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Oscillatory singular integrals on weighted Hardy spaces

by

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Abstract. Let

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^1} e^{iP(x-y)} \frac{f(y)}{x-y}\, dy\,,$$

where P is a real polynomial on $\mathbb{R}$. It is proved that T is bounded on the weighted $H^1(wdx)$ space with $w \in A_1$.

1. Introduction. Let ψ be a Schwartz function, $\psi \in \mathcal{S}(\mathbb{R})$, $\int_{\mathbb{R}} \psi(x)\, dx \neq 0$. Set

$$\psi_t(x) = t^{-1}\psi(x/t)\,, \quad t > 0\,,\ x \in \mathbb{R}\,.$$

For each distribution $f \in \mathcal{S}'(\mathbb{R})$, define

$$f^*(x) = \sup_{t>0} |(f*\psi_t)(x)|, \quad x \in \mathbb{R}\,.$$

The weighted Hardy space $H^1_w(\mathbb{R})$, with weight function w, is defined to be the space of all f such that

$$\|f^*\|_{L^1_w} = \int_{\mathbb{R}} f^*(x) w(x)\, dx < \infty\,.$$

If $f \in H^1_w$, we define $\|f\|_{H^1_w} = \|f^*\|_{L^1_w}$.

An operator T on the weighted Hardy space $H^1_w(\mathbb{R})$ is said to be bounded if there exists a constant C such that for each $f \in H^1_w$,

$$\|Tf\|_{H^1_w} \le C\|f\|_{H^1_w}\,.$$

Let $P(x)$ be a real polynomial on $\mathbb{R}$. Consider the oscillatory singular integral

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}} e^{iP(x-y)} \frac{f(y)}{x-y}\, dy\,. \tag{1}$$

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Operators of this type arose in the study of singular integrals on lower dimensional varieties. Their properties have been used in several kinds of problems ([6], [7], [9], [11]). By the theory of singular integrals, these operators are well defined for almost every $x \in \mathbb{R}$ provided $f \in L^p$, $1 \le p < \infty$, ([1], [8]). Their L^p $(1 < p < \infty)$ and weak $(1,1)$ boundedness have been established by F. Ricci, E. M. Stein, S. Chanillo, M. Christ and others ([1], [7], [9]). For an operator T defined by (1), a pointwise estimate for the sharp function of Tf was obtained ([4]), and as a consequence the weighted L^p $(1 < p < \infty)$ inequality for T followed. For general oscillatory integrals the weighted L^p estimates were obtained in [5]. The purpose of this paper is to show that T is bounded on the weighted Hardy space H^1_w, with $w \in A_1$.

For a positive locally integrable function w, we say that w satisfies the A_1 *condition* if there exists a constant A_w such that for all intervals $I \subset \mathbb{R}$,

$$\frac{1}{|I|}\int_I w(x)dx \le A_w(\operatorname*{ess\,inf}_{x\in I} w(x)) \tag{2}$$

where $|I|$ denotes the Lebesgue measure of I. When (2) holds, we write $w \in A_1$, and call A_w the A_1 *constant* of w.

The main results of this paper are as follows:

THEOREM 1. *Let $P(x)$ be a real polynomial of degree k, and let $Tf(x)$ be defined by* (1). *If $P'(0) = 0$ and $w \in A_1$, then there exists a constant C_k such that for each $f \in H^1_w$, we have $Tf \in L^1_w$ and*

$$\|Tf\|_{L^1_w} \le C_k\|f\|_{H^1_w}, \tag{3}$$

where C_k depends only on k and the A_1 constant A_w, but not on f and the coefficients of $P(x)$.

Since the operator we study here is a convolution operator, the characterization of H^1_w in terms of singular integral operators enables us to get the following stronger result.

THEOREM 2. *Let T be defined by* (1), *$P'(0) = 0, w \in A_1$. Then $f \in H^1_w$ implies $Tf \in H^1_w$, and there exists a constant C_k, depending only on A_w and the degree k of P, such that*

$$\|Tf\|_{H^1_w} \le C_k\|f\|_{H^1_w}, \tag{4}$$

for all $f \in H^1_w$.

Remark. The condition $P'(0) = 0$ is necessary for these theorems. To see this, assume $P(x) = ax$, $a \ne 0$. Then $Tf(x) = \pi e^{iax}\mathcal{H}(F)(x)$, where $\mathcal{H}$ is Hilbert transform and $F(y) = e^{-ayi}f(y)$. Take $w \equiv 1$, $f \in H^1$. Then $F \in L^1$. If the theorems were true in this case, then $\mathcal{H}(F)(x) \in L^1$ or H^1. This would imply

$$F(y) = e^{-ayi}f(y) \in H^1$$

for any $a \in \mathbb{R}$, $f \in H^1$. But this is obviously not right. For polynomials $P(x)$ with degree higher than one and $P'(0) \ne 0$, the theorems still cannot hold. This will be clear from the proofs.

2. Preliminaries. Weighted Hardy spaces H^p_w have been extensively studied in [12]. The proof of the following theorem can be found there.

Let $w \in A_1$. A real-valued function b is called an H^1_w *atom* if

(1) $b(x)$ is supported in an interval I;
(2) $\int b(x)\,dx = 0$;
(3) $\|b\|_\infty \le w(I)^{-1}$, where $w(I) = \int_I w(x)\,dx$.

THEOREM A. *For each $f \in H^1_w$, there exist atoms $\{b_j\}$ and coefficients $\{\lambda_j\}$ such that*

$$f(x) = \sum_{j=1}^{\infty} \lambda_j b_j(x) \tag{5}$$

and $\sum_{j=1}^{\infty} |\lambda_j| \le C\|f\|_{H^1_w}$, where C depends only on A_w. The sum in (5) *is both in the sense of distributions and in the H^1_w norm.*

THEOREM B. *If $1 < p < \infty$ and $w \in A_p$, then $\|Tf\|_{L^p_w} \le C_k\|f\|_{L^p_w}$, where C_k depends only on the A_1 constant A_w of w and the degree k of $P(x)$.*

This result was proved in [4]. Details about the A_p condition can be found in [2], [3] and [13].

The next theorem will play a key role in the proof of Theorem 2. It was proved in [12] in the general case. We restate it here in a special form which is sufficient for our use.

THEOREM C. *Let $\mathcal{H}$ be the Hilbert transform:*

$$\mathcal{H}f(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy\,.$$

If $f \in H^1_w$ and $w \in A_1$, then

(1) *$\mathcal{H}f \in H^1_w$ and $\|\mathcal{H}f\|_{H^1_w} \le C_1\|f\|_{H^1_w}$;*
(2) *$\|f\|_{H^1_w} \le C_2\|\mathcal{H}f\|_{L^1_w} + \|f\|_{L^1_w}$,*

where C_i, $i = 1, 2$, depend only on the A_1 constant A_w of w.

The following fact, useful for us, is a direct consequence of the definition of the A_1 condition.

THEOREM D. *Let I be an interval with centre at the origin, $w \in A_1$ and $\alpha \in L^1(\mathbb{R})$. If α is symmetric, nonnegative, and decreasing on $(0,\infty)$, then there exists a constant C, independent of α, such that*

$$\int_{|y|>|I|/2} \alpha(y)w(x-y)\,dy \le C\|\alpha\|_{L^1} \operatorname*{ess\,inf}_{y\in I} w(x-y)\,. \tag{6}$$

3. Some lemmas. We begin by proving the following lemma.

LEMMA 1. *Suppose $P(x)$ is a real polynomial of degree k with $P'(0)=0$, T is defined by (1), and $w \in A_1$. Then for each H^1_w atom b, there exists a constant C, depending only on the A_1 constant A_w and on the degree k of $P(x)$, such that*

$$\int |Tb(x)|w(x)\,dx \le C\,. \tag{7}$$

Proof. It is easily seen that if $w \in A_1$, then for each fixed $a \in \mathbb{R}$ and $t > 0$, the functions $w(t+a)$ and $w(tx)$ are still in the weight class A_1 with the same A_1 constant as w. Furthermore, if b is an H^1_w atom with weight $w(x)$ then $b(x+a)$ and $(1/t)b(x/t)$ are also atoms with weights $w(x+a)$ and $w(x/t)$ respectively. These facts enable us to use translation and dilation on the operator T to change the coefficient of the highest order term of $P(x)$ and the location of the support of the atom b. Therefore, we may assume that $P(x)=x^k+Q(x)$, with $Q(x)$ being a $(k-1)$th degree polynomial, and the centre of the support I of b is at the origin. We denote the length of I by $\delta = |I|$, let $w(I)=\int_I w(x)dx$, and use λI $(\lambda > 0)$ to represent the interval with the same centre as I but λ times as long.

The proof of (7) is by induction on the degree k of $P(x)$. Since $P'(0)=0$, we begin with $k=2$, that is, $P(x)=x^2$. Rewrite the integral on the left hand side of (7) as

$$\int_{\mathbb{R}} |Tb(x)|w(x)\,dx = \int_{2I} |Tb(x)|w(x)\,dx + \int_{\mathbb{R}\setminus 2I} |Tb(x)|w(x)\,dx.$$

Since $A_1 \subset A_2$, by Theorem B, the first term on the right hand side is dominated by

$$w(2I)^{1/2}\Big(\int_{2I} |Tb(x)|^2 w(x)\,dx\Big)^{1/2} \le Cw(I)^{1/2}\Big(\int_{2I} |b(x)|^2 w(x)\,dx\Big)^{1/2}$$
$$\le Cw(I)^{1/2}w(I)^{-1}w(I)^{1/2} = C\,.$$

Here we have used the fact that w satisfies the "doubling condition": $w(2I)$

$\le Cw(I)$. For the second term, if $\delta < 1$, then

$$\int_{\mathbb{R}\setminus 2I} |Tb(x)|w(x)\,dx = \int_{\mathbb{R}\setminus 2I} \Big|\int_I \frac{e^{iP(x-y)}}{x-y} b(y)\,dy\Big| w(x)\,dx$$
$$= \Big(\int_{\delta<|x|<\delta^{-1}} + \int_{\delta^{-1}\le|x|}\Big)\Big|\int_I \frac{e^{iP(x-y)}}{x-y} b(y)\,dy\Big| w(x)\,dx = I_1 + I_2\,.$$

Observe that $\int b(y)\,dy = 0$. It follows that

$$I_1 \le \int_{\delta<|x|<\delta^{-1}} \Big|\int_I \frac{e^{i(x-y)^2}-e^{ix^2}}{x-y} b(y)\,dy\Big| w(x)\,dx$$
$$+ \int_{\delta<|x|<\delta^{-1}} \Big|\int_I \Big(\frac{1}{x-y}-\frac{1}{x}\Big) b(y)\,dy\Big| w(x)\,dx$$
$$\le C\int_{\delta<|x|<\delta^{-1}} \delta \int_I |b(y)|\,dy\, w(x)\,dx + C\int_{\delta<|x|<\delta^{-1}} \frac{\delta}{|x|^2}\int_I |b(y)|\,dy\, w(x)\,dx\,.$$

Since $\int_I |b(y)|\,dy \le C|I|w(I)^{-1}$, by Theorem D it follows that

$$I_1 \le C|I|w(I)^{-1} \operatorname*{ess\,inf}_I w \le C\,,$$

where C depends only on the A_1 constant A_w of w.

Since $w \in A_1$, by a "reverse Hölder inequality" there exists $p > 1$ such that $w^p \in A_1$ ([2], [3], [13]). Thus

$$I_2 \le \int_{|x|>\delta^{-1}} \Big|\int e^{i(x-y)^2}\Big(\frac{1}{x-y}-\frac{1}{x}\Big) b(y)\,dy\Big| w(x)\,dx$$
$$+ C\int_{|x|>\delta^{-1}} \frac{1}{|x|}\Big|\int_I e^{i(x-y)^2} b(y)\,dy\Big| w(x)\,dx$$
$$\le C\int_{|x|>\delta^{-1}} \frac{\delta}{|x|^2} w(x)\,dx \int_I |b(y)|\,dy$$
$$+ C\Big(\int_{|x|>\delta^{-1}} \frac{w(x)^p}{|x|^p}\,dx\Big)^{1/p}\Big(\int_{|x|>\delta^{-1}} \Big|\int e^{i(x-y)^2} b(y)\,dy\Big|^q\,dx\Big)^{1/q}$$
$$\le C|I|w(I)^{-1} \operatorname*{ess\,inf}_I w + C\delta^{(p-1)/p} w(I)^{-1}(\operatorname*{ess\,inf}_I w)\delta^{1/p} \le C\,,$$

where we have used Theorem D and the boundedness of the Fourier transform. If $\delta \ge 1$, then

$$\int_{\mathbb{R}\setminus 2I} \Big| \int_I \frac{e^{i(x-y)^2}}{x-y} b(y)\,dy \Big| w(x)\,dx \le \int_{1\le|x|} \Big| \int_I \frac{e^{i(x-y)^2}}{x-y} b(y)\,dy \Big| w(x)\,dx\,.$$

The method used for I_2 can be applied to this integral, and it yields that the integral is dominated by a constant C depending only on Λ_w. The proof for $k=2$ is then complete.

Now supposing that Lemma 1 holds for all polynomials of degree $< k$ ($k>2$), we shall prove that it still holds for a polynomial of degree k.

Assume $P(x) = x^k + Q(x)$, where $Q(x)$ is a polynomial of degree $k-1$. Let $\widehat{\delta} = \max\{\delta^{-1/(k-1)}, \delta\}$. We have

$$\int_{\mathbb{R}} |Tb(x)|w(x)\,dx = \Big(\int_{2I} + \int_{\delta<|x|\le\widehat{\delta}} + \int_{\widehat{\delta}<|x|}\Big)|Tb(x)|w(x)\,dx$$
$$= D_1 + D_2 + D_3\,.$$

If $\delta \ge 1$, D_2 will disappear. D_1 can be treated in the same way as in the case $k=2$. When $\delta < 1$,

$$D_2 = \int_{\delta<|x|<\delta^{-1/(k-1)}} \Big| \int_I e^{iQ(x-y)} \Big(\frac{e^{i(x-y)^k} - e^{ix^k}}{x-y}\Big) b(y)\,dy \Big| w(x)\,dx$$
$$+ \int_{\delta<|x|<\delta^{-1}} \Big| \int_I e^{iQ(x-y)} \frac{b(y)}{x-y}\,dy \Big| w(x)\,dx.$$

By the induction hypothesis, the second term is bounded by a constant C. The first term is dominated by

$$C \int_{\delta<|x|<\delta^{-1/(k-1)}} \Big(\int |b(y)|\,dy\Big)|x|^{k-2}\delta w(x)\,dx$$
$$\le C\delta\delta^{-(k-2)/(k-1)} \int_{|x|<\delta^{-1/(k-1)}} w(x)\,dx\, w(I)^{-1}|I| \le C\,.$$

Therefore, $D_2 \le C$.

We now deal with D_3. Let $\phi \in C_0^\infty$, $\operatorname{supp}\phi \subset \{x : 1/4 < |x| < 1\}$, and $\sum_{j=0}^{\infty} \phi_j(x) \equiv 1$ for $|x| \ge 1/2$, where $\phi_j(x) = \phi(x/2^j)$. Set

$$T_j b(x) = \int e^{iP(x-y)} \phi_j(x-y) \frac{b(y)}{x-y}\,dy\,.$$

Then $Tb(x) = \sum_{j=0}^{\infty} T_j b(x)$. Let $\Gamma = \{|x| \ge 1 : |P''(x)| \le (|x|/2)^{k-2}\}$.

It is easy to see that there exist at most $3(k-2)$ functions ϕ_j such that $\operatorname{supp}\phi_j \cap \Gamma \ne \emptyset$. In fact, suppose that $z_1, \ldots, z_{k-2}$ are the roots of $P''(x)$, and $2^{j_i} \le |z_i| < 2^{j_{i+1}}$ ($i = 1, \ldots, k-2$). Then $|x - z_i| > |x|/2$ if $|x| \le 2^{j_i - 1}$ or $|x| > 2^{j_i+2}$. So, if $|j - j_i| \ge 3$ for $1 \le i \le k-2$, then $\operatorname{supp}\phi_j \cap \Gamma = \emptyset$. Let

$$\sigma = \{j \in \mathbb{Z}^+ : \operatorname{supp}\phi_j \cap \Gamma = \emptyset\}\,.$$

Then there are at most $3(k-2)$ elements in $\mathbb{Z}^+ \setminus \sigma$. On the other hand, for each $j \ge 0$ we have

$$\int_{\widehat{\delta}<|x|} |T_j b(x)|w(x)\,dx \le C\,.$$

In fact, if $2^j \ge 3\widehat{\delta}$, then

$$\int_{\widehat{\delta}<|x|} |T_j b(x)|w(x)\,dx \le C \int_{2^j/6<|x|<3\cdot 2^j} \frac{w(x)}{|x|}\,dx \int_I |b(y)|\,dy \le C\,.$$

If $2^j < 3\widehat{\delta}$, then

$$\int_{\widehat{\delta}<|x|} |T_j b(x)|w(x)\,dx \le C \int_{\widehat{\delta}<|x|<7\widehat{\delta}} \frac{w(x)}{|x|}\,dx \int_I |b(y)|\,dy \le C\,.$$

Therefore,

$$\sum_{j\in\mathbb{Z}^+\setminus\sigma} \int_{\widehat{\delta}<|x|} |T_j b(x)|w(x)\,dx \le C\,.$$

To estimate $\int_{|x|>\widehat{\delta}} |\sum_{j\in\sigma} T_j b(x)|w(x)\,dx$, we need the following lemma:

LEMMA 2. *Define*

$$T_p f(x) = \sum_{j\in\sigma} \int e^{iP(x-y)} \phi_j(x-y)(x-y)^{(k-2)(p-1)/p} f(y)\,dy\,.$$

Then

$$\|T_p f\|_{(L^{p'},\,dx)} \le C\|f\|_{(L^p,\,dx)}$$

where $1 \le p \le 2$, $1/p + 1/p' = 1$, *and* C *is independent of the coefficients of* P.

This lemma was proved in [4]. We shall give a proof at the end of this section for convenience.

Let us continue the proof of Lemma 1. Let $s = (k-2)(p-1)/p$.

Since

$$\int\limits_{|x|>\hat\delta}\Big|\sum_{j\in\sigma}T_jb(x)\Big|w(x)\,dx$$

$$= \int\limits_{|x|>\hat\delta}\Big|\int e^{iP(x-y)}\sum_{j\in\sigma}\phi_j(x-y)(x-y)^s \times\Big[\frac{1}{(x-y)^{s+1}}-\frac{1}{x^{s+1}}\Big]b(y)\,dy\Big|w(x)\,dx$$

$$+\int\limits_{|x|>\hat\delta}\frac{1}{|x|^{s+1}}\Big|\int e^{iP(x-y)}\sum_{j\in\sigma}\phi_j(x-y)(x-y)^s b(y)\,dy\Big|w(x)\,dx$$

$$= E_1+E_2\,.$$

Obviously, E_1 is dominated by

$$\int\limits_{|x|>\hat\delta}\frac{\delta}{|x|^2}w(x)\,dx\int|b(y)|\,dy\le C\,.$$

Since there exists $p>1$ such that $w^p\in A_1$, from Lemma 2 it follows that

$$E_2\le\Big(\int\limits_{|x|>\hat\delta}\Big(\frac{w(x)}{|x|^{s+1}}\Big)^p dx\Big)^{1/p} \times\Big(\int\Big|\int e^{iP(x-y)}\sum_{j\in\sigma}\phi_j(x-y)(x-y)^s b(y)\,dy\Big|^{p'}dx\Big)^{1/p'}$$

$$\le C(\operatorname*{ess\,inf}_I w)\hat\delta^{\,((s+1)p+1)/p}\Big(\int|b(y)|^p\,dy\Big)^{1/p}$$

$$\le C(\operatorname*{ess\,inf}_I w)\delta^{1-1/p}\delta^{1/p}w(I)^{-1}\le C\,.$$

The proof of Lemma 1 is complete.

Proof of Lemma 2. We introduce an auxiliary operator

$$A_z(f)(x)=\int e^{iP(x-y)}\sum_{j\in\sigma}\phi_j(x-y)(x-y)^z f(y)\,dy$$

with a complex parameter $z=s+it$, $0\le s\le(k-2)/2$. Observe that $\{A_z\}$ is an analytic family of linear operators in the sense of [10]. It is clear that if $s=0$, then

$$(8)\qquad \|A_{it}(f)\|_{(L^\infty,dx)}\le C\|f\|_{(L^1,dx)}\,.$$

Now consider $s=(k-2)/2$. We are going to show that

$$\|A_{(k-2)/2+it}(f)\|_{(L^2,dx)}\le C(1+|t|)\|f\|_{(L^2,dx)}$$

with C independent of the coefficients of P. After this is done, Lemma 2 follows by interpolation of the analytic family of operators $\{A_z\}$ ([10]).

To prove (8), using the Fourier transform we only have to show that for all $\eta\in\mathbb{R}$

$$(9)\qquad \Big|\int e^{i(-\eta x+P(x))}\sum_{j\in\sigma}\phi_j(x)x^{(k-2)/2+it}\,dx\Big|\le C(1+|t|)\,.$$

For each fixed $\eta\in\mathbb{R}$, let $P_1(x)$ be the derivative of the phase function: $P_1(x)=-i\eta+iP'(x)$. We rewrite it as

$$P_1(x)=ik(x-u_1)\dots(x-u_{k-1})\,,$$

where $\{u_l\}$ are the roots of P_1. Let β_l denote the real part of u_l. Then for each β_l, $|\beta_l|\ge 1/2$, there exists an integer $j_l\ge-1$ satisfying $2^{j_l}\le|\beta_l|<2^{j_l+1}$. Divide the index set σ into two subsets: $\sigma=\sigma_1\cup\sigma_2$, where

$$\sigma_1=\{j\in\sigma:|j-j_l|\le 1\text{ for some }j_l\}$$

and $\sigma_2=\sigma\setminus\sigma_1$. So, the real parts of all roots of P_1 keep some "proper distance" from $\bigcup_{j\in\sigma_2}\operatorname{supp}\phi_j$, and σ_1 is a finite set whose number of elements depends only on k. The left hand side of (9) is dominated by

$$\sum_{j\in\sigma_1}\Big|\int e^{i(-\eta x+P(x))}\phi_j(x)x^{(k-2)/2+it}\,dx\Big| +\Big|\int e^{i(-\eta x+P(x))}\Big(\sum_{j\in\sigma_2}\phi_j(x)\Big)x^{(k-2)/2+it}\,dx\Big|=E+F\,.$$

We deal with E first. By rescaling

$$E=\sum_{j\in\sigma_1}\Big|2^{j(k/2+it)}\int e^{i(-2^j\eta x+P(x))}\phi_j(x)x^{(k-2)/2+it}\,dx\Big|\,.$$

Observe that by the definition of σ, the second derivative of the phase function satisfies

$$|4^jP''(2^jx)|\ge C2^{jk}\qquad\text{for } x\in\operatorname{supp}\phi\,.$$

Applying the van der Corput lemma ([14, p. 197]), we obtain

$$E\le C\sum_{j\in\sigma_1}2^{jk/2}2^{-jk/2}(1+|t|)\le C(1+|t|)\,.$$

We now estimate F. By the definition of σ_2, if $x\in\operatorname{supp}(\sum_{j\in\sigma_2}\phi_j(x))$ then the first derivative of the phase function satisfies

$$|P_1(x)|=|(-\eta x+P(x))'|\ge C|x|^{k-1}\,.$$

Integrating by parts, we have

$$F \le \sum_{j\in\sigma_2} \left| \int e^{i(-\eta x + P(x))} \left(\frac{\phi_j(x) x^{(k-2)/2+it}}{P_1'(x)} \right)' dx \right|$$
$$\le \sum_{j\in\sigma_2} C(1+|t|)2^{-2j(k-1)}2^{j(3k/2-2)}$$
$$= C(1+|t|)\sum_{j\in\sigma_2} 2^{-jk/2} = C(1+|t|).$$

This completes the proof of (9); then Lemma 2 follows.

4. Proof of Theorem 1. Let $\varepsilon > 0$, $\Phi_\varepsilon \in C_0^\infty$, and let $\Phi_\varepsilon(x) = 1$ if $\varepsilon < |x| < 1/\varepsilon$, and $= 0$ if $|x| < \varepsilon/2$ and $|x| > 2/\varepsilon$. Set

$$T_\varepsilon f(x) = \int \Phi_\varepsilon(x-y) e^{iP(x-y)} \frac{f(y)}{x-y}\, dy, \quad \varepsilon > 0.$$

Checking the proof of Lemma 1, we will see that if the operator T is replaced by T_ε inequality (7) still holds with C independent of ε. Let f be an arbitrary H_w^1 function. By Theorem A, it has an atomic decomposition

$$f(x) = \sum_{j=0}^\infty \lambda_j b_j(x).$$

Since for each $\varepsilon > 0$, $\Phi_\varepsilon(x)e^{iP(x)}(1/x) \in C_0^\infty(\mathbb{R})$, it follows that

$$T_\varepsilon f(x) = \sum_{j=0}^\infty \lambda_j T_\varepsilon b_j(x).$$

Therefore,

$$\int |T_\varepsilon f(x)| w(x)\, dx \le \sum_{j=0}^\infty |\lambda_j| \int |T_\varepsilon b_j(x)| w(x)\, dx \le C\sum_{j=0}^\infty |\lambda_j| = C\|f\|_{H_w^1},$$

where C is independent of ε. Note that $\lim_{\varepsilon\to 0} T_\varepsilon f(x) = Tf(x)$ a.e. An application of Fatou's lemma implies that

$$\|Tf\|_{L_w^1} \le C\|f\|_{H_w^1},$$

which completes the proof of Theorem 1.

5. Proof of Theorem 2. Let $f \in H_w^1$. We prove first that for each $\varepsilon > 0$, $T_\varepsilon f \in H_w^1$. In fact, taking $\psi \in C_0^\infty$, $\int \psi(x)\, dx \ne 0$, $\psi_t = (1/t)\psi(\cdot/t)$, we have

$$(\psi_t * T_\varepsilon f)(x) = \left((\psi_t * f) * \Phi_\varepsilon(\cdot) \frac{e^{iP(\cdot)}}{|\cdot|} \right)(x).$$

Let $F_\varepsilon(x) = \Phi_\varepsilon(x)e^{iP(x)}/|x|$, and observe that $\int |F_\varepsilon(x-y)| w(y)\, dy \le C_\varepsilon w(x)$, which is a consequence of Theorem D. We get

$$\int (T_\varepsilon f)^*(x) w(x)\, dx \le \int \left(\int f^*(y) |F_\varepsilon(x-y)|\, dy \right) w(x)\, dx$$
$$\le C_\varepsilon \int f^*(x) w(x)\, dx = C_\varepsilon \|f\|_{H_w^1}.$$

This implies $T_\varepsilon f \in H_w^1$, for $\varepsilon > 0$. We can now apply Theorem C and Theorem 1 to $T_\varepsilon f$. It follows that

$$\|T_\varepsilon f\|_{H_w^1} \le C\|\mathcal{H}T_\varepsilon f\|_{L_w^1} + \|T_\varepsilon f\|_{L_w^1} = C\|T_\varepsilon(\mathcal{H}f)\|_{L_w^1} + \|T_\varepsilon f\|_{L_w^1}$$
$$\le C\|\mathcal{H}f\|_{H_w^1} + \|T_\varepsilon f\|_{L_w^1} \le C\|f\|_{H_w^1},$$

where the constant C is independent of ε. Obviously, for each $t > 0$,

$$\lim_{\varepsilon\to 0} (\psi_t * T_\varepsilon f)(x) = (\psi_t * Tf)(x).$$

Since $Tf \in L_w^1$, by the theory of maximal functions $\sup_{t>0}(\psi_t * Tf)(x)$ is finite almost everywhere. It follows that for each $\delta > 0$, and almost every $x \in \mathbb{R}$, there exists $t(x) > 0$ such that

$$\sup_{t>0} |(\psi_t * Tf)(x)| \le |(\psi_{t(x)} * Tf)(x)| + \delta = \lim_{\varepsilon\to 0} |(\psi_{t(x)} * T_\varepsilon f)(x)| + \delta.$$

So, for each $\alpha > 0$, applying Fatou's lemma and $\|T_\varepsilon f\|_{H_w^1} \le C\|f\|_{H_w^1}$, we get

$$\int_{|x|<\alpha} (Tf)^*(x) w(x)\, dx$$
$$\le \int_{|x|<\alpha} \lim_{\varepsilon\to 0} |(\psi_{t(x)} * T_\varepsilon f)(x)| w(x)\, dx + \delta \int_{|x|<\alpha} w(x)\, dx$$
$$\le \liminf_{\varepsilon\to 0} \int |(\psi_{t(x)} * T_\varepsilon f)(x)| w(x)\, dx + \delta \int_{|x|<\alpha} w(x)\, dx$$
$$\le \liminf_{\varepsilon\to 0} \int (T_\varepsilon f)^*(x) w(x)\, dx + \delta \int_{|x|<\alpha} w(x)\, dx$$
$$\le C\|f\|_{H_w^1} + \delta \int_{|x|<\alpha} w(x)\, dx.$$

Letting $\delta \to 0$, and then $\alpha \to \infty$, we thus obtain

$$\|Tf\|_{H_w^1} \le C\|f\|_{H_w^1}.$$

This completes the proof.

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On the multiplicity function of ergodic group extensions of rotations

by

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Abstract. For an arbitrary set $A \subseteq \mathbb{N}$ satisfying $1 \in A$ and $\operatorname{lcm}(m_1, m_2) \in A$ whenever $m_1, m_2 \in A$, an ergodic abelian group extension of a rotation for which the range of the multiplicity function equals A is constructed.

Introduction. In this paper we study the set $\mathcal{M}_T$ of all essential spectral multiplicities of an ergodic measure preserving the transformation T of a Lebesgue space $(X, \mathcal{B}, \mu)$. $\mathcal{M}_T$ is defined as the essential range of the multiplicity function with respect to the maximal spectral type of the associated unitary operator

$$U_T : L^2(X, \mu) \to L^2(X, \mu), \quad (U_T f)(x) = f(Tx), \quad x \in X.$$

Thus $\mathcal{M}_T$ is a subset of the set $\overline{\mathbb{N}}$ of all positive integers and infinity. Many examples in ergodic theory have $\mathcal{M}_T = \{1\}$ (e.g. irrational rotations), $\mathcal{M}_T = \{\infty\}$ (e.g. Kolmogorov automorphisms), $\mathcal{M}_T = \{1, \infty\}$ (e.g. affine transformations). Transformations with $\mathcal{M}_T = \{1, k\}$ have been constructed ([16]), for each positive integer k, and also with $\mathcal{M}_T = \{1, 2k\}$, where $2k$ corresponds to the multiplicity of the Lebesgue component ([1], [9], [12]).

The problem of whether for an arbitrary nonempty set $A \subseteq \mathbb{N}$ there exists an ergodic transformation T with $\mathcal{M}_T = A$ seems to be open. Toward the full solution of this question, Robinson in [18] has proved that for each finite set A of positive integers satisfying:

(i) $1 \in A$,
(ii) $\operatorname{lcm}(m_1, m_2) \in A$ whenever $m_1, m_2 \in A$,

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