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A weighted Plancherel formula II. The case of the ball

by

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Abstract. The group SU$(1,d)$ acts naturally on the Hilbert space $L^2(B, d\mu_\alpha)$ ($\alpha > -1$), where B is the unit ball of $\mathbb{C}^d$ and $d\mu_\alpha$ the weighted measure $(1-|z|^2)^\alpha\, dm(z)$. It is proved that the irreducible decomposition of the space has finitely many discrete parts and a continuous part. Each discrete part corresponds to a zero of the generalized Harish-Chandra **c**-function in the lower half plane. The discrete parts are studied via invariant Cauchy–Riemann operators. The representations on the discrete parts are equivalent to actions on some holomorphic tensor fields.

0. Introduction. A weighted Plancherel formula in the case of the disk of the complex plane was obtained in [9], where it is proved that a natural unitary group action of the Möbius group on $L^2(D, \mu_\alpha)$, where D is the unit disk and $d\mu_\alpha(z) = (1-|z|^2)^\alpha\, dx\, dy$, gives rise to a decomposition of the space into continuous and discrete parts. The representations of the continuous part are in the principal series, while those of the discrete parts are in the discrete holomorphic series. Note that, in particular, the first space of the discrete parts is the usual weighted Bergman space in $L^2(D, \mu_\alpha)$.

In the present paper, we give the corresponding Plancherel formula in the case of the ball in $\mathbb{C}^d$. We use the general approach via the Harish-Chandra **c**-function. It turns out that in our case (the analogue of) the **c**-function has zeros in the lower half plane, each zero giving one discrete part in the decomposition. When $\alpha = -d-1$, the weighted measure is just the invariant measure on the unit ball; in this case the Plancherel formula is studied in [7]. The discrete parts are studied via invariant Cauchy–Riemann operators. That is, we identify them with some components in an increasing sequence of kernel spaces of iterated invariant Cauchy–Riemann operators. The representations on the discrete spaces are equivalent to actions on suitable holomorphic tensor fields. Finally, we describe some bases for the discrete spaces. However, we have not been able to find an orthogonal basis.

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This problem involves orthogonal polynomials related to the ones studied by Appell and Kampé de Fériet [1].

In §1 we fix some notations and present our main theorems. The proofs are given in §2. In §3 we study the discrete parts and give some bases in these spaces.

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1. The Plancherel formula. Let B be the unit ball in $\mathbb{C}^d$ with dm the Lebesgue measure on it, and let $S=\partial B$ be the unit sphere with $d\sigma$ the normalized area measure. For $\alpha>-1$, we consider the weighted measure $d\mu_\alpha(z)=(1-|z|^2)^\alpha\,dm(z)$. The group $\mathrm{SU}(1,d)$ consists of all block matrices of the form

$$g=\begin{pmatrix} A & B\\ C & D\end{pmatrix},$$

where $D\in\mathbb{C}$, A, B, C are $d\times d$, $d\times 1$, $1\times d$ matrices, respectively with complex entries, such that g is unitary with respect to the indefinite metric $|z_1|^2+|z_2|^2+\ldots+|z_d|^2-|z_{d+1}|^2$ on $\mathbb{C}^{d+1}$ and $\det g=1$. It acts on the unit ball via the formula

$$gz=(Az+B)(Cz+D)^{-1}.$$

We define an action of $\mathrm{SU}(1,d)$ on $L^2(B,\mu_\alpha)$ by

$$T_gf(z)=f(gz)(\det g'(z))^{\frac{\nu}{d+1}},$$

where $\nu=\alpha+d+1$, and $g'(z)$ is the complex Jacobian. Then T is a unitary representation of $\mathrm{SU}(1,d)$ (with the same convention as in [9] concerning the ambiguity of the definition of the power). The corresponding invariant Laplacian was found in [9]. It is of the form

$$\Box_\nu=(1-|z|^2)\Big(\sum_{i=1}^d\frac{\partial^2}{\partial\bar z_i\partial z_i}-\overline{R}R-\nu\overline{R}\Big),$$

where $R=\sum_{i=0}^d z_i\partial/\partial z_i$. We will find eigenfunctions of the invariant Laplacian.

LEMMA 1.1. *The functions*

(1.1) $$e_{\lambda,\omega}(z)=\Big(\frac{1-|z|^2}{|1-\langle z,\omega\rangle|^2}\Big)^{\frac{-\nu+d+i\lambda}{2}}\frac{1}{(1-\langle z,\omega\rangle)^\nu},\quad \omega\in S,$$

are eigenfunctions of $\Box_\nu$ *with eigenvalues* $-\frac14((\alpha+1)^2+\lambda^2)$.

Proof. We follow the method in Helgason [8], pp. 402–403 (see also [10], Theorem 4.2.4). Put

$$P(z,\omega)=\Big(\frac{1-|z|^2}{|1-\langle z,\omega\rangle|^2}\Big)^d,$$

the invariant Poisson kernel, and

$$K(z,\omega)=(1-\langle z,\omega\rangle)^{-\nu},$$

the Bergman reproducing kernel. In this notation, we have

$$e_{\lambda,\omega}(z)=P(z,\omega)^{\frac{-\nu+d+i\lambda}{2d}}K(z,\omega).$$

If $g\in\mathrm{SU}(1,d)$, by Theorem 3.3.5 in [10], we have

$$P(gz,g\omega)=P(z,\omega)P(g0,g\omega).$$

It is also known that

$$K(gz,g\omega)=(\det g'(z))^{\frac{\nu}{d+1}}(\det\overline{g'(z)})^{\frac{\nu}{d+1}}K(z,\omega).$$

From this it follows that

$$e_{\lambda,\omega}(gz)(\det g'(z))^{\frac{\nu}{d+1}}=e_{\lambda,g^{-1}\omega}(z)(\det g'(0))^{\frac{\nu}{d+1}}e_{\lambda,\omega}(g0).$$

Since $\Box_\nu$ commutes with the group action T_g, we get

$$\Box_\nu e_{\lambda,\omega}(gz)(\det g'(z))^{\frac{\nu}{d+1}}=\Box_\nu e_{\lambda,g^{-1}\omega}(z)(\det g'(0))^{\frac{\nu}{d+1}}e_{\lambda,\omega}(g0).$$

Putting $z=0$, we see that

$$\Box_\nu e_{\lambda,\omega}(g0)=\Box_\nu e_{\lambda,g^{-1}\omega}(0)e_{\lambda,\omega}(g0).$$

That is, the $e_{\lambda,\omega}$ are eigenfunctions of $\Box_\nu$. The eigenvalue $\Box_\nu e_{\lambda,\omega}(0)$ can be calculated directly and we find it is $-\frac14((\alpha+1)^2+\lambda^2)$. ■

The functions

(1.2) $$\phi_\lambda(z)=\int_S e_{\lambda,\omega}(z)\,d\sigma(\omega)$$

are then radial eigenfunctions of $\Box_\nu$ with the same eigenvalues.

Let $\mathcal{D}(B)$ be the space of C^∞-functions on B with compact supports and let $\mathcal{D}^\natural(B)$ be the space of radial functions in $\mathcal{D}(B)$. For $f\in\mathcal{D}^\natural(B)$, we define the spherical transform as follows:

(1.3) $$\widetilde f(\lambda)=\int_B f(z)\phi_{-\lambda}(z)\,d\mu_\alpha(z),$$

and extend it to an entire function of λ. For $f\in\mathcal{D}(B)$, we define the generalized Fourier transform by

$$\widetilde f(\lambda,\omega)=\int_B f(z)e_{-\lambda,\omega}(z)\,d\mu_\alpha(z),\qquad \lambda\in\mathbb{R},\ \omega\in S,$$

and likewise extend it to an entire function of λ.

In what follows we will only consider the case when α is not an integer and we then define

$$k = [(\alpha+1)/2]. \tag{1.4}$$

The following Plancherel-type theorems will be proved in this paper.

THEOREM 1. *Let $\alpha > -1$ and not an integer. For $f \in \mathcal{D}^\sharp(B)$ we have*

(i) *the inversion formula*

$$f(z) = \frac{\Gamma(d)}{2^{\alpha+2}\pi^{d+1}} \int_{\mathbb{R}} \widetilde{f}(\lambda)\phi_\lambda(z)|\mathbf{c}(\lambda)|^{-2}\,d\lambda + \sum_{l=0}^{k} \mathbf{c}_l \widetilde{f}(-i(\alpha+1-2l)),$$

where

$$\mathbf{c}_l = \frac{\Gamma(d+l)\Gamma(k+1-l)\Gamma(\alpha+1+d)(\alpha+1-2l)}{\pi^d\Gamma(d)\Gamma(\alpha+2-l)(-1)^l\prod_{l'=0,\,l'\neq l}^{k}(l'-l)}, \tag{1.5}$$

$$\mathbf{c}(\lambda) = \frac{2^{-\nu+d-i\lambda}\Gamma(d)\Gamma(i\lambda)}{\Gamma\left(\frac{-\nu+d+i\lambda}{2}\right)\Gamma\left(\frac{\nu+d+i\lambda}{2}\right)}, \tag{1.6}$$

and (ii) *the Plancherel formula*

$$\int_B |f(z)|^2\,d\mu_\alpha(z) = \frac{\Gamma(d)}{2^{\alpha+2}\pi^{d+1}} \int_{\mathbb{R}} |\widetilde{f}(\lambda)|^2|\mathbf{c}(\lambda)|^{-2}\,d\lambda + \sum_{l=0}^{k} \mathbf{c}_l|\widetilde{f}(-i(\alpha+1-2l))|^2. \tag{1.7}$$

THEOREM 2. *Let $\alpha > -1$ and not an integer. If $f \in \mathcal{D}(B)$, we have*

(i) *the inversion formula*

$$f(z) = \frac{\Gamma(d)}{2^{\alpha+2}\pi^{d+1}} \int_S \int_{\mathbb{R}} \widetilde{f}(\lambda,\omega)e_{\lambda,\omega}(z)|\mathbf{c}(\lambda)|^{-2}\,d\lambda\,d\sigma(\omega) + \sum_{l=0}^{k} \mathbf{c}_l \int_S \widetilde{f}(-i(\alpha+1-2l),\omega)e_{-i(\alpha+1-2l),\omega}(z)\,d\sigma(\omega),$$

and (ii) *the Plancherel formula*

$$\int_B |f(z)|^2\,d\mu_\alpha(z) = \frac{\Gamma(d)}{2^{\alpha+2}\pi^{d+1}} \int_S \int_{\mathbb{R}} |\widetilde{f}(\lambda,\omega)|^2|\mathbf{c}(\lambda)|^{-2}\,d\lambda\,d\sigma(\omega) + \sum_{l=0}^{k} \mathbf{c}_l \int_S \widetilde{f}(-i(\alpha+1-2l),\omega)\overline{\widetilde{f}(i(\alpha+1-2l),\omega)}\,d\sigma(\omega).$$

Moreover, the quadratic forms

$$\mathbf{c}_l \int_S \widetilde{f}(-i(\alpha+1-2l),\omega)\overline{\widetilde{f}(i(\alpha+1-2l),\omega)}\,d\sigma(\omega)$$

are positive definite. Let P_l be the operators defined by

$$P_l f(z) = \mathbf{c}_l \int_S \widetilde{f}(-i(\alpha+1-2l),\omega)e_{-i(\alpha+1-2l),\omega}(z)\,d\sigma(\omega). \tag{1.8}$$

Then P_l can be extended to pairwise orthogonal projections on $L^2(B,\mu_\alpha)$. Define $A_l^{\alpha,2}(B) = P_l L^2(B,\mu_\alpha)$. The map $f \mapsto \widetilde{f}$ then extends to an isometry from

$$L^2(B,\mu_\alpha) \ominus \sum_{l=0}^{k}{}^{\oplus} A_l^{\alpha,2}(B)$$

into

$$L^2\left(\mathbb{R}\times S, \frac{\Gamma(d)}{2^{\alpha+2}\pi^{d+1}}|\mathbf{c}(\lambda)|^{-2}\,d\lambda\,d\sigma\right).$$

2. The c-function and the proof of the Plancherel theorem.

We start with the following

LEMMA 2.1. *The function ϕ_λ is given by*

$$\phi_\lambda(z) = (1-|z|^2)^{\frac{-\nu+d-i\lambda}{2}} F\left(\frac{\nu+d-i\lambda}{2}, \frac{-\nu+d-i\lambda}{2}; d; |z|^2\right), \tag{2.1}$$

where F is the hypergeometric function.

Proof. By definition (1.2) we have

$$\begin{aligned}
\phi_{-\lambda}(z) &= \int_S \left(\frac{1-|z|^2}{|1-\langle z,\omega\rangle|^2}\right)^{\frac{-\nu+d-i\lambda}{2}} \frac{1}{(1-\langle z,\omega\rangle)^\nu}\,d\sigma(\omega)\\
&= (1-|z|^2)^{\frac{-\nu+d-i\lambda}{2}} \int_S \frac{1}{|1-\langle z,\omega\rangle|^{\frac{-\nu+d-i\lambda}{2}}} \frac{1}{(1-\langle z,\omega\rangle)^\nu}\,d\sigma(\omega)\\
&= (1-|z|^2)^{\frac{-\nu+d-i\lambda}{2}} \int_S \frac{1}{(1-\langle z,\omega\rangle)^{\frac{\nu+d-i\lambda}{2}}} \frac{1}{(1-\langle \omega,z\rangle)^{\frac{-\nu+d-i\lambda}{2}}}\,d\sigma(\omega)\\
&= (1-|z|^2)^{\frac{-\nu+d-i\lambda}{2}} \sum_{n,m=0}^{\infty} \binom{\frac{-\nu+d-i\lambda}{2}}{n}\binom{\frac{\nu+d-i\lambda}{2}}{m}\\
&\quad\times \int_S (\langle z,\omega\rangle)^n(\langle \omega,z\rangle)^m\,d\sigma(\omega).
\end{aligned}$$

By Rudin [10], p. 18,

$$\int_S (\langle z,\omega\rangle)^n(\langle \omega,z\rangle)^m d\sigma(\omega) = \begin{cases} 0 & \text{if } n\neq m, \\ |z|^{2n} n!/(d)_n & \text{if } n=m. \end{cases}$$

Here we have used the Pochhammer symbol

$$(d)_n = d(d+1)\dots(d+n-1).$$

Writing $\binom{s}{n} = (-1)^n(-s)_n/n!$, we get

$$\begin{aligned}\phi_{-\lambda}(z) &= (1-|z|^2)^{\frac{-\nu+d-i\lambda}{2}}\sum_{n=0}^{\infty}\left(\frac{\nu+d-i\lambda}{2}\right)_n\left(\frac{-\nu+d-i\lambda}{2}\right)_n\frac{1}{(d)_n n!}|z|^{2n}\\ &= (1-|z|^2)^{\frac{-\nu+d-i\lambda}{2}} F\left(\frac{\nu+d-i\lambda}{2},\frac{-\nu+d-i\lambda}{2};d;|z|^2\right).\end{aligned}$$

It follows from the known property of the hypergeometric function (Erdélyi [3], Vol. 1, p. 64) that $\phi_\lambda(z)=\phi_{-\lambda}(z)$. The lemma is proved. ■

COROLLARY 2.2. *If* $\mathrm{Re}(i\lambda)>0$, *then the limit*

$$\lim_{r\to\infty}\phi_\lambda(\tanh r)e^{r(-\nu+d-i\lambda)} = \mathbf{c}(\lambda)$$

exists and we have

$$\mathbf{c}(\lambda) = \frac{2^{-\nu+d-i\lambda}\Gamma(d)\Gamma(i\lambda)}{\Gamma\left(\frac{-\nu+d+i\lambda}{2}\right)\Gamma\left(\frac{\nu+d+i\lambda}{2}\right)}. \tag{2.2}$$

Proof. If $\mathrm{Re}(i\lambda)>0$, then by Erdélyi [3], Vol. 1, p. 61,

$$\begin{aligned}&\lim_{|z|\to 1} F\left(\frac{\nu+d-i\lambda}{2},\frac{-\nu+d-i\lambda}{2};d;|z|^2\right)\\ &= F\left(\frac{\nu+d-i\lambda}{2},\frac{-\nu+d-i\lambda}{2};d;1\right) = \frac{\Gamma(d)\Gamma(i\lambda)}{\Gamma\left(\frac{-\nu+d+i\lambda}{2}\right)\Gamma\left(\frac{\nu+d+i\lambda}{2}\right)}.\end{aligned}$$

Now if $z=\tanh r$, then $e^{2r}(1-|z|^2)\to 2^2$ as $r\to\infty$. From the above and (2.1), the corollary follows easily. ■

Remark. If we formally let $\nu=0$, that is, $\alpha=-(d+1)$, and $d\mu_\alpha$ the invariant measure, our $\mathbf{c}$-function turns out to be the Harish-Chandra $\mathbf{c}$-function (Helgason [7]).

If we use geodesic polar coordinates, writing $z=(\tanh r)\omega$, $\omega\in S$, then the radial eigenfunctions $\phi_\lambda(z)=\phi_\lambda(\tanh r)$ of $\Box_\nu$ satisfy

$$L\Phi = -\tfrac14((-\nu+d)^2+\lambda^2)\Phi \tag{2.3}$$

where

$$L = \frac14\frac{d^2}{dr^2} + \left(\frac14-\frac{\nu}{2}\right)\tanh r\frac{d}{dr} + \left(\frac{d}{2}-\frac14\right)\coth r\frac{d}{dr}$$

is the radial part of $\Box_\nu$. We omit the calculation.

Next we proceed as in Helgason [7] to find two linearly independent solutions of (2.3). Let

$$\Phi_\lambda(r) = \sum_{n=0}^{\infty} e^{-2nr}\Gamma_n(\lambda)e^{(i\lambda-d+\nu)r}. \tag{2.4}$$

Substituting

$$\tanh r = -1+2\sum_{n=0}^{\infty}(-1)^n e^{-2nr},\qquad \coth r = -1+2\sum_{n=0}^{\infty}e^{-2nr},$$

into (2.3), and comparing the coefficients of e^{-2nr}, we arrive at the following recursion formula for $\Gamma_n(\lambda)$, $n>0$:

$$\begin{aligned}&n(n-i\lambda)\Gamma_n(\lambda)\\ &\quad= \sum_{k=0}^{n-1}(i\lambda-d+\nu-2k)[(-d+\tfrac12)+(-1)^{n-k}(\nu-\tfrac12)]\Gamma_k(\lambda).\end{aligned} \tag{2.5}$$

Take $\Gamma_0=1$. If $i\lambda\notin\mathbb{Z}^+$, then (2.5) defines $\Gamma_n(\lambda)$ uniquely. Moreover, they are rational functions of λ. Using the same method as in Helgason [8], p. 63, we can prove the following estimate: For any $h>0$, there exists a constant $K_{\lambda,h}$ such that

$$|\Gamma_n(\lambda)|\le K_{\lambda,h}e^{nh}. \tag{2.6}$$

Since h can be arbitrarily small, we see that (2.4) defines a solution of (2.3) on $(0,\infty)$. However, if $-i\lambda\notin\mathbb{Z}^+$, $\Phi_{-\lambda}$ is another solution, and it is easy to see that Φ_λ, $\Phi_{-\lambda}$ are linearly independent if $i\lambda\notin\mathbb{Z}$. Hence $\phi_\lambda = s_1(\lambda)\Phi_\lambda + s_2(\lambda)\Phi_{-\lambda}$ for some constants $s_1(\lambda)$ and $s_2(\lambda)$. Multiplying this equation by $e^{(-\nu+d-i\lambda)r}$ and taking the limit as $r\to\infty$, we get $s_1(\lambda)=\mathbf{c}(\lambda)$ if $\mathrm{Re}(i\lambda)>0$, $i\lambda\notin\mathbb{Z}$. From the proof of Lemma 2.1 we know that $\phi_{-\lambda}=\phi_\lambda$. Therefore $s_2(\lambda)=\mathbf{c}(-\lambda)$. That is, we have

$$\phi_\lambda = \mathbf{c}(\lambda)\Phi_\lambda + \mathbf{c}(-\lambda)\Phi_{-\lambda},\quad \mathrm{Re}(i\lambda)>0,\ i\lambda\notin\mathbb{Z}. \tag{2.7}$$

By analytic continuation, this holds for all λ. From the known properties of the gamma function we see that $c(\lambda)^{-1}$ has poles at those λ for which

$$-\frac{\nu+d+i\lambda}{2}\in\mathbb{Z}^+ \quad\text{or}\quad -\frac{-\nu+d+i\lambda}{2}\in\mathbb{Z}^+.$$

This is equivalent to

$$\lambda\in(2i\mathbb{Z}^+ - i(\alpha+1))\cup(2i\mathbb{Z}^+ + i(\alpha+2d+1)).$$

In particular, since α is not an integer, we see that $\mathbf{c}(\lambda)^{-1}$ has poles in the lower half plane at

$$\lambda = -i(\alpha+1-2l)\,, \qquad l = 0, 1, \ldots, k\,,$$

where k is defined by (1.4).

LEMMA 2.3. *The* c*-function satisfies*

$$|\mathbf{c}(\lambda)^{-1}| \le \frac{C_1 + C_2|\lambda|^{k+d+1/2}}{\prod_{l=0}^{k} |\lambda + i(\alpha+1-2l)|}\,, \qquad \mathrm{Re}(i\lambda) > 0\,,$$

where C_1, C_2 are certain constants.

Proof. Substituting the duplication formula (Erdélyi [3], Vol. 1, p. 5),

$$\Gamma(i\lambda) = 2^{i\lambda-1}\pi^{-1/2}\Gamma\Big(\frac{i\lambda}{2}\Big)\Gamma\Big(\frac{i\lambda+1}{2}\Big)$$

into (2.2), we get

$$\mathbf{c}(\lambda)^{-1} = 2^{-\alpha-2}\frac{\Gamma\Big(\frac{-\nu+d+i\lambda}{2}\Big)\Gamma\Big(\frac{\nu+d+i\lambda}{2}\Big)}{\Gamma(d)\pi^{-1/2}\Gamma\Big(\frac{i\lambda}{2}\Big)\Gamma\Big(\frac{i\lambda+1}{2}\Big)}\,.$$

Recall that $\nu = \alpha + d + 1$, so

$$\text{(2.8)} \quad \Gamma\Big(\frac{-\nu+d+i\lambda}{2}\Big)\prod_{l=0}^{k}\frac{l-\nu+d+i\lambda}{2} = \Gamma\Big(\Big[\frac{\alpha+1}{2}\Big] + 1 - \frac{\alpha+1}{2} + \frac{i\lambda}{2}\Big)\,.$$

Now if $\mathrm{Re}(i\lambda) > 0$, then

$$\mathrm{Re}\Big(\Big[\frac{\alpha+1}{2}\Big] + 1 - \frac{\alpha+1}{2} + \frac{i\lambda}{2}\Big) > 0\,, \qquad \mathrm{Re}(\nu + d - i\lambda) > 0\,.$$

Using the Binet expression ([3], Vol. 1, p. 22), we get

$$\left|\frac{\Gamma\Big(\frac{\nu+d+i\lambda}{2}\Big)}{\Gamma\Big(\frac{i\lambda+1}{2}\Big)}\right| \le C_1 + C_2|i\lambda|^{\frac{\alpha+1}{2}+d-\frac{1}{2}}\,,$$

$$\left|\frac{\Gamma\Big(\Big[\frac{\alpha+1}{2}\Big] + 1 - \frac{\alpha+1}{2} + \frac{i\lambda}{2}\Big)}{\Gamma\Big(\frac{i\lambda}{2}\Big)}\right| \le C_1 + C_2|i\lambda|^{[\frac{\alpha+1}{2}]+1-\frac{\alpha+1}{2}}\,.$$

Substituting (2.8) into $\mathbf{c}(\lambda)^{-1}$ and using the above inequalities, we get the lemma. ■

LEMMA 2.4. *The Γ_n satisfy*

$$|\Gamma_n(\lambda)| \le c(1+n^d)(1+|\lambda|^e)\,,$$

for suitable constants c, d, and e.

Proof. See the proof of Lemma 4.11 in Helgason [8]. ■

LEMMA 2.5. *Let F be an even entire function of exponential type R. Then the function*

$$f(z) = \int_{\mathbb{R}} F(\lambda)\phi_\lambda(z)|\mathbf{c}(\lambda)|^{-2}\,d\lambda + \sum_{l=0}^{k}\frac{2^{\alpha+2}\pi^{d+1}}{\Gamma(d)}\mathbf{c}_l F(-i(\alpha+1-2l))\phi_{i(\alpha+1-2l)}(z)$$

satisfies

$$f(z) = 0 \quad \text{if } d(0,z) > R\,,$$

where $d(0,z)$ is the hyperbolic distance, and $\mathbf{c}_l$ are the constants in Theorem 1.

Proof. From (2.2) we see that if $\lambda \in \mathbb{R}$ or $i\lambda \in \mathbb{R}$, then

$$|\mathbf{c}(\lambda)|^2 = \mathbf{c}(\lambda)\mathbf{c}(-\lambda)\,.$$

Using (2.4) and (2.7), we find

$$\int_{\mathbb{R}} F(\lambda)\phi_\lambda(\tanh r)|c(\lambda)|^{-2}d\lambda = \int_{\mathbb{R}} F(\lambda)\sum_{n=0}^{\infty}(\Gamma_n(\lambda)e^{(i\lambda-d+\nu)r}e^{-2nr}c(-\lambda)^{-1} + \Gamma_n(-\lambda)e^{(-i\lambda-d+\nu)r}e^{-2nr}\mathbf{c}(\lambda)^{-1})\,d\lambda\,.$$

Lemmas 2.3 and 2.4 then guarantee that we can change the order of integration and summation. The above becomes

$$2\sum_{n=0}^{\infty}\int_{\mathbb{R}} F(\lambda)\Gamma_n(-\lambda)e^{(-i\lambda-d+\nu)r}e^{-2nr}\mathbf{c}(\lambda)^{-1}\,d\lambda\,.$$

We now compute the nth term. As we noted before, $\mathbf{c}(\lambda)^{-1}$ is a meromorphic function with simple poles in the lower half plane at $-i(\alpha+1-2l)$, where $l = 0, 1, \ldots, k$, while $\Gamma_n(-\lambda)$ is holomorphic in the lower half plane (it has poles at $\lambda \in i\mathbb{Z}^+$). By Lemma 2.3 and the assumption on F, using Cauchy's theorem for $\eta < 0$ and $|\eta|$ large enough we now obtain

$$(2.9)\quad 2\int_{\mathbb{R}} F(\lambda)\Gamma_n(-\lambda)e^{-i\lambda}e^{-2nr}\mathbf{c}(\lambda)^{-1}\,d\lambda - 2\int_{\mathbb{R}+i\eta} F(\lambda)\Gamma_n(-\lambda)e^{-i\lambda}e^{-2nr}\mathbf{c}(\lambda)^{-1}\,d\lambda = -\sum_{l=0}^{k} c_l' F(i(\alpha+1-2l))\Gamma_n(-i(\alpha+1-2l))e^{2lr}e^{-2nr},$$

where

$$c_l' = \frac{\pi 2^{2(\alpha+2)-2l}\Gamma(k+1-l)\Gamma(\alpha+1+d-l)}{\Gamma(d)\Gamma(\alpha+1-2l)\prod_{l'=0,l'\neq l}^{k}(l'-l)}.$$

Furthermore, the second integral in (2.9) can be dominated by

$$Ce^{R|\eta|}e^{\eta r} = Ce^{(r-R)\eta}.$$

So if $r > R$, letting $|\eta| \to -\infty$, we see that this integral is 0.

Summing (2.9) over n, we get

$$(2.10)\quad \int_{\mathbb{R}} F(\lambda)\phi_\lambda(\tanh r)|\mathbf{c}(\lambda)|^{-2}\,d\lambda = -c_l'\sum_{l=0}^{k} F(-i(\alpha+1-2l))\sum_{n=0}^{\infty}\Gamma_n(i(\alpha+1-2l))e^{2lr}e^{-2nr}.$$

On the other hand, by analytic continuation,

$$\mathbf{c}(-i(\alpha+1-2l)) = 0, \qquad \mathbf{c}(i(\alpha+1-2l)) = \frac{\Gamma(d)2^{2l}(-1)^l(\alpha+2-2l)_l}{\Gamma(d+l)},$$

so by (2.7) we have

$$\phi_{i(\alpha+1-2l)}(\tanh r) = \mathbf{c}(i(\alpha+1-2l))\sum_{n=0}^{\infty}\Gamma_n(i(\alpha+1-2l))e^{2lr}e^{-2nr}.$$

Substituting this to (2.10), we get for $r > R$,

$$\int_{\mathbb{R}} F(\lambda)\phi_\lambda(\tanh r)|\mathbf{c}(\lambda)|^{-2}\,d\lambda = -\sum_{l=0}^{k} c_l'\mathbf{c}(i(\alpha+1-2l))^{-1}F(i(\alpha+1-2l))\phi_{i(\alpha+1-2l)}(\tanh r).$$

As $\mathbf{c}_l = c_l'\mathbf{c}(i(\alpha+1-2l))$, this is just the lemma. ∎

Now we can prove the following inversion formula.

THEOREM. *Assume that $\alpha > -1$ and not an integer. Then for $f \in \mathcal{D}^\natural(B)$ we have*

$$cf(0) = \int_{\mathbb{R}} \widetilde{f}(\lambda)|\mathbf{c}(\lambda)|^{-2}\,d\lambda + \sum_{l=0}^{k}\frac{2^{\alpha+2}\pi^{d+1}}{\Gamma(d)}\mathbf{c}_l\widetilde{f}(-i(\alpha+1-2l)),$$

where the constants $c = 2^{\alpha+2}\pi^{d+1}/\Gamma(d)$, k and $\mathbf{c}_l$ are the same as in Theorem 1.

Proof. Choose a function $\beta \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\beta \subset [-1,1]$, such that

$$\psi(\lambda) = \int_{\mathbb{R}} \beta(r)e^{-i\lambda r}\,dr$$

satisfies

$$\psi \text{ is even}, \quad \psi \geq 0, \quad \psi(0) = 1.$$

Define the linear functional $T : \mathcal{D}(B) \to \mathbb{C}$ by

$$T(f) = \int_{\mathbb{R}} \widetilde{f^\natural}(\lambda)|\mathbf{c}(\lambda)|^{-2}\,d\lambda + \sum_{l=0}^{k}\mathbf{c}_l\widetilde{f^\natural}(-i(\alpha+1-2l))$$

where

$$f^\natural(z) = \int_{\mathrm{U}(d)} f(Uz)\,dU,$$

and $\mathrm{U}(d)$ is the unitary group on $\mathbb{C}^d$ and dU the normalized Haar measure on it.

Below we prove that T is a constant multiple of the delta function. First we notice that $\widetilde{f^\natural}(\lambda)$ is the (Euclidean) Fourier transform of the average of f over some hypersurfaces (horocycles) with respect to some weight functions (we omit the details here). So the integral in the definition of T is absolutely convergent and T is a distribution. Therefore we have

$$\begin{aligned} T(f) &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\psi(\varepsilon\lambda)\widetilde{f^\natural}(\lambda)|\mathbf{c}(\lambda)|^{-2}\,d\lambda + \lim_{\varepsilon\to 0}\sum_{l=0}^{k}\mathbf{c}_l\psi(-i\varepsilon(\alpha+1-2l))\widetilde{f^\natural}(-i(\alpha+1-2l)) \\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\Big(\int_B f(z)\phi_\lambda(z)\,d\mu_\alpha(z)\Big)\psi(\varepsilon\lambda)|\mathbf{c}(\lambda)|^{-2}\,d\lambda \\ &\quad + \lim_{\varepsilon\to 0}\sum_{l=0}^{k}\mathbf{c}_l\int_B f(z)\phi_{i(\alpha+1-2l)}(z)\,d\mu_\alpha(z)\,\psi(-i\varepsilon(\alpha+1-2l)) \\ &= \lim_{\varepsilon\to 0}\int_B f(z)T_\varepsilon(z)\,d\mu_\alpha(z), \end{aligned}$$

where

$$T_\varepsilon(z) = \int_{\mathbb{R}} \psi(\varepsilon\lambda)\phi_\lambda(z)|\mathbf{c}(\lambda)|^{-2}\,d\lambda + \sum_{l=0}^{k} \mathbf{c}_l \phi_{i(\alpha+1-2l)}(z)\psi(-i\varepsilon(\alpha+1-2l)).$$

By the Paley–Wiener theorem, we know that $\psi(\varepsilon\lambda)$ is an entire function of exponential type ε. Hence by Lemma 2.5, $T_\varepsilon(z)$ has compact support $(\tanh\varepsilon)\overline{B}$. Moreover, $T_\varepsilon(z)$ has uniformly bounded $L^1(d\mu_\alpha)$-norm. In fact,

$$\|T_\varepsilon\|_1 = \int_B |T_\varepsilon(z)|\,d\mu_\alpha(z) \le \mu_\alpha((\tanh\varepsilon)B)\|T_\varepsilon\|_\infty.$$

From (1.2) it is not difficult to see that $|\phi_\lambda(z)| \le C$, for $\lambda \in \mathbb{R}\cup\{i(\alpha+1-2l): l = 0,1,\ldots,k\}$ and $z \in (\tanh\varepsilon)B$. Therefore

$$\|T_\varepsilon\|_\infty \le C\int_{\mathbb{R}} \psi(\varepsilon\lambda)|\mathbf{c}(\lambda)|^{-2}\,d\lambda + C\sum_{l=0}^{k}|\psi(-i\varepsilon(\alpha+1-2l))|$$
$$\le C\varepsilon^{-1}\int_{\mathbb{R}} \psi(\lambda)|\mathbf{c}(\lambda/\varepsilon)|^{-2}\,d\lambda + C\sum_{l=0}^{k}\int_{\mathbb{R}} |\beta(r)|e^{\varepsilon(\alpha+1-2l)r}\,dr.$$

By Lemma 2.3, the first term can be dominated by $C\varepsilon^{-1}\varepsilon^{-2(d-1/2)} = C\varepsilon^{-2d}$. The second term can be dominated by a constant. Hence $\|T_\varepsilon\|_\infty \le C\varepsilon^{-2d}$. Since $\mu_\alpha(\tanh\varepsilon B) \asymp \varepsilon^{-2d}$, we get

$$\|T_\varepsilon\|_1 \le C.$$

So finally T is a distribution of support $\{0\}$ and is also L^∞-continuous. Hence $T(f) = cf(0)$ for some constant c. We select a sequence in $\mathcal{D}^\sharp(B)$ that approaches the constant function 1 in $L^2(B, d\mu_\alpha)$ and calculate both sides in the inversion formula. We find then $c = 2^{\alpha+2}\pi^{\alpha+1}/\Gamma(d)$. This proves the theorem. ■

The rest of the proof of Theorems 1 and 2 in §1 is the same as in [9], provided the following is proved.

LEMMA 2.6. *For $\lambda \in \mathbb{C}$, and $g \in \mathrm{SU}(1,d)$, we have*

$$\phi_\lambda(g^{-1}z) = \{\overline{\det g'(0)}\}^{\frac{\nu}{d+1}}\{\det(g^{-1})'(z)\}^{-\frac{\nu}{d+1}}\int_S e_{\lambda,\omega}(z)e_{-\lambda,\bar\omega}(\overline{g0})\,d\sigma(\omega).$$

Proof. We follow the proof of Lemma 1.1 and the notations given there. By definition

$$\phi_\lambda(g^{-1}z) = \int_S P(g^{-1}z,\omega)^{\frac{-\nu+d+i\lambda}{2d}}K(g^{-1}z,\omega)\,d\sigma(\omega).$$

Changing variables $\omega \mapsto g^{-1}\omega$, and noticing that the Poisson kernel $P(g0,\omega)$ is the Jacobian of the transformation, we get

$$\phi_\lambda(g^{-1}z) = \int_S P(g^{-1}z, g^{-1}\omega)^{\frac{-\nu+d+i\lambda}{2d}}K(g^{-1}z, g^{-1}\omega)P(g0,\omega)\,d\sigma(\omega).$$

Using the transformation formula in Lemma 1.1, we get

$$\phi_\lambda(g^{-1}z) = \int_S P(z,\omega)^{-\frac{\nu+d+i\lambda}{2d}}K(z,\omega)P(g0,\omega)^{-\frac{\nu+d-i\lambda}{2d}}$$
$$\times P(g0,\omega)^{\frac{\nu}{d}}\{\det(g^{-1})'(z)\}^{-\frac{\nu}{d+1}}\{\overline{\det(g^{-1})'(\omega)}\}^{-\frac{\nu}{d+1}}\,d\sigma(\omega)$$
$$= \{\det(g^{-1})'(z)\}^{-\frac{\nu}{d+1}}\int_S e_{\lambda,\omega}(z)e_{-\lambda,\bar\omega}(\overline{g0})$$
$$\times K(g0,g0)^{-1}K(g0,\omega)\{\overline{\det(g^{-1})'(\omega)}\}^{-\frac{\nu}{d+1}}\,d\sigma(\omega)$$
$$= \{\overline{\det g'(0)}\}^{\frac{\nu}{d+1}}\{\det(g^{-1})'(z)\}^{-\frac{\nu}{d+1}}\int_S e_{\lambda,\omega}(z)e_{-\lambda,\bar\omega}(\overline{g0})\,d\sigma(\omega)$$
$$= \{\det(g^{-1})'(0)\}^{\frac{\nu}{d+1}}\{\det(g^{-1})'(z)\}^{-\frac{\nu}{d+1}}\int_S e_{\lambda,\omega}(z)e_{-\lambda,\bar\omega}(\overline{g0})\,d\sigma(\omega).$$

This proves the lemma. ■

3. The discrete parts. In this section we will study the discrete parts via invariant Cauchy–Riemann operators.

By the definition of the projection P_l (see Theorem 2 in Sec. 1) and (1.8), we have

$$P_l f(z) = \mathbf{c}_l\int_S \widetilde{f}(-i(\alpha+1-2l),\omega)e_{-i(\alpha+1-2l),\omega}(z)\,d\sigma(\omega)$$
$$= \mathbf{c}_l\int_B f(w)\int_S e_{i(\alpha+1-2l),\bar\omega}(\overline{w})e_{-i(\alpha+1-2l),\omega}(z)\,d\sigma(\omega)\,d\mu_\alpha(w).$$

Therefore

$$K_l(z,w) = \mathbf{c}_l\int_S e_{i(\alpha+1-2l),\bar\omega}(\overline{w})e_{-i(\alpha+1-2l),\omega}(z)\,d\sigma(\omega)$$

is the reproducing kernel of the space $A_l^{\alpha,2}(B)$. Putting $w = 0$ in the above equality and using Lemma 2.1, we find

$$K_l(z,0) = \mathbf{c}_l F\left(-l, l-\alpha-1; d; -\frac{|z|^2}{1-|z|^2}\right).$$

Moreover, we have

$$K_l(\phi z, \phi w) = \{\det\phi'(z)\}^{\frac{\nu}{d+1}}\{\overline{\det\phi'(w)}\}^{\frac{\nu}{d+1}}K_l(z,w).$$

From this we get the formula for K_l,

$$K_l(z,w) = \mathbf{c}_l \frac{1}{(1-\langle z,w\rangle)^{\alpha+1+d}} F\left(-l, l-\alpha-1; d; -\frac{|\psi_w(z)|^2}{1-|\psi_w(z)|^2}\right),$$

where ψ_w is an automorphism in $\mathrm{SU}(1,d)$ sending 0 to w. We can also write this as

$$K_l(z,w) = \mathbf{c}_l \frac{1}{(1-\langle z,w\rangle)^{\alpha+1+d}} F\left(-l, l-\alpha-1; d; 1-\frac{|1-\langle z,w\rangle|^2}{(1-|z|^2)(1-|w|^2)}\right).$$

If $l=0$, this is the Bergman kernel. So $A_0^{\alpha,2}(B)$ is the weighted Bergman space $A^{\alpha,2}(B)$. The reproducing kernel $K_1(z,w)$ can be further written out explicitly,

$$\begin{aligned} K_1(z,w) &= \mathbf{c}_1 \frac{1}{(1-\langle z,w\rangle)^{\alpha+1+d}} F\left(-1, -\alpha; d; 1-\frac{|1-\langle z,w\rangle|^2}{(1-|z|^2)(1-|w|^2)}\right) \\ &= \mathbf{c}_1 \frac{1}{(1-\langle z,w\rangle)^{\alpha+1+d}} \left(\frac{d+\alpha}{d} - \frac{\alpha}{d}\frac{|1-\langle z,w\rangle|^2}{(1-|z|^2)(1-|w|^2)}\right). \end{aligned}$$

The spaces in question can be constructed from invariant Cauchy–Riemann operators. Define the operator $\overline{D}^j$,

$$\overline{D}^j f(z) = \sum_{k=0}^{d} \frac{\partial \psi_j(w)}{\partial w_k} \frac{\partial (f\circ\psi)(w)}{\partial \overline{w}_k}\bigg|_{w=0}, \tag{3.1}$$

where ψ is an element in $\mathrm{SU}(1,d)$ such that $\psi 0 = z$. This operator is calculated in [9]; one finds

$$\overline{D}^j f(z) = (1-|z|^2)\left(\frac{\partial f}{\partial \overline{z}_j} - z_j \overline{R} f\right), \quad j=1,\ldots,d.$$

From the definition we see that the operators have the following intertwining relations with the group action:

$$(\overline{D}^j f)(\psi z)\{\det\psi'(z)\}^{\frac{\nu}{d+1}} = \sum_{k=0}^{d} \overline{\frac{\partial \psi_j(z)}{\partial w_k}} \overline{D}^k (f(\psi z)\{\det\psi'(z)\}^{\frac{\nu}{d+1}}).$$

This is why we say that they are invariant. Define

$$\mathcal{B}_l = \Big(\bigcap_{j_1,\ldots,j_l=1}^{d} \mathrm{Ker}(\overline{D}^{j_1}\ldots\overline{D}^{j_l})\Big) \cap L^2(B,\mu_\alpha).$$

By the intertwining property we see that the $\mathcal{B}_l$ are invariant subspaces of $L^2(B,\mu_\alpha)$ under the group action. Let

$$\mathcal{A}_l = \mathcal{B}_{l+1} \ominus \mathcal{B}_l, \quad l=0,1,\ldots,$$

where we let $\mathcal{B}_0 = 0$.

Next we prove that $\mathcal{A}_l = A_l^{\alpha,2}$. We note that from the reproducing property it is easy to see that $A_l^{\alpha,2}$ is an irreducible component of the group action. If $f\in\mathcal{A}_0$, then

$$(1-|z|^2)\left(\frac{\partial f}{\partial \overline{z}_j} - z_j \overline{R} f\right) = 0, \quad j=0,1,\ldots,d.$$

Multiplying by $\overline{z}^j$ and summing up, we get $\overline{R}f=0$. Therefore the equations become $\partial f/\partial\overline{z}_j = 0$, that is, $f\in A_0^{\alpha,2}(B)$. It is also obvious that $A_0^{\alpha,2}(B)\subset\mathcal{A}_0$. So $\mathcal{A}_0 = A_0^{\alpha,2}(B)$.

Now if $f\in\mathcal{B}_1$, then

$$\overline{D}^i \overline{D}^j f = 0, \quad i,j=1,\ldots,d.$$

By the above calculation, we know that $\overline{D}^i f = g_i$, for some analytic functions g_i, that is,

$$(1-|z|^2)\left(\frac{\partial}{\partial \overline{z}_i} - z_i\overline{R}\right) f = g_i. \tag{3.2}$$

Multiplying by $\overline{z}^i$ and summing, we get

$$\overline{R} f = \frac{1}{(1-|z|^2)^2}\sum_{i=1}^{d} \overline{z}_i g_i.$$

Hence the above equations become

$$\frac{\partial f}{\partial \overline{z}^i} = \frac{g_i}{1-|z|^2} + \frac{z_i}{(1-|z|^2)^2}\sum_{j=1}^{d}\overline{z}_j g_j.$$

This can be solved by

$$f = \sum_{j=1}^{d} \frac{\overline{z}_j}{1-|z|^2} g_j + g_0,$$

where g_0 is an arbitrary analytic function.

By induction, we can prove that if $f\in\mathcal{B}_l$, then f has the form

$$f = \sum_{|I|=l} \frac{\overline{z}^I}{(1-|z|^2)^l} g_I + \sum_{|I|=l-1} \frac{\overline{z}^I}{(1-|z|^2)^{l-1}} g_I + \ldots + g_0, \tag{3.3}$$

where we write $I=(i_1,\ldots,i_d)$, $z^I = z_1^{i_1}\ldots z_d^{i_d}$, $|I| = i_1+\ldots+i_d$, all the i's are nonnegative integers and all the g_I are analytic functions. Therefore if $f\in\mathcal{A}_l = \mathcal{B}_l\ominus\mathcal{B}_{l-1}$, then

$$f = \sum_{|I|=l} \frac{\overline{z}^I}{(1-|z|^2)^l} g_I + h,$$

where h is a function in $\mathcal{B}_{l-1}$ uniquely (linearly) determined by $\{g_I : |I| = l\}$. Since $\mathcal{A}_l$ is a closed subspace of $L^{\alpha,2}(B)$, and g_I is uniquely determined by $f \in \mathcal{A}_l$, we see that (by the Banach theorem) $g_I \in A^{\alpha-2l,2}(B)$. From the intertwining property we see that the group action on $\mathcal{A}_l$ induces an action on the tensor fields g_I: if $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SU}(1,d)$, $A = (a_{ij})_{d\times d}$, $B = (b_i)_{d\times 1}$, then

$$g_J \mapsto \sum_{|I|=l} (\overline{a}_{i_1 j_1} + b_{i_1} z_{j_1}) \ldots (\overline{a}_{i_l j_l} + b_{i_l} z_{j_l}) g_I(\phi z) \{\det \phi'(z)\}^{\frac{\nu - l}{d+1}} .$$

Using similar arguments to Lemma 13.1.2 in Rudin [10], it is not difficult to see that $\mathcal{A}_l$ constitutes an irreducible component.

Now if $f \in A_l^{\alpha,2}(B)$, let us show that $f \in \mathcal{A}_l$. Without loss of generality, we may assume that $f \in P_l\mathcal{D}(B)$, so that by Theorem 2 in Sec. 1,

$$f(z) = \int_S h(\omega) e_{-i(\alpha+1-2l),\omega}(z)\, d\sigma(\omega). \tag{3.4}$$

However, by (1.1)

$$\begin{aligned} e_{-i(\alpha+1-2l),\omega}(z) &= \frac{|1 - \langle z, \omega\rangle|^{2l}}{(1 - |z|^2)^l} \frac{1}{(1 - \langle z, \omega\rangle)^\nu} \\ &= \frac{1}{(1 - \langle z, \omega\rangle)^{\nu - l}} \sum_{j=0}^{l} \binom{l}{j} (-1)^j \frac{1}{(1 - |z|^2)^l} \sum_{|I|=j} \omega^I \overline{z}^I , \end{aligned}$$

and by the binomial theorem we have

$$\frac{1}{(1 - |z|^2)^l} = \frac{1}{(1 - |z|^2)^j} + \binom{l-j}{1} \frac{|z|^2}{(1 - |z|^2)^{j+1}} + \ldots + \frac{|z|^{2(l-j)}}{(1 - |z|^2)^l} ,$$

for every $j \le l$. Using this we see that f has the form (3.3). Hence $f \in \mathcal{B}_l$. That is, $A_l^{\alpha,2}(B) \subset \mathcal{B}_l$. An induction argument now reveals that indeed $A_l^{\alpha,2}(B) \subset \mathcal{A}_l$. Since $A_l^{\alpha,2}(B)$ is irreducible we have $A_l^{\alpha,2}(B) = \mathcal{A}_l$. This proves our claim. ■

From this characterization we can find a basis for the space $A_l^{\alpha,2}(B)$. The space $\mathcal{B}_l$ is generated by the functions

$$\frac{\overline{z}^J}{(1 - |z|^2)^{|J|}} z^I , \quad |J| = 0, 1, \ldots, l .$$

Writing

$$\frac{\overline{z}^J}{(1 - |z|^2)^{|J|}} z^I = \frac{|z^J|^2}{(1 - |z|^2)^{|J|}} z^{I-J} ,$$

we observe that if $I' \ne I''$ then the two functions $|z^J|^2 z^{I'}/(1 - |z|^2)^{I'}$ and $|z^J|^2 z^{I''}/(1 - |z|^2)^{|J|}$ are orthogonal. So it is a question of orthogonalizing

the following system of functions in $L^2(B, \mu_\alpha)$:

$$z^I, \left\{ \frac{|z^J|^2}{1 - |z|^2} z^I \right\}_{|J|=1}, \ldots, \left\{ \frac{|z^J|^2}{(1 - |z|^2)^{|J|}} z^I \right\}_{|J|=l},$$

where $I = (i_1, \ldots, i_d)$ is fixed, $i_1, \ldots, i_d \ge -l$ are integers and $|I| \ge -l$. For the negative indices, say $i_1 < 0$, we have to remove functions in the above list that are not in $L^2(B, \mu_\alpha)$, i.e., the terms $|z^J|^2 z^I/(1 - |z|^2)^{|J|}$ for which $j_1 < i_1$. Changing variables

$$t_j = \frac{|z_j|^2}{1 - |z|^2} , \quad j = 1, \ldots, d ,$$

we are faced with the problem of orthogonalizing the system

$$t^S , \quad S = (s_1, \ldots, s_d) , \ |S| \le k , \tag{3.5}$$

on $\mathbb{R}_+^d$ with respect to the measure

$$t^I \Big(1 + \sum_{m=1}^{d} t_m \Big)^{-(|I|+\alpha+d+1)} dm(t) , \tag{3.6}$$

where $dm(t)$ is the Lebesgue measure on $\mathbb{R}_+^d$. For multi-indices I and J, we set $I! = i_1! \ldots i_d!$, $(I)_J = (i_1)_{j_1} \ldots (i_d)_{j_d}$. We define polynomials $P_{I,S}$ by a Rodrigues type formula

$$\begin{aligned} P_{I,S}(t) = {} & \frac{I!}{(S+I)!} t^{-I} \Big(1 + \sum_{m=1}^{d} t_m \Big)^{|I|+\alpha+d+1} \\ & \times \Big(\frac{\partial}{\partial t} \Big)^S \Big[t^{I+S} \Big(1 + \sum_{m=1}^{d} t_m \Big)^{|S|-\alpha-d-1-|I|} \Big] , \end{aligned}$$

where $(\partial/\partial t)^S$ is interpreted in the obvious way, that is, $(\partial/\partial t)^S = (\partial/\partial t_1)^{s_1} \ldots (\partial/\partial t_d)^{s_d}$. It is easy to check that if $|S| \ne |S'|$, then $P_{I,S}$ and $P_{I,S'}$ are orthogonal. But in general they are not orthogonal if $|S| = |S'|$ (see Erdélyi [3], Vol. 2, p. 270). Direct calculation gives for $|S| = l$

$$P_{I,S}(t) = \sum_{j_1=0}^{s_1} \ldots \sum_{j_d=0}^{s_d} \frac{(-S)_J (\alpha + 1 + d + |I| - l)_{|J|}}{(i_1 + 1)_{j_1} \ldots (i_d + 1)_{j_d} J!} t^J \Big(1 + \sum_{m=1}^{d} t_m \Big)^{l-|J|} .$$

This formula is only valid when the indices $i_1, \ldots, i_d$ are nonnegative. For general I, $|I| \ge -l$, the above formula should read

$$\begin{aligned} P_{I,S}(t) = {} & \sum_{j_1=\max(0,-i_1)}^{s_1} \ldots \sum_{j_d=\max(0,-i_d)}^{s_d} \frac{(-S)_J (\alpha + 1 + d + |I| - l)_{|J|}}{(i_1 + 1)_{j_1} \ldots (i_d + 1)_{j_d} J!} \\ & \times t^J \Big(1 + \sum_{m=1}^{d} t_m \Big)^{l-|J|} . \end{aligned}$$

Therefore we get a basis $\{Q_{I,S}(z)z^I : |S| = l,\ |I| \ge -l,\ i_1,\ldots,i_d \ge -l\}$ in $A_l^{\alpha,2}(B)$, where

$$Q_{I,S}(z) = (1-|z|^2)^{-l} \times \sum_{j_1=\max(0,-i_1)}^{s_1} \cdots \sum_{j_d=\max(0,-i_d)}^{s_d} \frac{(-S)_J(\alpha+1+d+|I|-l)_{|J|}}{(i_1+1)_{j_1}\ldots(i_d+1)_{j_d}J!}|z_1^{j_1}|^2\ldots|z_d^{j_d}|^2.$$

We have been unable to find such an explicit formula for orthogonal polynomials of several variables with respect to the measure (3.6). Still, the above basis may be good enough to study the Hankel operator from the Bergman space $A^{\alpha,2}(B)$ to the space $A_l^{\alpha,2}(B)$. We will not study this here.

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Rank and spectral multiplicity

by

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Abstract. For a dynamical system (X,T,μ), we investigate the connections between a metric invariant, the rank $r(T)$, and a spectral invariant, the maximal multiplicity $m(T)$. We build examples of systems for which the pair $(m(T),r(T))$ takes values (m,m) for any integer $m \ge 1$ or $(p-1,\ p)$ for any prime number $p \ge 3$.

Introduction. Given a measure-preserving dynamical system (X,T,μ) there is a corresponding Hilbert space automorphism, namely the action of $U_TF = F \circ T$ on the space $L^2(X,\mu)$. The link between these so-called *metric* and *spectral* structures is still only partially known. The spectral structure, of course, is completely defined by the *maximal spectral type* and the *multiplicity function* of the operator U_T. One particular invariant that we shall study here is the *maximal spectral multiplicity* $m(T)$ (see I.5).

Now a metric invariant closely related to $m(T)$ is the *rank* $r(T)$, introduced by Chacon [Cha1], though named only in [ORW]. The first known systems with $m(T) = 1$ (simple spectrum) were of rank one (this including the well-known discrete spectrum systems).

In general $m(T) \le r(T)$ [Cha2]. The nontrivial result of [Rob1], that there exist systems with any given value of $m(T)$, uses systems of finite rank. Also, the rare examples of finite multiplicity where the maximal spectral type is Lebesgue (plus a discrete or singular continuous part) fall into this category [Age], [Lem], [MaNa], [Que].

The question of which values the pair $(m(T),r(T))$ may take was asked by M. Mentzen [Men1]. He conjectured that each pair (j,n), $j \le n$, may be obtained. The pair $(1,1)$ was obtained by Chacon [Cha1], $(1,2)$ by del Junco [delJ], $(1,n)$ by Mentzen [Men1], $(1,\infty)$ by Ferenczi [Fer1], $(2,n)$ by

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