Therefore we get a basis $\{Q_{I,S}(z)z^I : |S| = l,\ |I| \ge -l,\ i_1, \ldots, i_d \ge -l\}$ in $A_l^{\alpha,2}(B)$, where

$$Q_{I,S}(z) = (1-|z|^2)^{-l} \times \sum_{j_1=\max(0,-i_1)}^{s_1} \cdots \sum_{j_d=\max(0,-i_d)}^{s_d} \frac{(-S)_J(\alpha+1+d+|I|-l)_{|J|}}{(i_1+1)_{j_1}\ldots(i_d+1)_{j_d}J!}|z_1^{j_1}|^2\ldots|z_d^{j_d}|^2.$$

We have been unable to find such an explicit formula for orthogonal polynomials of several variables with respect to the measure (3.6). Still, the above basis may be good enough to study the Hankel operator from the Bergman space $A^{\alpha,2}(B)$ to the space $A_l^{\alpha,2}(B)$. We will not study this here.

References

[1] P. Appell et J. Kampé de Fériet, *Fonctions hypergéometriques et hypersphériques, Polynomes d'Hermite*, Gauthier-Villars, Paris 1926.

[2] J. Arazy, S. Fisher and J. Peetre, *Membership in the Schatten–von Neumann classes and Hankel operators on Bergman space*, J. London Math. Soc., to appear.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vols. 1, 2, McGraw-Hill, New York 1953.

[4] I. M. Gel'fand and M. I. Graev, *The analogue of Plancherel's theorem for real unimodular groups*, Dokl. Akad. Nauk SSSR 92 (1953), 461–464 (in Russian).

[5] Harish-Chandra, *Plancherel formula for semi-simple Lie groups*, Trans. Amer. Math. Soc. 76 (1954) 485–528.

[6] D. Hejhal, *The Selberg Trace Formula for* PSL(2, ℝ), Vol. 1, Lecture Notes in Math. 548; Vol. 2, Lecture Notes in Math. 1001, Springer, Berlin 1976, 1983.

[7] S. Helgason, *Groups and Geometric Analysis*, Academic Press, New York 1984.

[8] —, *Topics in Harmonic Analysis on Homogeneous Spaces*, Progr. in Math. 13, Birkhäuser, Boston 1981.

[9] J. Peetre, L. Peng and G. Zhang, *A weighted Plancherel formula I. The case of the disk. Applications to Hankel operators*, technical report, Stockholm.

[10] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer, New York 1980.

[11] N. Ya. Vilenkin, *Special Functions and the Theory of Group Representations*, Nauka, Moscow 1965 (in Russian).

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Rank and spectral multiplicity

by

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Abstract. For a dynamical system (X, T, μ), we investigate the connections between a metric invariant, the rank $r(T)$, and a spectral invariant, the maximal multiplicity $m(T)$. We build examples of systems for which the pair $(m(T), r(T))$ takes values (m, m) for any integer $m \ge 1$ or $(p-1,\ p)$ for any prime number $p \ge 3$.

Introduction. Given a measure-preserving dynamical system (X, T, μ) there is a corresponding Hilbert space automorphism, namely the action of $U_T F = F \circ T$ on the space $L^2(X, \mu)$. The link between these so-called *metric* and *spectral* structures is still only partially known. The spectral structure, of course, is completely defined by the *maximal spectral type* and the *multiplicity function* of the operator U_T. One particular invariant that we shall study here is the *maximal spectral multiplicity* $m(T)$ (see I.5).

Now a metric invariant closely related to $m(T)$ is the *rank* $r(T)$, introduced by Chacon [Cha1], though named only in [ORW]. The first known systems with $m(T) = 1$ (simple spectrum) were of rank one (this including the well-known discrete spectrum systems).

In general $m(T) \le r(T)$ [Cha2]. The nontrivial result of [Rob1], that there exist systems with any given value of $m(T)$, uses systems of finite rank. Also, the rare examples of finite multiplicity where the maximal spectral type is Lebesgue (plus a discrete or singular continuous part) fall into this category [Age], [Lem], [MaNa], [Que].

The question of which values the pair $(m(T), r(T))$ may take was asked by M. Mentzen [Men1]. He conjectured that each pair (j, n), $j \le n$, may be obtained. The pair $(1, 1)$ was obtained by Chacon [Cha1], $(1, 2)$ by del Junco [delJ], $(1, n)$ by Mentzen [Men1], $(1, \infty)$ by Ferenczi [Fer1], $(2, n)$ by

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Goodson and Lemańczyk [GoLe], (n,n) by Robinson [Rob2], though in fact it is implied by [Rob1], and recently $(n,2n)$ by Mentzen [Men2]. Here we first give new examples for the pair (n,n), which are interesting as they give an explicit construction of n Rokhlin towers, and that was unknown previously. Then we proceed to our main result: we construct examples for the pairs $(p-1,p)$ for any prime number $p\geq 3$. The transformations used in our constructions are Morse automorphisms over cyclic groups. This class is convenient for investigating spectral multiplicity and rank. On the one hand, each automorphism of this type has a shift representation [Kea1], [Mar1], which helps to estimate the rank. On the other hand, the operator U_T has simple spectrum on each subspace of $L^2(X)$ determined by the characters. Hence our examples contain a discrete part in their spectrum. It would be interesting to know if they can be modified to give a weakly mixing case.

I. Preliminaries. Let $\mathbb{Z}_n=\{0,1,\ldots,n-1\}$, $n\geq 2$, be a cyclic group and let Ω be the space of all bi-infinite sequences over $\mathbb{Z}_n$.

I.1. *Blocks and operations on blocks.* A finite sequence $B=(B[0]\ldots\ldots B[k-1])$, $B[i]\in\mathbb{Z}_n$, $k\geq 1$, is called a *block over* $\mathbb{Z}_n$. All blocks and sequences considered in this paper are over a cyclic group and we will say shortly block or sequence if no confusion can arise. The number k is called the *length* of B and denoted by $|B|$. If $\omega\in\Omega$ and B is a block then $\omega[i,s]$, $B[i,s]$, $0\leq i\leq s\leq k-1$, denote the block $(\omega[i]\ldots\omega[s])$ and $(B[i]\ldots B[s])$ respectively. If $C=(C[0]\ldots C[m-1])$ is another block then the *concatenation* of B and C is the block

$$BC=(B[0]\ldots B[k-1]C[0]\ldots C[m-1]).$$

In the same manner we can define the concatenation of a higher number of blocks. If $v:\mathbb{Z}_n\to\mathbb{Z}_n$ is a group automorphism, then $v(B)$ is the block

$$v(B)=(v(B[0])\ldots v(B[k-1])).$$

By B_j, $j\in\mathbb{Z}_n$, we will denote the block

$$B_j=(B[0]+j)\ldots(B[k-1]+j).$$

Now, we can define the *product* $B\times C$ of B and C as follows:

$$B\times C=B_{C[0]}\ldots B_{C[m-1]}.$$

I.2. *Occurrences, frequencies, tiling.* The block B is said to *occur* at place i in ω (resp. in C as above $(k\leq m)$) if $\omega[i,i+|B|-1]=B$ (resp. $C[i,i+|B|-1]=B$). By the *frequency* of B in C (resp. in ω) we mean the numbers

$$\mathrm{fr}(B,C)=|C|^{-1}\#\{0\leq i\leq |C|-|B|\ ;\ B\text{ occurs at place } i \text{ in } C\},$$
$$\mathrm{fr}(B,\omega)=\lim_{s\to\infty}\mathrm{fr}(B,\omega[0,s-1]),$$

if this limit exists.

For an infinite subsequence of ω, $E=\{\omega[n]\ ;\ n\in J\subset\mathbb{Z}\}$, we call the *density* of E the density of the set J in $\mathbb{Z}$, whenever it exists.

Let $\delta>0$. We say that B *δ-occurs* at place i in C (resp. in ω) if

$$d(B,C[i,i+|B|-1])<\delta\quad(\text{resp. } d(B,\omega[i,i+|B|-1])<\delta),$$

where

$$d[(x_1,\ldots,x_n),(y_1,\ldots,y_n)]=n^{-1}\#\{i\ ;\ x_i\neq y_i\}.$$

d is called the normalized *Hamming distance* or d-bar distance between sequences. Denote by $\phi(B,C)$ the number $|C|^{-1}\cdot\{$the maximum number of disjoint occurrences of B in $C\}$, and by $\phi_\delta(B,C)$ the number $|C|^{-1}\cdot\{$the maximum number of disjoint δ-occurrences of B in $C\}$.

Further, we define

$$\phi(B,\omega)=\lim_s\phi(B,\omega[0,s-1]),\qquad \phi_\delta(B,\omega)=\lim_s\phi_\delta(B,\omega[0,s-1]),$$

if these limits exist. The numbers $\phi(B,\omega)$ and $\phi_\delta(B,\omega)$ are called the *tiling frequency* and the *δ-tiling frequency* of B in ω (see [Fer2]). Finally, set

$$t(B,\omega)=|B|\phi(B,\omega),\qquad t_\delta(B,\omega)=|B|\phi_\delta(B,\omega).$$

I.3. *The dynamical system associated with the sequence* ω. By σ we denote the left shift homeomorphism of Ω. If $\omega\in\Omega$ then $\omega[n]$ will denote the value of ω at $n\in\mathbb{Z}$ and $\Theta(\omega)$ the σ-orbit of ω. Let Ω_ω be the σ-orbit closure of ω in Ω. The topological flow (Ω_ω,σ) is called *minimal* if there is no proper closed and σ-invariant subset of Ω_ω. We say that (Ω_ω,σ) is *uniquely ergodic* if there is a unique borelian normalized σ-invariant measure μ_ω on Ω_ω. (Ω_ω,σ) is said to be *strictly ergodic* if it is minimal and uniquely ergodic. Suppose that ω is strictly ergodic. The unique σ-invariant measure μ_ω is determined by the condition

$$\mu_\omega(B)=\mathrm{fr}(B,\omega)$$

for each block B. It follows from the Ergodic Theorem that $\phi(B,\omega)$ and $\phi_\delta(B,\omega)$ are well defined.

I.4. *Rank and related notions.* We say that $(\Omega_\omega,\sigma,\mu_\omega)$ is of *rank at most* r if for any $\delta>0$ and for every n, there exist r blocks $B_1,\ldots,B_r$, $|B_i|\geq n$, such that for all N large enough, for a set of s of density $>1-\delta$, the fragment $\omega[s,s+N-1]$ of ω is of the form

$$\omega[s,s+N-1]=\varepsilon_1W'_{i_1}\varepsilon_1W'_{i_2}\ldots\varepsilon_kW'_{i_k}\varepsilon_{k+1},$$

where $|\varepsilon_1| + \ldots + |\varepsilon_k| + |\varepsilon_{k+1}| < \delta N$ and the distance d between W'_{i_j} and some B_m, $j = 1, \ldots, k$, $1 \le m \le r$, is less than δ. The system $(\Omega_\omega, \sigma, \mu_\omega)$ is of rank r if it is of rank at most r and not of rank at most $r-1$.

Now, we define numbers $F_* = F_{*\omega}$ and $F^* = F^*_\omega$ as follows:

(1) $F_* = \sup\{a$; for every n there exists a block B with $|B| \ge n$ and $t(B,\omega) \ge a\}$,

(2) $F^* = \sup\{a$; for every $\delta > 0$ and every n there exists a block B with $|B| \ge n$ and $t_\delta(B,\omega) \ge a\}$.

Note that $F^* = 1$ corresponds exactly to rank one. Of course $F_* \le F^*$. In the sequel we will need the following easy observation.

Remark 1. *If $F^* < 1/(m-1)$, $m \ge 2$, then the rank of $(\Omega_\omega, \sigma, \mu_\omega)$ is at least m.*

I.5. *Spectral multiplicity.* We recall that $(\Omega, \sigma, \mu_\omega)$ has *spectral multiplicity smaller than* m if, U being the operator $L^2(\Omega) \to L^2(\Omega)$, $UF = F \circ \sigma$, the space $L^2(\Omega)$ is the direct orthogonal sum of at most m spaces $H_1, \ldots, H_m$, where each H_i is the closed linear subspace generated by $(U^n F_i,\ n \in \mathbb{Z})$ for some F_i in H_i.

We say that (Ω, σ) has *maximal spectral multiplicity* m if it has spectral multiplicity smaller than m and not smaller than $m-1$.

I.6. *Adding machines and cocycles.* Let $T : (X, B, \mu) \to (X, B, \mu)$ be an $\{n_t\}$-adic *adding machine*, i.e. $n_t \,|\, n_{t+1}$, $\lambda_{t+1} = n_{t+1}/n_t \ge 2$ for $t \ge 0$, $\lambda_0 = n_0 \ge 2$ and

(3) $$X = \Big\{x = \sum_{t=0}^{\infty} q_t n_{t-1}\ ;\ 0 \le q_t \le \lambda_t - 1,\ n_{-1} = 1\Big\}$$

is the group of $\{n_t\}$-adic numbers and $Tx = x + \widehat{1}$, $\widehat{1} = (1, 0, 0, \ldots)$. The space X has the standard sequence ξ_t of T-*towers*. Namely,

$$\xi_t = (D_0^t, \ldots, D_{n_t-1}^t),$$

where

$$D_0^t = \{x \in X\ ;\ q_0 = \ldots = q_t = 0\}, \quad T^s(D_0^t) = D_s^t, \quad s = 1, \ldots, n_t - 1.$$

Then ξ_{t+1} refines ξ_t and the sequence of partitions $\{\xi_t\}$ converges to the point partition. By $C(T)$ we denote the *metric centralizer* of T, i.e.

$$C(T) = \{S : X \to X\ ;\ S \text{ is measure preserving and } TS = ST\}.$$

The centralizer $C(T)$ can be naturally identified with X in such a way that if $x_0 \in X$ then $S = S_{x_0}$ is defined by $S(x) = x + x_0$. By a *cocycle* we mean a measurable function $\varphi : X \to \mathbb{Z}_n$. A cocycle φ defines an automorphism T_φ on $(X \times \mathbb{Z}_n, \widetilde{\mu})$ by

$$T_\varphi(x, j) = (Tx, j + \varphi(x)), \quad x \in X,\ j \in \mathbb{Z}_n,$$

where $\widetilde{\mu} = \mu \times \overline{m}$ and $\overline{m}$ is the Haar measure of $\mathbb{Z}_n$. T_φ is ergodic iff for every nonzero $j \in \mathbb{Z}_n$ there is no measurable solution $f : X \to S^1$ of the functional equation

$$\exp[2\pi i \varphi(x) j/n] = \overline{f}(Tx)/\overline{f}(x), \quad x \in X \quad \text{[Par]}.$$

The metric centralizer $C(T_\varphi)$ consists of triples (S, f, v) satisfying

(4) $$f(Tx) - f(x) = \varphi(Sx) - v(\varphi(x)),$$

where $S \in C(T)$, $f : X \to \mathbb{Z}_n$ is a measurable function and $v : \mathbb{Z}_n \to \mathbb{Z}_n$ is a group automorphism (see [New]).

I.7. *M-cocycles.* We will define a special class of cocycles called M-cocycles (Morse cocycles). We say that $\varphi : X \to \mathbb{Z}_n$ is an *M-cocycle* if for every $t \ge 0$, φ is constant on each level D_i^t for $i = 0, 1, \ldots, n_t - 2$. Such a cocycle is defined by a sequence of blocks $\{A_t\}$,

$$A_t = A_t[0] \ldots A_t[n_t - 2], \quad \text{where } \varphi|D_i^t = A_t[i], \quad i = 0, \ldots, n_t - 2.$$

If we define

$$a^{t+1}[i] = A_{t+1}[(i+1)n_t - 1], \quad i = 0, \ldots, \lambda_{t+1} - 2,$$

then A_{t+1} is a concatenation of the blocks A_t and the symbols $a^{t+1}[i]$ as follows:

(5) $$A_{t+1} = A_t a^{t+1}[0] A_t a^{t+1}[1] \ldots A_t a^{t+1}[\lambda - 2] A_t, \quad \lambda = \lambda_{t+1}.$$

Now, assume that $b^0, b^1, \ldots$ are finite blocks with $|b^t| = \lambda_t$, $t \ge 0$, starting with 0. Then we may define a one-sided sequence by

$$\omega = b^0 \times b^1 \times \ldots$$

Such a sequence is called a *generalized Morse sequence* over $\mathbb{Z}_n$ if it is not periodic and if each of the sequences

(6) $$\omega_t = b^t \times b^{t+1} \times \ldots, \quad t \ge 0,$$

contains every symbol in $\mathbb{Z}_n$. By grouping some of the b_i's we can assume that each block b^t contains every symbol in $\mathbb{Z}_n$. It is known [Mar2] that (Ω_ω, σ) is strictly ergodic if $\mu_{\omega_t}(j) = 1/n$ for every $j \in \mathbb{Z}_n$ and $t \ge 0$. It is not hard to observe that the condition

$$\mathrm{fr}(j, b^t) \ge \varrho > 0$$

for every $j \in \mathbb{Z}_n$ and $t = 0, 1, \ldots$ implies ω is strictly ergodic.

A Morse sequence ω allows one to define an M-cocycle $\varphi = \varphi_\omega$ on X. Let

(7) $$B_t = b^0 \times \ldots \times b^t, \quad t \ge 0.$$

Of course $|B_t| = n_t = \lambda_0 \ldots \lambda_t$. Define a block $\widehat{B}_t$, $|\widehat{B}_t| = n_t - 1$, by

$$\widehat{B}_t[i] = B_t[i+1] - B_t[i], \quad i = 0, \ldots, n_t - 2.$$

Now, we put $A_t = \widehat{B}_t$. The sequence of blocks $\{A_t\}$ satisfies (5). Thus ω defines an M-cocycle $\varphi = \varphi_\omega$. It follows from [Kwi] and [Lem] that the dynamical systems $(\Omega_\omega, \mu_\omega, \sigma)$ and $(X \times \mathbb{Z}_n, \widetilde{\mu}, T_\varphi)$ are metrically isomorphic.

I.8. *Continuous Morse sequencs.* Now, let ω be a strictly ergodic Morse sequence over $\mathbb{Z}_n$, and let $\varphi = \varphi_\omega$ be the M-cocycle defined by ω. For $k \in \mathbb{Z}_n$ define

$$H_k = \{(x,j) \mapsto \bar{f}(x) \exp[2\pi i k j/n] \ ; \ \bar{f} \in L^2(X,\mu)\} \subset L^2(X \times \mathbb{Z}_n, \widetilde{\mu}).$$

The subspaces H_k are T_φ-invariant and we have a decomposition

$$L^2(X \times \mathbb{Z}_n, \widetilde{\mu}) = \bigoplus_{k \in \mathbb{Z}_n} H_k.$$

It is shown in [KwSi] that T_φ has simple spectrum on each H_k. Let μ_k be the spectral measure of T_φ on H_k. It follows from [Kea2] that any two of those μ_k are either orthogonal or equivalent. The subspace H_0 is generated by the eigenfunctions of T_φ corresponding to all n_t-roots of unity. A Morse sequence ω is called *continuous* if H_0 contains all eigenfunctions of T_φ, or equivalently if each measure μ_k, $k \neq 0$, is continuous.

I.9. *First observations.* Suppose that $v : \mathbb{Z}_n \to \mathbb{Z}_n$ is a group automorphism.

Remark 2. *If there exists $S \in C(T)$ such that the functional equation (4) has a solution then*

(8) $$\mu_k \text{ is equivalent to } \mu_{v(k)}$$

for every $k \in \mathbb{Z}_n$.

Proof. The triple (S, f, v) defines $\widetilde{S} \in C(T_\varphi)$ by

$$\widetilde{S}(x,j) = (Sx, f(x) + v(j)), \quad x \in X, \ j \in \mathbb{Z}_n.$$

We check directly that $\widetilde{S}(H_k) = H_{v(k)}$, which implies (8).

Let m_ω be the maximal spectral multiplicity of $(\Omega_\omega, \sigma, \mu_\omega)$ and let r_ω be the rank of this system.

Remark 3. *Under the same assumptions as in Remark 2, we have $m_\omega \geq$ the length of the biggest v-trajectory of $\mathbb{Z}_n$.*

Remark 4. $r_\omega \leq n$.

Proof. For every $t \geq 0$, the blocks $B_t + j$, $j \in \mathbb{Z}_n$ (see (7)) completely cover the sequence ω.

II. Examples of Morse sequences with spectral mutiplicity m and rank m. Now, suppose that $\lambda_t = k_t l_t$, $k_t, l_t \geq 2$. As a consequence of Theorem 2 of [KwRo] we obtain the following.

THEOREM 1. *Let $\omega = b^0 \times b^1 \times \ldots$ be a Morse sequence such that for every $t \geq 0$ there exists a block d^t with $|d^t| = l_t$ such that*

$$b^t = d^t v(d^t) \ldots v^{k_t - 1}(d^t)$$

and let $S \in C(T)$ be defined by $S(x) = x + x_0$, where $x_0 = \sum_{t=0}^{\infty} l_t n_{t-1}$. If $\sum_{t=0}^{\infty} 1/k_t < \infty$ then the functional equation (4) has a solution.

Theorem 1 is valid for M-cocycles with values in a compact metric abelian group [GKLL].

Let $m \geq 1$ be a positive integer. Now, we are in a position to construct an example of a Morse sequence ω such that $r_\omega = m_\omega = m$.

Choose a prime number n such that $m \mid (n-1)$. Such an n exists by the Dirichlet theorem. There exists an automorphism $v : \mathbb{Z}_n \to \mathbb{Z}_n$ such that the v-trajectory of each $j \in \mathbb{Z}_n$, $j \neq 0$, has length m (see [Rob2]).

Let E be a block over $\mathbb{Z}_n$, $|E| = q \geq 2$, $E[0] = 0$. Take a sequence of positive integers $\{m_t\}$ such that $m_t > n$ and $m_t \to \infty$. Next, set $l_t = n m_t$.

Let j be a generator of $\mathbb{Z}_n$. Define a block d^t by

(9) $$d^t = E E_j \ldots E_{(l_t - 1)j}.$$

Choose $k_t \geq 1$ such that

(10) $$\sum_{t=0}^{\infty} \frac{1}{k_t} < \infty$$

and define

(11) $$b^t = d^t v(d^t) \ldots v^{k_t - 1}(d^t).$$

THEOREM 2. *If ω is the Morse sequence $b^0 \times b^1 \times \ldots$ and the blocks $b^0, b^1, \ldots$ are defined by* (9)–(11) *then $m_\omega = r_\omega = m$.*

Proof. It is easy to check that ω is strictly ergodic. The blocks b^t, $t \geq 0$, satisfy the assumptions of Theorem 1. Then the choice of v and Remark 3 imply that

(12) $$m_\omega \geq m.$$

We will show that $r_\omega \leq m$. Put $D_0 = E E_j \ldots E_{(n-1)j}$ and

$$D_i = v^i(D_0) = v^i(E)(v^i(E) + v^i(j)) \ldots (v^i(E) + (n-1)v^i(j)), \quad i = 1, \ldots, m-1.$$

Fix $t \geq 0$. The sequence ω_t (see (6)) is the concatenation of the blocks b_h^t, $h \in \mathbb{Z}_n$. We have

(13) $$b_h^t = (d^t + h)(v(d^t) + h) \ldots (v^{k_t - 1}(d^t) + h).$$

We will cover b^t_h by the blocks $D_0, \ldots, D_{m-1}$. Since $v^m = \mathrm{id}$, b^t_h is a concatenation of the blocks

$$d^t + h,\ v(d^t) + h,\ \ldots\ , v^{m-1}(d^t) + h.$$

Take s, $0 \le s \le m-1$. From (9) we have

$$(14)\quad v^s(d^t) + h = (v^s(E) + h)(v^s(E) + v^s(j) + h) \ldots (v^s(E) + (l_t - 1)v^s(j) + h).$$

There exist s_0, s_1, $0 \le s_0, s_1 \le n-1$, such that

$$h + s_0 v^s(j) = 0, \qquad h - s_1 v^s(j) = 0,$$

because $v^s(j)$ is a generator of $\mathbb{Z}_n$. Let

$$C_1 = (v^s(E) + h) \ldots (v^s(E) + (s_0 - 1)v^s(j) + h),$$
$$C_2 = (v^s(E) + (l_t - s_1 + 1)v^s(j) + h) \ldots (v^s(E) + (l_t - 1)v^s(j) + k).$$

We have

$$(15)\qquad |C_1| \le qn, \qquad |C_2| \le qn.$$

Then (14) implies $v^s(d^t) + h = C_1 D_s \ldots D_s C_2$.

It follows from the above, (13) and (15) that b^t_h is covered by the blocks $D_0, \ldots, D_{m-1}$ except for at most $2|b^t|/m_t$ places.

Set $D^t_i = B_{t-1} \times D_i$, $i = 0, \ldots, m-1$. The equality $\omega = B_{t-1} \times \omega_t$ means that the blocks $D^t_0, \ldots, D^t_{m-1}$ cover a subsequence of ω of density at least $1 - 2/m_t$.

The condition $m_t \to \infty$ implies $r_\omega \le m$. Together with (12) this implies $r_\omega = m_\omega = m$. The theorem is proved.

Unfortunately, our example does not have continuous spectrum, since there are eigenvalues coming from the odometer (3). However, we can choose the block E in such a way that these are the only ones.

PROPOSITION 1. *There is a block E such that the sequence ω defined by* (9)–(11) *is a continuous Morse sequence.*

Proof. It follows from the above considerations that $\mu_k \perp \mu_{k'}$ whenever k and k' are in different v-trajectories. In particular, $\mu_k \perp \mu_0$ if $k \ne 0$. To guarantee that ω is a continuous Morse sequence it suffices to show that for each $k \in \mathbb{Z}_n$, $k \ne 0$, the sequence

$$(16)\qquad \int_X \chi_k(\varphi^{n_t}(x))\, \mu(dx), \qquad \varphi = \varphi_\omega,$$

has a limit point α_k such that $|\alpha_k| < 1$ (see [GKLL]), where

$$\varphi^{n_t}(x) = \varphi(x) + \varphi(Tx) + \ldots + \varphi(T^{n_t - 1}x),$$

and χ_k is the character of $\mathbb{Z}_n$ defined by

$$\chi_k(s) = \exp[2\pi i k s/n], \qquad s \in \mathbb{Z}_n.$$

Using the same arguments as in [GKLL] we obtain

$$(17)\qquad \Big| \int_X \chi_k(\varphi^{n_t}(x))\, \mu(dx) - \sum_{s \in \mathbb{Z}_n} \frac{1}{m} \sum_{l=0}^{m-1} \chi_{v^l(k)}(s)\, \omega_t(s) \Big| < \frac{1}{\lambda_{t+1}} + \frac{m}{k_t} + \frac{1}{q l_t},$$

where $\omega_t(s) = \mathrm{fr}(s, \widehat{d}_t)$. It follows from (9) that $|\omega_t(s) - \mathrm{fr}(x, \widehat{E})| \le 1/q$, which gives

$$(18)\qquad \Big| \int_X \chi_k(\varphi^{n_t}(x))\, \mu(dx) - \sum_{s \in \mathbb{Z}_n} \frac{1}{m} \sum_{l=0}^{m-1} \chi_{v^l(k)}(s)\, \mathrm{fr}(s, \widehat{E}) \Big| < \frac{1}{\lambda_{t+1}} + \frac{m}{k_t} + \frac{1}{q l_t} + \frac{n}{q}.$$

Repeating the arguments from [GKLL] (part III) we may find a probability vector $\overline{\omega} = (\omega(s), s \in \mathbb{Z}_n)$ such that

$$(19)\qquad \Big| \sum_{s \in \mathbb{Z}_n} \frac{1}{m} \sum_{l=0}^{m-1} \chi_{v^l(k)}(s)\, \omega(s) \Big| \le \delta < 1$$

whenever $k \ne 0$. Now, choose q large enough and a block E, $E[0] = 0$, $|E| = q$, in such a way that

$$(20)\qquad \frac{n}{q} < \frac{1-\delta}{4},$$

$$(21)\qquad \Big| \sum_{s \in \mathbb{Z}_n} \frac{1}{m} \sum_{l=0}^{m-1} \chi_{v^l(k)}(\omega(s) - \mathrm{fr}(s, \widehat{E})) \Big| < \frac{1-\delta}{4}.$$

Then (17)–(21) imply

$$\Big| \int_X \chi_k(\varphi^{(n_t)}(x))\, \mu(dx) \Big| \le \frac{1}{4}\delta + \frac{3}{4} < 1,$$

for t large enough and $k \ne 0$. Thus the sequences (16) for $k \ne 0$ have limit points α_k such that $|\alpha_k| < 1$. Notice that this condition implies the ergodicity of T_φ.

III. Examples of Morse sequences with spectral multiplicity $p-1$ and rank p. Let $p \ge 3$ be a prime number and let $v : \mathbb{Z}_p \to \mathbb{Z}_p$ be an automorphism such that the v-trajectory of 1 coincides with $\mathbb{Z}_p^* = \{1, \ldots, p-1\}$.

Let

$$u_i = v^i(1), \quad i = 0, \ldots, p-2,$$

and let $F = 0u_0 0u_1 \ldots 0u_{p-2}$. Now, define

$$b^t = \underbrace{F \ldots F}_{2^t \text{ times}}, \quad t = 0, 1, \ldots \tag{22}$$

We have

$$|F| = 2(p-1) = \lambda, \quad |b^t| = 2^{t+1}(p-1) = \lambda_t.$$

Set

$$\omega = b^0 \times b^1 \times \ldots \tag{23}$$

In this section, we prove

THEOREM 3. *$m_\omega = p - 1$ and $r_\omega = p$.*

PROPOSITION 2. *The Morse sequence* (23) *is continuous and $m_\omega = p-1$.*

The blocks (22) satisfy the assumptions of Theorem 1 with $d^t = 0u_0$, $k_t = 2^t(p-1)$. Theorem 1 and Remark 3 imply $m_\omega \geq p-1$. Of course $r_\omega \leq p$ (see Remark 4). The spectral measures $\mu_1, \ldots, \mu_{p-1}$ are equivalent but μ_0 is purely atomic. In order to show $m_\omega = p-1$ it suffices to prove that $\mu_1 \perp \mu_0$.

In the same way as before, we have

$$\Big| \int_X \chi_1(\varphi^{n_t}(x))\,\mu(dx) - \sum_{s \in \mathbb{Z}_p} \chi_1(s)\,\mathrm{fr}(s, \widehat{b^t}) \Big| \leq \frac{1}{\lambda_{t+1}}. \tag{24}$$

It is easy to check that

$$\Big| \sum \chi_1(s)\,\mathrm{fr}(s, \widehat{b^t}) + \frac{1}{p-1} \Big| \leq \frac{1}{\lambda_{t+1}}. \tag{25}$$

The inequalities (24) and (25) imply that

$$\lim_t \int_X \chi_1(\varphi^{n_t}(x))\,\mu(dx) = -\frac{1}{p-1}.$$

Thus μ_1 is a continuous measure so that $m_\omega = p-1$. At the same time we find that T_φ is ergodic. The proposition is proved.

In the sequel we will estimate the number F^* (see I.4). We will show that $F^* < 1/(p-1)$. This will be proved in Proposition 3. The main tool is Lemma 2, which allows us to know the tiling of a "long" block $B_t \times C$ if we know the tiling of the shorter block C. We then proceed to estimate the tiling of blocks of length 2 and 3 (Lemmas 3 to 5); Lemmas 6 and 7 give then some preliminary estimates on blocks of any length, and this leads to the proof of the proposition.

The main problem we meet in these computations is to check that a given block $(B_t + j)$ does not occur too often outside its "natural" position. In fact, some block close to $(B_t + j)$ (in the sense of the Hamming distance d) may appear in a position slightly shifted from the natural ones. But the important fact is that, except for the occurrences just mentioned, no (B_t+j) or no d-neighbour of (B_t+j) will appear under another (B_t+i) or (and this part requires the longest computations) under a junction $(B_t+i)(B_t+i)$.

We start with the following observation:

$$d(F, F_jF_k[l, \lambda - l - 1]) \geq \frac{p - 5/2}{2(p-1)} = \varrho \tag{26}$$

for each $j, k \in \mathbb{Z}_p$ and $l = 1, \ldots, \lambda - 1$.

The inequality (26) allows us to estimate the d-distance between the blocks b^t and $b^t_j b^t_k[l, l + \lambda_t - 1]$, $l = 0, \ldots, \lambda_t - 1$. We have

$$d(b^t, b^t_j) = 1 \quad \text{if } j \neq 0, \tag{27}$$

$$d(b^t, b^t_j b^t_k[l, l + \lambda_t - 1]) \geq \varrho \quad \text{if } l \not\equiv 0 \pmod{\lambda}, \tag{28}$$

$$d(b^t, b^t_j b^t_k[l, l + \lambda_t - 1]) = \begin{cases} 0 & \text{if } j = 0 = k, \\ l/\lambda_t & \text{if } j = 0,\ k \neq 0, \\ 1 - l/\lambda_t & \text{if } j \neq 0,\ k = 0, \\ 1 & \text{if } j \neq 0,\ k \neq 0 \end{cases} \tag{29}$$

for $l \equiv 0 \pmod{\lambda}$.

LEMMA 1. *If $\widetilde{C}$ is a block occurring in ω_t and $|\widetilde{C}| \geq 3$ then*

$$\sum_{j \in \mathbb{Z}_p} \mathrm{fr}(jj, \widetilde{C}) \leq 1/3. \tag{30}$$

Proof. The sequence ω_t is the concatenation of the blocks b^t_k, $k \in \mathbb{Z}_p$. If $\widetilde{C}$ occurs in ω_t then $\widetilde{C}$ is of the form

$$\widetilde{C} = b^t_{k_0}[l_1, \lambda_t - 1]\,b^t_{k_1} \ldots b^t_{k_{u-1}} b^t_{k_u}[0, l_2],$$

where $u \geq 1$, $0 \leq l_1, l_2 \leq \lambda_t - 1$ and $k_0, \ldots, k_u \in \mathbb{Z}_p$. The couple (jj) does not occur inside any block b^t_k, $k \in \mathbb{Z}_p$. It can occur in $\widetilde{C}$ only at a junction $b^t_{k_i} b^t_{k_{i+1}}$, $i = 0, \ldots, u-1$. The inequality (30) follows easily from the above observations.

LEMMA 2 (Main Lemma). *If $\delta < \min(\varrho, 1/8)$ then $t_\delta(C, \omega_{t+1}) = t_\delta(b^t \times C, \omega_t)$.*

Proof. The inequality $t_\delta(C, \omega_{t+1}) \leq t_\delta(b^t \times C, \omega_t)$ follows from the definition of $t_\delta(C, \omega)$. Assume that $|C| = q \geq 2$ and let $C = C[0] \ldots C[q-1]$. Suppose that $b^t \times C$ δ-occurs in ω_t, i.e. there exists a block $E = E[0] \ldots E[q]$ occurring in ω_{t+1} such that

$$d(b^t \times C, (b^t \times E)[l, l + \lambda_t q - 1]) < \delta \quad \text{(see Fig. 1)}, \tag{31}$$

where $0 \le l \le \lambda_t - 1$.

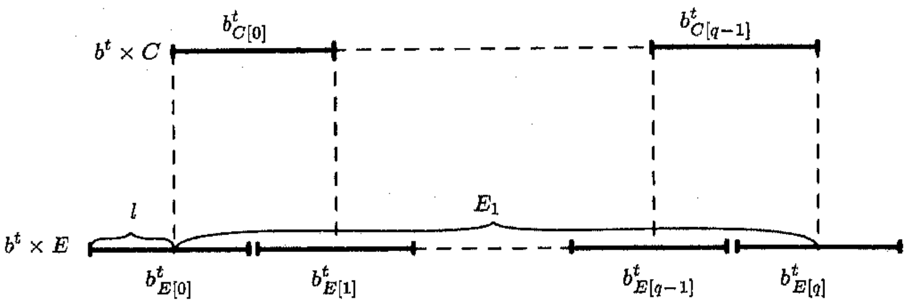

Fig. 1

We have

$$(32)\qquad d(b^t \times C, E_1) = \frac{1}{q}\sum_{u=0}^{q-1} d(b^t_{C[u]}, b^t_{E[u]} b^t_{E[u+1]}[l, l+\lambda_t - 1]).$$

The inequality (31) implies that at least one component in (32) is less than δ. Then (28) and $\delta < \varrho$ imply $l \equiv 0 \pmod{\lambda}$. Using (27) and (29) we obtain

$$(33)\qquad d(b^t \times C, E_1) = \left(1 - \frac{l}{\lambda_t}\right)\frac{a_0}{q} + \frac{l}{\lambda_t}\cdot\frac{a_1}{q} = \frac{a_0}{q} + \frac{l}{\lambda_t}\cdot\frac{a_1 - a_0}{q},$$

where

$$a_0 = \#\{0 \le u \le q-1\ ;\ C[u] \ne E[u]\},$$
$$a_1 = \#\{0 \le u \le q-1\ ;\ C[u] \ne E[u+1]\}.$$

Suppose that

$$(34)\qquad l \le \lambda_t/2.$$

Then (31), (33) and (34) imply $a_0 < 2\delta q$. Further, we have

$$(35)\qquad \begin{aligned} a_1 &\ge \#\{0 \le u \le q-1\ ;\ C[u] = E[u] \text{ and } C[u] \ne E[u+1]\} \\ &\ge \#\{0 \le u \le q-1\ ;\ E[u] \ne E[u+1]\} \\ &\quad - \#\{0 \le u \le q-1\ ;\ E[u] \ne C[u]\} \ge (q+1)\sum_{j \ne k} \mathrm{fr}(jk, E) - a_0. \end{aligned}$$

Applying Lemma 1 with $\widetilde{C} = E$ we obtain

$$a_1 \ge (q+1)\left(\frac{2}{3} - \frac{1}{q+1}\right) - a_0 = \frac{2}{3}q - \frac{1}{3} - a_0.$$

The above and (35) imply

$$(36)\qquad a_1 - a_0 \ge \left(\frac{2}{3} - 4\delta\right)q - \frac{1}{3} > 0$$

for $\delta < 1/8$. Using (31), (33) and (36) we get $a_0/q < \delta$. If $l/\lambda_t > 1/2$ then the same arguments imply $a_1/q < \delta$.

The above considerations mean that if

$$d(b^t \times C, \omega_t[i\lambda_t + l, (i+q-1)\lambda_t + l - 1]) < \delta, \qquad i \in \mathbb{Z},\ 0 \le l \le \lambda_t - 1,$$

then

$$d(b^t \times C, \omega_t[i\lambda_t, (i+q-1)\lambda_t - 1]) < \delta,$$

or

$$(37)\qquad d(b^t \times C, \omega_t[(i+1)\lambda_t, (i+q)\lambda_t - 1]) < \delta.$$

If $|C| = 1$ then it is easy to see that (31) implies $C[0] = E[0]$ or $C[0] = E[1]$ so that (37) is also satisfied. The condition (37) means that

$$t_\delta(b^t \times C, \omega_t) = t_\delta(C, \omega_{t+1}),$$

because

$$d(b^t \times C, \omega_t[i\lambda_t, (i+q-1)\lambda_t - 1]) = d(C, \omega_{t+1}[i, i+q-1]).$$

In this way Lemma 2 is proved.

Corollary 1. *If* $\delta < \min(\varrho, 1/8)$ *then* $t_\delta(C, \omega_{t+1}) = t_\delta(B_t \times C, \omega)$.

Proof. Applying Lemma 2 with C replaced by $b^t \times C$ and t replaced by $t-1$ we have

$$t_\delta(C, \omega_{t+1}) = t_\delta(b^t \times C, \omega_t) = t_\delta(b^{t-1} \times b^t \times C, \omega_{t-1}).$$

By induction we obtain

$$t_\delta(C, \omega_{t+1}) = t_\delta(B_t \times C, \omega), \qquad \omega = \omega_0.$$

Put

$$h_t(jk) = \frac{1}{p}\sum_{s=0}^{p-1} \mathrm{fr}(jk, b^t_s), \qquad j, k \in \mathbb{Z}_p.$$

Lemma 3. *We have*

$$(38)\qquad \mathrm{fr}(jk, \omega_t) = h_t(jk) + \frac{1}{\lambda_t} h_{t+1}(j - u_{p-2}, k) + \frac{1}{\lambda_t \lambda_{t+1}} h_{t+2}(j - 2u_{p-2}, k) + \ldots$$

Proof. Take $n \geq 1$. Then we have

$$(39)\quad \frac{1}{n\lambda_t}\#\{0 \leq u \leq n\lambda_t - 2\,;\ \omega_t[u,u+1] = jk\}$$
$$= \frac{1}{n\lambda_t}\sum_{s=0}^{p-1}\#\{0 \leq u \leq \lambda_t - 2\,;\ b_s^t[u,u+1] = jk\}$$
$$\times \#\{0 \leq u \leq n-1\,;\ \omega_{t+1}[u] = s\}$$
$$+ \frac{1}{n\lambda_t}\#\{0 \leq u \leq n-2\,;\ \omega_{t+1}[u,u+1] = (j - u_{p-2}, k)\}\,.$$

Since ω_{t+1} is a strictly ergodic Morse sequence,

$$(40)\quad \frac{1}{n}\#\{0 \leq u \leq n-1\,;\ \omega_{t+1}[u] = s\} \to \frac{1}{p} \quad \text{as } n \to \infty\,,$$

for every $s \in \mathbb{Z}_p$. If $n \to \infty$ then (39) and (40) imply

$$\mathrm{fr}(jk, \omega_t) = h_t(jk) + \frac{1}{\lambda_t}\,\mathrm{fr}((j - u_{p-2}, k), \omega_{t+1})\,.$$

This implies (38) by an induction argument.

Lemma 4. *For every $j, k \in \mathbb{Z}_p$, $\mathrm{fr}(jk, \omega_t) \leq 2/(p(\lambda-1))$.*

Proof. Let $j \neq k$. It is not hard to notice the following properties: the couple (jk) occurs 2^t times in b_j^t and 2^t times in b_k^t if $j \neq k + u_{p-2}$; it occurs $2^t - 1$ times in b_k^t if $j = k + u_{p-2}$ and it does not occur in b_s^t whenever $s \neq j$ and $s \neq k$. The above properties imply

$$\mathrm{fr}(jk, b_j^t) = 1/\lambda\,, \quad \mathrm{fr}(jk, b_k^t) \leq 1/\lambda\,,$$
$$\mathrm{fr}(jk, b_s^t) = 0 \quad \text{if } s \neq j \text{ and } s \neq k\,.$$

In this way

$$(41)\quad h_t(jk) \leq \frac{1}{p}\cdot\frac{2}{\lambda} = \frac{1}{p(p-1)}\,.$$

It is easy to check that

$$(42)\quad h_t(jj) = 0 \quad \text{for every } j \in \mathbb{Z}_p \text{ and } t \geq 0\,.$$

Combining (38), (41) and (42) we have

$$\mathrm{fr}(jk, \omega_t) \leq \frac{1}{p(p-1)}\left(1 + \frac{1}{\lambda_t} + \frac{1}{\lambda_t\lambda_{t+1}} + \ldots\right)$$
$$\leq \frac{1}{p(p-1)}\left(1 + \frac{1}{\lambda_t} + \frac{1}{\lambda_t^2} + \ldots\right)$$
$$= \frac{1}{p(p-1)}\cdot\frac{\lambda_t}{\lambda_t - 1} = \frac{1}{p(p-1)}\cdot\frac{\lambda}{\lambda - 1/2^t} \leq \frac{2}{p(\lambda-1)}\,.$$

If $j = k$ then the same arguments give

$$\mathrm{fr}(jj, \omega_t) \leq \frac{2}{p(\lambda-1)\lambda_t}\,,$$

which finishes the proof of the lemma.

An immediate consequence of Lemma 4 is the following.

Corollary 2. *If $\delta < 1/2$ then $t(jk, \omega_t) = t_\delta(jk, \omega_t) \leq 4/(p(\lambda-1))$.*

Lemma 5. *If C occurs in ω_t, $|C| = 3$ and $\delta < \min(1/(2p), \varrho)$ then*

$$(43)\quad t(C, \omega_t) = t_\delta(C, \omega_t) \leq \frac{3}{p(\lambda-1)}\,.$$

Proof. Since $\delta < 1/3$ we have $t(C, \omega_t) = t_\delta(C, \omega_t)$. Define

$$u_l^{(j)} = u_l + j = v^l(1) + j, \quad l = 0, \ldots, p-2,\ j \in \mathbb{Z}_p\,.$$

C has one of the following forms:

(A) $C = ju_l^{(j)}j$ for some $j \in \mathbb{Z}_p$ and $0 \leq l \leq p-2$,

(B) $C = ju_{l+1}^{(j)}$, $j \in \mathbb{Z}_p$, $0 \leq l \leq p-2$ (if $l = p-2$ then $l+1$ means 0),

(C) $C = ju_{p-2}^{(j)}$, $j \neq k$,

(D) $C = u_{p-2}^{(j)}ku_0^{(j)}$, $j \neq k$.

Case (A). C occurs in the block F_jj exactly once because the elements $u_0^{(j)}, u_{p-2}^{(j)}$ are pairwise different. At the same time C does not occur in any block F_ss if $j \neq s \in \mathbb{Z}_p$. The block C can occur in F_kF_s at positions $\lambda-2$, $\lambda-1$, λ or at $\lambda-1$, λ, $\lambda+1$, where $k, s \in \mathbb{Z}_p$ and either $k \neq j$ or $s \neq j$. If $k \neq j$ we have

$$C = ju_l^{(j)}j = ku_{p-2}^{(k)}s\,.$$

If $s \neq j$ then

$$ju_l^{(j)}j = u_{p-2}^{(k)}su_0^{(s)}\,,$$

which gives

$$(44)\quad j = k + u_{p-2}\,, \quad j + v^l(1) = s\,, \quad j = s+1\,.$$

There exists a unique l_0, $0 \leq l_0 \leq p-2$, such that $v^{l_0}(1) = p-1$. If $l \neq l_0$ then (44) is impossible. If $l = l_0$ then C occurs exactly once in F_kF_s at positions $\lambda-1, \lambda, \lambda+1$ if

$$(45)\quad k = j - u_{p-2}\,, \quad s = j - 1\,.$$

In view of the above considerations we have the following properties:

(46) C occurs at most 2^t times in b_j^t; C does not occur in b_s^t if $s \neq j$; C occurs exactly once in $b_k^t b_s^t$, where k, s satisfy (45), whenever $l = l_0$.

Thus we obtain

$$\operatorname{fr}(C, \omega_t) \leq 2^t \operatorname{fr}(j, \omega_{t+1}) \frac{1}{2^t \lambda} + \operatorname{fr}(ks, \omega_{t+1}) \frac{1}{2^t \lambda}.$$

By Lemma 4 and (40) we get

$$\operatorname{fr}(C, \omega_t) \leq \frac{1}{p\lambda} + \frac{1}{p(\lambda-1)\lambda_t} \leq \frac{1}{p(\lambda-1)}. \tag{47}$$

Now, (47) implies (43).

Case (B). The same arguments as in Case (A) lead to the properties (46) and C occurs exactly once in $b_k^t b_s^t$, where k, s satisfy (45), whenever $l = l_0 - 1$. Then we obtain (47). Note that if $l = l_0 - 1$ then C occurs exactly once in $F_k F_s$ at positions $\lambda - 2, \lambda - 1, \lambda$, i.e.

$$C = u_l^{(j)} j u_{l+1}^{(j)} = k u_{p-2}^{(k)} s. \tag{48}$$

Case (C). If there exists $s \in \mathbb{Z}_p$ such that C occurs in $F_s s u_0^{(s)}$ then C has the form as in (A) or (B). In this case (43) is satisfied. Assume that C does not occur in any $F_s s u_0^{(s)}$, $s \in \mathbb{Z}_p$. Then C can occur in $F_e F_s$ at positions $\lambda - 2$, $\lambda - 1$, λ or $\lambda - 1$, λ, $\lambda + 1$. The first possibility holds only if $e = j$ and $s = k$. The second case is possible only if

$$k - 1 = j + u_{p-2}, \quad e = j + u_{p+2} = s = b - 1. \tag{49}$$

The above considerations lead the following statements: C occurs exactly once in b_j^t, b_k^t at positions $\lambda_t - 2$, $\lambda_t - 1$, λ_t; C occurs exactly once in $b_e^t b_s^t$ if j, k and e, s satisfy (49). Hence, by Lemma 4,

$$\operatorname{fr}(C, \omega_t) \leq \operatorname{fr}(jk, \omega_{t+1}) \frac{1}{2\lambda_t} + \operatorname{fr}(es, \omega_{t+1}) \frac{1}{2\lambda_t} \leq \frac{2}{p(\lambda-1)} \cdot \frac{1}{\lambda_t} \tag{50}$$

and so

$$t(C, \omega_t) \leq \frac{6}{p(\lambda-1)\lambda_t} \leq \frac{3}{p(\lambda-1)}. \tag{51}$$

Case (D). Using the same arguments as in (C) we obtain (50) and (51). The proof of the lemma is finished.

LEMMA 6. *If* $|C| \geq 4$, C *occurs in* $F_j F_k[1, 2\lambda - 2]$ *and* $\delta < \min(\varrho, 1/(2\lambda))$, *then*

$$t_\delta(C, \omega_t) = t(C, \omega_t) \leq \max\left(\frac{1}{p}, \frac{4}{p(\lambda-1)}\right). \tag{52}$$

Proof. First consider the case $j = k$. It is clear that C does not occur in $F_s F_s$ if $s \neq j$. Let $|C| = 4$. Then C has the form

$$C = C_1 = j u_l^{(j)} j u_{l+1}^{(j)} \quad \text{or} \quad C = C_2 = u_l^{(j)} j u_{l+1}^{(j)} j.$$

In the same way as in the proof of Lemma 5 we verify that C_1 cannot occur in $F_e F_s$ whenever $e \neq j$ or $s \neq j$.

However, if C has the form C_2 then C occurs in $F_e F_s$ exactly once if

$$e = j - u_{p-2}, \quad s = j - 1, \quad v^{l+1}(1) = -1$$

(see (48)). Using the same arguments as in Cases (A) and (B) of Lemma 5 we obtain

$$\operatorname{fr}(C, \omega_t) \leq \frac{1}{p(\lambda-1)},$$

which implies

$$t(C, \omega_t) \leq \frac{4}{p(\lambda-1)}.$$

Now, we show the following property:

(53) *if* C *occurs in* $F_j F_j[1, 2\lambda - 2]$, $|C| \geq 5$, *then* C *cannot occur in* $F_e F_s$ *whenever* $e \neq j$ *or* $s \neq j$.

If $|C| = 5$ then C has one of the following forms:

$$C = C_1 = j u_l^{(j)} j u_{l+1}^{(j)} j \quad \text{or} \quad C = C_2 = u_l^{(j)} j u_{l+1}^{(j)} j u_{l+2}^{(j)}.$$

We check directly that if $C = C_1$ and C_1 occurs in $F_e F_s$ then either $e = j = s$ or $j = u_{p-3}^{(e)} = u_{p-2}^{(e)}$, which contradicts $v^{p-3}(1) \neq v^{p-2}(1)$.

In a similar way we verify that if $C = C_2$ then (53) is satisfied. It is obvious that (53) is satisfied even more easily if $|C| > 5$.

Now assume that $5 \leq |C| < \lambda$. It follows from (53) that C occurs at most 2^t times in b_j^t and C does not occur in b_s^t if $s \neq j$ and in $b_e^t b_s^t$ whenever $e \neq j$ or $s \neq j$. Thus we have

$$\operatorname{fr}(C, \omega_t) \leq \frac{2^t}{\lambda 2^t} \cdot \frac{1}{p} = \frac{1}{p\lambda}$$

and so

$$t(C, \omega_t) \leq |C| \frac{1}{p\lambda} < \lambda \frac{1}{p\lambda} = \frac{1}{p}.$$

If $|C| \geq \lambda$ then the maximal number of disjoint occurrences of C in b_j^t is at most 2^{t-1}. At the same time C does not occur in b_s^t if $s \neq j$ and in $b_e^t b_s^t$ whenever $e \neq j$ or $s \neq j$. This gives the estimate

$$\phi(C, \omega_t) \leq \frac{2^{t-1}}{\lambda 2^t} \cdot \frac{1}{p} = \frac{1}{2p\lambda}$$

and therefore

$$t(C,\omega_t) \le |C|\frac{1}{2p\lambda} = \frac{1}{p}\,.$$

We have proved (52) if $j=k$.

Now, suppose that C occurs in $F_jF_k[1,2\lambda-2]$ and $j \ne k$. If there exists $s \in \mathbb{Z}_p$ such that C occurs in $F_sF_s[1,2\lambda-2]$ then repeating the above reasoning we obtain (52).

If C does not occur in any $F_sF_s[1,2\lambda-2]$, $s \in \mathbb{Z}_p$, then in particular C occurs neither in F_j nor in F_k. Thus C contains the pair

$$u^{(j)}_{p-2}k = F_jF_k[\lambda-1,\lambda]\,.$$

Choose a subblock C_1 of C such that $|C_1| = 3$ and C_1 contains $u^{(j)}_{p-2}k$. Then using the same arguments as in Cases (C) and (D) of Lemma 5 we get

$$\mathrm{fr}(C,\omega_t) \le \frac{2}{p(\lambda-1)\lambda_t}\,.$$

Therefore

$$t(C,\omega_t) \le |C|\,\mathrm{fr}(C,\omega_t) < 2\lambda\frac{2}{p(\lambda-1)\lambda_t} \le \frac{4}{p(\lambda-1)}\,.$$

Lemma 6 is proved.

LEMMA 7. *If C occurs in $b^t_jb^t_k$, $j,k \in \mathbb{Z}_p$, and $\delta < \min(\varrho/2, 1/(8(p-1)))$, then*

$$t_\delta(C,\omega_t) \le \max\left(\frac{1}{p}\cdot\frac{\lambda+1}{\lambda}, \frac{4}{p(\lambda-1)}\right). \tag{54}$$

Proof. We can distinguish the following cases:

(I) C occurs in one of the blocks

$$F_jF_k[1,2\lambda-2]\,,\quad F_jF_j[1,2\lambda-2]\,,\quad F_kF_k[1,2\lambda-2]\,,$$

(II) $$C = \underbrace{F_j\ldots F_j}_{q\text{ times}} \quad\text{or}\quad C = \underbrace{F_k\ldots F_k}_{q\text{ times}},\qquad 1 \le q \le 2^t\,,$$

(III) $$C = \underbrace{F_j\ldots F_j}_{q_1}\underbrace{F_k\ldots F_k}_{q_2},\qquad 1 \le q_1+q_2 < 2\cdot 2^t\,,$$

(IV) $$C = \underbrace{\overline{F}_jF_j\ldots F_j}_{q_1}\underbrace{F_k\ldots F_k\overline{\overline{F}}_k}_{q_2},$$

where $\overline{F}_j = F_j[l_1,\lambda-1]$, $\overline{\overline{F}}_k = F_k[0,l_2]$, $0 < l_1 \le \lambda-1$, $0 \le l_2 < \lambda-1$ and $q_1+q_2 \ge 1$.

If C has the form (I) then Lemma 6 implies (54).

Case (II). Assume that C δ-occurs in ω_t at positions $l',\ldots,l'+|C|-1$, i.e.

$$d(C,\omega_t[l',l'+|C|-1]) < \delta\,.$$

It follows from (26) and the inequality $\delta < \varrho$ that $l' \equiv 0 \pmod{\lambda}$. This condition means that $\omega_t[l',l'+|C|-1]$ is a concatenation of blocks F_s, $s \in \mathbb{Z}_p$. Now, it is not hard to observe that $\omega_t[l',l'+|C|-1]$ is contained in a fragment C_1 of ω_t of the following form (see Fig. 2):

$$\begin{aligned} C_1 &= \omega_t[l\lambda_t - l\lambda, (l+1)\lambda_t + l_1\lambda - 1]\,,\quad \text{where}\\ \omega_t[l\lambda_t,(l+1)\lambda_t-1] &= b^t_j\,,\quad l_1 \le \delta q\,. \end{aligned} \tag{55}$$

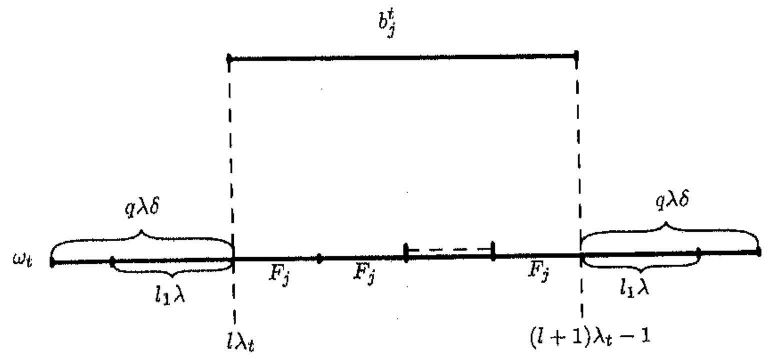

Fig. 2

The maximum number of disjoint δ-occurrences of C in such a fragment is estimated from above by

$$\frac{2^t\lambda + 2q\delta\lambda}{q\lambda} = \frac{2^t}{q} + 2\delta\,.$$

Since b^t_j occurs in ω_t with frequency $1/p$,

$$\phi_\delta(C,\omega_t) \le \left(\frac{2^t}{q}+2\delta\right)\cdot\frac{1}{2^t\lambda}\cdot\frac{1}{p} \le \frac{1}{p}\cdot\frac{1}{q\lambda}(1+2\delta)\,.$$

Mutiplying by $|C| = q\lambda$ we get

$$t_\delta(C,\omega_t) \le \frac{1}{p}(1+2\delta)\,. \tag{56}$$

Case (III). Assume that $q_1 \ge q_2$. If $q_2 < \delta q_1/(1-\delta)$ or equivalently $q_2 < \delta(q_1+q_2)$, then C is contained in a fragment of ω_t of the form (55). Repeating the same computations we come to the inequality

$$t_\delta(C,\omega_t) \le \frac{1}{p}\left(1+\frac{q_1+q_2}{2^t}2\delta\right) \le \frac{1}{p}(1+4\delta)\,. \tag{57}$$

If $q_2 \ge \delta q_1/(1-\delta)$ then C occurs in $b_j^t b_k^t$ exactly once. Thus

$$\phi_\delta(C,\omega_t) \le \mathrm{fr}(jk,\omega_{t+1})\frac{1}{2\lambda_t}$$

and therefore

$$t_\delta(C,\omega_t) \le |C|\,\mathrm{fr}(jk,\omega_{t+1})\frac{1}{2\lambda_t} \le \mathrm{fr}(jk,\omega_{t+1}).$$

Now, Lemma 4 gives

$$t_\delta(C,\omega_t) \le \frac{2}{p(\lambda-1)},$$

which implies (54).

Case (IV). First assume that $j = k$. Then C has the form

$$C = \overline{F}_j \underbrace{F_j \ldots F_j}_{q} \overline{\overline{F}}_j,$$

where $\overline{\overline{F}}_j = F_j[0,l_2]$, $0 \le l_2 < \lambda - 1$, $q \ge 1$. Define

$$C_1 = \underbrace{F_j \ldots F_j}_{q \text{ times}}.$$

If

$$d(C,\omega_t[l', l' + |C| - 1]) < \delta$$

then

$$d(C_1,\omega_t[l' + \lambda - l_1, l' + \lambda - l_1 + |C_1| - 1]) < 2\delta.$$

Since $\delta < \frac{1}{2}\varrho$, using (26) we again obtain $l' \equiv 0 \pmod{\lambda}$. Then C is contained in a fragment of ω_t of the form (55). The maximum number of disjoint δ-occurrences of C in such a fragment is not greater than

$$\frac{2^t\lambda + 2q\delta\lambda}{(q+2)\lambda}.$$

Repeating the same computations as in Case (II) we get (56).

If $j \ne k$ then we obtain (56) or (57) in the same way as previously. By (57) and the inequality $\delta < 1/(8(p-1))$ we have

$$t_\delta(C,\omega_t) \le \frac{1}{p}(1+4\delta) < \frac{1}{p}\left(1 + \frac{1}{2(p-1)}\right) = \frac{1}{p}\cdot\frac{\lambda+1}{\lambda}.$$

In this manner (54) is proved.

Now, we can estimate $t_\delta(E,\omega)$ for an arbitrary block E.

PROPOSITION 3. *There exist $\delta_0 > 0$ and $M_0 < 1/(p-1)$ such that for every block E occurring in ω we have $t_\delta(E,\omega) \le M_0$ whenever $\delta < \delta_0$.*

Proof. Let $\delta_1 = \min(\varrho/3, 1/(8(p-1)))$. If E occurs in $\omega = \omega_0$ in such a manner that there exist $j,k \in \mathbb{Z}_p$ for which E occurs in $b_j^0 b_k^0[1, 2\lambda_0 - 1]$ then in view of Lemma 7 we have

$$t_\delta(E,\omega) \le \max\left(\frac{1}{p}\cdot\frac{\lambda+1}{\lambda}, \frac{4}{p(\lambda-1)}\right) = M_1. \tag{58}$$

Suppose that E contains at least one block b_s^0, $s \in \mathbb{Z}_p$. We can find $t \ge 0$ such that E is of the form

$$E = E_1(B_t + j_1)\ldots(B_t + j_q)E_2, \tag{59}$$

where

$$E_1 = (B_t + j_0)[l_1, n_t - 1], \quad E_2 = (B_t + j_{q+1})[0, l_2],$$
$$0 < l_1 \le n_t - 1,\ 0 \le l_2 < n_t - 1,\ q \ge 1,$$

and the block $C = (j_1 \ldots j_q)$ occurs in $b_s^{t+1} b_q^{t+1}[1, 2\lambda_{t+1} - 2]$, $s, q \in \mathbb{Z}_p$. We have

$$E^* = (B_t + j_1)\ldots(B_t + j_q) = B_t \times C$$

and $q \le 2\lambda_{t+1} - 2$. It is evident that

$$t_\delta(E,\omega_0) \le \frac{|E|}{|E^*|} t_{3\delta}(E^*,\omega_0).$$

In view of Corollary 1, we obtain

$$t_{3\delta}(E^*,\omega_0) = t_{3\delta}(B_t \times C, \omega_0) = t_{3\delta}(C,\omega_{t+1})$$

whenever $\delta < \min(\varrho/3, 1/24)$. Applying Lemma 7 with $\delta < \delta_2$, where $\delta_2 < \min(\varrho/6, 1/(24(p-1)))$, we get

$$t_\delta(E,\omega) \le \frac{|E|}{|E^*|}M_1 = \left(1 + \frac{|E_1| + |E_2|}{|E^*|}\right)M_1$$
$$\le \left(1 + \frac{2n_t}{qn_t}\right)M_1 = \left(1 + \frac{2}{q}\right)M_1.$$

We have

$$M_1 = \begin{cases} \dfrac{4}{p(\lambda-1)} = \dfrac{4}{9} & \text{if } p = 3, \\[2mm] \dfrac{1}{p}\cdot\dfrac{\lambda+1}{\lambda} & \text{if } p \ge 5. \end{cases}$$

Let

$$q_0 = \begin{cases} 4p - 1, & p \ge 5, \\ 17, & p = 3. \end{cases}$$

If $q \geq q_0$ then

$$\left(1+\frac{2}{q}\right)M_1 \leq M_2 = \begin{cases} \frac{76}{156} < \frac{1}{2}, & p=3,\\ \frac{(4p+1)(2p-1)}{p(4p-1)(2p-2)} < \frac{1}{p-1}, & p \geq 5. \end{cases}$$

In this way we obtain $t_\delta(E,\omega) \leq M_2$ if $q \geq q_0$.

Let $q < q_0$ and let $\delta < 1/(3q_0)$. If E δ-occurs in ω_0 then E^* 3δ-occurs in ω_0. The condition $3\delta < 1/q_0$ implies that E^* occurs in ω_0. Consider the following cases:

(I$_1$) $$\frac{|E_1|}{(q_0+2)n_t} < \delta \quad \text{and} \quad \frac{|E_2|}{(q_0+2)n_t} < \delta\,,$$

(II$_1$) $$\frac{|E_1|}{(q_0+2)n_t} \geq \delta \quad \text{and} \quad \frac{|E_2|}{(q_0+2)n_t} < \delta\,,$$

(III$_1$) $$\frac{|E_1|}{(q_0+2)n_t} < \delta \quad \text{and} \quad \frac{|E_2|}{(q_0+2)n_t} \geq \delta\,,$$

(IV$_1$) $$\frac{|E_1|}{(q_0+2)n_t} \geq \delta \quad \text{and} \quad \frac{|E_2|}{(q_0+2)n_t} \geq \delta\,.$$

Case (I$_1$). We have

$$t_\delta(E,\omega_0) \leq \left(1+\frac{|E_1|+|E_2|}{qn_t}\right)t(E^*,\omega_0) \leq \left(1+\frac{2\delta(q_0+2)n_t}{qn_t}\right)t(E^*,\omega_0) \leq [1+2\delta(q_0+2)]t(E^*,\omega_0)\,.$$

In view of Lemmas 2 and 7 and (58)

(60) $$t_\delta(E,\omega_0) \leq [1+2\delta(q_0+2)]M_1\,.$$

Case (II$_1$). If E δ-occurs in ω_0 then the block $B_t \times (j_0C)$ occurs in ω_0. Further,

$$\begin{aligned} t_\delta(E,\omega_0) &\leq \frac{|E^*|+|E_1|+|E_2|}{|B_t\times(j_0C)|}t(B_t\times(j_0C),\omega_0)\\ &\leq \frac{(q+1)n_t+|E_2|}{(q+1)n_t}t(B_t\times(j_0C),\omega_0)\\ &= \left(1+\frac{|E_2|}{(q+1)n_t}\right)t(B_t\times(j_0C),\omega_0)\\ &\leq \left(1+\frac{\delta(q_0+2)}{2}\right)t(B_t\times(j_0C),\omega_0)\,. \end{aligned}$$

Using again Lemmas 2 and 7 we obtain (60).

Case (III$_1$). We get (60) by the same arguments.

Case (IV$_1$). We have the following property: whenever E of the form (59) δ-occurs in ω_0 then the block $B_t \times (j_0Cj_{q+1})$ occurs in ω_0. Thus

$$t_\delta(E,\omega_0) \leq t(B_t \times (j_0Cj_{q+1}),\omega_0)\,.$$

Lemmas 2 and 7 imply $t_\delta(E,\omega_0) \leq M_1$. Now, choose $\delta_3 > 0$ such that

$$M_3 = [1+2\delta_3(q_0+2)]M_1 < \frac{1}{p-1}\,.$$

Next, we put $\delta_0 = \min(\delta_1,\delta_2,\delta_3)$ and $M_0 = \max(M_2, M_3)$. The numbers δ_0 and M_0 satisfy the conclusion of Proposition 3.

As a consequence of Proposition 3 and the definition of F^* (see (1) and (2)) we get

COROLLARY 3. *For the Morse sequence ω defined by* (23) *we have $F^* < 1/(p-1)$.*

Now, applying Remarks 1 and 4 we have $r_\omega = p$. In Proposition 2, we have proved $m_\omega = p-1$. The proof of Theorem 3 is complete.

References

[Age] O. N. Ageev, *Dynamical systems with a Lebesgue component of even multiplicity in the spectrum*, Mat. Sb. 136 (178) (1988), 307–319 (in Russian).

[Cha1] R. V. Chacon, *A geometric construction of measure preserving transformations*, in: Proc. Fifth Berkeley Symposium on Mathematical Statistics and Probability, Vol. II, Part 2, Univ. of California Press, 1965, 335–360.

[Cha2] —, *Approximation and spectral multiplicity*, in: Contributions to Ergodic Theory and Probability, Lecture Notes in Math. 160, Springer, 1970, 18–27.

[Fer1] S. Ferenczi, *Systèmes localement de rang un*, Ann. Inst. H. Poincaré Probab. Statist. 20 (1984), 35–51.

[Fer2] —, *Tiling and local rank properties of the Morse sequence*, Theoret. Comput. Sci., to appear.

[GKLL] G. R. Goodson, J. Kwiatkowski, M. Lemańczyk and P. Liardet, *On the multiplicity function of ergodic group extensions of rotations*, this volume, 157–174.

[GoLe] G. R. Goodson and M. Lemańczyk, *On the rank of a class of bijective substitutions*, Studia Math. 96 (1990), 219–230.

[delJ] A. del Junco, *A transformation with simple spectrum which is not rank one*, Canad. J. Math. 29 (1977), 655–663.

[Kea1] M. Keane, *Generalized Morse sequences*, Z. Wahrsch. Verw. Gebiete 10 (1968), 335–353.

[Kea2] —, *Strongly mixing g-measures*, Invent. Math. 16 (1972), 309–353.

[Kwi] J. Kwiatkowski, *Isomorphism of regular Morse dynamical systems*, Studia Math. 72 (1982), 59–89.

[KwRo] J. Kwiatkowski and T. Rojek, *A method of solving a cocycle functional equation and applications*, ibid. 99 (1991), 69–86.

[KwSi] J. Kwiatkowski and A. Sikorski, *Spectral properties of G-symbolic Morse shifts*, Bull. Soc. Math. France 115 (1987), 19–33.

[Lem] M. Lemańczyk, *Toeplitz Z_2-extensions*, Ann. Inst. H. Poincaré Probab. Statist. 24 (1988), 1–43.

[Mar1] J. C. Martin, *Generalized Morse sequences on n symbols*, Proc. Amer. Math. Soc. 54 (1976), 379-383.

[Mar2] —, *The structure of generalized Morse minimal sets on n symbols*, Trans. Amer. Math. Soc. 232 (1977), 343-355.

[MaNa] J. Mathew and M. G. Nadkarni, *A measure preserving transformation whose spectrum has Lebesgue component of multiplicity two*, Bull. London Math. Soc. 16 (1984), 402–406.

[Men1] M. Mentzen, *Some examples of automorphisms with rank r and simple spectrum*, Bull. Polish Acad. Sci. Math. 35 (1987), 417–424.

[Men2] —, Thesis, preprint no. 2/89, Nicholas Copernicus University, Toruń 1989.

[New] D. Newton, *On canonical factors of ergodic dynamical systems*, J. London Math. Soc. 19 (1979), 129–136.

[ORW] D. S. Ornstein, D. J. Rudolph and B. Weiss, *Equivalence of measure preserving transformations*, Mem. Amer. Math. Soc. 262 (1982).

[Par] W. Parry, *Compact abelian group extensions of discrete dynamical systems*, Z. Wahrsch. Verw. Gebiete 13 (1969), 95–113.

[Que] M. Queffélec, *Substitution Dynamical Systems—Spectral Analysis*, Lecture Notes in Math. 1294, Springer, 1987.

[Rob1] E. A. Robinson, *Ergodic measure preserving transformations with arbitrary finite spectral multiplicities*, Invent. Math. 72 (1983), 299–314.

[Rob2] —, *Mixing and spectral multiplicity*, Ergodic Theory Dynamical Systems 5 (1985), 617–624.

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Oscillatory singular integrals on weighted Hardy spaces

by

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Abstract. Let
$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^1} e^{iP(x-y)} \frac{f(y)}{x-y}\, dy\,,$$
where P is a real polynomial on $\mathbb{R}$. It is proved that T is bounded on the weighted $H^1(wdx)$ space with $w \in A_1$.

1. Introduction. Let ψ be a Schwartz function, $\psi \in \mathcal{S}(\mathbb{R})$, $\int_{\mathbb{R}} \psi(x)\,dx \neq 0$. Set
$$\psi_t(x) = t^{-1}\psi(x/t)\,, \quad t > 0\,,\ x \in \mathbb{R}\,.$$
For each distribution $f \in \mathcal{S}'(\mathbb{R})$, define
$$f^*(x) = \sup_{t>0} |(f*\psi_t)(x)|, \quad x \in \mathbb{R}\,.$$
The weighted Hardy space $H^1_w(\mathbb{R})$, with weight function w, is defined to be the space of all f such that
$$\|f^*\|_{L^1_w} = \int_{\mathbb{R}} f^*(x)w(x)\,dx < \infty\,.$$
If $f \in H^1_w$, we define $\|f\|_{H^1_w} = \|f^*\|_{L^1_w}$.

An operator T on the weighted Hardy space $H^1_w(\mathbb{R})$ is said to be bounded if there exists a constant C such that for each $f \in H^1_w$,
$$\|Tf\|_{H^1_w} \le C\|f\|_{H^1_w}\,.$$

Let $P(x)$ be a real polynomial on $\mathbb{R}$. Consider the oscillatory singular integral
$$Tf(x) = \text{p.v.} \int_{\mathbb{R}} e^{iP(x-y)} \frac{f(y)}{x-y}\, dy\,. \tag{1}$$

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