we attempt the proof of Theorem 2.1 we need to establish a few lemmas. We will begin by introducing some more notation.

The sequences in $l_1(l_\infty)$ of the biorthogonal linear functionals associated with (e_{lk}) and (x_n) are denoted by (e^*_{lk}) and (x^*_n) respectively. For abbreviation we will write

$$e^*_{lk}(x_n) = b^n_{lk} \quad \text{and} \quad x^*_n(e_{lk}) = a^n_{lk}$$

for $l, k \in \mathbb{N}, 1 \le n \le \eta$. The Banach envelope of X, $\widehat{X}$, is a complemented subspace of $c_0(l_1)$ and $(x_n)_{n=1}^{\eta}$ is a K-unconditional basis of $\widehat{X}$. By (1.2), the basis $(x_n)_{n=1}^{\eta}$ is equivalent in $\widehat{X}$ to the normalized basis $(x_n/\|x_n\|_1)_{n=1}^{\eta}$. Therefore, Theorem 1.1 applies, hence there exist a constant D, depending only on K and $\|Q\|$, and a partition of $\{1,\ldots,\eta\}$ into disjoint subsets $(B_j)_{j=1}^{J}$ so that (1.1) holds, that is,

$$D^{-1} \max_{1\le j\le J} \sum_{n\in B_j} |a_n| \le \Big\| \sum_{n=1}^{\eta} a_n x_n \Big\|_1 \le D \max_{1\le j\le J} \sum_{n\in B_j} |a_n| \tag{2.1}$$

for any scalars $a_1, \ldots, a_\eta$.

Lemma 2.2 states a simple fact that will be useful in the sequel.

LEMMA 2.2. *Fix $0 < r < q < \infty$. For any $\varepsilon > 0$ there is C_ε so that*

$$\Big(\sum_{n=1}^{\infty} |a_n|^q \Big)^{1/q} \le C_\varepsilon \sup_n |a_n| + \varepsilon \Big(\sum_{n=1}^{\infty} |a_n|^r \Big)^{1/r}$$

for any choice of $a_1, \ldots, a_n, \ldots$

Proof. It is enough to prove this when $(\sum_{n=1}^{\infty} |a_n|^r)^{1/r} = 1$ and $(|a_n|)_{n\in\mathbb{N}}$ decreases. Then

$$n|a_n|^r \le \sum_{k=1}^{n} |a_k|^r \le 1$$

for all n. For each $\varepsilon > 0$ choose

$$n_\varepsilon = \Big[\Big(\frac{\delta_q}{(q/r-1)^{1/q}\varepsilon} \Big)^{qr/(q-r)} \Big] + 1$$

where $\delta_q = \max(1, 2^{1/q-1})$ and $[\cdot]$ denotes the *integer part* function. Then

$$\begin{aligned} \sum_{n=1}^{\infty} |a_n|^q &= \sum_{n=1}^{n_\varepsilon} |a_n|^q + \sum_{n=n_\varepsilon+1}^{\infty} |a_n|^q \\ &\le n_\varepsilon \sup_n |a_n|^q + \sum_{n=n_\varepsilon+1}^{\infty} \frac{1}{n^{q/r}} \\ &\le n_\varepsilon \sup_n |a_n|^q + \frac{1}{(q/r-1)n_\varepsilon^{q/r-1}} \\ &< n_\varepsilon \sup_n |a_n|^q + \varepsilon^q \delta_q^{-q}. \end{aligned}$$

Hence $C_\varepsilon \le \delta_q n_\varepsilon^{1/q} \le \delta_q^2((\delta_q(q/r-1)^{-1/q}\varepsilon^{-1})^{r/(q-r)} + 1)$. ■

Next, we will prove in Lemmas 2.3 and 2.4 that, for each j, $(x_n)_{n\in B_j}$ is equivalent to an l_q-basis in the q-Banach envelope $c_0(l_q)$ $(0 < p < q < 1)$. In the sequel, if A is a finite subset of $\mathbb{N}$, $|A|$ will denote the number of elements of A.

LEMMA 2.3. *There is a constant $\gamma > 0$ independent of j and N so that*

$$\Big\| \sum_{n\in A} x_n \Big\| \ge \gamma N^{1/p}$$

whenever $A \subseteq B_j$ with $|A| = N$.

Proof. Combining (2.1) and Theorem 1.2(ii) we can find a constant $0 < C < 1$, independent of j and N, so that whenever $A \subseteq B_j$ with $|A| = N$

$$CN < \Big\| \Big(\sum_{n\in A} |x_n|^2 \Big)^{1/2} \Big\|_1 = \sup_l \sum_{k=1}^{\infty} \Big(\sum_{n\in A} |b^n_{lk}|^2 \Big)^{1/2}.$$

We fix $l \in \mathbb{N}$ so that

$$CN \le \sum_{k=1}^{\infty} \Big(\sum_{n\in A} |b^n_{lk}|^2 \Big)^{1/2}. \tag{2.2}$$

On the other hand, $\sum_{k=1}^{\infty} |b^n_{lk}| \le \|x_n\| = 1$ for all $n \in A$ so, by Lemma 2.2 with $r = 1$, $q = 2$, and $\varepsilon = C/2$ (which yields $C_\varepsilon \le (2+C)/C \le 3/C$), we get

$$\begin{aligned} \sum_{k=1}^{\infty} \Big(\sum_{n\in A} |b^n_{lk}|^2 \Big)^{1/2} &\le \frac{3}{C} \sum_{k=1}^{\infty} \sup_{n\in A} |b^n_{lk}| + \frac{C}{2} \sum_{k=1}^{\infty} \sum_{n\in A} |b^n_{lk}| \\ &\le \frac{3}{C} \sum_{k=1}^{\infty} \sup_{n\in A} |b^n_{lk}| + \frac{C}{2} N. \end{aligned} \tag{2.3}$$

(2.2) and (2.3) together imply that

$$\sum_{k=1}^{\infty} \sup_{n\in A} |b^n_{lk}| \ge \frac{C^2}{6} N.$$

Therefore, since $\sup_{n\in A} |b^n_{lk}| \le 1$ and $p < 1$,

$$\begin{aligned} \Big\| \Big(\sum_{n\in A} |x_n|^2 \Big)^{1/2} \Big\| &= \sup_l \Big(\sum_{k=1}^{\infty} \Big(\sum_{n\in A} |b^n_{lk}|^2 \Big)^{p/2} \Big)^{1/p} \ge \Big(\sum_{k=1}^{\infty} (\sup_{n\in A} |b^n_{lk}|)^p \Big)^{1/p} \\ &\ge \Big(\sum_{k=1}^{\infty} \sup_{n\in A} |b^n_{lk}| \Big)^{1/p} \ge \Big(\frac{C^2}{6} \Big)^{1/p} N^{1/p}. \end{aligned}$$

Furthermore, by Theorem 1.2(ii), there is a constant A_2, depending only on p and K, such that

$$\Big\|\sum_{n\in A} x_n\Big\| \ge A_2^{-1}\Big\|\Big(\sum_{n\in A}|x_n|^2\Big)^{1/2}\Big\|.$$

Thus the result holds with $\gamma=(C^2/6)^{1/p}A_2^{-1}$. ■

LEMMA 2.4. *Fix q ($p<q<1$). There is a constant $\Gamma>0$, independent of j, so that*

$$\Big\|\sum_{n\in B_j} a_n x_n\Big\|_q \ge \Gamma\Big(\sum_{n\in B_j}|a_n|^q\Big)^{1/q} \tag{2.4}$$

for any scalars $(a_n)_{n\in B_j}$, $j=1,\dots,J$.

Proof. Fix $j\in\{1,\dots,J\}$, and call $\eta_j=|B_j|$. Let $(b_n)_{n=1}^{\eta_j}$ be a decreasing rearrangement of $(|a_n|)_{n\in B_j}$. From the K-unconditionality of $(x_n)_{n\in B_j}$ and Lemma 2.3 we deduce that

$$b_n\le K\gamma^{-1}n^{-1/p}\Big\|\sum_{m\in B_j}a_m x_m\Big\|$$

for $1\le n\le \eta_j$. Now,

$$\begin{aligned}\Big(\sum_{n\in B_j}|a_n|^q\Big)^{1/q} &= \Big(\sum_{n=1}^{\eta_j} b_n^q\Big)^{1/q}\\ &\le K\gamma^{-1}\Big(\sum_{n=1}^{\infty}\frac{1}{n^{q/p}}\Big)^{1/q}\Big\|\sum_{m\in B_j}a_m x_m\Big\|\\ &\le K\gamma^{-1}\Big(\frac{q}{q-p}\Big)^{1/q}\Big\|\sum_{m\in B_j}a_m x_m\Big\|.\end{aligned}\tag{2.5}$$

Thus we can define a map $T: X\to l_q$ by

$$T(x_n)=\begin{cases} e_n & \text{if } n\in B_j,\\ 0 & \text{if } n\notin B_j,\end{cases}$$

where $(e_n)_{n\in\mathbb{N}}$ represents the canonical basis of l_q, and (2.5) says that T is bounded with $\|T\|\le K\gamma^{-1}(q/(q-p))^{1/q}$. Every operator from X into a q-Banach space extends to the q-Banach envelope with preservation of norm; therefore,

$$\Big(\sum_{n\in B_j}|a_n|^q\Big)^{1/q}\le K\gamma^{-1}\Big(\frac{q}{q-p}\Big)^{1/q}\Big\|\sum_{n\in B_j}a_n x_n\Big\|_q$$

for any scalars $(a_n)_{n\in B_j}$, i.e. $\Gamma=K^{-1}\gamma((q-p)/q)^{1/q}$. ■

We now study the behaviour of the sequence $(x_n^*)_{n=1}^{\eta}$. We recall that for each $n=1,\dots,\eta$, $x_n^*\in l_1(l_\infty)$ and

$$\|x_n^*\|=\sum_{l=1}^{\infty}\sup_k|a_{lk}^n|\le K\|Q\|$$

where $a_{lk}^n=x_n^*(e_{lk})$ ($l,k\in\mathbb{N}$). We also recall that $(x_n)_{n=1}^{\eta}$ is a K-unconditional basis of $\widehat{X}$, the Banach envelope of X, which is complemented in $c_0(l_1)$; and that, by (1.2), $(\|Q\|K)^{-1}\le\|x_n\|_1\le 1$ for $n=1,\dots,\eta$. Therefore, the following result from [2] holds:

LEMMA 2.5 (Lemma 4.4 of [2]). *For every $\varepsilon>0$ there is a constant Θ, depending only on K and ε, and sequences $\alpha^j\in l_1$ with $\|\alpha^j\|_1\le\Theta$ so that*

$$\sum_{l=1}^{\infty}\big|\sup_k|a_{lk}^n|-\min(\alpha_l^j,\sup_k|a_{lk}^n|)\big|<\varepsilon$$

for all $n\in B_j$, and all $j=1,\dots,J$.

Lemma 2.5 implies that for each $j=1,\dots,J$ there is a finite set F_j of l's so that for each $n\in B_j$, the norm of x_n^* is concentrated in F_j. In Lemmas 2.6–2.8 we will see that, at the cost of ignoring some uniformly finite number of n's in each B_j, we can bound the numbers $|F_j|$ by a number depending only on K and $\|Q\|$.

LEMMA 2.6. *Fix $p<q<1$, suppose that $1,\dots,N\in B_j$, and choose a family $\{G_n\}_{n=1}^N$ of finite, mutually disjoint subsets of $\mathbb{N}$. Then there is a constant C, independent of j, N, and the choice of the G_n's, so that*

$$\Big(\sum_{n=1}^{N}\Big(\sum_{l\in G_n}\sup_k|a_{lk}^n|\Big)^q\Big)^{1/q}\le C. \tag{2.6}$$

Proof. For each $l\in G_n$ choose k_l and ε_l so that

$$\varepsilon_l a_{lk_l}^n=|a_{lk_l}^n|\ge\tfrac12\sup_k|a_{lk}^n|$$

Then

$$\begin{aligned}\Big(\sum_{n=1}^{N}\Big(\sum_{l\in G_n}\sup_k|a_{lk}^n|\Big)^q\Big)^{1/q} &\le 2\Big(\sum_{n=1}^{N}\Big(\sum_{l\in G_n}\varepsilon_l a_{lk_l}^n\Big)^q\Big)^{1/q}\\ &\le 2\Big(\sum_{n=1}^{N}\Big(\sum_{m=1}^{N}\Big|\sum_{l\in G_m}\varepsilon_l a_{lk_l}^n\Big|^2\Big)^{q/2}\Big)^{1/q}\\ &\le 2\Gamma^{-1}\Big\|\sum_{n=1}^{N}\Big(\sum_{m=1}^{N}\Big|\sum_{l\in G_m}\varepsilon_l a_{lk_l}^n\Big|^2\Big)^{1/2}x_n\Big\|_q\end{aligned}$$

where Γ is as in (2.4). Now, by Theorem 1.2, there are constants A_1 and A_2 so that

$$\Big\|\sum_{n=1}^{N}\Big(\sum_{m=1}^{N}\Big|\sum_{l\in G_m}\varepsilon_l a_{lk_l}^n\Big|^2\Big)^{1/2}x_n\Big\|_q \le A_2\Big\|\Big(\sum_{n=1}^{N}\sum_{m=1}^{N}\Big|\sum_{l\in G_m}\varepsilon_l a_{lk_l}^n x_n\Big|^2\Big)^{1/2}\Big\|_q$$

$$\le A_2A_1\int_0^1\int_0^1\Big\|\sum_{n=1}^{N}\Big(\sum_{m=1}^{N}r_m(t)\sum_{l\in G_m}\varepsilon_l a_{lk_l}^n\Big)x_n r_n(s)\Big\|_q\,dt\,ds$$

$$\le A_2A_1K\int_0^1\Big\|\sum_{m=1}^{N}\Big(\sum_{n=1}^{N}\sum_{l\in G_m}\varepsilon_l a_{lk_l}^n x_n\Big)r_m(t)\Big\|_q\,dt$$

$$= A_2A_1K\int_0^1\Big\|\sum_{m=1}^{N}P\Big(\sum_{l\in G_m}\varepsilon_l e_{lk_l}\Big)r_m(t)\Big\|_q\,dt$$

where $(r_m)_{m\in\mathbb{N}}$ is the sequence of Rademacher functions on $[0,1]$, and P is the projection associated with the basis $(x_n)_{n=1}^{\eta}$ of $\widehat{X}_q$ that maps $c_0(l_q)$ onto the subspace spanned by $\{x_1,\ldots,x_N\}$. The norm of P is at most $K\|Q\|$; therefore,

$$\Big\|\sum_{n=1}^{N}\Big(\sum_{m=1}^{N}\Big|\sum_{l\in G_m}\varepsilon_l a_{lk_l}^n\Big|^2\Big)^{1/2}x_n\Big\|_q$$

$$\le A_2A_1K^2\|Q\|\int_0^1\Big\|\sum_{m=1}^{N}\sum_{l\in G_m}\varepsilon_l e_{lk_l}r_m(t)\Big\|_q\,dt = A_2A_1K^2\|Q\|\,.$$

Hence (2.6) holds with $C=2A_2A_1K^2\|Q\|/\Gamma$. ∎

LEMMA 2.7. *There is a constant N_0, independent of j, so that if we define the sets $H_n(N_0)=\{l\in\mathbb{N}:\sup_k|a_{lk}^n|<1/(2N_0)\}$ then*

$$\Big|\Big\{n\in B_j:\sum_{l\in H_n(N_0)}\sup_k|a_{lk}^n|\ge 1/2\Big\}\Big|<N_0 \tag{2.7}$$

for all $j=1,\ldots,J$.

Proof. Fix q ($p<q<1$), and let C be as in (2.6). We will prove that (2.7) holds with $N_0=(8C)^{q/(1-q)}$. For this N_0, define

$$\mathcal{C}_j=\Big\{n\in B_j:\sum_{l\in H_n(N_0)}\sup_k|a_{lk}^n|\ge 1/2\Big\},$$

$$\mathcal{L}_j=\{l\in\mathbb{N}:\sup_k|a_{lk}^n|<1/(2N_0)\text{ for some }n\in\mathcal{C}_j\}\,.$$

For $l\in\mathcal{L}_j$, $n\in\mathcal{C}_j$, we define

$$A_l^n=\begin{cases}\sup_k|a_{lk}^n| & \text{if }\sup_k|a_{lk}^n|<1/(2N_0),\\ 0 & \text{otherwise.}\end{cases}$$

We claim that if $N=\min(|\mathcal{C}_j|,N_0)$ we can find $G_1,\ldots,G_N$, finite and mutually disjoint subsets of $\mathcal{L}_j$, and $i_1,\ldots,i_N\in\mathcal{C}_j$ so that

$$\sum_{l\in G_n}\sup_k|a_{lk}^{i_n}|\ge\frac{1}{4N_0}$$

for $n=1,\ldots,N$. To see this, we proceed as follows: For each $n\in\mathcal{C}_j$ we have

$$\sum_{l\in\mathcal{L}_j}A_l^n\ge\frac12$$

so we can find $G_1^n\subseteq\mathcal{L}_j$ so that

$$\frac{1}{2N_0}>\sum_{l\in G_1^n}A_l^n\ge\frac{1}{4N_0}\quad\text{and}\quad\min_{l\in G_1^n}A_l^n\ge\max_{l\in\mathcal{L}_j\setminus G_1^n}A_l^n\,.$$

Let $i_1\in\mathcal{C}_j$ be so that $|G_1^{i_1}|$ is minimal, and set $G_1=G_1^{i_1}$.

Now, for each $n\in\mathcal{C}_j\setminus\{i_1\}$ we have

$$\sum_{l\in\mathcal{L}_j\setminus G_1}A_l^n\ge\frac12-\frac{1}{2N_0}=\frac{1}{2N_0}(N_0-1)$$

so we can find $G_2^n\subseteq\mathcal{L}_j\setminus G_1$ so that

$$\frac{1}{2N_0}>\sum_{l\in G_2^n}A_l^n\ge\frac{1}{4N_0}\quad\text{and}\quad\min_{l\in G_2^n}A_l^n\ge\max_{l\in(\mathcal{L}_j\setminus G_1)\setminus G_2^n}A_l^n\,.$$

Let $i_2\in\mathcal{C}_j\setminus\{i_1\}$ be so that $|G_2^{i_2}|$ is minimal, and set $G_2=G_2^{i_2}$.

Now, for each $n\in\mathcal{C}_j\setminus\{i_1,i_2\}$ we have

$$\sum_{l\in\mathcal{L}_j\setminus(G_1\cup G_2)}A_l^n\ge\frac12-\frac{2}{2N_0}=\frac{1}{2N_0}(N_0-2)$$

and we find i_3 and G_3 as above.

We can repeat the above process $N=\min(|\mathcal{C}_j|,N_0)$ times to find $i_1,\ldots,i_N$ and $G_1,\ldots,G_N$ satisfying our claim. Furthermore, we can apply Lemma 2.6 to get

$$\Big(\sum_{i=1}^{N}\Big(\sum_{l\in G_n}\sup_k|a_{lk}^{i_n}|\Big)^q\Big)^{1/q}\le C \tag{2.8}$$

where C is the constant in (2.6).

Finally, we recall that N_0 and $G_1, \ldots, G_N$ have been chosen so that $C = N_0^{(1-q)/q}/8$ and $\sum_{l \in G_n} \sup_k |a_{lk}^{i_n}| \geq 1/(4N_0)$ $(n = 1, \ldots, N)$. Therefore, (2.8) implies that

$$\frac{N^{1/q}}{4N_0} \leq \frac{N_0^{(1-q)/q}}{8}$$

or $N \leq N_0/2^q < N_0$. Hence $|\mathcal{C}_j| < N_0$ for all $j = 1, \ldots, J$. ∎

From now on we fix N_0 as in Lemma 2.7 and define

$$B_j' = B_j \setminus \mathcal{C}_j = \Big\{ n \in B_j : \sum_{l \in H_n(N_0)} \sup_k |a_{lk}^n| < 1/2 \Big\} \quad (j = 1, \ldots, J).$$

LEMMA 2.8. *For each $j = 1, \ldots, J$ there is a finite subset of $\mathbb{N}$, F_j, so that*

$$\sum_{l \notin F_j} \sup_k |a_{lk}^n| < 1/2 \tag{2.9}$$

for all $n \in B_j'$. Moreover, $|F_j| < 4N_0\Theta$ for all $j = 1, \ldots, J$.

Proof. For each j, let

$$F_j = \{ l \in \mathbb{N} : \sup_k |a_{lk}^n| > 1/(2N_0) \text{ for some } n \in B_j' \}.$$

Then (2.9) clearly holds.

In order to bound $|F_j|$ we recall that, by Lemma 2.5, there is a constant Θ, independent of j, and sequences $\alpha^j \in l_1$, with $\|\alpha^j\|_1 \leq \Theta$, so that

$$\sum_{l=1}^{\infty} |\sup_k |a_{lk}^n| - \min(\alpha_l^j, \sup_k |a_{lk}^n|)| < 1/(4N_0)$$

for $n \in B_j$, $j = 1, \ldots, J$. Now, if $l \in F_j$ there is $n \in B_j'$ such that $\sup_k |a_{lk}^n| > 1/(2N_0)$, and

$$\begin{aligned} \alpha_l^j &\geq \min(\sup_k |a_{lk}^n|, \alpha_l^j) \geq \sup_k |a_{lk}^n| - |\sup_k |a_{lk}^n| - \min(\sup_k |a_{lk}^n|, \alpha_l^j)| \\ &> \frac{1}{2N_0} - \frac{1}{4N_0} = \frac{1}{4N_0}. \end{aligned}$$

Hence $|F_j| < 4N_0\Theta$ for all $j = 1, \ldots, J$. ∎

Finally, we observe that the sequence $\{x_n : n \in \mathcal{C}_j, j = 1, \ldots, J\}$ is equivalent in $c_0(l_p)$ to a c_0-basis. This will be a consequence of the following theorem:

THEOREM 2.9. *Let Y be p-convex $(0 < p < 1)$ and so that Y^* has cotype 2. Then there is a constant κ, depending only on p and the cotype-2 constant of Y^*, so that $\|y\| \leq \kappa \|y\|_1$ for all $y \in Y$.*

Proof. This is a consequence of Theorem 3.3 of [4]. ∎

COROLLARY 2.10. *There is a constant λ, depending only on K and $\|Q\|$, so that*

$$\lambda^{-1} \max_n |a_n| \leq \Big\| \sum_{\substack{n \in \mathcal{C}_j \\ 1 \leq j \leq J}} a_n x_n \Big\| \leq \lambda \max_n |a_n|$$

for any choice of scalars $\{a_n : n \in \mathcal{C}_j,\ j = 1, \ldots, J\}$.

Proof. Let Y be the closed linear span in $c_0(l_p)$ of $\{x_n : n \in \mathcal{C}_j, j = 1, \ldots, J\}$. The Banach envelope $\widehat{Y}$ of Y is the closed linear span in $c_0(l_1)$ of $\{x_n : n \in \mathcal{C}_j, j = 1, \ldots, J\}$. We proved in Lemma 2.7 that $|\mathcal{C}_j| < N_0$ for all $j = 1, \ldots, J$; therefore, by (2.1),

$$D^{-1} \max_n |a_n| \leq \Big\| \sum_{\substack{n \in \mathcal{C}_j \\ 1 \leq j \leq J}} a_n x_n \Big\|_1 \leq D N_0 \max_n |a_n|$$

for any scalars (a_n). This implies that the cotype-2 constant of Y^* depends only on D and N_0, which depend only on K and $\|Q\|$, and by Theorem 2.9

$$\Big\| \sum_{\substack{n \in \mathcal{C}_j \\ 1 \leq j \leq J}} a_n x_n \Big\| \leq D N_0 \kappa \max_n |a_n|.$$

Furthermore, by the K-unconditionality of $(x_n)_{n=1}^{\eta}$ we also have

$$K^{-1} \max_n |a_n| \leq \Big\| \sum_{\substack{n \in \mathcal{C}_j \\ 1 \leq j \leq J}} a_n x_n \Big\|. \ ∎$$

Our last lemma is a quite elementary observation that will simplify much the proof of Theorem 2.1.

LEMMA 2.11. *Suppose $\{\Lambda_i : i = 1, \ldots, N\}$ is a partition of $\{1, \ldots, J\}$ and that for each $j = 1, \ldots, J$, $\{\Omega_j^m : m = 1, \ldots, M\}$ is a partition of B_j. Suppose there is a constant ϱ so that for each i and m*

$$\varrho^{-1} \sup_{j \in \Lambda_i} \Big(\sum_{n \in \Omega_j^m} |a_n|^p \Big)^{1/p} \leq \Big\| \sum_{j \in \Lambda_i} \sum_{n \in \Omega_j^m} a_n x_n \Big\| \leq \varrho \sup_{j \in \Lambda_i} \Big(\sum_{n \in \Omega_j^m} |a_n|^p \Big)^{1/p}$$

for any sequence of scalars $(a_n)_{n \in \mathbb{N}}$. Further suppose that N, M, and ϱ depend only on K and $\|Q\|$. Then Theorem 2.1 holds.

Proof. By the p-subadditivity of $\|\cdot\|$,

$$\begin{aligned} \Big\| \sum_{j=1}^{J} \sum_{n \in B_j} a_n x_n \Big\| &= \Big\| \sum_{i=1}^{N} \sum_{j \in \Lambda_i} \sum_{m=1}^{M} \sum_{n \in \Omega_j^m} a_n x_n \Big\| \\ &\leq \Big(\sum_{i=1}^{N} \sum_{m=1}^{M} \Big\| \sum_{j \in \Lambda_i} \sum_{n \in \Omega_j^m} a_n x_n \Big\|^p \Big)^{1/p} \end{aligned}$$

$$\le N^{1/p}M^{1/p}\sup_{i=1}^{N}\sup_{m=1}^{M}\Big\|\sum_{j\in\Lambda_i}\sum_{n\in\Omega_j^m}a_nx_n\Big\|$$

$$\le N^{1/p}M^{1/p}\varrho\sup_{i=1}^{N}\sup_{m=1}^{M}\sup_{j\in\Lambda_i}\Big(\sum_{n\in\Omega_j^m}|a_n|^p\Big)^{1/p},$$

and by the K-unconditionality of the basis $(x_n)_{n=1}^{\eta}$,

$$\Big\|\sum_{j=1}^{J}\sum_{n\in B_j}a_nx_n\Big\|\ge K^{-1}\sup_{i=1}^{N}\sup_{m=1}^{M}\Big\|\sum_{j\in\Lambda_i}\sum_{n\in\Omega_j^m}a_nx_n\Big\|$$

$$\ge K^{-1}\varrho^{-1}\sup_{i=1}^{N}\sup_{m=1}^{M}\sup_{j\in\Lambda_i}\Big(\sum_{n\in\Omega_j^m}|a_n|^p\Big)^{1/p},$$

which proves the claim. ■

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let us first remark that by Corollary 2.10 and Lemma 2.11 it is enough to prove the result for $\{x_n : n\in B_j',\ j=1,\ldots,J\}\subseteq\{x_n : n=1,\ldots,\eta\}$.

For each $j=1,\ldots,J$, $n\in B_j'$, (2.9) holds, therefore

$$1=x_n^*(x_n)=\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}a_{lk}^nb_{lk}^n\le\sum_{l\in F_j}\sum_{k=1}^{\infty}|a_{lk}^nb_{lk}^n|+\sum_{l\notin F_j}\sum_{k=1}^{\infty}|a_{lk}^nb_{lk}^n|$$

$$\le\sum_{l\in F_j}\sum_{k=1}^{\infty}|a_{lk}^nb_{lk}^n|+\Big(\sum_{l\notin F_j}\sup_k|a_{lk}^n|\Big)\Big(\sup_l\Big(\sum_{k=1}^{\infty}|b_{lk}^n|^p\Big)^{1/p}\Big)$$

$$<\sum_{l\in F_j}\sum_{k=1}^{\infty}|a_{lk}^nb_{lk}^n|+\frac12\|x_n\|.$$

Since $\|x_n\|=1$ we get

$$\sum_{l\in F_j}\sum_{k=1}^{\infty}|a_{lk}^nb_{lk}^n|>1/2$$

for all $n\in B_j'$, $j=1,\ldots,J$. Thus for each $j=1,\ldots,J$ we can define a function $f_j: B_j'\to F_j$ so that

$$\sum_{k=1}^{\infty}|a_{f_j(n)k}^nb_{f_j(n)k}^n|\ge\frac{1}{2|F_j|}$$

for $n\in B_j'$. We now write

$$B_j'=\bigcup_{\substack{l\in F_j\\ \text{disjoint}}}f_j^{-1}(l)\quad(j=1,\ldots,J),$$

and recall that $|F_j|\le 4N_0\Theta$ for all j.

Fix $l_j\in F_j$ and call $B_j''=f_j^{-1}(l_j)$ $(j=1,\ldots,J)$. Recall that

$$\sum_{k=1}^{\infty}|a_{l_jk}^nb_{l_jk}^n|\ge\frac{1}{2|F_j|}\ge\frac{1}{8N_0\Theta}$$

for all $n\in B_j''$, $j=1,\ldots,J$. Now suppose that there are M different j's, $j_1,\ldots,j_M$, so that $l_{j_1}=\ldots=l_{j_M}=\bar l$. We pick $n_i\in B_{j_i}''$ $(i=1,\ldots,M)$; then, since $|a_{lk}^n|\le\|Q\|K$ for all l,k, and n,

$$M^{1/2}\le 8N_0\Theta\Big(\sum_{i=1}^{M}\Big(\sum_{k=1}^{\infty}|a_{\bar lk}^{n_i}b_{\bar lk}^{n_i}|\Big)^2\Big)^{1/2}$$

$$\le 8N_0\Theta K\|Q\|\sum_{k=1}^{\infty}\Big(\sum_{i=1}^{M}|b_{\bar lk}^{n_i}|^2\Big)^{1/2}$$

$$\le 8N_0\Theta K\|Q\|\sup_l\sum_{k=1}^{\infty}\Big(\sum_{i=1}^{M}|b_{lk}^{n_i}|^2\Big)^{1/2}$$

$$=8N_0\Theta K\|Q\|\Big\|\Big(\sum_{i=1}^{M}|x_{n_i}|^2\Big)^{1/2}\Big\|_1.$$

Moreover, combining (2.1) and Theorem 1.2(ii) gives

$$\Big\|\Big(\sum_{i=1}^{M}|x_{n_i}|^2\Big)^{1/2}\Big\|_1\le A_2D.$$

Therefore, $M\le(8N_0\Theta K\|Q\|A_2D)^2$, which depends only on K and $\|Q\|$, and there is a partition of $\{1,\ldots,J\}$ into M disjoint subsets $S_1,\ldots,S_M$, and a map $\sigma:\mathbb N\to\mathbb N$ whose restriction to each S_i is one-to-one so that

$$\sum_{k=1}^{\infty}|a_{\sigma(j)k}^nb_{\sigma(j)k}^n|\ge\frac{1}{8N_0\Theta}\tag{2.10}$$

for each $n\in B_j''$. On the other hand,

$$\Big(\sum_{k=1}^{\infty}|a_{\sigma(j)k}^nb_{\sigma(j)k}^n|^p\Big)^{1/p}\le\sup_k|a_{\sigma(j)k}^n|\Big(\sum_{k=1}^{\infty}|b_{\sigma(j)k}^n|^p\Big)^{1/p}\le K\|Q\|\tag{2.11}$$

for each $n \in B_j''$. (2.10) and (2.11) imply, by Lemma 2.2, that there is $\delta > 0$ such that

$$\sup_k |a^n_{\sigma(j)k} b^n_{\sigma(j)k}| > \delta$$

for each $n \in B_j''$. So for each $n \in B_j''$ we can pick $k_n \in \mathbb{N}$ so that

$$|a^n_{\sigma(j)k_n} b^n_{\sigma(j)k_n}| > \delta.$$

Now suppose there are N different n's, $n_1, \ldots, n_N \in B_j''$, so that $k_{n_1} = \ldots = k_{n_N} = \bar{k}$. Then, since $|b^n_{lk}| \le 1$ for all l, k, and n,

$$N \le \delta^{-1} \sum_{m=1}^{N} |a^{n_m}_{\sigma(j)\bar{k}} b^{n_m}_{\sigma(j)\bar{k}}| \le \delta^{-1} \sum_{m=1}^{N} |a^{n_m}_{\sigma(j)\bar{k}}| .$$

Moreover, by (2.1) and the K-unconditionality of the basis $(x_n)_{n=1}^{\eta}$ in $c_0(l_1)$,

$$\sum_{m=1}^{N} |a^{n_m}_{\sigma(j)\bar{k}}| \le D\Big\| \sum_{m=1}^{N} a^{n_m}_{\sigma(j)\bar{k}} x_{n_m}\Big\|_1 \le DK\Big\| \sum_{n=1}^{\eta} a^{n}_{\sigma(j)\bar{k}} x_n\Big\|_1$$
$$= DK\|Q(e_{\sigma(j)\bar{k}})\|_1 = DK\|Q\| .$$

Therefore, $N \le \delta^{-1} DK\|Q\|$, which depends only on K and $\|Q\|$.

For each $j = 1, \ldots, J$ there is now a partition of B_j'' into N disjoint subsets, $R^j_1, \ldots, R^j_J$, and a map $\nu_j : B_j'' \to \mathbb{N}$ whose restriction to each R^j_m is one-to-one, so that

$$|a^n_{\sigma(j)\nu_j(n)} b^n_{\sigma(j)\nu_j(n)}| > \delta \tag{2.12}$$

for each $n \in B_j''$. This implies by Theorem 1.3 that $\{x_n : n \in R^j_m,\ j \in S_i\}$ is equivalent to $\{e_{\sigma(j)\nu_j(n)} : n \in R^j_m,\ j \in S_i\}$. The result follows from Lemma 2.11. ∎

As a straightforward consequence of Theorem 2.1 we get the following infinite-dimensional results:

THEOREM 2.12. *Every normalized unconditional basis of an infinite-dimensional complemented subspace of* $c_0(l_p)$ *is equivalent to a permutation of the unit vector basis of one of the following spaces*: c_0, l_p, $c_0 \oplus l_p$, $c_0(l_p^n)_{n=1}^{\infty}$, $l_p \oplus c_0(l_p^n)_{n=1}^{\infty}$, $c_0(l_p)$.

THEOREM 2.13. *The following quasi-Banach spaces have unique unconditional basis up to permutation*: $c_0 \oplus l_p$, $c_0(l_p^n)_{n=1}^{\infty}$, $l_p \oplus c_0(l_p^n)_{n=1}^{\infty}$, $c_0(l_p)$.

The results of this paper form part of the thesis of the author ([7]). The author wants to thank her thesis advisor, Professor N. J. Kalton, for his guidance during the realization of this work.

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Received August 20, 1990
Revised version September 20, 1991
(2713)