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# On the multiplicity function of ergodic group extensions of rotations

by

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**Abstract.** For an arbitrary set $A \subseteq \mathbb{N}$ satisfying $1 \in A$ and $\operatorname{lcm}(m_1, m_2) \in A$ whenever $m_1, m_2 \in A$, an ergodic abelian group extension of a rotation for which the range of the multiplicity function equals $A$ is constructed.

**Introduction.** In this paper we study the set $\mathcal{M}_T$ of all essential spectral multiplicities of an ergodic measure preserving the transformation $T$ of a Lebesgue space $(X, \mathcal{B}, \mu)$. $\mathcal{M}_T$ is defined as the essential range of the multiplicity function with respect to the maximal spectral type of the associated unitary operator

$$U_T : L^2(X, \mu) \to L^2(X, \mu), \quad (U_T f)(x) = f(Tx), \quad x \in X.$$

Thus $\mathcal{M}_T$ is a subset of the set $\overline{\mathbb{N}}$ of all positive integers and infinity. Many examples in ergodic theory have $\mathcal{M}_T = \{1\}$ (e.g. irrational rotations), $\mathcal{M}_T = \{\infty\}$ (e.g. Kolmogorov automorphisms), $\mathcal{M}_T = \{1, \infty\}$ (e.g. affine transformations). Transformations with $\mathcal{M}_T = \{1, k\}$ have been constructed ([16]), for each positive integer $k$, and also with $\mathcal{M}_T = \{1, 2k\}$, where $2k$ corresponds to the multiplicity of the Lebesgue component ([1], [9], [12]).

The problem of whether for an arbitrary nonempty set $A \subseteq \mathbb{N}$ there exists an ergodic transformation $T$ with $\mathcal{M}_T = A$ seems to be open. Toward the full solution of this question, Robinson in [18] has proved that for each finite set $A$ of positive integers satisfying:

(i) $1 \in A$,
(ii) $\operatorname{lcm}(m_1, m_2) \in A$ whenever $m_1, m_2 \in A$,

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there exists a weakly mixing transformation $T$ such that $\mathcal{M}_T = A$. The transformation used by Robinson was a group extension (in fact nonabelian) of an automorphism $T_0$ which admits a good cyclic approximation. However, his example is based on generic arguments and it is not constructive. He showed that a dense $G_\delta$ set of group extensions $T$ of $T_0$ satisfies $\mathcal{M}_T = A$. The main result of the present paper is

THEOREM 1. *Let $A$ be a subset of positive integers (finite or not) satisfying* (i) *and* (ii). *Then there exists an ergodic transformation $T$ with $\mathcal{M}_T = A$.*

The transformations employed in the proof of Theorem 1 are abelian group extensions of the so-called adding machines. If $A$ is a finite set then these transformations turn out to be Morse automorphisms over a finite abelian group (in the sense of [10]). Our transformations are described in a constructive way and moreover, each of them has a shift representation. This made it possible to compare the spectral multiplicity and the rank of special examples of such transformations ([2]). The classical Morse symbolic dynamical systems over the group $\mathbb{Z}_2 = \{0,1\}$, defined by Keane in [6], have simple spectra ([7]). Goodson in [3] has constructed examples of Morse automorphisms over cyclic groups with $\mathcal{M}_T = \{1,2\}$. A similar result has been obtained in [8]. A conjecture arose that the multiplicity function of all Morse automorphisms over cyclic groups is upper bounded by 2 (formally the question was raised in [4]). As a consequence of our considerations we answer that question negatively.

THEOREM 2. *Let $A$ be a finite set of positive integers satisfying* (i) *and* (ii). *There exists a Morse automorphism $T$ over a finite cyclic group such that $\mathcal{M}_T = A$.*

In particular, for every natural number $k \geq 1$, there exists a Morse automorphism $T$ over a cyclic group whose maximal spectral multiplicity is $k$. Robinson in [18] has proved the same result using Morse automorphisms, but over nonabelian groups. It is interesting to know what kind of spectral measures appear in our construction. Let $A = \{n_1, n_2, \ldots\}$ satisfy (i) and (ii) and let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be the transformation from the proof of Theorem 1. Then

$$L^2(X,\mu) = \bigoplus_{i\geq 1} (Z(h_1^{(i)}) \oplus \ldots \oplus Z(h_{n_i}^{(i)})),$$

where $Z(h_j^{(i)})$, $1 \leq j \leq n_i$, $i \geq 1$, are pairwise orthogonal $U_T$-cyclic subspaces and if $\varrho_i^{(j)}$ denotes the maximal spectral type of $U_T : Z(h_j^{(i)}) \to Z(h_j^{(i)})$ then



(iii) $\varrho_1^{(j)} \sim \ldots \sim \varrho_{n_j}^{(j)}$, $j \geq 1$,

(iv) $\delta_z * \varrho_1^{(j)} \perp \varrho_1^{(k)}$ for every $z \in S^1$ and $j \neq k$, in particular, $\varrho_1^{(j)} \perp \varrho_1^{(k)}$,

(v) for each $z \in S^1$, $j \geq 1$ and $s \neq t$

$$\delta_z * \underbrace{(\varrho_1^{(j)} * \ldots * \varrho_1^{(j)})}_{s} \perp \underbrace{(\varrho_1^{(j)} * \ldots * \varrho_1^{(j)})}_{t}.$$

In our considerations, the *centralizer* $C(T)$ of $T$ plays a role. We recall that $C(T)$ consists of all measure preserving transformations commuting with $T$.

**I. Description of the method and results.** From now on, $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ denotes an ergodic rotation on a compact metric monothetic group $X$ with Haar measure $\mu$. Let $G$ be a compact metric abelian group with Haar measure $m$. By a *cocycle* we mean a measurable function $\phi : X \to G$. A cocycle $\phi$ defines an automorphism $T_\phi$ on $(X \times G, \widetilde{\mu})$ by $T_\phi(x, g) = (Tx, \phi(x) + g)$, $x \in X$, $g \in G$, where $\widetilde{\mu} = \mu \times m$. Such an automorphism is called a *G-extension* of $T$. It need not be ergodic. In fact, *it enjoys the ergodicity property iff for every nontrivial character $\chi \in \widehat{G}$ there is no measurable solution $f : X \to S^1$ of the functional equation* $\chi(\phi(x)) = f(Tx)/f(x)$ ([14]). The space $L^2(X \times G, \widetilde{\mu})$ can be decomposed as

$$L^2(X \times G, \widetilde{\mu}) = \bigoplus_{\chi \in \widehat{G}} L_\chi,$$

where $L_\chi = \{f \otimes \chi : f \in L^2(X, \mu)\}$. Notice that $U_{T_\phi} : L_\chi \to L_\chi$ is unitarily equivalent to the unitary operator $V_{\phi,T,\chi} : L^2(X,\mu) \to L^2(X,\mu)$, where $V_{\phi,T,\chi}(f)(x) = \chi(\phi(x))f(Tx)$, $x \in X$. Let $\varrho_\chi$ denote the maximal spectral type of $V_{\phi,T,\chi}$. We will construct $\phi$'s satisfying

(1) $V_{\phi,T,\chi}$ has simple spectrum for each $\chi \in \widehat{G}$,

(2) $\varrho_\chi$ and $\varrho_\gamma$ are either orthogonal or equivalent for each $\chi, \gamma \in \widehat{G}$.

Obviously if (1) and (2) hold, then $\mathcal{M}_{T_\phi}$ consists of all cardinalities of the equivalence classes of the relation $\sim$ on $\widehat{G} \times \widehat{G}$ defined as $\chi \sim \gamma$ if $\varrho_\chi \sim \varrho_\gamma$.

Now we present a way of showing that under certain circumstances $\varrho_\chi$ and $\varrho_\gamma$ are equivalent.

PROPOSITION 1. *Let $\chi, \gamma \in \widehat{G}$. Suppose that there exists a continuous group automorphism $v : G \to G$ and $S \in C(T)$ satisfying*

(i) $\gamma = \chi \circ v$,

(ii) *there exists a measurable solution $f : X \to G$ of the functional equation*

$$\phi(Sx) - v(\phi(x)) = f(Tx) - f(x). \tag{3}$$

*Then the unitary operator $W = V_{f,S,\chi}$ satisfies $WV_{\phi,T,\chi} = V_{\phi,T,\gamma}W$. Consequently, $V_{\phi,T,\chi}$ and $V_{\phi,T,\gamma}$ are unitarily equivalent.*

Notice that if (3) holds then the transformation $S_{f,v}$ acting on $X \times G$ by the formula

$$S_{f,v}(x,g) = (Sx, f(x) + v(g)) \tag{4}$$

preserves $\widetilde{\mu}$ and commutes with $T_\phi$. Consequently, $S_{f,v} \in C(T_\phi)$. Actually, when $T_\phi$ is ergodic, each element of the centralizer of $T_\phi$ is of the form (4) (see [13]). As an immediate consequence of Proposition 1 we obtain the following.

COROLLARY 1. *The maximal spectral multiplicity of $T_\phi$ is bounded from below by $\sup_{\chi\in\widehat{G}} \operatorname{card}\{\chi v : v \in \mathcal{A}\}$, where $\mathcal{A} = \{v : G \to G : v$ is a continuous group automorphism such that there exists $S_{f,v} \in C(T_\phi)\}$.*

Notice that if $v \in \mathcal{A}$, then certainly $v^n \in \mathcal{A}$ for each integer $n$. Under our standing assumption (1), *the measures $\varrho_\chi$ and $\varrho_\gamma$ are equivalent iff $V_{\phi,T,\chi}$ and $V_{\phi,T,\gamma}$ are unitarily equivalent.* The result below is in a sense the converse to Proposition 1 in the case of cyclic groups and seems to be of independent interest.

PROPOSITION 2. *Let $\chi, \gamma \in \widehat{G}$, where $G = \mathbb{Z}_n$, $n \geq 2$, and let $T_\phi$ be ergodic. Suppose that $V_{\phi,T,\chi}$ and $V_{\phi,T,\gamma}$ are unitarily equivalent via $W : L^2(X,\mu) \to L^2(X,\mu)$, a unitary operator of the form*

$$(Wf)(x) = h(x)f(Sx)$$

*for some measurable $h : X \to \mathbb{C}$ and $S : X \to X$. Then*

(i) $S \in C(T)$, $|h(x)| = 1$,

(ii) *there exists a continuous group automorphism $v : G \to G$ such that $\gamma = \chi \circ v$, and if $\chi$ is a generator of $\widehat{G}$ then $S_{f,v} \in C(T_\phi)$ for some measurable $f : X \to G$. Moreover, $W = cV_{f,S,\chi}$ for some $|c| = 1$.*

Proposition 2 combined with Theorem 4 below shows that a nontrivial multiplicity function as in the examples of [3] and [8] arises for reasons other than those appearing in this paper (the set $\mathcal{A}$ in those examples consists of the identity group automorphism).

Now, we show how to prove the mutual singularity of the measures $\varrho_\chi$ and $\varrho_\gamma$.

Let $H$ be a separable Hilbert space and let $U : H \to H$ be a unitary operator. Assume that $\alpha$ is a complex number, $|\alpha| \leq 1$. We say that $U$ is *$\alpha$-weakly mixing* if there exists a nondecreasing sequence $\{m_t\}$ of positive integers such that for each $h \in H$, we have

$$(U^{m_t}(h), h) \to \alpha\|h\|^2 \quad \text{as } t \to \infty. \tag{5}$$



We obtain the following.

PROPOSITION 3. *Let $U_i : H_i \to H_i$ be a unitary operator on a separable Hilbert space, $i = 1, 2$. Let $\varrho_i$ denote the maximal spectral type of $U_i$. If $U_i$, $i = 1, 2$, are $\alpha_i$-weakly mixing with respect to the same sequence $\{m_t\}$ then $\delta_z * \varrho_1 \perp \varrho_2$ (for each $z \in S_1$) provided $|\alpha_1| \neq |\alpha_2|$.*

We will also use the following.

PROPOSITION 4. *If $U : H \to H$ is $\alpha$-weakly mixing, $0 < |\alpha| < 1$, then $\delta_z * \varrho^{(m)} \perp \varrho^{(n)}$ for each $z \in S^1$ and $m \neq n$, where $\varrho$ is the maximal spectral type of $U$ and $\varrho^{(m)} = \varrho * \ldots * \varrho$ ($m$ times).*

We will apply the concept of $\alpha$-weak mixing to $T_\phi$, more precisely, to the family of unitary operators $\{V_{\phi,T,\chi} : \chi \in \widehat{G}\}$. We say that a sequence $\{m_t\}$ of positive integers is a *rigid time* for $T$ if for every $f \in L^2(X,\mu)$ we have

$$\|fT^{m_t} - f\|_2 \to 0 \quad \text{as } t \to \infty.$$

For each $n \geq 1$, $\phi^{(n)}$ denotes the cocycle

$$\phi^{(n)}(x) = \phi(x) + \phi(Tx) + \ldots + \phi(T^{n-1}x), \quad x \in X.$$

Here is our criterion for the $\alpha$-weak mixing of $V_{\phi,T,\chi}$.

PROPOSITION 5. *Assume that for each $\chi \in \widehat{G}$, as $t \to \infty$ we have*

$$\int_X \chi(\phi^{(m_t)}(x))\, d\mu \to \alpha,$$

*where $\{m_t\}$ is a rigid time for $T$. Then the operator $V_{\phi,T,\chi}$ is $\alpha$-weakly mixing along $\{m_t\}$.*

The main results of this paper are consequences of the following theorem.

THEOREM 3. *Let $G$ be a compact metric abelian group. Assume that $v : G \to G$ is a continuous group automorphism satisfying*

(6) *for all $\chi \in \widehat{G}$, $\operatorname{card}\{\chi \circ v^n : n \geq 0\} < \infty$.*

*Then there exists an adding machine $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$, an ergodic cocycle $\phi : X \to G$, and $S \in C(T)$ satisfying*

(7a) *for each $\chi \in \widehat{G}$ there exists a rigid time $\{n'_t\}$ for $T$ satisfying*

$$\lim_{t\to\infty} \int_X \chi(\phi^{(n'_t)}(x))\, d\mu(x) = \alpha'_\chi,$$

(7b) *for each pair $(\chi,\gamma) \in \widehat{G} \times \widehat{G}$ there exists a rigid time $\{n''_t\}$ for $T$ satisfying*

$$\lim_{t\to\infty} \int_X \omega(\phi^{(n''_t)}(x))\, d\mu(x) = \alpha''_\omega, \quad \omega = \chi, \gamma,$$

*and moreover*

(8) *for each* $\chi \in \widehat{G}$, $\chi \neq 1$, *we have* $0 < |\alpha'_\chi| < 1$,

(9) *for each* $(\chi,\gamma) \in \widehat{G}\times\widehat{G}$ *with* $\chi v^n \neq \gamma$ *for all* $n$, *we have* $|\alpha''_\chi| \neq |\alpha''_\gamma|$,

(10) $V_{\phi,T,\chi}$ *has simple spectrum for each* $\chi \in \widehat{G}$,

(11) *there exists a measurable solution* $f : X \to G$ *of the functional equation*

$$\phi(Sx) - v(\phi(x)) = f(Tx) - f(x).$$

Notice that (7a) and (8) directly imply the ergodicity of $T_\phi$. Let $v : G \to G$ satisfy the conditions of Theorem 3 and let $\mathcal{M}_v$ be the set of the cardinalities of the sets $\{\chi \circ v^n : n \geq 0\}$, $\chi \in \widehat{G}$. Let $\phi$ satisfy the conclusion of Theorem 3. Then, applying (7b) and Proposition 5, (9) and Proposition 3, (10), (11) and Proposition 1, we obtain $\mathcal{M}_{T_\phi} = \mathcal{M}_v$.

Finally, we would like to note that in the case of finite abelian groups our constructions are actually generalized Morse sequences (in the sense of Martin [10], [11]). The result below combined with [3] and [8] shows that the centralizer method applied in our paper is *not* the only one giving rise to a nontrivial (i.e. different from the constant function 1) multiplicity function.

THEOREM 4. *Let* $T : (X,\mathcal{B},\mu) \to (X,\mathcal{B},\mu)$ *be an* $\{n_t\}$*-adic adding machine with standard sequence of towers* $D^t = (D_0^t,\ldots,D_{n_t-1}^t)$. *Assume that* $G$ *is a finite abelian group and* $\phi : X \to G$ *is a Morse cocycle and put* $\sigma_g(x,h) = (x, g+h)$, $h \in G$. *Then if* (a) *and* (b) *below hold, the centralizer of* $T_\phi$ *is trivial, i.e.*

$$C(T_\phi) = \{(T_\phi)^n \sigma_g : n \in \mathbb{Z},\ g \in G\};$$

(a) *the sequence* $\{n_{t+1}/n_t\}$ *is bounded,*

(b) $(\exists \delta > 0)(\forall t)(\exists g_1, g_2 \in G)(g_1 \neq g_2)$

$$\mu(T^{n_t-1}(D_0^t) \cap \phi^{-1}(g_i)) \geq \delta\mu(D_0^t), \quad i = 1,2.$$

## II. Proofs

Proof of Proposition 1. The equality $WV_{\phi,T,\chi} = V_{\phi,T,\gamma}W$ can be checked using easy computations. ∎

Proof of Proposition 2. We prove (i) for an arbitrary group $G$. Indeed, $WV_{\phi,T,\chi}(k) = V_{\phi,T,\gamma}W(k)$ implies

(12) $$h(x)\chi(\phi(Sx))k(STx) = \gamma(\phi(x))h(Tx)k(TSx)$$

for each $k \in L^2(X,\mu)$. In particular, on putting $k = 1$ we get $|h(x)| = |h(Tx)|$ and by the ergodicity of $T$, $|h|$ is constant, so $|h(x)| = 1$ since $W$ is unitary. Moreover, $S$ has to preserve the measure. Now $h(x) \neq 0$, so by using the same argument, $|k(STx)| = |k(TSx)|$ for each $k \in L^2(X,\mu)$, and in particular for the characteristic functions of measurable sets. Hence $S \in C(T)$. ∎

We now prove three lemmas. Lemmas 1 and 2 do not require $G = \mathbb{Z}_n$ (i.e. the cyclic group of order $n$).

LEMMA 1. *Suppose* $|G| = n$. *Then* $h^n(x) = \text{const}$ *and hence* $h(x) = c\exp(2\pi i f(x)/n)$ *for some measurable* $f : X \to \mathbb{Z}_n$, *and* $|c| = 1$.

Proof. $\chi(G)$ is a subgroup of the $n$th roots of unity, so $(\chi(g))^n = 1$ for each $\chi \in \widehat{G}$, $g \in G$. By (12) (with $k = 1$) we have

$$(h(Tx)/h(x))^n = [\chi(\phi(Sx))/\gamma(\phi(x))]^n = 1,$$

so $h^n(Tx) = h^n(x)$. Then we use the ergodicity of $T$ to conclude that $h(x) = \exp(2\pi i\beta)\exp(2\pi i f(x)/n)$ for some $\beta \in [0,1)$ and a measurable function $f : X \to \mathbb{Z}_n$. ∎

LEMMA 2. *Let* $T_\phi$ *be ergodic. If* $V_{\phi,T,\chi}$ *and* $V_{\phi,T,\gamma}$ *are unitarily equivalent via* $W$ *then* $\chi^s = 1$ *iff* $\gamma^s = 1$.

Proof. Suppose $\chi^s = 1$. Then since $\chi(\phi(Sx))/\gamma(\phi(x)) = h(Tx)/h(x)$, we have $h^s(Tx)/h^s(x) = \gamma^{-s}(\phi(x))$. So by the ergodicity of $T_\phi$, $\gamma^s = 1$. The converse uses $W^{-1}$ instead of $W$. ∎

LEMMA 3. *Suppose that* $V_{\phi,T,\chi}$ *and* $V_{\phi,T,\gamma}$ *are unitarily equivalent via* $W$ *and let* $v : \mathbb{Z}_n \to \mathbb{Z}_n$ *be a group automorphism. Then* $V_{\phi,T,\chi v}$ *and* $V_{\phi,T,\gamma v}$ *are unitarily equivalent.*

Proof. Since $v$ is an automorphism, there exists $r \in \mathbb{Z}_n$, $(r,n) = 1$, with $v(g) = rg$, $g \in Z_n$. Therefore

$$\chi v(\phi(Sx))/\gamma v(\phi(x)) = \chi^r(\phi(Sx))/\gamma^r(\phi(x)) = h^r(Tx)/h^r(x)$$

and $W_r(k)(x) = h^r(x)k(Sx)$, where $k \in L^2(X,\mu)$, establishes the desired equivalence. ∎

Now we continue the proof of Proposition 2 and proceed to (ii). If $\chi$ is a generator of $\widehat{G}$, then by Lemma 3, we may assume that $\chi(g) = \exp(2\pi i g/n)$ and $\gamma(g) = \exp(2\pi i r g/n)$ for some $r \in \mathbb{Z}_n$. In view of Lemma 2, since $\chi^s = 1$ iff $s = n$, we have $(r,n) = 1$. Define an automorphism $v : \mathbb{Z}_n \to \mathbb{Z}_n$ by $v(g) = rg$. It follows that $\gamma = \chi v$ (a similar argument shows that this is true generally).

But $\chi(\phi(Sx))/\gamma(\phi(x)) = h(Tx)/h(x)$ implies, by Lemma 1,

$$\chi(\phi(Sx)) - v(\phi(x)) = \exp(2\pi i f(Tx)/n)/\exp(2\pi i f(x)/n),$$

or in other words $\phi(Sx) - v(\phi(x)) = f(Tx) - f(x)$ in $\mathbb{Z}_n$. ∎

Proof of Proposition 3. Let $\nu$ be a probability measure absolutely continuous with respect to $\delta_z * \varrho_1$ and $\varrho_2$. Then there exist $h_i \in H_i$, $\|h_i\| = 1$,

$i = 1,2$, such that $\widehat{\nu}[n] = \int_{S^1} z^n \, d\nu(z) = (U_2^n h_2, h_2)$ and $(\delta_{z^{-1}} * \nu)^\wedge[n] = \int_{S^1} z^n \, d(\delta_{z^{-1}} * \nu)(z) = (U_1^n h_1, h_1)$ since $\delta_{z^{-1}} * \nu \ll \varrho_1$. In view of (5), for $U_1$ and $U_2$, we obtain $\widehat{\nu}[m_t] \to \alpha_1$ and $(\delta_{z^{-1}} * \nu)^\wedge[m_t] \to \alpha_2$. Now

$$|(\delta_{z^{-1}} * \nu)^\wedge[m_t]| = |\widehat{\nu}[m_t]|,$$

which implies $|\alpha_1| = |\alpha_2|$, a contradiction. ■

Proof of Proposition 4. The conclusion follows directly from the proof of Lemma 3.9 in [5]. ■

Proof of Proposition 5. Denote $V_{\phi,T,\chi}$ by $V$. If $k$ is a constant function, then

$$(V^{m_t} k, k) \to \alpha \|k\|^2 . \tag{13}$$

Let $k \in L^2(X, \mu)$ be an eigenfunction of $T$, i.e. $kT = \lambda k$ with $\lambda \neq 1$, $\|k\|_2 = 1$. We will show that

$$\int_X \chi(\phi^{(m_t)}(x)) k(x) \, d\mu(x) \to 0 \quad \text{as } t \to \infty . \tag{14}$$

If (14) is not satisfied then there exists a subsequence of $\{m_t\}$ (denote it by $\{m_t\}$ for simplicity) such that

$$\int_X \chi(\phi^{(m_t)}(x)) k(x) \, d\mu(x) \to d \quad \text{and} \quad 0 < |d| \le 1 . \tag{15}$$

We have

$$\int_X \chi(\phi^{(m_t)}(x)) k(x) \, d\mu(x) = \int_X \chi(\phi^{(m_t)}(Tx)) k(Tx) \, d\mu(x) \tag{16}$$
$$= \lambda \int_X k(x) \chi(\phi^{(m_t)}(x)) \chi(\phi(T^{m_t} x)) \overline{\chi(\phi(x))} \, d\mu(x) .$$

Since $\{m_t\}$ is a rigid time for $T$, it follows that

$$\chi(\phi(T^{m_t} \cdot)) \overline{\chi(\phi(\cdot))} \to 1 \quad \text{in measure.} \tag{17}$$

Then (16) and (17) give us $\int_X \chi(\phi^{(m_t)}(x)) k(x) \, d\mu(x) \to \lambda d$. This contradicts (15) and therefore (14) must hold.

Now let $k_0 = 1, k_1, k_2, \ldots$, be an orthonormal basis of $L^2(X, \mu)$ consisting of eigenfunctions of $T$. Consider the function

$$k = \sum_{i=0}^{l} c_i k_i . \tag{18}$$

Then

$$\|k\|_2^2 = \sum_{i=0}^{l} |c_i|^2 . \tag{19}$$



It is clear that the conditions $\|k_i T^{m_t} - k_i\|_2 \to 0$ and $k_i T^{m_t} = \lambda_i^{m_t} k_i$ imply

$$\lambda_i^{m_t} \to 1 , \quad i = 0, 1, \ldots \tag{20}$$

It follows from (14) that

$$\int_X \chi(\phi^{(m_t)}(x)) k_i(x) \overline{k_j(x)} \, d\mu(x) \to 0 \quad \text{for } i \neq j . \tag{21}$$

Furthermore, we have

$$\begin{aligned}
(V^{m_t} k, k) &= \sum_{i,j=0}^{l} c_i \overline{c}_j (V^{m_t} k_i, k_j) \\
&= \sum_{i,j=0}^{l} c_i \overline{c}_j \int_X \chi(\phi^{(m_t)}(x)) k_i(T^{m_t} x) \overline{k_j(x)} \, d\mu(x) \\
&= \sum_{i,j=0}^{l} c_i \overline{c}_j \lambda_i^{m_t} \int_X \chi(\phi^{(m_t)}(x)) k_i(x) \overline{k_j(x)} \, d\mu(x) \\
&= \sum_{i=0}^{l} |c_i|^2 \lambda_i^{m_t} \int_X \chi(\phi^{(m_t)}(x)) \, d\mu(x) \\
&\quad + \sum_{i \neq j} c_i \overline{c}_j \lambda_i^{m_t} \int_X \chi(\phi^{(m_t)}(x)) k_i(x) \overline{k_j(x)} \, d\mu(x) .
\end{aligned}$$

Using the assumption of Proposition 5 and (19)–(21) we obtain $(V^{m_t} k, k) \to \alpha \|k\|^2$ as $t \to \infty$.

In order to complete the proof it remains to show (13) for every $k \in L^2(X, \mu)$. Let $k \neq 0$ and take $\varepsilon > 0$, $\varepsilon < \|k\|_2$. There exists a function $\overline{k}$ of the form (18) such that $\|k - \overline{k}\|_2 < \min(\varepsilon \|k\|_2 / 12, \varepsilon / 4)$. Then $|(V^{m_t} \overline{k}, \overline{k}) - \alpha \|\overline{k}\|_2^2| < \varepsilon / 4$ for $t$ large enough and hence

$$\begin{aligned}
|(V^{m_t} k, k) - \alpha \|k\|_2^2| &< |(V^{m_t} k, k) - (V^{m_t} k, \overline{k})| + |(V^{m_t} k, \overline{k}) - (V^{m_t} \overline{k}, \overline{k})| \\
&\quad + |(V^{m_t} \overline{k}, \overline{k}) - \alpha \|\overline{k}\|_2^2| + |\alpha| \, | \, \|k\|_2^2 - \|\overline{k}\|_2^2| \\
&\le \|k - \overline{k}\|_2 (\|k\|_2 + \|\overline{k}\|_2) + \varepsilon/4 + (\varepsilon \|k\|_2 / 12) 3 \|k\|_2 \\
&< \varepsilon
\end{aligned}$$

for $t$ large enough. ■

Before passing to the other proofs, we will need some auxiliary considerations. Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an $\{n_t\}$-*adic adding machine*, i.e. $n_t \mid n_{t+1}$, $\lambda_{t+1} = n_{t+1}/n_t \ge 2$ for $t \ge 0$, $\lambda_0 = n_0 \ge 2$ and

$$X = \Big\{ x = \sum_{t=0}^{\infty} q_t n_{t-1} : 0 \le q_t \le \lambda_t - 1, \ n_{-1} = 1 \Big\}$$

is the group of $\{n_t\}$-*adic numbers*, where $Tx = x + \hat{1}$, $\hat{1} = (1,0,0,\ldots)$. The centralizer $C(T)$ of $T$ can be naturally identified with $X$ as follows. Let $D^t = (D_0^t, \ldots, D_{n_t-1}^t)$ be the standard sequence of $T$-towers

$$D_0^t = \{x \in X : q_0 = q_1 = \ldots = q_t = 0\}, \qquad D_0^t = D_s^t$$

($s$ is taken mod $n_t$). Then $D^{t+1}$ refines $D^t$ and obviously the sequence of partitions $\{D^t\}$ converges to the point partition. Take $S \in C(T)$. As $S$ is determined by an $x \in X$, $S(D_j^t) = D_{j+j_t}^t$ ($j + j_t$ is taken mod $n_t$) for each $j = 0, 1, \ldots, n_t - 1$, $t \geq 0$, where $j_t = \sum_{i=0}^{t} q_i n_{i-1}$.

Let $G$ be a compact metric abelian group. We will define a special class of cocycles called M-cocycles. We say that $\phi : X \to G$ is an M-*cocycle* if for every $t \geq 0$ the cocycle $\phi$ is constant on each level $D_i^t$ for $i = 0, 1, \ldots, n_t - 2$. Such a $\phi$ is defined by a sequence of blocks $\{A_t\}$ ($A_t = A_t[0]A_t[1]\ldots A_t[n_t - 2]$), where

$$\phi | D_i^t = A_t[i], \qquad i = 0, 1, \ldots, n_t - 2.$$

Define

$$a^{t+1}[i] = A_{t+1}[(i+1)n_t - 1], \qquad i = 0, 1, \ldots, \lambda_{t+1} - 2,$$
$$a^{t+1} = a^{t+1}[0] \ldots a^{t+1}[\lambda_{t+1} - 2].$$

We obtain

$$A_{t+1} = A_t a^{t+1}[0] A_t a^{t+1}[1] A_t \ldots A_t a^{t+1}[\lambda_{t+1} - 2] A_t.$$

If, in addition, we write $a^0 = A_0$, we see that an M-cocycle is completely defined by the sequence of blocks $\{a^t\}$. Also notice in passing that if $G$ is a finite abelian group then the class of group extensions obtained from M-cocycles coincides with the class of automorphisms arising from generalized Morse sequences over $G$ (see [10] and [11]). Instead of the sequence $\{a^t\}$, we will consider another sequence of blocks which will also determine $\phi$. Namely, write

(22) $$b^0 = a^0, \qquad b^t[i] = a^t[i] + u_{t-1}, \qquad t \geq 1,$$

where $u_t = A_t[0] + A_t[1] + \ldots + A_t[n_t - 2]$, $t = 0, 1, \ldots$

Let $v : G \to G$ be a continuous group automorphism. If $C$ is a block over $G$ then $v(C)$ is the block $v(C[0])v(C[1])\ldots v(C[s-1])$, where $s = |C|$ is the length of $C$.

P r o o f  o f  T h e o r e m 3. Assume that an adding machine $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is given by $\lambda_t = q_t k_t + 1$, $q_t \geq 1$, $k_t \geq 2$, $t \geq 0$. Let $Sx = x + x_0$, where $x_0 = \sum_{t=0}^{\infty} q_t n_{t-1}$.

LEMMA 4. *Suppose that* $\sum_{t=0}^{\infty} 1/k_t < \infty$ *and let blocks* $\{b^t\}$, $t \geq 0$, $|b^t| = \lambda_t - 1$, *be of the form*

(23) $$b^t = d^t v(d^t) \ldots v^{k_t - 1}(d^t),$$



*where* $|d^t| = q_t$. *If* $\phi$ *is the cocycle determined by* $\{b^t\}$ *then there is a measurable solution* $f : X \to G$ *of the equation*

(24) $$\phi(Sx) - v(\phi(x)) = f(Tx) - f(x).$$

P r o o f. Let $j_t = \sum_{j=0}^{t} q_j n_{j-1}$, $t = 0, 1, \ldots$ Then

(25) $$j_{t+1} = q_{t+1} n_t + j_t.$$

Consider the cocycle $\psi(x) = \phi(Sx) - v(\phi(x))$, $x \in X$, on the tower $D^t$ (see Fig. 1).

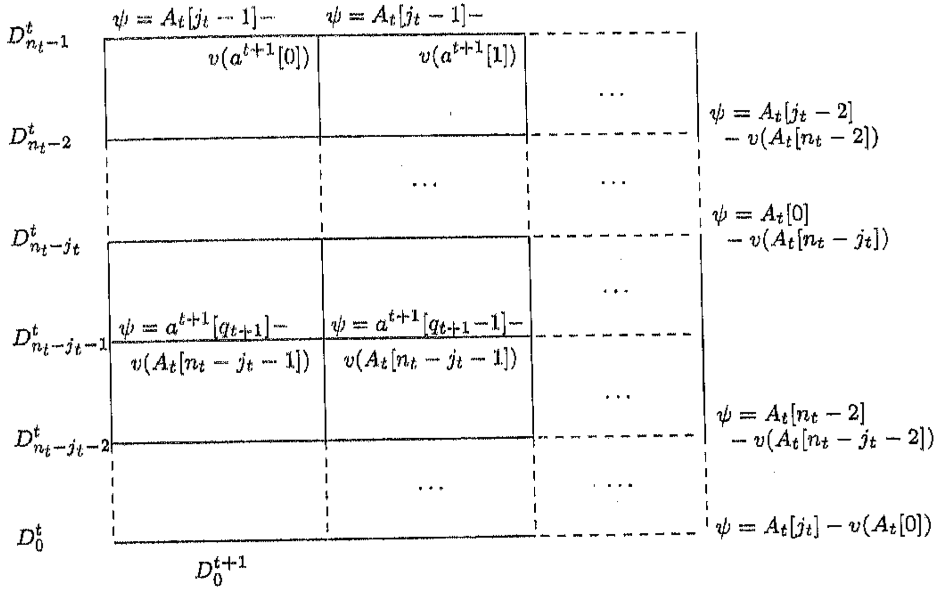


Fig. 1. The values of $\psi$ on $D^t$

Define a function $f_t$ on $C_t = \bigcup_{i=0}^{n_t - j_t - 1} D_i^t$ by $f_t(x) = 0$ for $x \in D_0^t$ and

$$f_t(x) = \psi(T^{-1}x) + \ldots + \psi(T^{-i}x)$$
$$= A_t[j_t] + \ldots + A_t[j_t + i - 1] - v(A_t[0] + \ldots + A_t[i-1])$$

for $x \in D_i^t$ and $i = 1, \ldots, n_t - j_t - 1$. Notice that $f_t$ satisfies (24) for $x \in \overline{C}_t = C_t \backslash D_{n_t - j_t - 1}^t$. Moreover,

(26) $$\mu(\overline{C}_t) = 1 - j_t/n_t - 1/n_t.$$

We will show that

(27) $$f_t(x) = f_{t+1}(x) \quad \text{if } x \in C_t \cap C_{t+1}.$$

Indeed, observe that $f_t(x) = f_{t+1}(x)$ for $x \in \bigcup_{j=0}^{n_t - j_t - 1} D_{in_t + j}^{t+1}$ provided $f_t(x) = f_{t+1}(x)$ for $x \in D_{in_t}^{t+1}$, $i = 0, 1, \ldots, \lambda_{t+1} - 1$. Therefore, in order to prove (27) it suffices to show that $f_t(x) = f_{t+1}(x)$ for $x \in D_0^{t+1}, D_{n_t}^{t+1}, \ldots, D_{(\lambda_{t+1}-1)n_t}^{t+1}$. Take $x \in D_{in_t}^{t+1}$. Then immediately from Fig. 1 we obtain

$$\begin{aligned} f_{t+1}(x) &= \psi(T^{-1}x) + \ldots + \psi(T^{-in_t}x) \\ &= (A_t[0] + \ldots + A_t[n_t-2]) - v(A_t[0] + \ldots + A_t[n_t-2]) \\ &\quad + a^{t+1}[q_{t+1}] - v(a^{t+1}[0]) + \ldots + (A_t[0] + \ldots + A_t[n_t-2]) \\ &\quad - v(A_t[0] + \ldots + A_t[n_t-2]) + a^{t+1}[q_{t+1}+i-1] - v(a^{t+1}[i-1]) \\ &= (u_t - v(u_t) + a^{t+1}[q_{t+1}] - v(a^{t+1}[0]) + \ldots \\ &\quad + (u_t - v(u_t) + a^{t+1}[q_{t+1}+i-1] - v(a^{t+1}[i-1])) . \end{aligned}$$

In view of (22),

$$\begin{aligned} f_{t+1}(x) &= u_t - v(u_t) + b^{t+1}[q_{t+1}] - u_t - v(b^{t+1}[0] - u_t) + \ldots \\ &\quad + u_t - v(u_t) + b^{t+1}[q_{t+1}+i-1] - u_t - v(b^{t+1}[i-1] - u_t) \\ &= b^{t+1}[q_{t+1}] - v(b^{t+1}[0]) + \ldots + b^{t+1}[q_{t+1}+i-1] - v(b^{t+1}[i-1]) \end{aligned}$$

and consequently by (23), $f_{t+1}(x) = f_t(x)$.

It follows from (26) that

$$\mu(\overline{C}_t \cap \overline{C}_{t+1} \cap \ldots) \geq 1 - \sum_{l=t}^{\infty} j_l/n_l - \sum_{l=t}^{\infty} 1/n_l \geq 1 - \sum_{l=t}^{\infty} 1/k_l - \sum_{l=t}^{\infty} 1/n_l$$

since (25) holds. In view of the convergence of $\sum_{l=0}^{\infty} 1/k_l$, we see that $\mu(\overline{C}_t \cap \overline{C}_{t+1} \cap \ldots) \nearrow 1$ and consequently that $f_t(x) = f_{t+1}(x) = \ldots$ if $x \in \overline{C}_t \cap \overline{C}_{t+1} \cap \ldots$ Therefore $\{f_t\}$ converges in measure to some $f : X \to G$. Since $f(x) = f_t(x)$ for $x \in \overline{C}_t \cap \overline{C}_{t+1} \cap \ldots$, $f$ satisfies (24) for a.e. $x \in X$. ∎

The following lemma is well known ([15], [8], [3]).

LEMMA 5. *If $\phi : X \to G$ is an* M-*cocycle and $\chi \in \widehat{G}$ then $V_{\phi,T,\chi}$ has simple spectrum.* ∎

Let $v : G \to G$ be a continuous group automorphism satisfying (6) of Theorem 3. Denote by $\widehat{v} : \widehat{G} \to \widehat{G}$ the dual automorphism. In view of (6), each orbit of $\widehat{v}$ is finite, i.e. the set $\{\widehat{v}^r(\gamma) : r \in \mathbb{Z}\}$ is finite for each $\gamma \in \widehat{G}$. Let $\widehat{G} = \bigcup_{i\geq 1} \Gamma_i$, where $\Gamma_i$ is an orbit of $\widehat{v}$, $i \geq 1$, $\Gamma_i \cap \Gamma_j = \emptyset$. Choose $\gamma_i \in \Gamma_i, i \geq 1$. Let $r(\gamma) = r$ be the smallest positive integer such that $\widehat{v}^r(\gamma) =$ identity. We will write $r_i$ instead of $r(\gamma_i)$, $i \geq 1$.

**III. The construction of $\phi$ and $T$ satisfying the conclusions of Theorem 3.** Take positive integers $k_t$, $t \geq 0$, satisfying $\sum_{t=0}^{\infty} 1/k_t < \infty$.

Step 1. Choose a countable dense subset $G' \subseteq G$, $G' = \{g_1, g_2, \ldots\}$, and set $G_n = \{g_1, \ldots, g_n\}$, $n = 1, 2, \ldots$



Step 2. Divide $\mathbb{N} = \{0, 1, \ldots\}$ into infinitely many pairwise disjoint *infinite* subsets $M_{ij}$, $i \neq j$, and $N_i$, $i \geq 0$, such that

$$\mathbb{N} = \bigcup_{i\neq j} M_{ij} \cup \bigcup_i N_i , \quad \text{where} \quad M_{ij} \cap N_i = \emptyset .$$

Step 3. Let $i \neq j$. There exist $n = n(i,j)$ and $g' \in G_n$ such that

$$\frac{1}{r_i} \sum_{l=0}^{r_i-1} \widehat{v}^l(\gamma_i)(g') - \frac{1}{r_j} \sum_{l=0}^{r_j-1} \widehat{v}^l(\gamma_j)(g') \neq 0 . \tag{28}$$

This is possible because the functions

$$g \overset{A_i}{\mapsto} \frac{1}{r_i} \sum_{l=0}^{r_i-1} \widehat{v}^l(\gamma_i)(g) \quad \text{and} \quad g \overset{A_j}{\mapsto} \frac{1}{r_j} \sum_{l=0}^{r_j-1} \widehat{v}^l(\gamma_j)(g) ,$$

$g \in G$, are orthogonal and nonzero in $L^2(G, m)$.

Step 4. Take the same number $n = n(i,j)$ as in Step 3 and consider the simplex $\Delta_n$ of all probability vectors $\overline{s} = (s(g))$, $g \in G_n$. Define a function $F_{ij} = F_{ij}(\overline{s})$, $\overline{s} \in \Delta_n$, by

$$F_{ij}(\overline{s}) = \sum_{g\in G_n} (A_i(g) - A_j(g)) s(g) . \tag{29}$$

Step 5. Choose $\overline{s}_0 = \overline{s}_0(i,j) \in \Delta_n$ such that

$$F_{ij}(\overline{s}_0) = d_{ij} \neq 0 \quad \text{and} \quad s_0(g) > 0 \text{ for every } g \in G_n . \tag{30}$$

Such an $\overline{s}_0$ exists because the equation (with complex coefficients)

$$F_{ij}(\overline{x}) = \sum_{g\in G_n} (A_i(g) - A_j(g)) x(g) = 0$$

determines the intersection of two vector subspaces in $\mathbb{R}^{|G_n|}$, at least one of which has dimension $|G_n| - 1$ (it follows from (28) that at least one coefficient is different from 0). These planes have Lebesgue measure zero. Define

$$\Delta_n' = \{\overline{s} = (s(g)) : F_{ij}(\overline{s}) = 0 \text{ and } s(g) > 0 \text{ for some } g \in G_n\} .$$

Then $\Delta_n \setminus \Delta_n'$ is nonempty and open in $\Delta_n$.

Step 6. Choose a ball $K(\overline{s}_0, \varepsilon)$ such that $K(\overline{s}_0, \varepsilon) \subset \Delta_n \setminus \Delta_n'$ and

$$|F_{ij}(\overline{s}) - F_{ij}(\overline{s}_0)| < (1/2)|d_{ij}| \quad \text{if } \overline{s} \in K(\overline{s}_0, \varepsilon) . \tag{31}$$

Step 7. Let $B$ be a block over $G_n$. By the average frequencies of the elements of $G_n$ the block $B$ determines an element of $\overline{s}(B) \in \Delta_n$, i.e.

$$\overline{s}(B)(g) = (1/|B|) \operatorname{card} \{0 \leq i \leq |B| - 1 : B[i] = g\} . \tag{32}$$

Choose a block over $G_n$ such that $\overline{s}(B) \in K(\overline{s}_0, \varepsilon)$. Let $q = q_{ij} = |B|$.

For every $t \in M_{ij}$ define $\lambda_t = k_t q + 1$ and let the block $b^t$, $|b^t| = \lambda_t$, be the following concatenation:

$$b^t = Bv(B)\ldots v^{k_t-1}(B). \tag{33}$$

We iterate Steps 3–7 for every pair $(i,j)$, $i \neq j$.

Step 8. Let $i \geq 0$. There exists $n' = n_i$ and two different elements $g_1, g_2 \in G_{n'}$ such that $A_i(g_1) \neq A_i(g_2)$. Choose a number $\beta$, $0 < \beta < 1$, satisfying $d = \beta A_i(g_1) + (1-\beta)A_i(g_2) \neq 0$. Then define $\overline{s}_\beta \in \Delta_n$, $\overline{s}_\beta = s_\beta(g)$, $g \in G_{n'}$, as follows:

$$s_\beta(g_1) = \beta; \quad s_\beta(g_2) = 1-\beta; \quad s_\beta(g) = 0 \text{ if } g \neq g_1, g_2. \tag{34}$$

Step 9. Define a function $F_i = F_i(\overline{s})$, $\overline{s} \in \Delta_{n'}$, by

$$F_i(\overline{s}) = \sum_{g \in G_{n'}} A_i(g)s(g). \tag{35}$$

It follows from Step 8 that $F_i(\overline{s}_\beta) = d \neq 0$ and $|F_i(\overline{s}_\beta)| < 1$, because $|F_i(\overline{s})| \leq 1$ for every $\overline{s} \in \Delta_{n'}$. There exists a ball $K(\overline{s}_\beta, \delta_1)$ in $\Delta_{n'}$ and $\delta > 0$ such that

$$\delta < |F_i(\overline{s})| < 1 - \delta \tag{36}$$

for every $\overline{s} \in K(\overline{s}_\beta, \delta_1)$. We can assume that $\overline{s}_\beta$ is an interior point of $\Delta_{n'}$.

Step 10. Choose a block $B_1$ over $G_{n'}$ such that $\overline{s}(B_1) \in K(\overline{s}_\beta, \delta_1)$. Then we put $q_i = q = |B_1|$. For every $t \in N_i$ define $\lambda_t = qk_t + 1$ and a block $b^t$ as

$$b^t = B_1 v(B_1)\ldots v^{k_t-1}(B_1). \tag{37}$$

We iterate Steps 8–10 for every $i \geq 0$.

By Steps 7 and 10, for each $t \in \mathbb{N}$, we have defined blocks $b^t$ over $G$, $|b^t| = \lambda_t - 1$, and then by (22) the blocks $a^t$, $t \geq 0$. These blocks determine an M-cocycle $\phi$.

Now we will prove (7a,b), (8) and (9) of Theorem 3. We will evaluate $\int_X \gamma(\phi^{(n_t)}(x))\,d\mu(x)$ for each $\gamma \in \widehat{G}$. We have

$$\phi^{(n_t)}(x) = A_t[0] + \ldots + A_t[n_t - 2] + a^{t+1}[0] \quad \text{if } x \in D_0^{t+1} \cup \ldots \cup D_{n_t-1}^{t+1},$$
$$\phi^{(n_t)}(x) = u_t + a^{t+1}[i]$$
$$\text{if } x \in D_{in_t}^{t+1} \cup D_{in_t+1}^{t+1} \cup \ldots \cup D_{in_t+n_t-1}^{t+1}, \; i = 1, \ldots, \lambda_{t+1} - 2.$$

Thus

$$\int_X \gamma(\phi^{(n_t)}(x))\,d\mu(x) = \frac{1}{\lambda_{t+1}-1} \sum_{g \in G} \gamma(g) \operatorname{card}\{0 \leq i \leq \lambda_{t+1} - 2 : u_t + a^{t+1}[i] = g\} + \varrho_t,$$



where $\varrho_t \leq 1/\lambda_{t+1}$ and only countably many summands are different from 0. Using (22) we can rewrite the above equality as

$$\int_X \gamma(\phi^{(n_t)}(x))\,d\mu(x) = w(b^{t+1}) + \varrho_t,$$

where $w(b^{t+1}) = \sum_{g \in G} \gamma(g)s(b^{t+1})(g)$ and $s(b^{t+1})(g)$ is the distribution of $g$ in $b^{t+1}$. Therefore

$$\Big|\int_X \gamma(\phi^{(n_t)}(x))\,d\mu - w(b^{t+1})\Big| \leq 1/\lambda_{t+1}. \tag{38}$$

Each block $b^t$ has the form (23). Now we will calculate $w(b^{t+1})$.

Put $\lambda = \lambda_t$, $k = k_t$, $q = q_t$. Then $\lambda - 1 = kq$. The following holds:

$$\begin{aligned}\operatorname{card}\{0 \leq i \leq \lambda - 2 : b^t[i] = g\} &= \sum_{l=0}^{k-1} \operatorname{card}\{lq \leq i < lq + q - 1 : b^t[i] = g\} \\ &= \sum_{l=0}^{k-1} \operatorname{card}\{0 \leq i \leq q-1 : d^t[i] = v^{-l}(g)\}.\end{aligned}$$

Hence

$$\begin{aligned}s(b^t)(g) &= (1/(\lambda-1)) \operatorname{card}\{0 \leq i \leq \lambda - 2 : b^t[i] = g\} \\ &= \frac{1}{k}\sum_{l=0}^{k-1} \frac{1}{q} \operatorname{card}\{0 \leq i \leq q-1 : d^t[i] = v^{-l}(g)\} \\ &= \frac{1}{k}\sum_{l=0}^{k-1} s(d^t)(v^{-l}(g)).\end{aligned}$$

As a consequence we obtain

$$w(b^t) = \frac{1}{k_t}\sum_{l=0}^{k_t-1}\sum_{g \in G} \gamma(g)s(d^t)(v^{-l}(g)) = \frac{1}{k_t}\sum_{l=0}^{k_t-1}\sum_{g \in G} \widehat{v}^l(\gamma)(g)s(d^t)(g).$$

Let $k_t = wr + r'$, where $r = r(\gamma)$and $0 \leq r' < r$. Then

$$\Big|\frac{1}{k_t}\sum_{l=0}^{k_t-1}\sum_{g \in G} \widehat{v}^l(\gamma)(g)s(d^t)(g) - \frac{1}{r}\sum_{l=0}^{r-1}\sum_{g \in G} \widehat{v}^l(\gamma)(g)s(d^t)(g)\Big| \leq r/k_t, \tag{39}$$

since $\widehat{v}^r(\gamma) = \gamma$. Using (38) and (39), we have

$$\Big|\int_X \gamma(\phi^{(n_t)}(x))\,d\mu(x) - \frac{1}{r}\sum_{l=0}^{r-1}\sum_{g \in G} \widehat{v}^l(\gamma)(g)s(d^t)(g)\Big| \leq 1/\lambda_{t+1} + r/k_{t+1}. \tag{40}$$

Fix $i$ and take $t \in N_i$ (see Steps 8–10). Then $b^t = B_1$, $\overline{s}(d^t) = \overline{s}(B_1)$ and

$$(41)\qquad \frac{1}{r_i}\sum_{l=0}^{r_i-1}\sum_{g\in G}\widehat{v}^l(\gamma_i)(g)s(d^t)(g) = F_i(\overline{s}(B_1)).$$

It follows from (35), (40) and (41) that

$$(42)\qquad \delta/2 < \Big|\int_X \gamma_i(\phi^{(n_t)}(x))\,d\mu(x)\Big| < 1-\delta/2 \quad \text{for } t \text{ large enough, } t \in N_i.$$

By taking subsequences if necessary, we may assume that

$$\int_X \gamma_i(\phi^{(n_t)}(x))\,d\mu(x) \to \alpha \quad\text{and}\quad \delta/2 < |\alpha| < 1-\delta/2.$$

The above properties imply (7a) and (8).

Now we will show (7b) and (9). Let $i \neq j$ and let $t \in M_{ij}$ (see Steps 3–7). Then $b^t = B$ and $\overline{s}(b^t) = \overline{s}(B)$. Using (30) and (31) we obtain $|F_{ij}(\overline{s}(B))| \geq (1/2)|d_{ij}|$. Then (40) gives us

$$\Big|\int_X \gamma_i(\phi^{(n_t)}(x))\,d\mu(x) - \int_X \gamma_j(\phi^{(n_t)}(x))\,d\mu(x)\Big| \geq (1/2)|d_{ij}| - 2/\lambda_{t+1} - r_i/k_{t+1} - r_j/k_{t+1}.$$

In this way

$$(43)\qquad \Big|\int_X \gamma_i(\phi^{(n_t)}(x))\,d\mu(x) - \int_X \gamma_j(\phi^{(n_t)}(x))\,d\mu(x)\Big| \geq \delta_2 > 0$$

for infinitely many $t$. Taking a subsequence of $\{n_t\}$ we may assume that $\int_X \gamma_k(\phi^{(n_t)}(x))\,d\mu(x) \to \alpha_k$ $(k = i, j)$. Thus (9) now follows, and so also does Theorem 3 since Lemmas 4 and 5 imply (10) and (11). ∎

We will need the following result proved in [17].

LEMMA 6. *Suppose $\mathcal{M}$ is a finite set of natural numbers such that*

(c) $1 \in \mathcal{M}$,

(d) *whenever* $m_1, m_2 \in \mathcal{M}$ *then* $\operatorname{lcm}(m_1, m_2) \in \mathcal{M}$.

*Then there exists a cyclic group $\mathbb{Z}_n$ and a group automorphism $\overline{v}$ of $\mathbb{Z}_n$ such that $\mathcal{M}_{\overline{v}} = \mathcal{M}$.* ∎

Proof of Theorem 1. Let $A = \{1, m_1, m_2, \ldots\}$ be a subset of the natural numbers satisfying (i) and (ii). Let $A_j$ be the smallest set containing $\{1, m_1, \ldots, m_j\}$ and satisfying (c) and (d). Then $A_j$ is finite and $A_1 \subseteq A_2 \subseteq \ldots$ Applying Lemma 6 we choose cyclic groups $\mathbb{Z}_{n_j}$ and automorphisms $v_j$ of $Z_{n_j}$ such that $\mathcal{M}_{v_j} = A_j$, $j \geq 1$. Let $\overline{G}$ be the product of $\mathbb{Z}_{n_j}$ and $\overline{v}$ the corresponding product automorphism. It is clear that $\mathcal{M}_{\overline{v}} = A$ and that $\overline{v}$ satisfies (6) (we recall that $\widehat{\overline{G}} = \bigoplus_j \widehat{\mathbb{Z}}_{n_j}$). Now we apply Theorem 3 to $\overline{v}$ and find $T$ and $\phi$ such that $A = \mathcal{M}_{T_\phi}$. ∎



Proof of Theorem 2. If $A$ is a finite set satisfying (i) and (ii) then applying Lemma 6 we choose a cyclic group $\mathbb{Z}_n$ and an automorphism $\overline{v}$ of $\mathbb{Z}_n$ such that $\mathcal{M}_{\overline{v}} = A$. Then we take $G = \widehat{\mathbb{Z}}_n = \mathbb{Z}_n$, $v = \widehat{\overline{v}}$ and construct a Morse cocycle $\phi$ over $G$ as in Steps 1–10. We then have

$$\mathcal{M}_{T_\phi} = \mathcal{M}_{\overline{v}} = \mathcal{M}_v = A. \ ∎$$

Proof of Theorem 4. Assume that $S_{f,v} \in C(T_\phi)$ and $S \notin \{T^m : m \in \mathbb{Z}\}$. This implies that there exists an infinity of $t$'s such that if $S(D_0^{t+1}) = D_{j_t}^{t+1}$ then $n_t \leq j_t \leq n_{t+1} - n_t$ (by (a)). Assume that

$$\phi(Sx) - v(\phi(x)) = f(Tx) - f(x).$$

Then

$$(44)\qquad \phi^{(n_t)}(Sx) - v(\phi^{(n_t)}(x)) = f(T^{n_t}x) - f(x).$$

Take $\varepsilon > 0$. Since $f : X \to G$ is measurable and $G$ is finite, $f$ is constant on most of the levels $D_i^t$ (except for an $\varepsilon$-fraction of such a good level). There exist $r_1, r_2$ such that

(45) $r_1 \geq n_{t+1} - n_t$, $S^{-1}(D_{r_1}^{t+1}) = D_{r_2}^{t+1}$ and $f$ is constant on $D_{r_2}^{t+1}$ except for an $\varepsilon$-fraction of the level.

From (44) and (45) we obtain

$$\phi^{(n_t)}(Sx) = v(\phi^{(n_t)}(x))$$

for $x \in D_{r_2}^{t+1}$ except for a set of measure $2\varepsilon\lambda\mu(D_{r_2}^{t+1})$, where $\lambda$ is an upper bound for $\{n_{t+1}/n_t\}$. But $\phi$ is a Morse cocycle, so $v(\phi^{(n_t)}(x))$ is constant for all $x \in D_{r_2}^{t+1}$ $(r_2 < n_{t+1} - n_t)$, while $\phi^{(n_t)}(Sx)$, by (b), varies. Taking $\varepsilon$ small enough we obtain a contradiction. ∎

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# Approximation of continuous convex-cone-valued functions by monotone operators

by

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**Abstract.** In this paper we study the approximation of continuous functions $F$, defined on a compact Hausdorff space $S$, whose values $F(t)$, for each $t$ in $S$, are convex subsets of a normed space $E$. Both quantitative estimates (in the Hausdorff semimetric) and Bohman Korovkin type approximation theorems for sequences of monotone operators are obtained.

**0. Introduction.** It is the purpose of this paper to discuss convergence results and quantitative estimates for the approximation by monotone operators of continuous functions $F$ defined on a compact Hausdorff space $S$, such that the value $F(t)$, for each $t \in S$, is an element of some convex cone $\mathcal{C}$ endowed with a semimetric $d_H$. In many applications $\mathcal{C}$ is a convex subcone of the convex cone $\mathcal{C}(E)$ of all convex nonempty bounded subsets of a normed space $E$ over the reals, the semimetric $d_H$ being the Hausdorff semimetric

$$d_H(K, L) = \inf\{\lambda > 0\,;\ K \subset L + \lambda B,\ L \subset K + \lambda B\},$$

where $B$ is the closed unit ball of $E$.

After giving the necessary definitions in §1 and §2, we consider in §3 the problem of quantitative estimates for the approximation by sequences $\{T_n\}_{n\geq 1}$ of monotone $\mathbb{R}_+$-linear operators on $C(S;\mathcal{C})$, and show how to extend to this context some of the local estimates of Shisha and Mond [5].

In §4 and §5 we give examples of monotone $\mathbb{R}_+$-linear operators on $C(S;\mathcal{C})$. In §4 we treat the case of operators of interpolation type and in §5 we consider two such operators, namely the Bernstein operators $B_n$, defined in $C([0,1];\mathcal{C})$ or in $C(S_m;\mathcal{C})$, where $S_m$ is the standard simplex in $\mathbb{R}^m$, and the Hermite–Fejér operators $H_n$, defined in $C([-1,1];\mathcal{C})$. Our Theorem 3 gives the estimates for the degree of approximation by $B_n$ on

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