On the multiplicity function of ergodic group extensions of rotations

by

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Abstract. For an arbitrary set $A \subseteq \mathbb{N}$ satisfying $1 \in A$ and lcm$(m_1,m_2) \in A$ whenever $m_1, m_2 \in A$, an ergodic abelian group extension of a rotation for which the range of the multiplicity function equals $A$ is constructed.

Introduction. In this paper we study the set $\mathcal{M}_T$ of all essential spectral multiplicities of an ergodic measure preserving the transformation $T$ of a Lebesgue space $(X, \mathcal{B}, \mu)$. $\mathcal{M}_T$ is defined as the essential range of the multiplicity function with respect to the maximal spectral type of the associated unitary operator

$$U_T : L^2(X, \mu) \to L^2(X, \mu), \quad (U_Tf)(x) = f(Tx), \quad x \in X.$$ 

Thus $\mathcal{M}_T$ is a subset of the set $\mathbb{N}$ of all positive integers and infinity. Many examples in ergodic theory have $\mathcal{M}_T = \{1\}$ (e.g. irrational rotations), $\mathcal{M}_T = \{\infty\}$ (e.g. Kolmogorov automorphisms), $\mathcal{M}_T = \{1, \infty\}$ (e.g. affine transformations). Transformations with $\mathcal{M}_T = \{1, k\}$ have been constructed ([16]), for each positive integer $k$, and also with $\mathcal{M}_T = \{1, 2k\}$, where $2k$ corresponds to the multiplicity of the Lebesgue component ([1], [9], [12]).

The problem of whether for an arbitrary nonempty set $A \subseteq \mathbb{N}$ there exists an ergodic transformation $T$ with $\mathcal{M}_T = A$ seems to be open. Toward the full solution of this question, Robinson in [18] has proved that for each finite set $A$ of positive integers satisfying:

(i) $1 \in A$,
(ii) lcm$(m_1,m_2) \in A$ whenever $m_1, m_2 \in A$,
there exists a weakly mixing transformation $T$ such that $\mathcal{M}_T = A$. The transformation used by Robinson was a group extension (in fact nonabelian) of an automorphism $T_0$ which admits a good cyclic approximation. However, his example is based on generic arguments and it is not constructive. He showed that a dense $G_\delta$ set of group extensions $T$ of $T_0$ satisfies $\mathcal{M}_T = A$. The main result of the present paper is

**Theorem 1.** Let $A$ be a subset of positive integers (finite or not) satisfying (i) and (ii). Then there exists an ergodic transformation $T$ with $\mathcal{M}_T = A$.

The transformations employed in the proof of Theorem 1 are abelian group extensions of the so-called adding machines. If $A$ is a finite set then these transformations turn out to be Morse automorphisms over a finite abelian group (in the sense of [10]). Our transformations are described in a constructive way and moreover, each of them has a shift representation. This made it possible to compare the spectral multiplicity and the rank of special examples of such transformations ([2]). The classical Morse symbolic dynamical systems over the group $Z_2 = \{0, 1\}$, defined by Keane in [6], have simple spectra ([7]). Goodson in [3] has constructed examples of Morse automorphisms over cyclic groups with $\mathcal{M}_T = \{1, 2\}$. A similar result has been obtained in [8]. A conjecture arose that the multiplicity function of all Morse automorphisms over cyclic groups is upper bounded by 2 (formally the question was raised in [4]). As a consequence of our considerations we answer that question negatively.

**Theorem 2.** Let $A$ be a finite set of positive integers satisfying (i) and (ii). There exists a Morse automorphism $T$ over a finite cyclic group such that $\mathcal{M}_T = A$.

In particular, for every natural number $k \geq 1$, there exists a Morse automorphism $T$ over a cyclic group whose maximal spectral multiplicity is $k$. Robinson in [18] has proved the same result using Morse automorphisms, but over nonabelian groups. It is interesting to know what kind of spectral measures appear in our construction. Let $A = \{n_1, n_2, \ldots\}$ satisfy (i) and (ii) and let $T : (X, B, \mu) \to (X, B, \mu)$ be the transformation from the proof of Theorem 1. Then

$$L^2(X, \mu) = \bigoplus_{i \geq 1} (Z(h_{j_i}^{(i)})) \oplus \cdots \oplus (Z(h_{n_i}^{(i)})),$$

where $Z(h_{j_i}^{(i)})$, $1 \leq j \leq n_i$, $i \geq 1$, are pairwise orthogonal $U_T$-cyclic subspaces and if $g_{j_i}^{(i)}$ denotes the maximal spectral type of $U_T : Z(h_{j_i}^{(i)}) \to Z(h_{j_i}^{(i)})$ then

(iii) $g_{j_1}^{(1)} \sim \cdots \sim g_{j_n}^{(1)}$, $j \geq 1$,

(iv) $\delta_x \ast g_{j}^{(i)} \simeq g_{j}^{(i)}$ for every $x \in S^1$ and $j \neq k$, in particular, $g_{j}^{(i)} \simeq g_{j}^{(k)}$,

(v) for each $x \in S^1$, $i \geq 1$ and $s \neq t$,

$$\delta_x \ast (g_{j_1}^{(i)} \ast \cdots \ast g_{j_m}^{(i)}) \simeq \delta_x \ast (g_{j_1}^{(s)} \ast \cdots \ast g_{j_m}^{(s)}).$$

In our considerations, the centralizer $C(T)$ of $T$ plays a role. We recall that $C(T)$ consists of all measure preserving transformations commuting with $T$.

I. Description of the method and results. From now on, $T : (X, B, \mu) \to (X, B, \mu)$ denotes an ergodic rotation on a compact metric monotonic group $X$ with Haar measure $\mu$. Let $G$ be a compact metric abelian group with Haar measure $\mu$. By a cocycle we mean a measurable function $\phi : X \to G$. A cocycle $\phi$ defines an automorphism $T_\phi$ on $(X \times G, \mu)$ by $T_\phi(x, g) = (Tx, \phi(x) + g)$, $x \in X$, $g \in G$, where $\mu = \mu \times \mu$. Such an automorphism is called a G-extension of $T$. It need not be ergodic. In fact, it enjoys the ergodic property iff for every nontrivial character $\chi \in G$ there is no measurable solution $f : X \to S^1$ of the functional equation $\chi(\phi(x)) = f(Tx) - f(x)$ ([14]). The space $L^2(X \times G, \mu)$ can be decomposed as

$$L^2(X \times G, \mu) = \bigoplus_{\chi \in G} L_X,$$

where $L_X = \{f \otimes \chi : f \in L^2(X, \mu)\}$. Notice that $U_{T_\phi} : L_X \to L_X$ is unitarily equivalent to the unitary operator $V_\phi : L^2(X, \mu) \to L^2(X, \mu)$, where $V_\phi(f)(x) = \chi(\phi(x))f(Tx)$, $x \in X$. Let $\phi_T$ denote the maximal spectral type of $V_{\phi_T}$. We will construct $\phi_T$ satisfying

(1) $V_{\phi_T}$ has simple spectrum for each $\chi \in \hat{G}$,

(2) $\phi_T$ and $\gamma_T$ are either orthogonal or equivalent for each $\chi, \gamma \in \hat{G}$.

Obviously if (1) and (2) hold, then $\mathcal{M}_{T_\phi}$ consists of all cardinalities of the equivalence classes of the relation $\sim$ on $\hat{G} \times \hat{G}$ defined as $\chi \sim \gamma$ if $\phi_T \sim \phi_T$.

Now we present a way of showing that under certain circumstances $\phi_T$ and $\gamma_T$ are equivalent.

**Proposition 1.** Let $\chi, \gamma \in \hat{G}$. Suppose that there exists a continuous group automorphism $v : G \to G$ and $S \in C(T)$ satisfying

(i) $\gamma = \chi \circ v$,

(ii) there exists a measurable solution $f : X \to G$ of the functional equation

$$\phi(Sx) - \phi(x) = f(Tx) - f(x).$$

(3) $\phi(Sx) - \phi(x) = f(Tx) - f(x).$
Then the unitary operator \( W = V_{f, S, X} \) satisfies \( W V_{\phi, T, X} = V_{\phi, T, \gamma} W \). Consequently, \( V_{\phi, T, X} \) and \( V_{\phi, T, \gamma} \) are unitarily equivalent.

Notice that if (3) holds then the transformation \( S_{f,v} \) acting on \( X \times G \) by the formula

\[
S_{f,v}(x, g) = (Sx, f(x) + v(g))
\]

preserves \( \bar{\mu} \) and commutes with \( T_\phi \). Consequently, \( S_{f,v} \in C(T_\phi) \). Actually, when \( T_\phi \) is ergodic, each element of the centralizer of \( T_\phi \) is of the form (4) (see [13]). As an immediate consequence of Proposition 1 we obtain the following.

**Corollary 1.** The maximal spectral multiplicity of \( T_\phi \) is bounded from below by \( \sup_{\chi \in \hat{G}} \text{Card}\{\chi : v \in A\} \), where \( A = \{v : G \to G : v \text{ is a continuous group automorphism such that there exists } S_{f,v} \in C(T_\phi)\} \).

Notice that if \( v \in A \), then certainly \( v^n \in A \) for each integer \( n \). Under our standing assumption (1), the measures \( \varphi_x \) and \( \varphi_x \) are equivalent if \( V_{\phi, T, X} \) and \( V_{\phi, T, \gamma} \) are unitarily equivalent. The result below is in a sense the converse to Proposition 1 in the case of cyclic groups and seems to be of independent interest.

**Proposition 2.** Let \( X, \gamma \in \hat{G} \), where \( G = \mathbb{Z}_n \), \( n \geq 2 \), and let \( T_\phi \) be ergodic. Suppose that \( V_{\phi, T, X} \) and \( V_{\phi, T, \gamma} \) are unitarily equivalent via \( W : L^2(X, \mu) \to L^2(X, \mu) \), a unitary operator of the form

\[
(Wf)(x) = h(x)f(Sx)
\]

for some measurable \( h : X \to \mathbb{C} \) and \( S : X \to X \). Then

(i) \( S \in C(T) \), \( |h(x)| = 1 \),

(ii) there exists a continuous group automorphism \( \varphi : G \to G \) such that \( \gamma = \chi \circ \varphi \), and if \( \chi \) is a generator of \( \hat{G} \) then \( S f = \varphi \) for some measurable \( f : X \to G \). Moreover, \( W = cf_{S, X} \) for some \( |c| = 1 \).

Proposition 2 combined with Theorem 4 below shows that a nontrivial multiplicity function as in the examples of [3] and [8] arises for reasons other than those appearing in this paper (the set \( A \) in those examples consists of the identity group automorphism).

Now, we show how to prove the mutual singularity of the measures \( \varphi_X \) and \( \varphi_\gamma \).

Let \( H \) be a separable Hilbert space and let \( U : H \to H \) be a unitary operator. Assume that \( \alpha \) is a complex number, \( |\alpha| \leq 1 \). We say that \( U \) is \( \alpha \)-weakly mixing if there exists a nondecreasing sequence \( \{m_t\} \) of positive integers such that for each \( h \in H \), we have

\[
(U^{m_t}(h), h) \to |\alpha|^2 \|h\|^2 \quad \text{as } t \to \infty.
\]

We obtain the following.

**Proposition 3.** Let \( U_t : H_t \to H_t \) be a unitary operator on a separable Hilbert space, \( t = 1, 2 \). Let \( \varphi_t \) denote the maximal spectral type of \( U_t \). If \( U_t \), \( t = 1, 2 \), are \( \alpha \)-weakly mixing with respect to the same sequence \( \{m_t\} \) then \( \delta_\alpha \ast \varphi_1 \perp \varphi_2 \) (for each \( n \in S_1 \)) provided \( |\alpha_1| \neq |\alpha_2| \).

We will also use the following.

**Proposition 4.** If \( U : H \to H \) is \( \alpha \)-weakly mixing, \( 0 < |\alpha| < 1 \), then \( \delta_\alpha \ast g^{(m)} \perp g^{(n)} \) for each \( \alpha \in S_1 \) and \( m \neq n \), where \( q \) is the maximal spectral type of \( U \) and \( g^{(n)} = g \ast \ldots \ast g \) (\( m \) times).

We will apply the concept of \( \alpha \)-weak mixing to \( T_\phi \), more precisely, to the family of unitary operators \( \{V_{\phi, T, X} : \chi \in \hat{G} \} \). We say that a sequence \( \{m_t\} \) of positive integers is a rigid time for \( T \) if for every \( f \in L^2(X, \mu) \) we have

\[
\|f T^{m_t} - f\|_2 \to 0 \quad \text{as } t \to \infty.
\]

For each \( n \geq 1 \), \( \varphi^{(n)} \) denotes the cocycle

\[
\varphi^{(n)}(x) = \varphi(x) + \varphi(T(x)) + \ldots + \varphi(T^{n-1}(x)) \quad x \in X.
\]

Here is our criterion for the \( \alpha \)-weak mixing of \( V_{\phi, T, X} \).

**Proposition 5.** Assume that for each \( \chi \in \hat{G} \), as \( t \to \infty \) we have

\[
\int_X \chi(\varphi^{(m_t)}(x)) \, d\mu(x) = \alpha,
\]

where \( \{m_t\} \) is a rigid time for \( T \). Then the operator \( V_{\phi, T, X} \) is \( \alpha \)-weakly mixing along \( \{m_t\} \).

The main results of this paper are consequences of the following theorem.

**Theorem 3.** Let \( G \) be a compact metric abelian group. Assume that \( \varphi : G \to G \) is a continuous group automorphism satisfying

\[
(6) \quad \text{for all } x \in \hat{G}, \text{card}\{x \circ v^n : n \geq 0\} < \infty.
\]

Then there exists an adding machine \( T : (X, B, \mu) \to (X, B, \mu) \), an ergodic cocycle \( \varphi : X \to G \), and \( S \in C(T) \) satisfying

\[
\text{(7a) for each } \chi \in \hat{G} \text{ there exists a rigid time } \{n_t\} \text{ for } T \text{ satisfying }
\]

\[
\lim_{t \to \infty} \int_X \chi(\varphi^{(n_t)}(x)) \, d\mu(x) = \alpha_x,
\]

\[
\text{(7b) for each pair } (\chi, \gamma) \in \hat{G} \times \hat{G} \text{ there exists a rigid time } \{n_t\} \text{ for } T \text{ satisfying }
\]

\[
\lim_{t \to \infty} \int_X \chi(\omega(\varphi^{(n_t)}(x)) \, d\mu(x) = \alpha_{\omega}, \quad \omega = \chi, \gamma.
\]
and moreover

(8) for each \( \chi \in \widehat{G} \), \( \chi \neq 1 \), we have \( 0 < |\alpha_{\chi}^{\prime}| < 1 \),

(9) for each \( (\chi, \gamma) \in \widehat{G} \times \widehat{G} \) with \( \chi \gamma = \gamma \) for all \( n \), we have \( |\alpha_{\chi}^{\prime} n| \neq |\alpha_{\gamma}^{\prime} n| \),

(10) \( V_{\phi, T, \chi} \) has simple spectrum for each \( \chi \in \widehat{G} \),

(11) there exists a measurable solution \( f : X \to G \) of the functional equation

\[
\phi(Sx) - v(\phi(x)) = f(Tx) - f(x).
\]

Notice that (7a) and (8) directly imply the ergodicity of \( T_{\phi} \). Let \( v : G \to G \) satisfy the conditions of Theorem 3 and let \( M_{\phi} \) be the set of the cardinalities of the sets \( \{\chi v^n : n \geq 0\} \), \( \chi \in \widehat{G} \). Let \( \phi \) satisfy the conclusion of Theorem 3. Then, applying (7b) and Proposition 5, (9) and Proposition 3, (10), (11) and Proposition 1, we obtain \( M_{\phi, T, \chi} = M_{\phi} \).

Finally, we would like to note that in the case of finite abelian groups our constructions are actually generalized Morse sequences (in the sense of Martin [10], [11]). The result below combined with [3] and [8] shows that the centralizer method applied in our paper is not the only one giving rise to a nontrivial (i.e. different from the constant function 1) multiplicity function.

**Theorem 4.** Let \( T : (X, B, \mu) \to (X, B, \mu) \) be an \( \{n_i\} \)-adic adding machine with standard sequence of towers \( D^x = (D_0^x, \ldots, D_{n_i-1}^x) \). Assume that \( G \) is a finite abelian group and \( \phi : X \to G \) is a Morse cocycle and put \( \sigma_\phi(x, h) = (x, g + h), \ x \in G \). Then if (a) and (b) below hold, the centralizer of \( T_\phi \) is trivial, i.e.

\[
C(T_\phi) = \{(\phi(g)^n)^{\sigma_\phi} : g \in B \};
\]

(a) the sequence \( \{n_{i+1}/n_i\} \) is bounded,

(b) \( (3\beta > 0)(\forall \epsilon)(3\beta_1, \beta_2 \in G)(\beta_1 \neq \beta_2) \)

\[
\mu(T^{-1}(D_0^x) \cap \phi^{-1}(\phi(x))) \geq \beta \mu(D_0^x), \quad i = 1, 2.
\]

**II. Proofs**

**Proof of Proposition 1.** The equality \( W V_{\phi, T, \chi} = V_{\phi, T, \gamma} W \) can be checked using easy computations. ■

**Proof of Proposition 2.** We prove (i) for an arbitrary group \( G \). Indeed, \( W V_{\phi, T, \gamma}(k) \) implies

\[
h(x)\chi(\phi(Sx))k(STx) = \gamma(\phi(x))h(Tx)k(TSx)
\]

for each \( \epsilon \in L^2(X, \mu) \). In particular, putting \( k = 1 \) we get \( h(x) = h(Tx) \) and by the ergodicity of \( T, \ h \) is constant, so \( h(x) = 1 \) since \( W \) is unitary. Moreover, \( S \) has to preserve the measure. Now \( h(x) \neq 0 \), so by the same argument, \( |k(STx)| = |k(TSx)| \) for each \( k \in L^2(X, \mu) \), and in particular for the characteristic functions of measurable sets. Hence \( S \in C(T) \).

We now prove three lemmas. Lemmas 1 and 2 do not require \( G = \mathbb{Z}_n \) (i.e. the cyclic group of order \( n \)).

**Lemma 1.** Suppose \( |G| = n \). Then \( h^n(x) = \text{const and hence } h(x) = \exp(2\pi i f(x)/n) \) for some measurable \( f : X \to \mathbb{Z}_n \) and \( |c| = 1 \).

**Proof.** \( \chi(G) \) is a subgroup of the \( n \)th roots of unity, so \( (\chi(g)) = 1 \) for each \( \chi \in \widehat{G}, \ g \in G \). By (12) (with \( k = 1 \)) we have

\[
(\chi(Tx)/\chi(x)) = [\chi(\phi(Sx))/\gamma(\phi(x))] = 1,
\]

so \( h^n(x) = h^n(Tx) \). Then we use the ergodicity of \( T \) to conclude that \( h(x) = \exp(2\pi i \beta) \exp(2\pi i f(x)/n) \) for some \( \beta \in [0, 1) \) and a measurable function \( f : X \to \mathbb{Z}_n \).

**Lemma 2.** Let \( T_0 \) be ergodic. If \( V_{\phi, T, \chi} \) and \( V_{\phi, T, \gamma} \) are unitarily equivalent via \( W \) then \( \gamma = 1 \) if \( \gamma = 1 \).

**Proof.** Suppose \( \gamma = 1 \). Then since \( \chi(\phi(Sx)) = \gamma(\phi(x)) = h(Tx)/h(x) \), we have \( h(Tx)/h(x) = \gamma(x)^{-1}(\phi(x)) \). So by the ergodicity of \( T_\phi \), \( \gamma = 1 \). The converse uses \( W^{-1} \) instead of \( W \).

**Lemma 3.** Suppose that \( V_{\phi, T_\chi} \) and \( V_{\phi, T_\gamma} \) are unitarily equivalent via \( W \) and let \( v : \mathbb{Z}_n \to \mathbb{Z}_n \) be a group automorphism. Then \( V_{\phi, T_\chi v} \) and \( V_{\phi, T_\gamma v} \) are unitarily equivalent.

**Proof.** Since \( v \) is an automorphism, there exists \( r \in \mathbb{Z}_n, \ (r, n) = 1 \), with \( v(g) = r g, \ g \in \mathbb{Z}_n \). Therefore

\[
\chi(\phi(Sx))/\chi(\phi(x)) = \chi(\phi(Sx))/\gamma(\phi(x)) = h(Tx)/h(x)
\]

and \( W_r(k)x = h(x)k(Sx) \), where \( k \in L^2(X, \mu) \), establishes the desired equivalence.

Now we continue the proof of Proposition 2 and proceed to (ii). If \( \chi \) is a generator of \( \mathbb{G} \), then by Lemma 3, we may assume that \( \chi(g) = \exp(2\pi i g/n) \) and \( \gamma(g) = \exp(2\pi i r g/n) \) for some \( r \in \mathbb{Z}_n \). In view of Lemma 2, since \( \chi^{s} = 1 \) iff \( s = n \), we have \( (r, n) = 1 \). Define an automorphism \( v : \mathbb{Z}_n \to \mathbb{Z}_n \) by \( v(g) = r g \). It follows that \( \gamma = \chi v \) (a similar argument shows that this is true generally).

But \( \chi(\phi(Sx))/\gamma(\phi(x)) = h(Tx)/h(x) \) implies, by Lemma 1,

\[
\chi(\phi(Sx)) - \chi(\phi(x)) = \exp(2\pi i f(Tx)/n) / \exp(2\pi i f(x)/n),
\]

or in other words \( \phi(Sx) - \phi(x) = f(Tx) - f(x) \) in \( \mathbb{Z}_n \).

**Proof of Proposition 3.** Let \( \nu \) be a probability measure absolutely continuous with respect to \( \delta_s g_1 \) and \( g_2 \). Then there exist \( h_i \in H_i, \ ||h_i|| = 1, ||h_i|| = 1, \)
It is clear that the conditions $\|k_i T^m - k_i\|_2 \to 0$ and $k_i T^m = \lambda_i^{m_i} k_i$ imply
\begin{equation}
\lambda_i^{m_i} \to 1, \quad i = 0, 1, \ldots
\end{equation}

It follows from (14) that
\begin{equation}
\int_X \chi(\varphi(m_i)(x)) k_i(x) k_j(x) d\mu(x) \to 0 \quad \text{for } i \neq j.
\end{equation}

Furthermore, we have
\begin{equation}
(V^{m_i} k, k) = \sum_{i,j=0}^{l} c_i \bar{c}_j (V^{m_i} k_i, k_j)
\end{equation}
\begin{equation}
= \sum_{i,j=0}^{l} c_i \bar{c}_j \int_X \chi(\varphi(m_i)(x)) k_i(T^{m_i} x) \overline{k_j(x)} d\mu(x)
\end{equation}
\begin{equation}
= \sum_{i,j=0}^{l} c_i \bar{c}_j \lambda_i^{m_i} \int_X \chi(\varphi(m_i)(x)) k_i(x) \overline{k_j(x)} d\mu(x)
\end{equation}
\begin{equation}
= \sum_{i=0}^{l} |c_i|^2 \lambda_i^{m_i} \int_X \chi(\varphi(m_i)(x)) d\mu(x)
\end{equation}
\begin{equation}
+ \sum_{i \neq j} c_i \bar{c}_j \lambda_i^{m_i} \int_X \chi(\varphi(m_i)(x)) k_i(x) \overline{k_j(x)} d\mu(x).
\end{equation}

Using the assumption of Proposition 5 and (19)-(21) we obtain $\lim_{t \to \infty} \| V^{m_i} k, k \|_2 \to 0$ as $t \to \infty$.

In order to complete the proof it remains to show (13) for every $k \in L^2(X, \mu)$. Let $k \neq 0$ and take $\varepsilon > 0$, $\varepsilon < \|k\|_2$. There exists a function $\tilde{k}$ of the form (18) such that $\|k - \tilde{k}\|_2 < \min(\varepsilon, \|k\|_2/12, \varepsilon/4)$. Then $\|V^{m_i} \tilde{k}\|_2 \to 0$ as $t \to \infty$ and hence
\begin{equation}
\|V^{m_i} k, k\|_2 < \|V^{m_i} k\|_2 < \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + (\varepsilon/12, \varepsilon/12) \|k\|_2
\end{equation}
for $t$ large enough.

Before passing to the other proofs, we will need some auxiliary considerations. Let $T : (X, B, \mu) \to (X, B, \mu)$ be an $\{n_t\}$-adic adding machine, i.e. $n_t | n_{t+1}$, $\lambda_{t+1} = n_{t+1}/n_t \geq 2$ for $t \geq 0$, $\lambda_0 = n_0 \geq 2$ and $X = \{ x = \sum_{t=0}^{\infty} q_t n_{t-1} : 0 \leq q_t \leq n_t - 1, n_{-1} = 1 \}$.
is the group of \( \{n_t\} \)-adic numbers, where \( T \sigma = x + \bar{1}, \bar{1} = (1, 0, 0, \ldots) \). The centralizer \( C(T) \) of \( T \) can be naturally identified with \( X \) as follows. Let \( D^t = (D^t_0, \ldots, D^t_{n_t - 1}) \) be the standard sequence of \( T \)-towers

\[
D^t_0 = \{ x \in X : q_0 = q_1 = \ldots = q_t = 0 \}, \quad D^t_1 = D^t_0
\]

(\( t \) is taken mod \( n_t \)). Then \( D^{t+1} \) refines \( D^t \) and obviously the sequence of partitions \( \{D^t\} \) converges to the point partition. Take \( S \in C(T) \). As \( S \) is determined by an \( x \in X \), \( S(D^t_j) = D^t_{j+t} \) (\( j + t \) is taken mod \( n_t \)) for each \( j = 0, 1, \ldots, n_t - 1, t \geq 0 \), where \( j_t = \sum_{i=0}^{t} g_i n_{t-1} \).

Let \( G \) be a compact metric abelian group. We will define a special class of cocycles called \( M \)-cocycles. We say that \( \phi : X \to G \) is an \( M \)-cocycle if for every \( t \geq 0 \) the cocyle \( \phi \) is constant on each level \( D^t_i \) for \( i = 0, 1, \ldots, n_t - 2 \). Such a \( \phi \) is defined by a sequence of blocks \( \{A_t\} \) (\( A_t = A_t[0]A_t[1] \ldots A_t[n_t - 2] \)), where

\[
\phi_D^i = A_t[i], \quad i = 0, 1, \ldots, n_t - 2.
\]

Define

\[
a^{t+1}[i] = A_{t+1}[i+1,n_t - 1], \quad i = 0, 1, \ldots, \lambda_{t+1} - 2,
\]

\[
a^{t+1} = a^{t+1}[0] \ldots a^{t+1}[\lambda_{t+1} - 2].
\]

We obtain

\[
A_{t+1} = A_t a^{t+1}[0] A_t a^{t+1}[1] A_t \ldots A_t a^{t+1}[\lambda_t - 2] A_t.
\]

If, in addition, we write \( a^0 = A_0 \), we see that an \( M \)-cocyle is completely defined by the sequence of blocks \( \{a^t\} \). Also notice in passing that if \( G \) is a finite abelian group then the class of group extensions obtained from \( M \)-cocycles coincides with the class of automorphisms arising from generalized Morse sequences over \( G \) (see [10] and [11]). Instead of the sequence \( \{a^t\} \), we will consider another sequence of blocks which will also determine \( \phi \). Namely, write

\[
b^0 = a^0, \quad b^t[i] = a^t[i] + u_{t-1}, \quad t \geq 1,
\]

where \( u_t = A_t[0] + A_t[1] + \ldots + A_t[n_t - 2], t = 0, 1, \ldots \).

Let \( v : G \to G \) be a continuous group automorphism. If \( C \) is a block over \( G \) then \( v(C) \) is the block \( v[C[0]]v[C[1]] \ldots v[C[|C| - 1]] \), where \( |C| = \text{ length of } C \).

**Proof of Theorem 3.** Assume that an adding machine \( T : (X, B, \mu) \to (X, B, \mu) \) is given by \( \lambda_t = q_t k_t + 1, q_t \geq 1, k_t \geq 2, t \geq 0 \). Let \( S \sigma = x + x_0 \), where \( x_0 = \sum_{t=0}^{\infty} g_t n_{t-1} \).

**Lemma 4.** Suppose that \( \sum_{t=0}^{\infty} 1/k_t < \infty \) and let blocks \( \{b^t\}, t \geq 0, |b^t| = \lambda_t - 1, \) be of the form

\[
b^t = d^t v(d^t) \ldots v^{\lambda_t-1}(d^t),
\]

where \( |d^t| = q_t \). If \( \phi \) is the cocycle determined by \( \{b^t\} \) then there is a measurable solution \( f : X \to G \) of the equation

\[
\phi(Sx) - v(\phi(x)) = f(Tx) - f(x).
\]

**Proof.** Let \( j_t = \sum_{j=0}^{t} g_j n_{j-1}, t = 0, 1, \ldots \) Then

\[
j_{t+1} = q_t n_t + j_t.
\]

Consider the cocycle \( \psi(x) = \phi(Sx) - v(\phi(x)), x \in X \), on the tower \( D^t \) (see Fig. 1).

**Fig. 1.** The values of \( \psi \) on \( D^t \)

Define a function \( f_t \) on \( C_t = \bigcup_{s=0}^{n_t-1} D^t_s \) by \( f_t(x) = 0 \) for \( x \in D^t_0 \) and

\[
f_t(x) = \psi(T^{-1}x) + \ldots + \psi(T^{-s}x)
\]

\[
= A_t[0] + \ldots + A_t[j_t + s - 1] - v(A_t[0] + \ldots + A_t[j_t - 1])
\]

for \( x \in D^t_s \) and \( s = 1, \ldots, n_t - j_t - 1 \). Notice that \( f_t \) satisfies (24) for \( x \in C_t \setminus D^t_{n_t-j_t-1} \). Moreover,

\[
\mu(C_t) = 1 - j_t/n_t - 1/n_t.
\]

We will show that

\[
f_t(x) = f_{t+1}(x) \quad \text{if } x \in C_t \cap C_{t+1}.
\]

Indeed, observe that \( f_t(x) = f_{t+1}(x) \) for \( x \in \bigcup_{j=0}^{n_t-j_t-1} D^{t+1}_{n_t-j_t} \) provided \( f_t(x) = f_{t+1}(x) \) for \( x \in D^{t+1}_{n_t-j_t}, t = 0, 1, \ldots, \lambda_{t+1} - 1 \). Therefore,
in order to prove (27) it suffices to show that \( f_i(x) = f_{i+1}(x) \) for \( x \in D^{i+1}_{\text{int}}, D^{i+1}_{\text{ext}}, \ldots, D^{i+1}_{(a_{i+1}-1)\text{int}} \). Take \( x \in D^{i+1}_{\text{int}} \). Then immediately from Fig. 1 we obtain

\[
\begin{align*}
  f_{i+1}(x) &= \psi(T^{-1}x) + \ldots + \psi(T^{-i+1}x) \\
  &= (A_i[0] + \ldots + A_i[n_i - 2]) - v(A_i[0] + \ldots + A_i[n_i - 2]) \\
  &\quad + a^{i+1}[q_{i+1}] - v(a^{i+1}[0]) + \ldots + (A_i[0] + \ldots + A_i[n_i - 2]) \\
  &\quad - v(A_i[0] + \ldots + A_i[n_i - 2]) + a^{i+1}[q_{i+1} + i - 1] - v(a^{i+1}[i - 1]) \\
  &= (u_2 - v(u_2) + a^{*+1}[q_{i+1}] - v(a^{*+1}[0]) + \ldots \\
  &\quad + (u_2 - v(u_2) + a^{*+1}[q_{i+1} + i - 1] - v(a^{*+1}[i - 1])).
\end{align*}
\]

In view of (22),

\[
\begin{align*}
  f_{i+1}(x) &= u_2 - v(u_2) + b^{*+1}[q_{i+1}] - u_2 - v(b^{*+1}[0] - u_2) + \ldots \\
  &\quad + u_2 - v(u_2) + b^{*+1}[q_{i+1} + i - 1] - u_2 - v(b^{*+1}[i - 1] - u_2) \\
  &= b^{*+1}[q_{i+1}] - v(b^{*+1}[0]) + \ldots + b^{*+1}[q_{i+1} + i - 1] - v(b^{*+1}[i - 1])
\end{align*}
\]

and consequently by (23), \( f_{i+1}(x) = f_i(x) \).

It follows from (26) that

\[
\mu(C_t \cap C_{t+1} \cap \ldots) \geq 1 - \sum_{l=1}^{\infty} j_l / n_l - \sum_{l=1}^{\infty} 1 / k_l - \sum_{l=1}^{\infty} 1 / n_l
\]

since (25) holds. In view of the convergence of \( \sum_{l=0}^{\infty} 1 / k_l \), we see that \( \mu(C_t \cap C_{t+1} \cap \ldots) / \mu(C_t) \to 1 \) and consequently that \( f_t(x) = f_{t+1}(x) \) if \( x \in C_t \cap C_{t+1} \cap \ldots \). Therefore \( \{f_t\} \) converges in measure to some \( f : X \to G \). Since \( f(x) = f_t(x) \) for \( x \in C_t \cap C_{t+1} \cap \ldots \), \( f \) satisfies (24) for a.e. \( x \in X \).

The following lemma is well known ([15], [8], [3]).

**Lemma 5.** If \( \phi : X \to G \) is an M-cocycle and \( \chi \in \mathcal{G} \) then \( V_{\phi, \pi, \chi} \) has simple spectrum.

Let \( \psi : G \to G \) be a continuous group automorphism satisfying (6) of Theorem 3. Denote by \( \psi : \mathcal{G} \to \mathcal{G} \) the dual automorphism. In view of (6), each orbit of \( \psi \) is finite, i.e. the set \( \{\psi(\gamma) : r \in \mathbb{Z}\} \) is finite for each \( \gamma \in \mathcal{G} \). Let \( \mathcal{G} = \bigcup_{i=1}^{\infty} \mathcal{F}_i \), where \( \mathcal{F}_i \) is an orbit of \( \psi \), \( i \geq 1 \), \( \mathcal{F}_i \cap \mathcal{F}_j = \emptyset \). Choose \( \gamma_i \in \mathcal{F}_i, i \geq 1 \). Let \( r(\gamma) = r \) be the smallest positive integer such that \( \psi^r(\gamma) = \gamma \). We will write \( r(\gamma) \) instead of \( \psi(\gamma) \), \( i \geq 1 \).

### III. The construction of \( \phi \) and \( T \) satisfying the conclusions of Theorem 3

Take positive integers \( k_t, t \geq 0 \), satisfying \( \sum_{t=0}^{\infty} 1 / k_t < \infty \).

**Step 1.** Choose a countable dense subset \( G' \subseteq G \), \( G' = \{g_1, g_2, \ldots\} \), and set \( G_n = \{g_1, \ldots, g_n\}, n = 1, 2, \ldots \)

**Step 2.** Divide \( \mathbb{N} = \{0, 1, \ldots\} \) into infinitely many pairwise disjoint infinite subsets \( M_{ij}, i \neq j \), and \( N_i, i \geq 0 \), such that

\[
\mathbb{N} = \bigcup_{i,j} M_{ij} \cup \bigcup_i N_i, \quad M_{ij} \cap N_i = \emptyset.
\]

**Step 3.** Let \( i \neq j \). There exist \( n = n(i, j) \) and \( g' \in G_n \) such that

\[
\frac{1}{r_t} \sum_{i=0}^{r_t-1} \psi^i(\gamma_j)(g') - \frac{1}{r_j} \sum_{i=0}^{r_j-1} \psi^i(\gamma_j)(g') \neq 0.
\]

This is possible because the functions

\[
g \mapsto \frac{1}{r_t} \sum_{i=0}^{r_t-1} \psi^i(\gamma_j)(g) \quad \text{and} \quad g \mapsto \frac{1}{r_j} \sum_{i=0}^{r_j-1} \psi^i(\gamma_j)(g),
\]

\( g \in G \), are orthogonal and nonzero in \( L^2(G, \mu) \).

**Step 4.** Take the same number \( n = n(i, j) \) as in Step 3 and consider the simplex \( \Delta_n \) of all probability vectors \( \psi = (s(g), g \in G_n) \). Define a function \( F_{ij}(\psi) = F_{ij}^t(\psi), \psi \in \Delta_n \), by

\[
F_{ij}(\psi) = \sum_{g \in G_n} (A_i(g) - A_j(g)) s(g).
\]

**Step 5.** Choose \( \tilde{\psi}_0 = \tilde{\psi}_0(i, j) \in \Delta_n \) such that

\[
F_{ij}(\tilde{\psi}_0) = d_{ij} \neq 0 \quad \text{and} \quad s_0(g) > 0 \quad \text{for every} \quad g \in G_n.
\]

Such an \( \tilde{\psi}_0 \) exists because the equation (with complex coefficients)

\[
F_{ij}^t(\tilde{\psi}) = \sum_{g \in G_n} (A_i(g) - A_j(g)) s(g) = 0
\]

determines the intersection of two vector subspaces in \( \mathbb{R}^{G_n - 1} \), at least one of which has dimension \( |G_n| - 1 \) (it follows from (28) that at least one coefficient is different from 0). These planes have Lebesgue measure zero. Define

\[
\Delta'_n = \{\tilde{\psi} = (s(g)) : F_{ij}(\tilde{\psi}) = 0 \text{ and } s(g) > 0 \text{ for some } g \in G_n\}.
\]

Then \( \Delta_n \cap \Delta'_n \) is nonempty and open in \( \Delta_n \).

**Step 6.** Choose a ball \( K(\tilde{\psi}_0, \epsilon) \) such that \( K(\tilde{\psi}_0, \epsilon) \subseteq \Delta_n \setminus \Delta'_n \) and

\[
|F_{ij}(\tilde{\psi}) - F_{ij}(\tilde{\psi}_0)| < (1/2)|d_{ij}| \quad \text{if} \quad \tilde{\psi} \in K(\tilde{\psi}_0, \epsilon).
\]

**Step 7.** Let \( B \) be a block over \( G_n \). By the average frequencies of the elements of \( G_n \) the block \( B \) determines an element of \( \mathcal{F}(B) \in \Delta_n \), i.e.

\[
\mathcal{F}(B)(g) = (1/|B|) \text{card} \{0 \leq i \leq |B| - 1 : B[i] = g\}.
\]

Choose a block over \( G_n \) such that \( \mathcal{F}(B) \in K(\tilde{\psi}_0, \epsilon) \). Let \( q = q_f = |B| \).
For every \( i \in M_{ij} \) define \( \lambda_i = k_i q + 1 \) and let the block \( b_i^* \), \( |b_i^*| = \lambda_i \), be the following concatenation:

\[
(33) \quad b_i^* = B_{v}(B) \ldots v^{k_i-1}(B).
\]

We iterate Steps 3–7 for every pair \((i, j)\), \(i \neq j\).

**Step 8.** Let \( i \geq 0 \). There exists \( n' = n_i \) and two different elements \( g_1, g_2 \in G_{n'} \) such that \( A_{i}(g_1) \neq A_{i}(g_2) \). Choose a number \( \beta \), \( 0 < \beta < 1 \), satisfying \( d = \beta A_i(g_1) + (1-\beta)A_i(g_2) \neq 0 \). Then define \( s_{\beta} \in \Delta_{n'}, s_{\beta} = s_{\beta}(g), g \in G_{n'}, \) as follows:

\[
(34) \quad s_{\beta}(g_1) = \beta; \quad s_{\beta}(g_2) = 1 - \beta; \quad s_{\beta}(g) = 0 \text{ if } g \neq g_1, g_2.
\]

**Step 9.** Define a function \( F_i = F_i(s_{\beta}), s_{\Delta_{n'}} \), by

\[
(35) \quad F_i(s_{\beta}) = \sum_{g \in G_{n'}} A_i(g) s_{\beta}(g).
\]

It follows from Step 8 that \( F_i(s_{\beta}) = d \neq 0 \) and \( |F_i(s_{\beta})| \leq 1 \), because \( |F_i(s)| \leq 1 \) for every \( s \in \Delta_{n'} \). There exists a ball \( K(s_{\beta}, d_1) \) in \( \Delta_{n'} \) and \( \delta > 0 \) such that

\[
(36) \quad \delta < |F_i(s_{\beta})| < 1 - \delta
\]

for every \( s \in K(s_{\beta}, d_1) \). We can assume that \( s_{\beta} \) is an interior point of \( \Delta_{n'} \).

**Step 10.** Choose a block \( B_1 \) over \( G_{n'} \) such that \( s(B_1) \in K(s_{\beta}, d_1) \). Then we put \( g_i = q = |B_1| \). For every \( i \in N_i \) define \( \lambda_i = q_i k_i + 1 \) and a block \( b_i^* \) as

\[
(37) \quad b_i^* = B_{v}(B_1) \ldots v^{k_i-1}(B_1).
\]

We iterate Steps 8–10 for every \( i \geq 0 \).

By Steps 7 and 10, for each \( t \in N \), we have defined blocks \( b^t \) over \( G \), \( |b^t| = \lambda_t - 1 \), and then by (22) the blocks \( a^t, t \geq 0 \). These blocks determine an M-cocycle \( \phi \).

Now we will prove (7a,b), (8) and (9) of Theorem 3. We will evaluate \( \int_X \gamma(\phi^{(n_i)}(x)) \ d\mu(x) \) for each \( \gamma \in \mathcal{G} \). We have

\[
\phi^{(n_i)}(x) = A_i[0] + \ldots + A_i[n_i - 2] + a^{t+1}[0]
\]

if \( x \in D_0^{t+1} \cup \ldots \cup D_{n_i-1}^{t+1}, \)

\[
\phi^{(n_i)}(x) = u_t + a^{t+1}[i]
\]

if \( x \in D_{n_i}^{t+1} \cup D_{n_i+1}^{t+1} \cup \ldots \cup D_{\lambda_i+1}^{t+1}, t = 1, \ldots, \lambda_{t+1} - 2 \).

Thus

\[
\int_X \gamma(\phi^{(n_i)}(x)) \ d\mu(x)
\]

\[
= \frac{1}{\lambda_{t+1} - 1} \sum_{g \in G} \gamma(g) \text{card } \{0 \leq i \leq \lambda_{t+1} - 2: u_t + a^{t+1}[i] = g\} + g_t,
\]

where \( g_t \leq 1/\lambda_{t+1} \) and only countably many summands are different from 0. Using (22) we can rewrite the above equality as

\[
\int_X \gamma(\phi^{(n_i)}(x)) \ d\mu(x) = w(b^{t+1}) + g_t,
\]

where \( w(b^{t+1}) = \sum_{g \in G} \gamma(g)s(b^{t+1})(g) \) and \( s(b^{t+1})(g) \) is the distribution of \( g \) in \( b^{t+1} \). Therefore

\[
(38) \quad \left| \int_X \gamma(\phi^{(n_i)}(x)) \ d\mu - w(b^{t+1}) \right| \leq 1/\lambda_{t+1}.
\]

Each block \( b^t \) has the form (23). Now we will calculate \( w(b^{t+1}) \).

Put \( \lambda = \lambda_t, k = k_t, q = q_t \). Then \( \lambda - 1 = kq \). The following holds:

\[
\text{card}\{0 \leq i \leq \lambda - 2: b^t[i] = g\} = \sum_{l=0}^{k-1} \text{card}\{lq \leq i < lq + q - 1: b^t[i] = g\}
\]

\[
= \sum_{l=0}^{k-1} \text{card}\{0 \leq i < q - 1: d^l[i] = v^{-l}(g)\}.
\]

Hence

\[
s(b^t)(g) = (1/(\lambda - 1)) \text{card}\{0 \leq i \leq \lambda - 2: b^t[i] = g\}
\]

\[
= \frac{1}{k} \sum_{l=0}^{k-1} 1\text{ card}\{0 \leq i < q - 1: d^l[i] = v^{-l}(g)\}
\]

\[
= \frac{1}{k} \sum_{l=0}^{k-1} s(d^l)(v^{-l}(g)).
\]

As a consequence we obtain

\[
w(b^{t+1}) = \frac{1}{k_t} \sum_{l=0}^{k_t-1} \sum_{g \in G} \gamma(g)s(d^l)(v^{-l}(g)) = \frac{1}{k_t} \sum_{l=0}^{k_t-1} \sum_{g \in G} \overline{\delta}^l(\gamma)(g)s(d^l)(g).
\]

Let \( k_t = w_t + r' \), where \( r = r(\gamma) \) and \( 0 \leq r' < r \). Then

\[
(39) \quad \left| \frac{1}{k_t} \sum_{l=0}^{k_t-1} \sum_{g \in G} \overline{\delta}^l(\gamma)(g)s(d^l)(g) - \frac{1}{r} \sum_{l=0}^{r-1} \sum_{g \in G} \overline{\delta}^l(\gamma)(g)s(d^l)(g) \right| \leq r/k_t,
\]

since \( \overline{\delta}^l(\gamma) = \gamma \). Using (38) and (39), we have

\[
(40) \quad \left| \int_X \gamma(\phi^{(n_i)}(x)) \ d\mu(x) - \frac{1}{r} \sum_{l=0}^{r-1} \sum_{g \in G} \overline{\delta}^l(\gamma)(g)s(d^l)(g) \right| \leq 1/\lambda_{t+1} + r/k_{t+1}.
\]
Fix \( i \) and take \( t \in N_i \) (see Steps 8–10). Then \( b' = B_1, \overline{s}(d') = \overline{s}(B_1) \) and
\[
\frac{1}{r_1} \sum_{i = 0}^{n_i - 1} \sum_{g \in G} \overline{\phi}(\gamma_i)(g)s(d')(g) = F_i(\overline{s}(B_1)).
\]
It follows from (35), (40) and (41) that
\[
\delta/2 < \left| \int_X \gamma_i (\phi^{(n_i)})(x) \, d\mu(x) \right| < 1 - \delta/2 \text{ for } t \text{ large enough, } t \in N_i.
\]
By taking subsequences if necessary, we may assume that
\[
\int_X \gamma_i (\phi^{(n_i)})(x) \, d\mu(x) \to \alpha \text{ and } \delta/2 < |\alpha| < 1 - \delta/2.
\]
The above properties imply (7a) and (8).

Now we will show (7b) and (9). Let \( i \neq j \) and let \( t \in M_{ij} \) (see Steps 3–7). Then \( b' = B \) and \( \overline{s}(b') = \overline{s}(B) \). Using (30) and (31) we obtain
\[
|F_{ij}(\overline{s}(B))| \geq (1/2)|d_{ij}|. \text{ Then (40) gives us}
\]
\[
\left| \int_X \gamma_i (\phi^{(n_i)})(x) \, d\mu(x) - \int_X \gamma_j (\phi^{(n_j)})(x) \, d\mu(x) \right| \geq (1/2)|d_{ij}| - 2/\lambda t_{i+1} - r_i/k_{i+1} - r_j/k_{i+1}.
\]
In this way
\[
\int_X \gamma_i (\phi^{(n_i)})(x) \, d\mu(x) - \int_X \gamma_j (\phi^{(n_j)})(x) \, d\mu(x) \geq \delta_2 > 0
\]
for infinitely many \( t \). Taking a subsequence of \( \{n_t\} \) we may assume that \( \int_X \gamma_i (\phi^{(n_k)})(x) \, d\mu(x) \to \alpha_k (k = i, j) \). Thus (9) now follows, and so also does Theorem 3 since Lemmas 4 and 5 imply (10) and (11). \( \blacksquare \)

Lemma 6. Suppose \( \mathcal{M} \) is a finite set of natural numbers such that
\[
(c) \quad 1 \in M,
\]
\[
(d) \quad \text{whenever } m_1, m_2 \in M \text{ then } \text{lcm}(m_1, m_2) \in M.
\]
Then there exists a cyclic group \( Z_n \) and a group automorphism \( \overline{v} \) of \( Z_n \) such that \( M_{\overline{v}} = \mathcal{M} \). \( \blacksquare \)

Proof of Theorem 1. Let \( A = \{1, m_1, m_2, \ldots\} \) be a subset of the natural numbers satisfying (i) and (ii). Let \( A_j \) be the smallest set containing \( \{1, m_1, \ldots, m_j\} \) and satisfying (c) and (d). Then \( A_j \) is finite and \( A_1 \subseteq A_2 \subseteq \ldots \). Applying Lemma 6 we choose cyclic groups \( Z_{n_j} \) and automorphisms \( v_j \) of \( Z_{n_j} \) such that \( M_{v_j} = A_j, j \geq 1 \). Let \( \overline{G} \) be the product of \( Z_{n_j} \) and \( \overline{v} \) the corresponding product automorphism. It is clear that \( M_{\overline{v}} = A \) and that
\( \overline{v} \) satisfies (6) (we recall that \( \overline{G} = \overline{\bigoplus_j Z_{n_j}} \)). Now we apply Theorem 3 to \( \overline{v} \) and find \( T \) and \( \phi \) such that \( A = M_{T\phi} \).

Proof of Theorem 2. If \( A \) is a finite set satisfying (i) and (ii) then applying Lemma 6 we choose a cyclic group \( Z_n \) and an automorphism \( \overline{v} \) of \( Z_n \) such that \( M_{\overline{v}} = A \). Then we take \( G = Z_n = Z_n, v = \overline{v} \) and construct a Morse cocycle \( \phi \) over \( G \) as in Steps 1–10. We then have
\[
M_{T\phi} = M_{\overline{v}} = M_v = A \text{.}
\]

Proof of Theorem 4. Assume that \( S_t \in C(T_1) \) and \( S \notin \{T_m : m \in \mathbb{Z}\} \). This implies that there exists an infinity of \( t \)'s such that if \( S(D_{t}^{n+1}) = D_{j}^{n+1} \) then \( n_i \leq j \leq n_{i+1} - n_i \) by (a). Assume that
\[
\phi(S_t x) - \phi(x) = f(T_n x) - f(x).
\]
Then
\[
\phi^{(n_i)}(S_t x) - \phi^{(n_i)}(x) = f(T_n x) - f(x).
\]
Take \( \varepsilon > 0 \). Since \( f : X \to G \) is measurable and \( G \) is finite, \( f \) is constant on most of the levels \( D_t \) (except for an \( \varepsilon \)-fraction of such a good level). There exist \( r_1, r_2 \) such that
\[
r_1 \geq n_{i+1} - n_i, S^{-1}(D_{r_1}^{n+1}) = D_{r_2}^{n+1} \text{ and } f \text{ is constant on } D_{r_2}^{n+1} \text{ except for an } \varepsilon \text{-fraction of the level}.
\]
From (44) and (45) we obtain
\[
\phi^{(n_i)}(S_t x) = \phi^{(n_i)}(x)
\]
for \( x \in D_t^{n+1} \) except for a set of measure \( 2\varepsilon \lambda \mu(D_t^{n+1}) \), where \( \lambda \) is an upper bound for \( \{n_{i+1}/n_i\} \). But \( \phi \) is a Morse cocycle, so \( \phi^{(n_i)}(x) \) is constant for all \( x \in D_t^{n+1} \) \((r_2 < n_{i+1} - n_i)\), while \( \phi^{(n_i)}(S_t x) \), by (b), varies. Taking \( \varepsilon \) small enough we obtain a contradiction. \( \blacksquare \)

References

Approximation of continuous convex-cone-valued functions by monotone operators

by

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Abstract. In this paper we study the approximation of continuous functions \( F \), defined on a compact Hausdorff space \( S \), whose values \( F(t) \), for each \( t \) in \( S \), are convex subsets of a normed space \( E \). Both quantitative estimates (in the Hausdorff semimetric) and Bohman–Korovkin type approximation theorems for sequences of monotone operators are obtained.

0. Introduction. It is the purpose of this paper to discuss convergence results and quantitative estimates for the approximation by monotone operators of continuous functions \( F \) defined on a compact Hausdorff space \( S \), such that the value \( F(t) \), for each \( t \) in \( S \), is an element of some convex cone \( C \) endowed with a semimetric \( d_H \). In many applications \( C \) is a convex subcone of the convex cone \( C(E) \) of all convex nonempty bounded subsets of a normed space \( E \) over the reals, the semimetric \( d_H \) being the Hausdorff semimetric.

\[
d_H(K, L) = \inf \{\lambda > 0 ; K \subset L + \lambda B, L \subset K + \lambda B\},
\]

where \( B \) is the closed unit ball of \( E \).

After giving the necessary definitions in §1 and §2, we consider in §3 the problem of quantitative estimates for the approximation by sequences \( \{T_n\}_{n \geq 1} \) of monotone \( R^1 \)-linear operators on \( C(S; C) \), and show how to extend to this context some of the local estimates of Shisha and Mond [5].

In §4 and §5 we give examples of monotone \( R^1 \)-linear operators on \( C(S; C) \). In §4 we treat the case of operators of interpolation type and in §5 we consider two such operators, namely the Bernstein operators \( B_n \), defined in \( C([0,1]; C) \) or in \( C(S_m; C) \), where \( S_m \) is the standard simplex in \( R^m \), and the Hermite–Fejér operators \( H_n \), defined in \( C([-1,1]; C) \). Our Theorem 3 gives the estimates for the degree of approximation by \( B_n \) on

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