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Approximation of continuous convex-cone-valued functions by monotone operators

by

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Abstract. In this paper we study the approximation of continuous functions F, defined on a compact Hausdorff space S, whose values $F(t)$, for each t in S, are convex subsets of a normed space E. Both quantitative estimates (in the Hausdorff semimetric) and Bohman–Korovkin type approximation theorems for sequences of monotone operators are obtained.

0. Introduction. It is the purpose of this paper to discuss convergence results and quantitative estimates for the approximation by monotone operators of continuous functions F defined on a compact Hausdorff space S, such that the value $F(t)$, for each $t \in S$, is an element of some convex cone $\mathcal{C}$ endowed with a semimetric d_H. In many applications $\mathcal{C}$ is a convex subcone of the convex cone $\mathcal{C}(E)$ of all convex nonempty bounded subsets of a normed space E over the reals, the semimetric d_H being the Hausdorff semimetric

$$d_H(K, L) = \inf\{\lambda > 0\,;\ K \subset L + \lambda B,\ L \subset K + \lambda B\},$$

where B is the closed unit ball of E.

After giving the necessary definitions in §1 and §2, we consider in §3 the problem of quantitative estimates for the approximation by sequences $\{T_n\}_{n\geq 1}$ of monotone $\mathbb{R}_+$-linear operators on $C(S;\mathcal{C})$, and show how to extend to this context some of the local estimates of Shisha and Mond [5].

In §4 and §5 we give examples of monotone $\mathbb{R}_+$-linear operators on $C(S;\mathcal{C})$. In §4 we treat the case of operators of interpolation type and in §5 we consider two such operators, namely the Bernstein operators B_n, defined in $C([0,1];\mathcal{C})$ or in $C(S_m;\mathcal{C})$, where S_m is the standard simplex in $\mathbb{R}^m$, and the Hermite–Fejér operators H_n, defined in $C([-1,1];\mathcal{C})$. Our Theorem 3 gives the estimates for the degree of approximation by B_n on

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$C(S_m;\mathcal{C})$. We show that all the classical estimates from $C(S_m;\mathbb{R})$ remain true. Our Theorem 3 extends and generalizes Theorem 1 of R. A. Vitale [6], where $E = \mathbb{R}^d$ and $m = 1$. When the space E is infinite-dimensional, then the representation theory used in [6] and [4] is not applicable, since the surface of the unit ball of E is not compact.

In §6 we begin to develop the theory of Korovkin systems in $C(S;\mathcal{C})$, where $\mathcal{C}$ is an arbitrary Hausdorff convex cone, equipped with its Hausdorff semimetric d_H. We were able to extend several of the Bohman–Korovkin type theorems from the *linear space* $C(S;\mathbb{R})$ to our *convex cone* $C(S;\mathcal{C})$, in the case of monotone $\mathbb{R}_+$-linear operators on $C(S;\mathcal{C})$ that are regular, i.e., that map functions of type fK to functions of the same type. Our Corollary 9 and our Theorem 6 should be compared, respectively, with Theorem 2 of Vitale [6] and Theorem 3.1 of Keimel and Roth [4]. In our results we can take $\mathcal{C} = \mathcal{C}(E)$, the set of all convex nonempty bounded subsets of an infinite-dimensional Banach space E. Notice that the operators of interpolation type given in §4 and §5 are regular. Our Theorem 10 is the extension of Theorem 1 of Grossman [3] to our context.

1. Hausdorff convex cones. We start by reviewing some of the properties of convex cones.

DEFINITION 1. An (abstract) *convex cone* is a nonempty set $\mathcal{C}$ such that to every $K, L \in \mathcal{C}$ there corresponds an element $K + L$, called the *sum* of K and L, in such a way that addition is commutative and associative, and there exists in $\mathcal{C}$ a unique element 0, called the *vertex* of $\mathcal{C}$, such that $K+0 = K$, for every $K \in \mathcal{C}$. Moreover, to every pair, λ and K, where $\lambda \geq 0$ is a nonnegative real number and $K \in \mathcal{C}$, there corresponds an element λK, called the *product* of λ and K, in such a way that multiplication is associative: $\lambda(\mu K) = (\lambda\mu)K$; $1 \cdot K = K$ and $0K = 0$ for every $K \in \mathcal{C}$; the distributive laws are satisfied: $\lambda(K+L) = \lambda K + \lambda L$, $(\lambda+\mu)K = \lambda K + \mu K$, for all $K, L \in \mathcal{C}$ and $\lambda \geq 0$, $\mu \geq 0$.

It follows that $\lambda \cdot 0 = 0$, for every $\lambda \geq 0$.

DEFINITION 2. An *ordered convex cone* is a pair $(\mathcal{C}, \leq)$, where $\mathcal{C}$ is an (abstract) convex cone and $\leq$ is an ordering of its elements, i.e., $\leq$ is a reflexive, transitive and antisymmetric relation on $\mathcal{C}$ such that

(2.1) $K \leq L$ implies $K + M \leq L + M$,
(2.2) $K \leq L$, $\lambda \geq 0$ implies $\lambda K \leq \lambda L$,
(2.3) $K \leq K + L$, for every $L \geq 0$.

It follows that $\lambda \leq \mu$ implies $\lambda K \leq \mu K$, for every $K \geq 0$.

DEFINITION 3. A nonempty subset $\mathcal{K}$ of an (abstract) convex cone $\mathcal{C}$ is called a *convex subcone* if $K, L \in \mathcal{K}$ and $\lambda \geq 0$ imply $K + L \in \mathcal{K}$ and $\lambda K \in \mathcal{K}$. When equipped with the induced operations, a convex subcone $\mathcal{K} \subset \mathcal{C}$ becomes a convex cone (resp. an ordered convex cone if $(\mathcal{C}, \leq)$ is an ordered convex cone).

EXAMPLE 1. Let E be a vector space over the reals. Let $\mathcal{C} = \mathcal{C}onv(E)$ be the set of all convex nonempty subsets of E. If $K, L \in \mathcal{C}onv(E)$ and $\lambda \geq 0$, define

$$K + L = \{u + v\,;\, u \in K, v \in L\},$$
$$\lambda K = \{\lambda u\,;\, u \in K\},$$
$$0 = \{\theta\}, \quad \text{where } \theta \text{ is the origin of } E,$$
$$K \leq L \quad \text{if and only if} \quad K \subset L.$$

With this definition, $(\mathcal{C}onv(E), \leq)$ is an ordered convex cone.

EXAMPLE 2. If $\mathbb{R}$ is equipped with the usual operations and ordering, it becomes an ordered convex cone, with $\{0\}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_-$ as convex subcones.

EXAMPLE 3. Let S be a nonempty set and let $(\mathcal{C}, \leq)$ be an ordered convex cone. The set $\mathcal{F}(S;\mathcal{C})$ of all mappings $F : S \to \mathcal{C}$, with pointwise operations and ordering, is an ordered convex cone.

Before stating our next definition let us recall that a function d satisfying all the usual requirements for a metric, except that $d(x,y) = 0$ may happen with $x \neq y$, is called a *semimetric*.

DEFINITION 4. Let $(\mathcal{C}, \leq)$ be an ordered convex cone and let d_H be a semimetric on $\mathcal{C}$. We say that d_H is a *Hausdorff semimetric* on $\mathcal{C}$ if there exists $B \geq 0$ in $\mathcal{C}$ such that

(a) for all $K, L \in \mathcal{C}$ and $\lambda > 0$, the following is true: $d_H(K,L) \leq \lambda$ if and only if $K \leq L + \lambda B$ and $L \leq K + \lambda B$,
(b) $\lambda B \leq \mu B$ implies $\lambda \leq \mu$.

If d_H is a Hausdorff semimetric on $\mathcal{C}$, we say that $(\mathcal{C}, d_H)$, or $\mathcal{C}$, is a *Hausdorff convex cone*.

EXAMPLE 4. If $\mathcal{C} = \mathbb{R}_+$ with the usual operations and ordering, then the usual distance $d_H(x,y) = |x - y|$ is a Hausdorff metric on $\mathbb{R}_+$, with $B = 1$. Notice that we can also take $\mathcal{C} = \mathbb{R}$, and the usual distance is still a Hausdorff metric on $\mathbb{R}$.

EXAMPLE 5. Let E be a normed space over the reals. Let $\mathcal{C}(E)$ be the convex subcone of $\mathcal{C}onv(E)$, consisting of those elements of $\mathcal{C}onv(E)$ that are *bounded* sets, and let B be the closed unit ball of E. Define on $\mathcal{C}(E)$ the usual Hausdorff semimetric

$$d_H(K,L) = \inf\{\lambda > 0\,;\, K \subset L + \lambda B, L \subset K + \lambda B\}$$

for all $K, L \in \mathcal{C}(E)$. Then d_H satisfies properties (a) and (b) of Definition 4. Hence, $\mathcal{C}(E)$ is a Hausdorff convex cone when equipped with the usual Hausdorff semimetric.

EXAMPLE 6. If $\mathcal{K} = \mathcal{K}(E)$, the set of all *compact* and convex nonempty subsets of E, then $\mathcal{K}$ is a convex subcone of $\mathcal{C}(E)$, and since the elements of $\mathcal{K}$ are *closed* sets, $(\mathcal{K}, d_H)$ is a metric space.

DEFINITION 5. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two convex cones. An operator $T : \mathcal{C}_1 \to \mathcal{C}_2$ is called $\mathbb{R}_+$-*linear* if

$$T(F+G) = TF + TG, \quad T(\lambda F) = \lambda TF,$$

for all $F, G \in \mathcal{C}_1$ and $\lambda \geq 0$.

DEFINITION 6. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two ordered convex cones. An operator $T : \mathcal{C}_1 \to \mathcal{C}_2$ is called *monotone* if $F \leq G$ implies $TF \leq TG$, for all $F, G \in \mathcal{C}_1$.

2. Spaces of continuous functions. Let S be a compact Hausdorff space. Let $(\mathcal{C}, d_H)$ be a Hausdorff convex cone. We introduce in $\mathcal{F}(S;\mathcal{C})$ the following notion of convergence: given a sequence $\{F_n\}_{n\geq 1}$ in $\mathcal{F}(S;\mathcal{C})$ and an element $F \in \mathcal{F}(S;\mathcal{C})$, then $F_n \to F$ if and only if $d_H(F_n(s), F(s)) \to 0$, uniformly in $s \in S$.

We denote by $C(S;\mathcal{C})$ the convex subcone of $\mathcal{F}(S;\mathcal{C})$ consisting of all continuous mappings $F : S \to \mathcal{C}$. In $C(S;\mathcal{C})$ we consider the topology of uniform convergence over S, determined by the semimetric defined by

$$d(F,G) = \sup\{d_H(F(s), G(s))\,;\ s \in S\}$$

for all $F, G \in C(S;\mathcal{C})$. Hence $F_n \to F$ in $C(S;\mathcal{C})$ if and only if $d(F_n, F) \to 0$.

If $\mathcal{K} \subset \mathcal{C}$ is a convex subcone such that d_H is a metric on $\mathcal{K}$, then on the convex subcone $C(S;\mathcal{K})$ of $C(S;\mathcal{C})$ the semimetric $d(F,G)$ defined above is a metric.

Denote by $C(S)$ the real Banach space of all continuous real-valued functions $f : S \to \mathbb{R}$, equipped wih the sup-norm

$$\|f\| = \sup\{|f(s)|\,;\ s \in S\}.$$

If we write $C_+(S) = \{f \in C(S)\,;\ f \geq 0\}$, then $C_+(S) = C(S;\mathbb{R}_+)$. Notice that, when $f \in C_+(S)$ and $K \in \mathcal{C}$, the function F defined on S by $s \mapsto f(s)K$ belongs to $C(S;\mathcal{C})$. The set $C_+(S)K$ of all such functions, when $K \in \mathcal{C}$ is fixed, is a convex subcone of $C(S;\mathcal{C})$.

Notice that $\mathcal{C}$ being a semimetric space and S being compact, every $F \in C(S;\mathcal{C})$ is in fact *uniformly continuous*. In the particular case that S is a compact metric space, say with metric d, this means that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $t, x \in S$, $d(t,x) < \delta$ implies $d_H(F(t), F(x)) < \varepsilon$. The *modulus of continuity* of $F \in C(S;\mathcal{C})$ is then defined as

$$\omega(F;\delta) = \sup\{d_H(F(s), F(t))\,;\ s,t \in S,\ d(s,t) \leq \delta\}$$

for every $\delta > 0$. By uniform continuity of F, we have $\omega(F;\delta) \to 0$ as $\delta \to 0$. Notice also that $\omega(F;\delta)$ is monotonically increasing, i.e., $\delta_1 \leq \delta_2$ implies $\omega(F;\delta_1) \leq \omega(F;\delta_2)$.

PROPOSITION 1. *Let S be a compact and convex subset of a normed space E. Then*

$$\omega(F;\lambda\delta) \leq (1+\lambda)\omega(F;\delta)$$

for every $F \in C(S;\mathcal{C})$ and $\lambda \geq 0$.

Proof. The proof is standard and hence not given here. ∎

DEFINITION 7. For each $K \in \mathcal{C}$, we define $K^* \in C(S;\mathcal{C})$ by $K^*(t) = K$, for all $t \in S$.

LEMMA 1. *Let $\mathcal{K}$ be a convex subcone of a Hausdorff convex cone $(\mathcal{C}, d_H)$. Let $\{T_n\}_{n\geq 1}$ be a sequence of monotone $\mathbb{R}_+$-linear operators on the convex cone $C(S;\mathcal{C})$ such that $T_n K^* \to K^*$, for every $K \in \mathcal{K} \cup \{B\}$. If $F \in C(S;\mathcal{K})$, then $(T_n[F(x)^*], x) \to F(x)$, uniformly in $x \in S$.*

Proof. Let $0 < \varepsilon < 1$ be given. By compactness, there exists a finite set $\{x_1, \ldots, x_m\} \subset S$ such that, given $x \in S$ there is $i = 1, \ldots, m$ such that $x \in N(x_i)$, where for each $t \in S$, $N(t) = \{s \in S\,;\ d_H(F(t), F(s)) < \varepsilon\}$. Choose n_0 so that $n \geq n_0$ implies

$$d_H((T_n[F(x_i)^*], t), F(x_i)) < \varepsilon, \quad d_H((T_n B^*, t), B) < \varepsilon,$$

for all $t \in S$ and all $i = 1, \ldots, m$.

Now let $x \in S$. Choose $i = 1, \ldots, m$ so that $x \in N(x_i)$. Then for all $n \geq n_0$ we have

$$F(x) \leq F(x_i) + \varepsilon B, \quad F(x_i) \leq F(x) + \varepsilon B.$$

Hence

$$\begin{aligned}(T_n[F(x)^*], x) &\leq (T_n[F(x_i)^*], x) + \varepsilon(T_n B^*, x)\\ &\leq F(x_i) + \varepsilon B + \varepsilon(B + \varepsilon B) \leq F(x) + 4\varepsilon B.\end{aligned}$$

Similarly, $F(x) \leq (T_n[F(x)^*], x) + 4\varepsilon B$. Hence $d_H((T_n[F(x)^*], x), F(x)) \leq 4\varepsilon$ for all $x \in S$. ∎

DEFINITION 8. An $\mathbb{R}_+$-linear operator $T : C(S;\mathcal{C}) \to C(S;\mathcal{C})$ is said to *preserve the constants* if $TK^* = K^*$, for every $K \in \mathcal{C}$.

DEFINITION 9. Let $\mathcal{K}$ be a convex subcone of a Hausdorff convex cone $(\mathcal{C}, d_H)$. Let $T : C(S;\mathcal{C}) \to C(S;\mathcal{C})$ be a monotone $\mathbb{R}_+$-linear operator. We say that T is *regular over* $\mathcal{K}$ if there exists a monotone $\mathbb{R}_+$-linear operator $\widetilde{T} : C_+(S) \to C_+(S)$ such that

$$(*) \qquad T(fK) = \widetilde{T}(f)K, \quad \text{for all } f \in C_+(S) \text{ and } K \in \mathcal{K}.$$

This is equivalent to saying that there exists a positive linear operator $\widetilde{T} : C(S;\mathbb{R}) \to C(S;\mathbb{R})$ such that $(*)$ holds.

When $\mathcal{K} = \mathcal{C}$ and T is regular over $\mathcal{K}$, we say simply that T is *regular*.

3. Quantitative estimates for monotone operators. Throughout this section S is a compact and convex subset of a normed space E, and $(\mathcal{C}, d_H)$ is some Hausdorff convex cone. Hence $\omega(F;\delta)$ is defined for each $F \in C(S;\mathcal{C})$, and Proposition 1 is true.

We follow the argument of Shisha and Mond [5], extending some of their results from the linear structure to the convex cone structure.

LEMMA 2. *Let $F \in C(S;\mathcal{C})$ and $\delta > 0$ be given. Then*

$$d_H(F(t),F(x)) \le [1 + \|t-x\|^2/\delta^2]\omega(F;\delta)$$

for all $t, x \in S$.

Proof. If $\|t-x\| \ge \delta$, then

$$d_H(F(t),F(x)) \le \omega(F;\|t-x\|) \le (1+\|t-x\|/\delta)\omega(F;\delta) \le (1+\|t-x\|^2/\delta^2)\omega(F;\delta).$$

If $\|t-x\| \le \delta$, then

$$d_H(F(t),F(x)) \le \omega(F;\delta) \le \omega(F;\delta)[1+\|t-x\|^2/\delta^2]. \ \blacksquare$$

DEFINITION 10. Let $\{T_n\}_{n\ge1}$ be a sequence of $\mathbb{R}_+$-linear operators on the convex cone $C(S;\mathcal{C})$. Define

$$A_n(x) = (T_nP_x, x), \quad \text{for all } x \in S,$$

where $P_x(t) = \|t-x\|^2B$, for all $t \in S$. We write

$$A_n(x) = O(n^{-1}), \quad \text{uniformly in } x \in S,$$

if there is some constant $k > 0$ such that $nA_n(x) \le kB$, $n = 1, 2, \ldots$, for every $x \in S$.

THEOREM 1. *Let $\mathcal{K}$ be a convex subcone of a Hausdorff convex cone $(\mathcal{C}, d_H)$. Let $\{T_n\}_{n\ge1}$ be a sequence of monotone $\mathbb{R}_+$-linear operators on $C(S;\mathcal{C})$. Assume that*

(1) $T_nK^* \to K^*$, *for every $K \in \mathcal{K} \cup \{B\}$,*
(2) $A_n(x) = O(n^{-1})$, *uniformly in $x \in S$.*

Then $T_nF \to F$, for every $F \in C(S;\mathcal{K})$.

Proof. Let $F \in C(S;\mathcal{K})$ and $\varepsilon > 0$ be given. By (1) and Lemma 1, choose n_1 so that $n \ge n_1$ implies

(i) $(T_nB^*, x) \le B + (\varepsilon/2)B$,
(ii) $(T_n[F(x)^*], x) \le F(x) + (\varepsilon/2)B$,

for every $x \in S$. By (2), choose $k > 0$ so that

(iii) $nA_n(x) \le kB$

for every $x \in S$, and then choose n_2 so that $n \ge n_2$ implies

(iv) $\omega(F; 1/\sqrt{n})) \le (\varepsilon/2)(1+k+\varepsilon/2)^{-1}$.

Let $t, x \in S$ be given. By Lemma 2, we have

$$F(t) \le F(x) + [1 + \|t-x\|^2/\delta^2]\omega(F;\delta)B$$

for every $\delta > 0$. Hence

$$(T_nF, x) \le (T_n[F(x)^*], x) + \omega(F;\delta)[(T_nB^*, x) + A_n(x)/\delta^2].$$

Taking $\delta = 1/\sqrt{n}$, we get

$$(T_nF, x) \le (T_n[F(x)^*], x) + \omega(F; 1/\sqrt{n})[(T_nB^*, x) + nA_n(x)].$$

By (i)–(iv), it follows that for $n \ge n_0 = \max(n_1, n_2)$

$$(T_nF, x) \le F(x) + (\varepsilon/2)B + \omega(F;1/\sqrt{n})[B + (\varepsilon/2)B + kB] \le F(x) + (\varepsilon/2)B + (\varepsilon/2)B = F(x) + \varepsilon B,$$

for every $x \in S$. In a similar way, we get the twin inequality for $n \ge n_0$:

$$F(x) \le (T_nF, x) + \varepsilon B$$

for every $x \in S$. Hence $d_H((T_nF, x), F(x)) < \varepsilon$ for all $n \ge n_0$ and $x \in S$. $\blacksquare$

COROLLARY 1. *Let $\{T_n\}_{n\ge1}$ be a sequence of monotone $\mathbb{R}_+$-linear operators on the convex cone $C(S;\mathcal{C})$. If they preserve the constants and*

$$A_n(x) = O(n^{-1}), \quad \textit{uniformly in } x \in S,$$

then $T_nF \to F$, for every $F \in C(S;\mathcal{C})$.

DEFINITION 11. If $\{T_n\}_{n\ge1}$ is a sequence of $\mathbb{R}_+$-linear operators that are regular, we define $\alpha_n \in C_+(S)$, $n = 1, 2, \ldots$, by

$$\alpha_n(x) = (\widetilde{T}_n\|t-x\|^2, x), \quad \text{for all } x \in S.$$

THEOREM 2. *Let $\{T_n\}_{n\ge1}$ be a sequence of monotone $\mathbb{R}_+$-linear operators on the convex cone $C(S;\mathcal{C})$. Assume that they preserve the constants and are regular. Then*

$$d_H((T_nF, x), F(x)) \le [1 + \alpha_n(x)/\delta^2]\omega(F;\delta)$$

for every $F \in C(S;\mathcal{C})$, $x \in S$ and $\delta > 0$.

Proof. From Lemma 2 and the hypothesis made, we have

$$(T_nF, x) \le F(x) + \omega(F;\delta)[B + A_n(x)/\delta^2].$$

By regularity, $A_n(x) = \alpha_n(x)B$. Hence

$$(T_nF, x) \le F(x) + [1 + \alpha_n(x)/\delta^2]\omega(F;\delta)B.$$

The twin inequality

$$F(x) \le (T_nF, x) + [1 + \alpha_n(x)/\delta^2]\omega(F;\delta)B$$

is obtained similarly, and therefore

$$d_H((T_nF,x), F(x)) \le [1 + \alpha_n(x)/\delta^2]\omega(F;\delta)$$

for every $x \in S$ and $\delta > 0$. ∎

COROLLARY 2. *Let $\{T_n\}_{n\ge 1}$ be as in Theorem 2. Assume that for some $\varphi \in C_+(S)$ and $\beta > 0$*

$$\alpha_n(x) \le \varphi(x)n^{-2\beta}$$

for every $x \in S$ and $n = 1, 2, \ldots$ Then

$$d_H((T_nF,x), F(x)) \le [1 + \varphi(x)]\omega(F; n^{-\beta})$$

for every $F \in C(S;\mathcal{C})$, $x \in S$, and $n = 1, 2, \ldots$

Proof. Take $\delta = n^{-\beta}$ in Theorem 2. ∎

COROLLARY 3. *Let $\{T_n\}_{n\ge 1}$ be as in Theorem 2. For every $x \in S$ where $\alpha_n(x) > 0$, we have*

$$d_H((T_nF,x), F(x)) \le 2\omega(F; \alpha_n(x)^{1/2})$$

for every $F \in C(S;\mathcal{C})$ and $n = 1, 2, \ldots$

Proof. Take $\delta = \alpha_n(x)^{1/2}$ in Theorem 2. ∎

4. Operators of interpolation type. Let S be a compact Hausdorff space and let $(\mathcal{C}, d_H)$ be a Hausdorff convex cone.

Let J be a *finite* set, and for each $k \in J$, let $t_k \in S$ and $\varphi_k \in C_+(S)$ be given. Define an operator $T : C(S;\mathcal{C}) \to C(S;\mathcal{C})$ by setting

$$(*) \qquad (TF, x) = \sum_{k\in J} \varphi_k(x)F(t_k)$$

for every $F \in C(S;\mathcal{C})$ and $x \in S$. Then T is a monotone $\mathbb{R}_+$-linear operator defined on $C(S;\mathcal{C})$. On the other hand, if $K \in \mathcal{C}$ and $F(t) = f(t)K$, for all $t \in S$, where $f \in C_+(S)$, then

$$(TF, x) = \sum_{k\in J} \varphi_k(x)[f(t_k)K] = \Big[\sum_{k\in J} \varphi_k(x)f(t_k)\Big]K.$$

Hence T is regular.

Assume that, for every $x \in S$,

$$(**) \qquad \sum_{k\in J} \varphi_k(x) = 1.$$

We claim that T preserves the constants. Indeed, let $K \in \mathcal{C}$ be given. Then for $F = K^*$ we have

$$(TF,x) = \sum_{k\in J} \varphi_k(x)F(t_k) = \sum_{k\in J}(\varphi_k(x)K) = \Big[\sum_{k\in J}\varphi_k(x)\Big]K = K = K^*(x)$$

for every $x \in S$. Thus Theorem 2 is true for any sequence of monotone $\mathbb{R}_+$-linear operators of the interpolation type $(*)$ such that $(**)$ is true whenever S is a compact and convex subset of some normed space E. Notice that in this case (see Definition 11) we have

$$\alpha_n(x) = \sum_{k\in J(n)} \varphi_{k,n}(x)\|t_{k,n} - x\|^2$$

for every $x \in S$.

5. Operators of Bernstein and of Hermite–Fejér type. Suppose that the compact Hausdorff space S is the standard simplex S_m, i.e.,

$$S = S_m = \Big\{(x_1,\ldots,x_m) \in \mathbb{R}^m ; \sum_{i=1}^m x_i \le 1,\ x_i \ge 0,\ i = 1,\ldots,m\Big\}.$$

The nth *Bernstein operator* B_n on the simplex S_m is an operator of interpolation type defined as follows. Let $J(n)$ be the set of all m-tuples of nonnegative integers $k = (k_1,\ldots,k_m)$ such that $k_1 + \ldots + k_m \le n$. Now if $k \in J(n)$, the point $t_{k,n} \in S_m$ is given by

$$t_{k,n} = k/n = (k_1/n, \ldots, k_m/n).$$

The function $\varphi_{k,n} \in C_+(S_m)$ is defined by

$$\varphi_{k,n}(x) = \binom{n}{k} x^k (1 - |x|)^{n-|k|}$$

for every $x \in S_m$, where

$$x^k = x_1^{k_1}\cdot\ldots\cdot x_m^{k_m}, \qquad \binom{n}{k} = n!/((k_1)!\ldots(k_m)!(n-|k|)!),$$
$$|k| = k_1 + \ldots + k_m, \qquad |x| = x_1 + \ldots + x_m$$

if $k = (k_1,\ldots,k_m)$ and $x = (x_1,\ldots,x_m) \in S_m$. Hence, for any $F \in C(S;\mathcal{C})$,

$$(B_nF, x) = \sum_{|k|\le n} \varphi_{k,n}(x)F(k/n)$$

for every $x \in S_m$. We know that (see, e.g. Ditzian [2], p. 297)

(i) $\displaystyle\sum_{|k|\le n} \varphi_{k,n}(x) = 1,$

(ii) $$\sum_{|k|\le n}\left\|\frac{k}{n}-x\right\|^2\varphi_{k,n}(x)=\frac{1}{n}\sum_{i=1}^{m}x_i(1-x_i)$$

for every $x=(x_1,\ldots,x_m)\in S_m$. Hence

$$\alpha_n(x)=\frac{1}{n}\sum_{i=1}^{m}x_i(1-x_i)$$

and we may apply Theorem 2 and Corollary 2 to get the following results.

THEOREM 3. *For every $F\in C(S_m;\mathcal{C})$, $\delta>0$ and $x\in S_m$, we have*

(1) $$d_H((B_nF,x),F(x))\le\Big[1+\frac{1}{\delta^2}\cdot\frac{1}{n}\sum_{i=1}^{m}x_i(1-x_i)\Big]\omega(F;\delta),$$

(2) $$d_H((B_nF,x),F(x))\le\Big[1+\sum_{i=1}^{m}x_i(1-x_i)\Big]\omega(F;1/\sqrt{n}),$$

(3) $$\sup_{x\in S_m}d_H((B_nF,x),F(x))\le(1+m/4)\omega(F;1/\sqrt{n}).$$

Proof. For each $i=1,\ldots,m$, $x_i(1-x_i)\le 1/4$. ∎

THEOREM 4. *For every $F\in C(S_m;\mathcal{C})$, we have $B_nF\to F$.*

Proof. Each $F\in C(S_m;\mathcal{C})$ is uniformly continuous and therefore $\omega(F;1/\sqrt{n})\to 0$ as $n\to\infty$. ∎

Remark. Let S be a compact Hausdorff space and let $\mathcal{C}$ be a Hausdorff convex cone contained in $Conv(E)$, where E is some normed space. Given $F\in C(S;\mathcal{C})$, can we find a single-valued function $f\in C(S;E)$ such that $f(x)\in F(x)$, for every $x\in S$? When the elements of $\mathcal{C}$ are closed and E is a Banach space, the answer is yes, by Michael's selection theorem, since any $F\in C(S;\mathcal{C})$ is lower semicontinuous. When $S=[0,1]$, we can use the Bernstein operators to construct, for each $\varepsilon>0$, an ε-approximate continuous selection for F, i.e., to construct $f\in C([0,1];E)$ such that $f(x)\in F(x)+\varepsilon B$, for all $x\in[0,1]$. Indeed, for each $n=1,2,\ldots$ and each $k=0,\ldots,n$, choose vectors $v_{k,n}\in F(k/n)$ and define

$$f_n(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}v_{k,n},$$

for all $0\le x\le 1$. Clearly, $f_n\in C([0,1];E)$ and $f_n(x)\in(B_nF,x)$, for all $n=1,2,\ldots$ Now, given $\varepsilon>0$, choose n_0 so that $n\ge n_0$ implies $d(B_nF,F)<\varepsilon$. Then $n\ge n_0$ implies $(B_nF,x)\subset F(x)+\varepsilon B$, for all $x\in[0,1]$. Hence $f_n(x)\in F(x)+\varepsilon B$, for all $x\in[0,1]$, if $n\ge n_0$, and f_n is an ε-approximate continuous selection for F. ∎

Let $S=[-1,1]$. Let $T_n(x)=\cos(n\arccos x)$ be the Chebyshev polynomial of degree n, and let $t_{k,n}$, $k=1,\ldots,n$, be the zeros of T_n. Hence

$$t_{k,n}=\cos\Big(\frac{2k-1}{2n}\pi\Big),\qquad k=1,\ldots,n.$$

The nth *Hermite–Fejér operator* H_n on $C([-1,1];\mathcal{C})$ is an operator of interpolation type defined as follows:

$$(H_nF,x)=\sum_{k=1}^{n}(1-xt_{k,n})\Big[\frac{T_n(x)}{n(x-t_{k,n})}\Big]^2F(t_{k,n})$$

for every $x\in[-1,1]$. We know that (see, e.g., DeVore [1], pp. 42–43)

(i) $$\sum_{k=1}^{n}(1-xt_{k,n})\Big[\frac{T_n(x)}{n(x-t_{k,n})}\Big]^2=1,$$

(ii) $$(H_nP_x,x)=\frac{1}{n}[T_n(x)]^2B,$$

for all $x\in[-1,1]$, where $P_x(t)=(t-x)^2B$, for all $t\in[-1,1]$. Hence

$$\alpha_n(x)=\frac{1}{n}[T_n(x)]^2.$$

Therefore we can apply Theorem 2 and its corollaries to get the following results.

THEOREM 5. *For every $F\in C([-1,1];\mathcal{C})$, $\delta>0$ and $x\in[-1,1]$, we have*

(1) $$d_H((H_nF,x),F(x))\le\Big(1+\frac{1}{\delta^2}\cdot\frac{1}{n}[T_n(x)]^2\Big)\omega(F;\delta),$$

(2) $$d_H((H_nF,x),F(x))\le[1+(T_n(x))^2]\omega(F;1/\sqrt{n}),$$

(3) $$\sup_{x\in[-1,1]}d_H((H_nF,x),F(x))\le 2\omega(F;1/\sqrt{n}).$$

Proof. $|T_n(x)|\le 1$ for every $x\in[-1,1]$. ∎

COROLLARY 4. *For every $F\in C([-1,1];\mathcal{C})$ and $x\in[-1,1]$, we have*

$$d_H((H_nF,x),F(x))\le 2\omega(F;|T_n(x)|/\sqrt{n}).$$

Proof. For $x\ne t_{k,n}$, take $\delta=|T_n(x)|/\sqrt{n}$ in Theorem 5. When $x=t_{k,n}$, $(H_nF,x)=F(x)$. ∎

6. Korovkin systems. Throughout this section S is a compact Hausdorff space, $(\mathcal{C},d_H)$ is a Hausdorff convex cone and $\mathcal{K}$ is a convex subcone of $\mathcal{C}$ such that there exists an element $K_0\in\mathcal{K}$ with $d_H(\lambda K_0,\mu K_0)=|\lambda-\mu|$. For example, if $\mathcal{C}=\mathcal{C}(E)$ (see Example 5) and $\mathcal{K}$ is any convex subcone of

$\mathcal{C}(E)$ containing an element K_0 of the form $K_0 = \{v\}$, where $v \in E$ is such that $\|v\| = 1$, then $d_H(\lambda K_0, \mu K_0) = \|\lambda v - uv\| = |\lambda - \mu|$.

The convex cone $\mathcal{K} = \mathcal{C}$ has the above property. Take $K_0 = B$ and notice that $d_H(\lambda B, \mu B) = |\lambda - \mu|$ for all $\lambda \geq 0$ and $\mu \geq 0$.

DEFINITION 12. A subset $\mathcal{G} \subset C(S;\mathcal{K})$ is called a *Korovkin system for* $C(S;\mathcal{K})$ if, for any sequence $\{T_n\}_{n\geq 1}$ of monotone $\mathbb{R}_+$-linear operators on $C(S;\mathcal{C})$,

$$(*)\qquad T_nG \to G\,,\ \text{for all } G \in \mathcal{G}\,,\ \text{implies } T_nF \to F\,,\ \text{for all } f \in C(S;\mathcal{K})\,.$$

When $(*)$ holds only for sequences of monotone $\mathbb{R}_+$-linear operators on $C(S;\mathcal{C})$ that are regular over $\mathcal{K}$ (see Definition 9), then we say that $\mathcal{G}$ is a *regular Korovkin system for* $C(S;\mathcal{K})$.

Remark. Let $\mathcal{C} = \mathbb{R}_+$. Then a Korovkin system for $C(S;\mathbb{R}_+)$ is a Korovkin system in the linear space $C(S)$. Indeed, let $\mathcal{F} \subset C_+(S)$ be a Korovkin system according to Definition 12. Let $\{T_n\}_{n\geq 1}$ be a sequence of positive linear operators $T_n : C(S) \to C(S)$ such that $T_ng \to g$, for every $g \in \mathcal{F}$. When we restrict T_n to $C_+(S)$ it becomes a monotone $\mathbb{R}_+$-linear operator on $C(S;\mathbb{R}_+)$. Hence $T_nf \to f$, for every $f \in C(S;\mathbb{R}_+)$. Now let $f \in C(S)$ be given. Write $f = f_+ - f_-$, where $f_+ = \max(f, 0)$, $f_- = \max(-f, 0)$. Then $T_nf = T_n(f_+) - T_n(f_-)$, because T_n is a linear operator. Consequently, $T_nf \to f_+ - f_- = f$, and therefore $\mathcal{F}$ is a Korovkin system in $C(S)$. Conversely, if $\mathcal{F}$ is a Korovkin system in $C(S)$ and $\mathcal{F} \subset C_+(S)$, then $\mathcal{F}$ is a Korovkin system for $C(S;\mathbb{R}_+)$, since every monotone $\mathbb{R}_+$-linear operator on $C(S;\mathbb{R}_+)$ can be extended to $C(S)$ as a positive linear operator.

LEMMA 3. *Let $\{T_n\}_{n\geq 1}$ be a sequence of monotone $\mathbb{R}_+$-linear operators on the convex cone $C(S;\mathcal{C})$ such that $T_nK^* \to K^*$ for every $K \in \mathcal{K} \cup \{B\}$. Let $F \in C(S;\mathcal{K})$ be such that for each $\varepsilon > 0$, there is some bounded function $P : S \times S \to \mathbb{R}_+$ such that*

$$(*)\qquad d_H(F(t), F(x)) \leq \varepsilon + P(t, x)$$

for all $t, x \in S$ and $P_x(t) := P(t, x)$, $t \in S$, belongs to $C_+(S)$, for each $x \in S$. If $(T_n(P_xB), x) \to 0$ uniformly in $x \in S$, then $T_nF \to F$.

Proof. Let $0 < \varepsilon < 1$ be given. By hypothesis, there exists a function $P : S \times S \to \mathbb{R}_+$ such that $(*)$ holds for all $t, x \in S$. Hence

$$F(t) \leq F(x) + \varepsilon B + P_x(t)B$$

and, consequently,

$$(T_nF, x) \leq (T_n[F(x)^*], x) + \varepsilon(T_nB^*, x) + A_n(x)\,,$$

where $A_n(x) = (T_n(P_xB), x)$, for every $x \in S$. By Lemma 1, choose n_0 so that $n \geq n_0$ implies for every $x \in S$,

$$(T_n[F(x)^*], x) \leq F(x) + \varepsilon B\,,\quad (T_nB^*, x) \leq B + \varepsilon B\,,\quad A_n(x) \leq \varepsilon B\,.$$

Then $(T_nF, x) \leq F(x) + 4\varepsilon B$ for all $x \in S$. Similarly, one gets $F(x) \leq (T_nF, x) + 4\varepsilon B$ for all $x \in S$. Hence $d_H((T_nF, x), F(x)) \leq 4\varepsilon$, for all $x \in S$ and $n \geq n_0$. ■

LEMMA 4. *Let S be a compact Hausdorff space and let $\mathcal{F} \subset C_+(S)$ be a nonempty subset that separates the points of S. For each $R \in C_+(S \times S)$ that vanishes on the diagonal $\Delta = \{(t, s) \in S \times S\,;\, t = s\}$, and for each $\varepsilon > 0$, there exists $\delta > 0$ and $\varphi_1, \ldots, \varphi_m \in \mathcal{F}$ such that*

$$R(t, x) \leq \varepsilon + \frac{M}{\delta^2}\sum_{i=1}^{m}(\varphi_i(t) - \varphi_i(x))^2$$

for all $t, x \in S$, where $M = \sup\{R(t, x)\,;\,(t, x) \in S \times S\}$.

Proof. Since $\mathcal{F}$ separates the points of S, and S is a compact Hausdorff space, the weak topology determined by $\mathcal{F}$ coincides with the initial topology on S. Hence each $R \in C_+(S\times S)$ is uniformly continuous with respect to the uniform structure determined by the weak topology. Therefore, given $\varepsilon > 0$, there exist $\delta > 0$ and $\varphi_1, \ldots, \varphi_m \in \mathcal{F}$ such that $|R(t, x) - R(u, y)| < \varepsilon$, for all (t, x) and (u, y) in $N(\delta)$, where for each $\delta > 0$,

$$N(\delta) = \{(v, z) \in S \times S\,;\, |\varphi_i(v) - \varphi_i(z)| < \delta\,,\ i = 1, \ldots, m\}\,.$$

Since $\Delta \subset N(\delta)$ and $R(u, y) = 0$ if $u = y$, it follows that $R(t, x) < \varepsilon$ whenever $|\varphi_i(t) - \varphi_i(x)| < \delta$ for all $i = 1, \ldots, m$.

Now let $(t, x) \in S \times S$ be such that $|\varphi_i(t) - \varphi_i(x)| \geq \delta$ for some index $1 \leq i \leq m$. Then

$$1 \leq \frac{1}{\delta^2}\sum_{i=1}^{m}(\varphi_i(t) - \varphi_i(x))^2\,,$$

and therefore

$$R(t, x) \leq \frac{M}{\delta^2}\sum_{i=1}^{m}(\varphi_i(t) - \varphi_i(x))^2\,,$$

where $M = \sup\{R(t, x)\,;\,(t, x) \in S \times S\}$.

In any case,

$$R(t, x) \leq \varepsilon + \frac{M}{\delta^2}\sum_{i=1}^{m}(\varphi_i(t) - \varphi_i(x))^2\,,$$

for all $t, x \in S$. ■

THEOREM 6. *Let $\mathcal{F} \subset C(S;\mathbb{R}_+)$ be a Korovkin system for $C(S;\mathbb{R}_+)$. Choose $K_0 \in \mathcal{K}$ such that $d_H(\lambda K_0, \mu K_0) = |\lambda - \mu|$, for all $\lambda \geq 0, \mu \geq 0$. Then $\mathcal{G} = \{fK_0\,;\, f \in \mathcal{F}\}$ is a regular Korovkin system for $C(S;\mathcal{K})$.*

Proof. Let $\{T_n\}_{n\geq 1}$ be a sequence of monotone $\mathbb{R}_+$-linear operators that are regular. Assume that $T_n G \to G$, for all $G \in \mathcal{G}$.

Let $F \in C(S;\mathcal{K})$ and $\varepsilon > 0$ be given. Let $R(t,x) = d_H(F(t), F(x))$, for $t, x \in S$. Clearly, $R \in C_+(S \times S)$ and $R(x,x) = 0$ for $x \in S$. Now S being a compact Hausdorff space, $C_+(S)$ separates the points of S, and by an obvious modification of Lemma 4, there exist $\delta > 0$ and $\varphi_1, \ldots, \varphi_m \in C_+(S)$ such that

$$d_H(F(t), F(x)) \leq \varepsilon + \frac{M}{\delta}\sum_{i=1}^{m}|\varphi_i(t) - \varphi_i(x)|$$

for all $t, x \in S$, where $M = \sup\{d_H(F(t), F(x))\,;\,(t,x) \in S \times S\}$.

Let $P(t,x) = (M/\delta)\sum_{i=1}^{m}|\varphi_i(t) - \varphi_i(x)|$, $(t,x) \in S \times S$. Notice that $P(x,x) = 0$. Then $P : S \times S \to \mathbb{R}_+$ is bounded and $P_x(t) := P(t,x)$, $t \in S$, belongs to $C_+(S)$ for each $x \in S$. By Lemma 3, to prove that $T_n F \to F$, it suffices to show that $T_n K^* \to K^*$, for every $K \in \mathcal{K} \cup \{B\}$, and that $(T_n(P_x B), x) \to 0$, uniformly in $x \in S$.

Now for each $x \in S$,

$$d_H((T_n(fK_0), x), f(x)K_0) = d_H((\widetilde{T}_n f, x)K_0, f(x)K_0) = |(\widetilde{T}_n f, x) - f(x)|\,.$$

Hence $\|\widetilde{T}_n f - f\| = d(T_n(fK_0), fK_0) \to 0$, for every $f \in \mathcal{F}$. Since $\mathcal{F}$ is a Korovkin system for $C(S;\mathbb{R}_+)$, this implies $\widetilde{T}g \to g$ for every $g \in C_+(S)$. In particular, $\widetilde{T}_n 1 \to 1$ and $\widetilde{T}_n P_x \to P_x$, for each $x \in S$. Now $\widetilde{T}_n 1 \to 1$ implies $T_n K^* \to K^*$ for every $K \in \mathcal{K} \cup \{B\}$, and in fact for every $K \in \mathcal{C}$. It remains to show that $(\widetilde{T}_n P_x, x) \to 0$ uniformly in $x \in S$. Let $\varepsilon > 0$ be given. For each $x \in S$, let $N(x) = \{t \in S\,;\, P_x(t) < \varepsilon\}$. By compactness of S, there is a finite set $\{x_1, \ldots, x_k\} \subset S$ such that $S \subset N(x_1) \cup \ldots \cup N(x_k)$. Choose n_0 so that $n \geq n_0$ implies $|(\widetilde{T}_n P_{x_j}, t) - P_{x_j}(t)| < \varepsilon$, and $|(\widetilde{T}_n 1, t) - 1| < \varepsilon$, for all $t \in S$, $j = 1, \ldots, k$.

Now let $x \in S$ be given. Then $P_{x_j}(x) < \varepsilon$ for some $j \in \{1, \ldots, k\}$. Now $P_x(t) \leq P_{x_j}(t) + P_{x_j}(x) \leq P_{x_j}(t) + \varepsilon$ for all $t \in S$. Hence

$$\begin{aligned}(\widetilde{T}_n P_x, x) &\leq \widetilde{T}_n P_{x_j}, x) + \varepsilon(\widetilde{T}_n 1, x)\\ &\leq P_{x_j}(x) + \varepsilon + \varepsilon(1+\varepsilon) < \varepsilon + \varepsilon + \varepsilon(1+\varepsilon)\end{aligned}$$

for all $n \geq n_0$, and so $(\widetilde{T}_n P_x, x) \to 0$, uniformly in $x \in S$. ∎

COROLLARY 5. *Let $\mathcal{F} \subset C(S;\mathbb{R}_+)$ be a Korovkin system in $C(S;\mathbb{R}_+)$. Then $\mathcal{G} = \{fB\,;\, f \in \mathcal{F}\}$ is a regular Korovkin system in $C(S;\mathcal{C})$.*

Proof. Take $\mathcal{K} = \mathcal{C}$ in Theorem 6 and notice that in any Hausdorff convex cone $d_H(\lambda B, \mu B) = |\lambda - \mu|$ for all $\lambda \geq 0$ and $\mu \geq 0$. ∎

COROLLARY 6. *Let E be a normed space over the reals and let $v \in E$ be chosen with $\|v\| = 1$. If $K_0 = \{v\}$, and $\mathcal{F} \subset C(S;\mathbb{R}_+)$ is a Korovkin system for $C(S;\mathbb{R}_+)$, then $\mathcal{G} = \{fK_0\,;\, f \in \mathcal{F}\}$ is a regular Korovkin system for $C(S;\mathcal{K})$ where $\mathcal{K}$ is any convex subcone of $\mathcal{C}(E)$ that contains K_0. In particular, $\mathcal{G}$ is a regular Korovkin system for $C(S;\mathcal{K}(E))$ and for $C(S;\mathcal{C}(E))$.*

THEOREM 7. *Let S be a compact nonempty subset of a normed space E. Let $\{f_1, \ldots, f_m\}$ be a finite subset of $C_+(S)$ such that there are bounded real-valued functions $\{a_1, \ldots, a_m\}$ defined on S and some constant $M > 0$ such that*

$$Q(t,x) := \sum_{j=1}^{m} a_j(x) f_j(t) \geq M\|t - x\|^2, \qquad Q(x,x) = 0\,,$$

for all $t, x \in S$. Assume that for each $j = 1, \ldots, m$, the function a_j does not change sign in S. Then $\{1, f_1, \ldots, f_m\}$ is a Korovkin system for $C(S;\mathbb{R}_+)$.

Proof. Let $\{T_n\}_{n\geq 1}$ be a sequence of monotone $\mathbb{R}_+$-linear operators on $C(S;\mathbb{R}_+)$. Assume that $T_n f_j \to f_j$, for each $1 \leq j \leq m$, and $T_n 1 \to 1$.

Let $F \in C(S;\mathbb{R}_+)$ be given. We claim that $T_n F \to F$. Let $\varepsilon > 0$ be given. By the uniform continuity of F, there is some $\delta > 0$ such that

$$|F(t) - F(x)| \leq \varepsilon + \frac{2\|F\|}{\delta^2}\|t - x\|^2$$

for all $t, x \in S$. Hence $|F(t) - F(x)| \leq \varepsilon + P(t,x)$, where

$$P(t,x) = \frac{2\|F\|}{\delta^2} \cdot \frac{1}{M} Q(t,x)\,.$$

By Lemma 3, to prove that $T_n F \to F$, it suffices to show that $(T_n P_x, x) \to 0$, uniformly in $x \in S$, and for this it suffices to show that $(T_n Q_x, x) \to 0$ uniformly in $x \in S$, where, for each $x \in S$,

$$Q_x(t) = Q(t,x) = \sum_{j=1}^{m} a_j(x) f_j(t)$$

for all $t \in S$.

Without loss of generality, we may assume that $a_j(x) \geq 0$, for all $x \in S$, if $j = 1, \ldots, k$; and $a_j(x) \leq 0$, for all $x \in S$, if $j = k+1, \ldots, m$.

Let

$$Q^+(t,x) = \sum_{j=1}^{k} a_j(x) f_j(t)\,, \qquad Q^-(t,x) = \sum_{j=k+1}^{m}(-a_j(x)) f_j(t)\,.$$

Then $Q^+ \geq 0$, $Q^- \geq 0$ and $Q + Q^- = Q^+$. Hence $(T_n Q_x, x) + (T_n Q_x^-, x) = (T_n Q_x^+, x)$ for every $x \in S$. Now $T_n f_j \to f_j$ for each $j = 1, \ldots, m$. Hence $(T_n Q_x^-, x) \to Q_x^-(x)$, and $(T_n Q_x^+, x) \to Q_x^+(x)$. Therefore $(T_n Q_x, x) \to$

$Q_x^+(x) - Q_x^-(x)$, uniformly in $x \in S$. Since $Q_x^+(x) - Q_x^-(x) = Q_x(x) = 0$, this ends the proof. ∎

COROLLARY 7. *The set $\{1, t, t^2\}$ is a Korovkin system for $C([0, 1]; \mathbb{R}_+)$.*

COROLLARY 8. *The set $\{1, \pi_1, \ldots, \pi_m, \pi_1^2 + \ldots + \pi_m^2\}$ is a Korovkin system for $C([0, 1]^m; \mathbb{R}_+)$, where $\pi_j(t) = t_j$ $(j = 1, \ldots, m)$ for $t = (t_1, \ldots, t_m) \in [0, 1]^m$.*

Proof. If $t, x \in [0, 1]$, then $(t-x)^2 = t^2 - 2xt + x^2 \cdot 1$. Now $-2x \le 0$ and $x^2 \ge 0$ for all $x \in [0, 1]$. If $t, x \in [0, 1]^m$, then

$$\|t - x\|^2 = \sum_{i=1}^{m} (t_i - x_i)^2$$

$$= (\pi_1^2 + \ldots + \pi_m^2)(t) - \sum_{i=1}^{m} 2x_i \pi_i(t) + (\pi_1^2 + \ldots + \pi_m^2)(x) \cdot 1,$$

and $-2x_i \le 0$ $(i = 1, \ldots, m)$, $(\pi_1^2 + \ldots + \pi_m^2)(x) \ge 0$ for all $x \in [0, 1]^m$. ∎

THEOREM 8. *Let S and the family $\{f_1, \ldots, f_m\}$ be as in Theorem 7. Choose $K_0 \in \mathcal{K}$ such that $d_H(\lambda K_0, \mu K_0) = |\lambda - \mu|$, for all $\lambda \ge 0, \mu \ge 0$. Then $\mathcal{G} = \{f_1 K_0, \ldots, f_m K_0\}$ is a regular Korovkin system for $C(S; \mathcal{K})$.*

Proof. Apply Theorems 6 and 7. ∎

COROLLARY 9. *Let $S = [0, 1]$. Choose $K_0 \in \mathcal{K}$ such that $d_H(\lambda K_0, \mu K_0) = |\lambda - \mu|$, for all $\lambda \ge 0$, $\mu \ge 0$. Then $\mathcal{G} = \{1 \cdot K_0, tK_0, t^2 K_0\}$ is a regular Korovkin system for $C(S; \mathcal{K})$.*

COROLLARY 10. *Let $S = [0, 1]^m$, and let K_0 be as in Corollary 8. Then*

$$\mathcal{G} = \{1K_0, \pi_1 K_0, \ldots, \pi_m K_0, (\pi_1^2 + \ldots + \pi_m^2) K_0\}$$

is a regular Korovkin system for $C(S; \mathcal{K})$.

Remark. If a and b are real numbers with $0 \le a < b$, then the closed interval $[a, b] \subset \mathbb{R}_+$ can be substituted for $[0, 1]$ in Corollaries 6 and 8. Similarly, if S is any compact and convex nonempty subset of $(\mathbb{R}_+)^m$, then S can be substituted for $[0, 1]^m$ in Corollaries 7 and 9.

THEOREM 9. *Let $\mathcal{F} \subset C_+(S)$ be a subset that separates the points of S. Then $\mathcal{G} = \{\varphi^k ; \varphi \in \mathcal{F}$ and $k = 0, 1, 2\}$ is a Korovkin system for $C(S; \mathbb{R}_+)$.*

Proof. Let $\{T_n\}_{n \ge 1}$ be a sequence of monotone $\mathbb{R}_+$-linear operators on $C(S; \mathbb{R}_+)$. Assume that $T_n g \to g$, for all $g \in \mathcal{G}$.

Let $f \in C(S; \mathbb{R}_+)$ and $\varepsilon > 0$ be given. The mapping $R(t, x) = |f(t) - f(x)|$ belongs to $C_+(S \times S)$ and $R(x, x) = 0$. By Lemma 4, there exist $\delta > 0$ and $\varphi_1, \ldots, \varphi_m \in \mathcal{F}$ such that

$$|f(t) - f(x)| \le \varepsilon + \frac{M}{\delta^2} P(t, x)$$

for all $t, x \in S$, where

$$P(t, x) = \sum_{i=1}^{m} (\varphi_i(t) - \varphi_i(x))^2$$

and $M = \sup\{|f(t) - f(x)| ; (t, x) \in S \times S\}$.

By Lemma 3, to prove that $T_n f \to f$, it suffices to show that $(T_n P_x, x) \to 0$, uniformly in $x \in S$. Since $P_x(t) = P(t, x)$ is a finite sum, it suffices to show that $(T_n[(\varphi_i - \varphi_i(x))^2], x) \to 0$, uniformly in $x \in S$, for each $i = 1, \ldots, m$. Notice that

$$(T_n[(\varphi_i - \varphi_i(x))^2], x) + 2\varphi_i(x)(T_n \varphi_i, x) = (T_n \varphi_i^2, x) + \varphi_i^2(x)(T_n 1, x)$$

for each $x \in S$. Now $T_n g \to g$, for all $g \in \mathcal{G}$, implies $T_n \varphi_i^k \to \varphi_i^k$ for $k = 0, 1, 2$. Hence $(T_n \varphi_i^k, x) \to \varphi_i^k(x)$, uniformly in $x \in S$, for each $k = 0, 1, 2$. Hence $(T_n[(\varphi_i - \varphi_i(x))^2], x) \to 0$, uniformly in $x \in S$. ∎

THEOREM 10. *Let $\mathcal{F} \subset C_+(S)$ be a subset that separates the points of S. Let K_0 be as in Theorem 6. Then $\mathcal{G} = \{\varphi^k K_0 ; \varphi \in \mathcal{F}$ and $k = 0, 1, 2\}$ is a regular Korovkin system for $C(S; \mathcal{K})$.*

Proof. Apply Theorems 6 and 9. ∎

COROLLARY 11. *Let $\mathcal{F} \subset C_+(S)$ be a subset that separates the points of S. Then $\mathcal{G} = \{\varphi^k B ; \varphi \in \mathcal{F}$ and $k = 0, 1, 2\}$ is a regular Korovkin system in $C(S; \mathcal{C})$.*

Proof. Apply Corollary 5 and Theorem 9. ∎

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