STUDIA MATHEMATICA 102 (1) (1992)

Linear topological properties of the Lumer–Smirnov class of the polydisc

by

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Abstract. Linear topological properties of the Lumer–Smirnov class $LN_*(\mathbb{U}^n)$ of the unit polydisc $\mathbb{U}^n$ are studied. The topological dual and the Fréchet envelope are described. It is proved that $LN_*(\mathbb{U}^n)$ has a weak basis but it is nonseparable in its original topology. Moreover, it is shown that the Orlicz–Pettis theorem fails for $LN_*(\mathbb{U}^n)$.

1. Introduction. Let $\varphi : [-\infty, \infty) \to [0, \infty)$ be a nondecreasing convex function not identically 0 and let f be a holomorphic function on the unit disc $\mathbb{U}$ in the complex plane $\mathbb{C}$. It is well known that the following assertions are equivalent:

(Hφ1) $$\|f\|_\varphi = \sup_{0<r<1} \int_{-\pi}^{\pi} \varphi(\log|f(re^{it})|)\,dt < \infty,$$

(Hφ2) $$\varphi(\log|f|) \le u \quad \text{for some } u \text{ harmonic in } \mathbb{U}.$$

The family of all f satisfying (Hφ1) or (Hφ2) is called the *Hardy class* H_φ. In particular, if $\varphi(t) = t^+ = \max\{t, 0\}$ then H_φ is the *Nevanlinna class* $N(\mathbb{U})$ of the unit disc while the set-theoretical sum of all H_φ with φ *strongly convex* (i.e., $\varphi(t)/t \to \infty$ as $t \to \infty$) is the *Smirnov class* $N_*(U)$ of the unit disc. It is known that $N(\mathbb{U})$ equipped with the topology defined by the metric

$$d(f,g) = \sup_{0<r<1} \int_{-\pi}^{\pi} \log(1 + |(f-g)(re^{it})|)\,dt$$

is a topological vector group and $N_*(\mathbb{U})$ is the largest linear subspace in $N(\mathbb{U})$ which is a topological vector space in the relative topology (see [24]). The topological vector space structure of the Smirnov class $N_*(\mathbb{U})$ was extensively studied by N. Yanagihara (see [25, 26]).

When one passes to several complex variables, say to the polydisc $\mathbb{U}^n$ in $\mathbb{C}^n$, then it is possible to use in (Hφ2) at least two nonequivalent types

1991 *Mathematics Subject Classification*: 46E10, 46J15, 32A35.

of harmonicity; namely n-harmonicity and pluriharmonicity. The first one leads to the standard Hardy, Nevanlinna and Smirnov classes of the polydisc (see [18]). Many functional-analytic properties of the Smirnov class $N_*(\mathbb{U}^n)$ were described in [12].

The present paper is devoted to a study of the topological vector space structure of the Lumer–Smirnov class $LN_*(\mathbb{U}^n)$ of the unit polydisc $\mathbb{U}^n$, i.e., the space of holomorphic functions on $\mathbb{U}^n$ defined just like $N_*(\mathbb{U}^n)$ but with u in (Hφ2) being pluriharmonic. $LN_*(\mathbb{U}^n)$ equipped with the vector topology defined by the F-norm

$$|||f||| = \inf\{u(0) : u \text{ is pluriharmonic and } \log(1+|f|) \le u\}$$

is an F-space (complete metrizable t.v.s.).

The paper is organized as follows. In the next section we collect basic definitions and notation, and we show a few equivalent descriptions of the Lumer–Smirnov classes $LN_*(\Omega)$ of arbitrary balanced bounded simply connected domains Ω in $\mathbb{C}^n$. We observe that, as in the case of the disc, $LN_*(\Omega)$ is the largest linear subspace of $LN(\Omega)$ which is a topological vector space in the relative topology.

In Section 3 we prove that each $f \in LN_*(\Omega)$ can be written in the form $f = h\cdot k$, where $k = e^g$, h, g are holomorphic, h is bounded, $k \in LN_*(\Omega)$ and $\operatorname{Re} g > 1$. This factorization theorem is crucial for the rest of the paper but it has also some independent interest. In particular, it implies that the modulus of any function f in $LN_*(\Omega)$ can be majorized by the modulus of another function k in $LN_*(\Omega)$ which is zero-free. An example due to W. Rudin ([17], Example) shows that a similar majorization is impossible with f, k belonging to the Lumer–Hardy space $LH_p(\Omega)$ at least when $p = 2/m$, $m \in \mathbb{N}$.

In Section 4 we prove that if $n > 1$ then $LN_*(\mathbb{U}^n)$ contains an isomorphic copy of the Banach space ℓ^∞ of all bounded complex sequences. In particular, this implies that $LN_*(\mathbb{U}^n)$ is nonseparable and so the space of all polynomials is not dense in $LN_*(\mathbb{U}^n)$.

In Section 5 we describe all continuous linear functionals on the closed subspace $LN_0(\mathbb{U}^n)$ of $LN_*(\mathbb{U}^n)$ spanned by the polynomials. Moreover, we construct the Fréchet envelope of $LN_0(\mathbb{U}^n)$, i.e., a Fréchet space $LF_*(\mathbb{U}^n)$ of holomorphic functions on $\mathbb{U}^n$ which contains $LN_0(\mathbb{U}^n)$ as a dense subspace and has the same topological dual as $LN_0(\mathbb{U}^n)$. These results form a very important step in the proof of our main theorem (Theorem 6.1) which states that the space of all polynomials is always weakly dense in $LN_*(\mathbb{U}^n)$. This theorem allows us to identify the dual space of $LN_0(\mathbb{U}^n)$ with the dual of $LN_*(\mathbb{U}^n)$ as well as to show that $LF_*(\mathbb{U}^n)$ is the Fréchet envelope of the entire space $LN_*(\mathbb{U}^n)$. In addition, it turns out that $LN_*(\mathbb{U}^n)$ is an F-space with separating dual which has a weak Schauder basis though it is

nonseparable in its own topology. This phenomenon was first observed in weak-L_p sequence spaces $l(p,\infty)$ for $0 < p < 1$ (see [11]) and, of course, it is possible because of the failure of the Hahn–Banach theorem in these settings. N. J. Kalton [6] proved that the Hahn–Banach theorem must fail in every F-space X which is not locally convex. He also showed that if, moreover, X is separable and has a separating dual then it contains a proper closed weakly dense (PCWD) subspace [7]. $LN_*(\mathbb{U}^n)$ is nonseparable but it still has a PCWD-subspace which is very regular. We show that $LN_0(\mathbb{U}^n)$ is a Möbius invariant PCWD-subspace of $LN_*(\mathbb{U}^n)$.

Theorem 6.1 has other interesting consequences. Using it we prove that if $n \ne m$ then $LN_*(\mathbb{U}^n)$ is not isomorphic to $LN_*(\mathbb{U}^m)$. Moreover, we show that the copy of ℓ^∞ constructed in Section 4 as well as every infinite-dimensional locally bounded subspace of $LN_*(\mathbb{U}^n)$ must be uncomplemented.

We finish the paper by proving that if $n > 1$ then there exists a series in $LN_*(\mathbb{U}^n)$ which is weakly subseries convergent but is not convergent in the original topology of $LN_*(\mathbb{U}^n)$, i.e., $LN_*(\mathbb{U}^n)$ does not have the Orlicz–Pettis property (OPP). Thus, contrary to the case of separable F-spaces with separating duals (see [5]), the Orlicz–Pettis theorem fails in our setting. The author proved in [11, 15] that if $0 < p < 1$ and $n > 1$ then also the Lumer–Hardy space $LH_p(\mathbb{B}_n)$ of the unit ball in $\mathbb{C}^n$ as well as the weak-L_p sequence space $l(p,\infty)$ do not have the OPP.

2. Preliminaries.

Let Ω be a bounded balanced simply connected domain in $\mathbb{C}^n$ and let B be its Bergman–Shilov boundary. Throughout this paper $H(\Omega)$ will denote the space of all functions holomorphic in Ω while $RP(\Omega)$ will be the set of all pluriharmonic functions in Ω (= real parts of functions in $H(\Omega)$). $H^\infty(\Omega)$ will denote the space of all bounded functions belonging to $H(\Omega)$ and $A(\Omega)$ the space of all functions holomorphic in Ω and continuous on $\overline{\Omega}$.

Let $\varphi : [-\infty,\infty) \to [0,\infty)$ be a nondecreasing convex function not identically 0 such that $\varphi(-\infty) = 0$. The *Lumer–Hardy class* $LH_\varphi(\Omega)$ is defined to consist of all $f \in H(\Omega)$ such that $\varphi(\log|f|) \le u$ for some $u \in RP(\Omega)$ (see [10]). Taking $\varphi(t) = \exp(t^p)$, $0 < p < \infty$, we obtain the standard Lumer–Hardy space $LH_p(\Omega)$.

The *Lumer–Nevanlinna class* $LN(\Omega)$ is the space $LH_\varphi(\Omega)$ defined by the function $\varphi(t) = t^+ = t \vee 0$. For each $f \in LN(\Omega)$ define

$$|||f||| = \inf\{u(0) : u \in RP(\Omega),\ \log(1+|f|) \le u\}.$$

It is easily seen that $|||f|||$ is finite for each $f \in LN(\Omega)$, $LN(\Omega)$ is a vector space, and $d(f,g) = |||f-g|||$ is a translation invariant metric on $LN(\Omega)$.

In fact, the metric d defines on $LN(\Omega)$ the group topology ν which has a base at zero consisting of balanced sets (i.e., $(LN(\Omega),\nu)$ is a topological vector group). It was shown in [10] (see also [20], Chap. 7) that

$$|||f||| = \sup_{r\in(0,1)} \sup_{\varrho\in M_0(B)} \int_B \log(1+|f(r\zeta)|)\,d\varrho(\zeta)$$

for each $f \in LN(\Omega)$, where $M_0(B)$ is the set of all probability Borel measures ϱ on B such that $f(0) = \int_B f(\zeta)\,d\varrho(\zeta)$ for all $f \in A(\Omega)$.

The *Lumer–Smirnov class* $LN_*(\Omega)$ is the set-theoretical sum of all Lumer–Hardy classes $LH_\varphi(\Omega)$ with φ strongly convex. In the sequel we give two other descriptions of $LN_*(\Omega)$.

We will say that a set Λ of Borel functions on B is $M_0(B)$-*uniformly integrable* whenever

(a) there exists a $K > 0$ such that $\int_B |f|\,d\varrho \le K$ for all $f \in \Lambda$ and $\varrho \in M_0(B)$, and

(b) for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\int_E |f|\,d\varrho < \varepsilon$ for every Borel set $E \subseteq B$ such that $\varrho(E) < \delta$, $f \in \Lambda$, and $\varrho \in M_0(B)$.

LEMMA 2.1. *A set Λ of Borel functions on B is $M_0(B)$-uniformly integrable if and only if there are a $K > 0$ and a strongly convex function φ such that $\int_B \varphi(|f|)\,d\varrho \le K$ for all $f \in \Lambda$ and $\varrho \in M_0(B)$.*

The proof of this lemma is essentially the same as the proof of Theorem 3.1.2 in [18] and so it may be omitted.

PROPOSITION 2.2. *For an arbitrary subset $\mathcal{G}$ of $LN(\Omega)$ the following assertions are equivalent:*

(a) *$\mathcal{G}$ is absorbed by each neighbourhood of zero in $LN(\Omega)$;*

(b) *the family $\{\log(1+|f_r|) : r \in (0,1), f \in \mathcal{G}\}$ is $M_0(B)$-uniformly integrable, where f_r is the function on B defined by $f_r(\zeta) = f(r\zeta)$;*

(c) *there exists a $K > 0$ and a strongly convex function φ such that $\int \varphi(\log|f_r|)\,d\varrho \le K$ for all $r \in (0,1)$ and $\varrho \in M_0(B)$.*

Proof. The equivalence (b)⇔(c) follows immediately from Lemma 2.1 while (a)⇔(b) can be proved in a similar way to [2], Theorem 4.1 (see also [13], Proposition 1.2).

COROLLARY 2.3. (a) $LN_*(\Omega) = \{f \in LN(\Omega) : \lim_{a\to 0} |||af||| = 0\} = \{f \in LN(\Omega) :$ *the family* $\{\log(1+|f_r|) : r \in (0,1)\}$ *is $M_0(B)$-uniformly integrable*$\}$.

(b) *$LN_*(\Omega)$ is the largest linear subspace of $LN(\Omega)$ which is a topological vector space in the relative topology.*

(c) *$LN_*(\Omega)$ is a topological algebra.*

Proof. (a), (b) It is easy to see that a locally balanced group topology τ on a vector space X is linear if and only if every element $x \in X$ is absorbed by each τ-neighbourhood of zero in X. Thus (a) and (b) follow easily from Proposition 2.2.

(c) The inequality $\log(1+ts) \le \log(1+t)+\log(1+s)$, $t, s \ge 0$, implies that $LN_*(\Omega)$ is an algebra and that the multiplication operation $\mathcal{M}$ is continuous at the point $(0,0)$. However, $LN_*(\Omega)$ is also a topological vector space, so $\mathcal{M}$ is continuous at every point of $LN_*(\Omega) \times LN_*(\Omega)$.

3. A factorization theorem

THEOREM 3.1. *For each $f \in LN_*(\Omega)$ there are $b \in H^\infty(\Omega)$ and $g \in H(\Omega)$ such that*

(a) $f = b \cdot e^g$,

(b) $\operatorname{Re} g > 1$,

(c) $e^g \in LN_*(\Omega)$.

For any sequence $\mathcal{P} = \{p_k\} \subset (0,1)$ we define the function

$$(*)\qquad \mathcal{G}_{\mathcal{P}}(z) = \sum_{k=1}^{\infty} 2^{-k} z^{p_k} \quad \text{for } z \in \mathbb{C},\ \operatorname{Re} z > 1\,.$$

For the proof of Theorem 3.1 we need some lemmas.

LEMMA 3.2. *For every positive, strongly convex, and strictly increasing function φ on $[1,\infty)$ satisfying $\varphi(t) \ge t$ there exists a sequence $\mathcal{P} = \{p_k\} \subset (0,1)$ such that*

$$\varphi^{-1}(\operatorname{Re} z) \le \operatorname{Re} \mathcal{G}_{\mathcal{P}}(z) \quad \textit{for all } z \in \mathbb{C},\ \operatorname{Re} z > 1\,.$$

Proof. Since φ is strictly increasing and strongly convex, φ^{-1} exists and is strongly concave, i.e., $\varphi^{-1}(t)/t \to 0$ as $t \to \infty$. Let $\{t_k\}_{k=0}^{\infty}$ be a strictly increasing sequence of positive numbers such that $t_0 = 1$ and $\varphi^{-1}(t) \le 2^{-k-1}t$ for all $t \ge t_k$, $k = 1, 2, \ldots$ We can find a sequence $\mathcal{P} = \{p_k\} \subset (0,1)$ such that $t^{p_k} \ge t/2$ for all $1 \le t \le t_{k+1}$, $k = 1, 2, \ldots$ Let $\mathcal{G}_{\mathcal{P}}$ be the function associated with $\mathcal{P}$ according to $(*)$. Fix $z \in \mathbb{C}$ with $t_k < \operatorname{Re} z \le t_{k+1}$. Using the inequality $(\operatorname{Re} z)^p \le \operatorname{Re} z^p$, which holds for all $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$ and any $p \in (0,1)$, we obtain

$$\begin{aligned}\varphi^{-1}(\operatorname{Re} z) &\le 2^{-k-1}\operatorname{Re} z \le 2^{-k}(\operatorname{Re} z)^{p_k} \le 2^{-k}\operatorname{Re} z^{p_k}\\ &\le \sum_{k=1}^{\infty} 2^{-k}\operatorname{Re} z^{p_k} = \operatorname{Re}\Big(\sum_{k=1}^{\infty} 2^{-k} z^{p_k}\Big) = \operatorname{Re}\mathcal{G}_{\mathcal{P}}(z)\,.\end{aligned}$$

The proof is finished.

LEMMA 3.3. *Let $h \in H(\Omega)$, $\operatorname{Re} h > 1$. For every sequence $\mathcal{P} = \{p_k\} \subset (0,1)$ the function $\exp(\mathcal{G}_{\mathcal{P}} \circ h)$ belongs to $LN_*(\Omega)$.*

Proof. Observe first that $\exp(h^p) \in LN_*(\Omega)$ for every $p \in (0,1)$. Indeed, fix $p \in (0,1)$ and choose $q > 1$ such that $qp < 1$. Let ψ be any nonincreasing convex function on $[-\infty, \infty)$ such that $\psi(t) = t^q$ for $t \in [1, \infty)$. Then ψ is strongly convex and

$$\psi(\log|\exp(h^p)|) = (\operatorname{Re} h^p)^q \le |h^p|^q = |h^{pq}| \le C_{pq} \operatorname{Re} h^{pq},$$

where C_{pq} is a positive constant which depends only on pq (the range of h^{pq} lies in the wedge $\{|\arg z| < \pi pq/2\}$). Consequently, $\exp(h^p) \in LH_\psi \subset LN_*$.

Let now $\mathcal{P} = \{p_k\}$ be an arbitrary sequence contained in $(0,1)$ and set $\mathcal{G} = \mathcal{G}_{\mathcal{P}} \circ h$. Each function $\operatorname{Re} h^{p_k}$, $k = 1, 2, \ldots$, is pluriharmonic, so

$$\sup_k \Big\{ \int_B \operatorname{Re} h^{p_k}(r\zeta)\, d\varrho(\zeta) : \varrho \in M_0(\Omega),\ r \in (0,1) \Big\} = \sup_k \operatorname{Re} h^{p_k}(0) = K < \infty.$$

For each $r \in (0,1)$ define $A_r = \{\zeta \in B : \operatorname{Re} \mathcal{G}(r\zeta) \ge 1\}$ and $C_r = B \setminus A_r$. Then, using the inequality $\log(1+x) \le 2x$, which holds for each $x \ge 1$, we obtain

$$\begin{aligned}
&\int_E \log(1 + |(e^{\mathcal{G}})_r|)\, d\varrho \\
&\le \int_{E\cap A_r} \log(1 + |(e^{\mathcal{G}})_r|)\, d\varrho + \int_{E\cap C_r} \log(1 + |(e^{\mathcal{G}})_r|)\, d\varrho \\
&\le 2 \int_E \operatorname{Re} \mathcal{G}_r\, d\varrho + \varrho(E) \log 2 \\
&\le 2 \sum_{j=1}^{m} 2^{-j} \int_E \log(1 + |(\exp(h^{p_j}))_r|)\, d\varrho \\
&\quad + 2 \sum_{j=m+1}^{\infty} 2^{-j} \int_E \operatorname{Re} h^{p_j}\, d\varrho + \varrho(E) \log 2 \\
&\le \sum_{j=1}^{m} 2^{-j+1} \int_E \log(1 + |(\exp(h^{p_j}))_r|)\, d\varrho + K 2^{-m+1} + \varrho(E) \log 2
\end{aligned}$$

for all $r \in (0,1)$, $\varrho \in M_0(\Omega)$, and $m = 1, 2, \ldots$ We know that each function $\exp(h^{p_j})$, $j = 1, 2, \ldots$, belongs to $LN_*(\Omega)$, so we see that the family $\{\log(1 + |(e^{\mathcal{G}})_r|) : r \in (0,1)\}$ is $M_0(B)$-uniformly integrable. Finally, by Proposition 2.2, $e^{\mathcal{G}} \in LN_*(\Omega)$.

Proof of Theorem 3.1. Fix $f \in LN_*$. There is a strongly convex φ such that $f \in LH_\varphi$. Thus, $\varphi(\log|f|) \le \operatorname{Re} h$ for some $h \in H(\Omega)$. We may assume that $\varphi(t) \ge t$ for all $t \ge 1$ and that $\operatorname{Re} h > 1$. Therefore, by Lemma 3.2, there is a sequence $\mathcal{P} = \{p_k\} \in (0,1)$ such that $\varphi^{-1}(\operatorname{Re} z) \le \operatorname{Re} \mathcal{G}_{\mathcal{P}}(z)$ for all $z \in \mathbb{C}$, $\operatorname{Re} z > 1$. Consequently, taking $g = \mathcal{G}_{\mathcal{P}} \circ h$ we obtain

$$|f| \le \exp(\varphi^{-1}(\operatorname{Re} h)) \le \exp(\operatorname{Re} \mathcal{G}_{\mathcal{P}} \circ h) = |e^g|.$$

Finally, $f = b \cdot e^g$, where $b = fe^{-g} \in H^\infty(\Omega)$ and, by Lemma 3.3, $e^g \in LN_*(\Omega)$.

Remark 3.4. In [17] it was shown that for any $p > 0$, $\varepsilon > 0$ and any $f \in LH_p(\Omega)$ there is a zero-free function $g \in LH_{p-\varepsilon}(\Omega)$ such that $|f| \le |g|$. This theorem and Theorem 3.1 suggest the following problem. Suppose that φ and ψ are strongly convex functions such that $\lim_{t\to\infty} \varphi(t)/\psi(t) = \infty$, and let $f \in LH_\varphi(\Omega)$. Does there exist a zero-free function $g \in LH_\psi(\Omega)$ such that $|f| \le |g|$?

4. The Lumer–Smirnov class of the polydisc. Throughout the rest of the paper we will assume that Ω is the unit polydisc $\mathbb{U}^n$ in $\mathbb{C}^n$, i.e., the n-fold product of the unit disc $\mathbb{U}$ in $\mathbb{C}$. It is convenient to equip $\mathbb{C}^n$ with the supnorm $|\cdot|_\infty$, i.e., $|z|_\infty = \max\{|z_i| : 1 \le i \le n\}$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Moreover, let $\mathbb{T}$ denote the unit circle, dm the normalized Lebesgue measure on $\mathbb{T}$, and $\mathbb{Z}_+$ the set of all nonnegative integers. For a natural number n, $\mathbb{T}^n$, dm_n, $\mathbb{Z}_+^n$ will denote the n-fold products of $\mathbb{T}$, dm, and $\mathbb{Z}_+$ respectively. Obviously, $\mathbb{T}^n$ is the Bergman–Shilov boundary of $\mathbb{U}^n$.

For $f \in H(\mathbb{U}^n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, let $\widehat{f}(\alpha)$ denote the αth Taylor coefficient of f.

If $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $r \in (0,1)$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, we will frequently use the notation $rz := (rz_1, \ldots, rz_n)$, $z^\alpha := z_1^{\alpha_1} \ldots z_n^{\alpha_n}$.

Since for each positive pluriharmonic function u on $\mathbb{U}^n$ and $\zeta \in \overline{\mathbb{U}}^n$ the *slice function* u_ζ, defined by $u_\zeta(\lambda) = u(\zeta\lambda)$, $\lambda \in \mathbb{U}$, is a positive harmonic function on $\mathbb{U}$, we have

$$u_\zeta(\lambda) \le 2u(0)/(1 - |\lambda|). \tag{4.1}$$

Consequently,

$$|f(z)| \le \exp\Big(\frac{2|||f|||}{1 - |z|_\infty}\Big) \quad \text{for each } f \in LN_*(\mathbb{U}) \text{ and } z \in \mathbb{U}^n. \tag{4.2}$$

The above estimate shows that the topology ν of $LN(\Omega)$ is stronger than the compact-open topology κ. This implies easily that $LN_*(\mathbb{U}^n)$ is an F-space whose topological dual separates points.

Proposition 4.1. *Let $n > 1$. There is a sequence $\{f_k\}$ of homogeneous polynomials on $\mathbb{C}^n$ such that:*

(a) *for each sequence $\gamma = \{\gamma_k\}$ in the Banach space ℓ^∞ of all bounded complex sequences the series $\sum \gamma_k f_k$ is κ-convergent to a function $f_\gamma \in H^\infty(\mathbb{U}^n)$, and*

(b) *the linear operator* $\ell^\infty \ni \gamma \to f_\gamma \in LN_*(\mathbb{U}^n)$ *is a topological (into) isomorphism.*

Proof. This result and its proof are close in spirit to [19], Theorem (see also [20], Theorem 7.4.6).

If $n < m$ then $LN_*(\mathbb{U}^n)$ is isomorphic to a subspace of $LN_*(\mathbb{U}^m)$. Therefore, without lost of generality we may assume that $n = 2$.

It is easily seen that if $z, w \in \mathbb{U}$ and $|(z+w)/2|$ is close to 1, then z must be close to w. Consequently, there exists a sequence $\{\varepsilon_j\} \subset (1/2, 1)$, $\lim \varepsilon_j = 1$, such that the sets $V_j = \{(z,w) \in \mathbb{U}^2 : |z + e^{i/j}w| > 2\varepsilon_j\}$, $j = 1, 2, \ldots$, are pairwise disjoint. Find a strictly increasing sequence $\{n_j\}$ of positive integers such that $\varepsilon_j^{n_j} < 2^{-j}$ and define $g_j(z,w) = ((z + e^{i/j}w)/2)^{n_j}$ for $z, w \in \mathbb{U}$ and $j = 1, 2, \ldots$

For each $(z, w) \in \mathbb{U}^2$ there exists at most one set V_k such that $(z, w) \in V_k$. Therefore, for $\gamma = \{\gamma_j\} \in \ell^\infty$ we have

$$\Big|\sum \gamma_j g_j(z,w)\Big| \le \sum_{j\ne k} |\gamma_j| \varepsilon_j^{n_j} + |\gamma_k g_k(z,w)| \le 2\|\gamma\|_\infty .$$

Consequently, the series $\sum \gamma_k g_k$ is κ-convergent to a function g_γ in $H^\infty(\mathbb{U}^n)$ and the mapping $T : \ell^\infty \ni \gamma \to g_\gamma \in H^\infty(\mathbb{U}^n)$ is continuous. This implies immediately that T is a continuous operator from ℓ^∞ into $LN_*(\mathbb{U}^n)$. Observe that the sequence $\{Te_j\} = \{g_j\}$ is not convergent to zero in $LN_*(\mathbb{U}^n)$, where e_j is the jth unit vector in ℓ^∞. Indeed, if we take $\zeta = (1, e^{-i/j}) \in \mathbb{T}^2$ then the slice function $(g_j)_\zeta$ belongs to $N_*(\mathbb{U})$ and, obviously

$$|||g_j||| \ge \sup_r \int_{\mathbb{T}} \log(1 + |g_j(r\zeta\omega)|)\, dm(\omega) = 1 .$$

Applying [1], Corollaire, we can find a subset $M = \{n_j\}$ of $\mathbb{N}$ such that the mapping T restricted to the subspace $\ell^\infty(M)$ ($\approx \ell^\infty$) of ℓ^∞ consisting of all sequences whose supports are contained in M is a topological isomorphism. Taking $f_j = g_{n_j}$ we conclude the proof.

In Section 7 we show that the copy of ℓ^∞ constructed above as well as every infinite-dimensional locally bounded subspace of $LN_*(\mathbb{U}^n)$ must be uncomplemented.

Proposition 4.1 implies that the space $LN_*(\mathbb{U}^n)$ is nonseparable if $n > 1$. In particular, the polydisc algebra is not dense in $LN_*(\mathbb{U}^n)$. Throughout the rest of the paper we will denote by $LN_0(\mathbb{U}^n)$ the closure of $A(\mathbb{U}^n)$ in $LN_*(\mathbb{U}^n)$.

We recall that a set $X \subset H(\mathbb{U}^n)$ is *Möbius invariant* (i.e., invariant with respect to the group $\mathrm{Aut}(\mathbb{U}^n)$ of all holomorphic automorphisms of $\mathbb{U}^n$) if $f \circ \Phi \in X$ whenever $f \in X$ and $\Phi \in \mathrm{Aut}(\mathbb{U}^n)$.

PROPOSITION 4.2. (a) $LN_0(\mathbb{U}^n) = \{f \in LN_*(\mathbb{U}^n) : \nu\text{-}\lim_{r\to 1-} f_r = f\}$.

(b) *The set of all polynomials is dense in* $LN_0(\mathbb{U}^n)$.

(c) $LN_0(\mathbb{U}^n)$ *is Möbius invariant.*

Proof. Easy proofs of (a) and (b) may be omitted.

(c) Fix $\Phi \in \mathrm{Aut}(\mathbb{U}^n)$ and define a linear endomorphism T_Φ of $LN_*(\mathbb{U}^n)$ by $T_\Phi f = f \circ \Phi$. It is obvious that T_Φ is continuous if we equip $LN_*(\mathbb{U}^n)$ with the pointwise convergence topology. This and the closed graph theorem imply that T_Φ is a continuous automorphism of $LN_*(\mathbb{U}^n)$. Since $\mathrm{Aut}(\mathbb{U}^n) \subset A(\mathbb{U}^n)$ we have $T_\Phi(A(\mathbb{U}^n)) \subset A(\mathbb{U}^n)$, and the result follows from the density of $A(\mathbb{U}^n)$ in $LN_0(\mathbb{U}^n)$.

5. Linear functionals on $LN_0(\mathbb{U}^n)$. In this section we describe all continuous linear functionals on $LN_0(\mathbb{U}^n)$ and we construct a Fréchet space (locally convex F-space) of holomorphic functions on $\mathbb{U}^n$ which contains $LN_0(\mathbb{U}^n)$ as a dense subspace and has the same topological dual as $LN_0(\mathbb{U}^n)$. This space is unique up to isomorphism and, in fact, it is isomorphic to the Fréchet envelope of $LN_0(\mathbb{U}^n)$. We recall that the *Fréchet envelope* $\widehat{E}$ of an F-space $E = (E, \tau)$ whose topological dual separates points is the completion of the space (E, τ^c), where τ^c is the strongest locally convex topology on E which is weaker than τ. Obviously, if $\mathcal{B}$ is a base of neighbourhoods of zero for τ, then the family $\{\mathrm{conv}\, U : U \in \mathcal{B}\}$ is a base at zero for τ^c. It is known that τ^c coincides with the Mackey topology of the dual pair (E, E'_τ), where E'_τ is the topological dual of (E, τ). We refer to [9, 21, 22] for information on Fréchet envelopes.

Taking as a model the Fréchet envelope F_* of $N_* = LN_*(\mathbb{U})$ constructed by N. Yanagihara [25], we define the space $LF_*(\mathbb{U}^n) = \{f \in H(\mathbb{U}^n) : \|f\|_k = \sup_\alpha |\widehat{f}(\alpha)| \exp(-|\alpha|^{1/2}/k) < \infty$ for all $k \in N\}$. $LF_*(\mathbb{U}^n)$ equipped with the topology φ defined by the sequence of norms $\{\|\cdot\|_k : k \in \mathbb{N}\}$ is a Fréchet space. Using essentially the same arguments as in [25], Theorem 1 and Theorem 4(ii), one can prove assertions (a) and (b) of the following proposition.

PROPOSITION 5.1. (a) *For each* $f \in LF_*(\mathbb{U}^n)$, $\varphi\text{-}\lim f_r = f$.

(b) $LF_*(\mathbb{U}^n) = \{f \in H(\mathbb{U}^n) : \sup_{z\in\mathbb{U}^n} |f(z)| \exp(c/(1 - |z|_\infty)) < \infty$ *for each* $c > 0\}$.

In particular,

(c) $A(\mathbb{U}^n)$ *is a dense subspace of* $LF_*(\mathbb{U}^n)$;

(d) $LN_*(\mathbb{U}^n) \subset LF_*(\mathbb{U}^n)$ *and the inclusion mapping is continuous.*

In the above proposition, (c) is a direct consequence of (a) while (d) follows from (b) and (4.2).

Observe that the mapping $f \to \{\widehat{f}(\alpha)\}_{\alpha\in\mathbb{Z}_+^n}$ is a linear bijection of $LF_*(\mathbb{U}^n)$ onto the space $\Lambda_1(|\alpha|^{1/2})$ consisting of all complex families $x =$

$\{x(\alpha)\}_{\alpha\in\mathbb{Z}_+^n}$ such that $|x(\alpha)| = O(\exp(c|\alpha|^{1/2}))$ for all $c > 0$. Fixing any bijection j of $\mathbb{Z}_+^n$ onto $\mathbb{Z}_+$ such that $\mathrm{j}(\alpha) < \mathrm{j}(\alpha')$ if $|\alpha| < |\alpha'|$, we introduce an order in $\mathbb{Z}_+^n$. Just like in [14] we can check that the sequence $\{\mathrm{j}(\alpha)\}$ has the same rate of growth as the sequence $\{|\alpha|^n\}$ as $|\alpha| \to \infty$. Consequently, the mapping $x \to x \circ \mathrm{j}^{-1}$ is an isomorphism of $\Lambda_1(|\alpha|^{1/2})$ onto the space $\Lambda_1(j^{1/2n})$ consisting of all complex sequences $x = \{x(j)\}_{j\in\mathbb{Z}_+}$ such that $|x(j)| = O(\exp(cj^{1/2n}))$ for all $c > 0$. Therefore, we have just shown the following result.

PROPOSITION 5.2. *$LF_*(\mathbb{U}^n)$ is isomorphic to the nuclear power series space $\Lambda_1(j^{1/2n})$.*

The reader is referred to [3, 16] for information on nuclear power series spaces.

THEOREM 5.3. *Each continuous linear functional T on the space $LN_0(\mathbb{U}^n)$ or on the space $LF_*(\mathbb{U}^n)$ is of the form*

$$(*)\qquad Tf = T_\lambda f := \sum \widehat{f}(\alpha)\lambda(\alpha)\,,$$

where $f \in LN_0(\mathbb{U}^n)$ or $f \in LF_*(\mathbb{U}^n)$ respectively and $\{\lambda(\alpha)\}_{\alpha\in\mathbb{Z}_+^n}$ is a family of complex numbers such that

$$(**)\qquad \sup_\alpha |\lambda(\alpha)| \exp(|\alpha|^{1/2}/m) < \infty$$

for some $m \in \mathbb{N}$.

For the proof of this theorem we need the following lemma.

LEMMA 5.4. *Let $\mathcal{O}(\mathbb{U}^n)$ be the set of all restrictions to $\mathbb{U}^n$ of functions defined and holomorphic on some neighbourhood of $\overline{\mathbb{U}}^n$ in $\mathbb{C}^n$. Then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\inf_{\alpha\in\mathbb{Z}_+^n} \{\sup |\widehat{f}(\alpha)| \exp(-\delta|\alpha|^{1/2}) : f \in \mathcal{O}(\mathbb{U}^n),\ |||f||| \le \varepsilon\} > 0\,.$$

Proof. For $n = 1$ this lemma was proved in [14], Lemma 3.3 (see also [12], Lemma 4.4).

In general, fix an $\varepsilon > 0$, find $\delta > 0$ and a sequence $\{f_k\} \subset \mathcal{O}(\mathbb{U})$ such that $|\widehat{f}_k(k)| \ge \delta\exp(\delta|k|^{1/2})$ and $|||f_k||| \le \varepsilon$ for $k = 0, 1, 2, \ldots$ For each $\alpha \in \mathbb{Z}_+^n$ choose $i_\alpha \in \{1, \ldots, n\}$ such that $\alpha_{i_\alpha} = \max\{\alpha_i : i = 1, \ldots, n\}$ and set $k_\alpha = \alpha_{i_\alpha}$. Then the formula

$$g_\alpha(z) = f_{k_\alpha}(z_{i_\alpha}) \cdot \prod_{i\ne i_\alpha} z_i^{\alpha_i}$$

defines a function which belongs to $\mathcal{O}(\mathbb{U}^n)$ and $|||g_\alpha||| \le \varepsilon$, for each $\alpha \in \mathbb{Z}_+^n$. Obviously, for every $\alpha \in \mathbb{Z}_+^n$,

$$|\widehat{g}_\alpha(\alpha)| = |\widehat{f}_{k_\alpha}(k_\alpha)| \ge \delta\exp(\delta k_\alpha^{1/2}) \ge \delta\exp(\delta|\alpha|^{1/2}/n)\,.$$

Proof of Theorem 5.3. $LN_0(\mathbb{U}^n)$ is a dense subspace of $LF_*(\mathbb{U}^n)$ and its topology is stronger than the relative topology (see Proposition 5.1). Therefore, every continuous linear functional on $LF_*(\mathbb{U}^n)$ is the extension of a unique continuous linear functional on $LN_0(\mathbb{U}^n)$. If we identify $LF_*(\mathbb{U}^n)$ with $\Lambda_1(|\alpha|^{1/2})$ as we did in our discussion following Proposition 5.1, using [16], Proposition 7.4.8, we easily see that $(*)$ together with $(**)$ describe all continuous linear functionals on $LF_*(\mathbb{U}^n)$. Thus, for the proof of the theorem it is enough to show that every continuous linear functional T on $LN_0(\mathbb{U}^n)$ can be represented by $(*)$ with some sequence $\{\lambda(\alpha)\}$ satisfying $(**)$.

Fix a continuous linear functional T on $LN_0(\mathbb{U}^n)$ and set $\lambda(\alpha) = T(z^\alpha)$ for each $\alpha \in \mathbb{Z}_+^n$. There is an $\varepsilon > 0$ such that

$$(+)\qquad |Tf| \le 1 \quad \text{for each } f \in LN_0(\mathbb{U}^n),\ |||f||| \le \varepsilon\,.$$

For each $f \in \mathcal{O}(\mathbb{U}^n)$ (i.e., f holomorphic in a neighbourhood of $\overline{\mathbb{U}}^n$) define a function T_f on $\mathbb{T}^n$ by $T_f(\zeta) = T(f(\zeta\cdot))$. The Taylor series of $f(\zeta\cdot)$ is uniformly convergent on $\mathbb{U}^n$, so it is convergent in $LN_0(\mathbb{U}^n)$. Consequently, $T(f(\zeta\cdot)) = \sum \widehat{f}(\alpha)\lambda(\alpha)\zeta^\alpha$, and so $\widehat{f}(\alpha)\lambda(\alpha)$ is the αth Fourier coefficient of T_f with respect to the orthonormal system $\{\zeta^\alpha\}$ in the Hilbert space $L_2(\mathbb{T}^n, m_n)$. $(+)$ implies that $\sup\{|T_f(\zeta)| : \zeta \in \mathbb{T}^n\} \le 1$ for each $f \in \mathcal{O}(\mathbb{U}^n)$ with $|||f||| \le \varepsilon$. Using the Bessel inequality we obtain $|\widehat{f}(\alpha)\lambda(\alpha)| \le \|T_f\|_{L_2} \le \|T_f\|_\infty \le 1$ for all $f \in \mathcal{O}(\mathbb{U}^n)$, $|||f||| \le \varepsilon$. It follows that $|\lambda(\alpha)| \le \inf\{|\widehat{f}(\alpha)|^{-1} : f \in \mathcal{O}(\mathbb{U}^n),\ |||f||| \le \varepsilon\}$ for each $\alpha \in \mathbb{Z}_+^n$. Thus, using Lemma 5.4 we see that the family $\lambda = \{\lambda(\alpha)\}$ must satisfy $(**)$. Finally, λ defines by $(*)$ a continuous linear functional on $LN_0(\mathbb{U}^n) \subset LF_*(\mathbb{U}^n)$, which coincides with T on $\mathcal{O}(\mathbb{U}^n)$ (the Taylor series of each $f \in \mathcal{O}(\mathbb{U}^n)$ is convergent in $LN_0(\mathbb{U}^n)$). However, $\mathcal{O}(\mathbb{U}^n)$ is dense in $LN_0(\mathbb{U}^n)$, so $Tf = T_\lambda f$ for each $f \in LN_0(\mathbb{U}^n)$. The proof is finished.

Theorem 5.3 and Proposition 5.2 immediately imply the following corollary.

COROLLARY 5.5. *$LF_*(\mathbb{U}^n)$ is the Fréchet envelope of $LN_0(\mathbb{U}^n)$.*

6. The Fréchet envelope of $LN_*(\mathbb{U}^n)$.

In this section we prove the main result of the paper.

THEOREM 6.1. (a) *The polydisc algebra $A(\mathbb{U}^n)$ is weakly ($=$ Mackey) dense in $LN_*(\mathbb{U}^n)$.*

(b) *Each continuous linear functional on $LN_*(\mathbb{U}^n)$ is the restriction of a unique continuous linear functional on $LF_*(\mathbb{U}^n)$. Consequently, Theorem 5.3 describes all continuous linear functionals on $LN_*(\mathbb{U}^n)$.*

(c) *$LF_*(\mathbb{U}^n)$ is the Fréchet envelope of $LN_*(\mathbb{U}^n)$.*

For the proof of this theorem we need a few lemmas.

LEMMA 6.2. *Let $\mu = \mu_n$ be the Mackey topology of $LN_*(\mathbb{U}^n)$.*

(a) *$(LN_*(\mathbb{U}^n), \mu)$ is a topological algebra.*

(b) *If $\mathcal{G}$ is a bounded subset of $LN_*(\mathbb{U}^n)$, then the family $\mathcal{M}$ of multiplication operators $M_g : LN_*(\mathbb{U}^n) \to LN_*(\mathbb{U}^n)$, $g \in \mathcal{G}$, defined by $M_g f = fg$ is μ-equicontinuous.*

(c) *Define $\zeta_k(z) = z_n^k$, for $z = (z', z_n) \in \mathbb{U}^{n-1} \times \mathbb{U} = \mathbb{U}^n$, $k = 1, 2, \ldots$ Then the sequence $\{(k+1)\zeta_k\}$ is μ-convergent to zero.*

Proof. (a) From the inequality $\log(1 + ts) \le \log(1+t) + \log(1+s)$, $t, s > 0$, it follows that $V_\varepsilon \cdot V_\varepsilon \subseteq V_{2\varepsilon}$, where $V_\varepsilon = \{f : |||f||| \le \varepsilon\}$ is the ε-ball in $LN_*(\mathbb{U}^n)$, $\varepsilon > 0$. Consequently, $(\operatorname{conv} V_\varepsilon) \cdot (\operatorname{conv} V_\varepsilon) \subseteq \operatorname{conv} V_{2\varepsilon}$, and so the multiplication operation is $\mu(= \nu^c)$-continuous.

(b) Fix an $\varepsilon > 0$. Since $\mathcal{G}$ is bounded, we can find a $t > 0$ such that $|||tg||| < \varepsilon/2$ for all $g \in \mathcal{G}$. Using the inequality $\log(1 + |fg|) \le \log(1 + |f/t|) + \log(1 + |tg|)$ we see that $|||M_g f||| < \varepsilon$ for all $f \in LN_*(\mathbb{U}^n)$ with $|||f/t||| < \varepsilon/2$ and all $g \in \mathcal{G}$, and so $\mathcal{M}$ is ν-equicontinuous. Obviously, $\mathcal{M}$ remains equicontinuous if we equip $LN_*(\mathbb{U}^n)$ with the strongest locally convex topology which is weaker than ν, i.e., with $\nu^c = \mu$.

(c) Obviously, $\{\zeta_k\} \subset LN_0(\mathbb{U}^n) \subset LF_*(\mathbb{U}^n)$. It is easily seen that the sequence $\{(k+1)\zeta_k\}$ tends to zero in the topology induced on LN_0 by LF_*. This topology coincides with the Mackey topology of LN_0 (see Corollary 5.5). However, the Mackey topology of a subspace is always stronger than the Mackey topology of the entire space. Hence $\{(k+1)\zeta_k\}$ is μ-convergent to zero.

LEMMA 6.3. *$H^\infty(\mathbb{U}^n)$ is contained in the μ_n-closure of $A(\mathbb{U}^n)$.*

Proof. We will prove the lemma by induction. If $n = 1$ then it is well known that $A(\mathbb{U})$ is ν-dense in $LN_*(\mathbb{U}) = N_*$. Suppose now that $n > 1$, $H^\infty(\mathbb{U}^{n-1}) \subset \overline{A(\mathbb{U}^{n-1})}^{\mu_{n-1}}$ and fix $f \in H^\infty(\mathbb{U}^n)$. We will show that $f \in \overline{A(\mathbb{U}^n)}^{\mu_n}$. Of course, we may assume that $\|f\|_\infty (= \|f\|_{H^\infty(\mathbb{U}^n)}) \le 1$.

For each $z' \in \mathbb{U}^{n-1}$, $z_n \in \mathbb{U}$ and $j \in \mathbb{Z}_+$ define

$$H_j(z') = \sum_{\beta \in \mathbb{Z}_+^{n-1}} \widehat{f}(\beta, j) z'^\beta \quad \text{and} \quad h_j(z', z_n) = z_n^j H_j(z').$$

For $z' \in \mathbb{U}^{n-1}$ the function $f(z', \cdot)$ is holomorphic in $\mathbb{U}$. It is easy to check that $H_j(z') = \widehat{f}(z', j)$ for each $z' \in \mathbb{U}^{n-1}$ and $j \in \mathbb{Z}_+$, where $\widehat{f}(z', j)$ is the jth Taylor coefficient of $f(z', \cdot)$. Obviously, $\|f(z', \cdot)\|_{H^\infty(\mathbb{U})} \le \|f\|_\infty \le 1$ for each $z' \in \mathbb{U}^{n-1}$. Consequently, $|\widehat{f}(z', j)| \le 1$ for each $j \in \mathbb{Z}_+$ and $z' \in \mathbb{U}^{n-1}$, so

$$(*) \qquad H_j \in H^\infty(\mathbb{U}^{n-1}) \quad \text{and} \quad \|H_j\|_\infty \le 1 \quad \text{for } j = 0, 1, 2, \ldots$$

Let $\mathcal{J} : LN_*(\mathbb{U}^{n-1}) \to LN_*(\mathbb{U}^n)$ be the mapping defined by $(\mathcal{J}f)(z', z_n) = f(z')$ for $z' \in \mathbb{U}^{n-1}$ and $z_n \in \mathbb{U}$. It is clear that $\mathcal{J}$ is linear, (ν_{n-1}, ν_n)-continuous, and $\mathcal{J}(A(\mathbb{U}^{n-1})) \subset A(\mathbb{U}^n)$. Consequently, $\mathcal{J}$ is (μ_{n-1}, μ_n)-continuous and, by the induction hypothesis, we obtain

$$(**) \qquad \mathcal{J}(H^\infty(\mathbb{U}^{n-1})) \subset \mathcal{J}(\overline{A(\mathbb{U}^{n-1})}^{\mu_{n-1}}) \subset \overline{\mathcal{J}(A(\mathbb{U}^{n-1}))}^{\mu_n} \subset \overline{A(\mathbb{U}^n)}^{\mu_n}.$$

Let ζ_j be defined as in Lemma 6.2(c). Then $h_j = (\mathcal{J}H_j) \cdot \zeta_j$, so, by $(*)$, $(**)$, and Lemma 6.2(a), $h_j \in \overline{A(\mathbb{U}^n)}^{\mu_n}$ for $j = 0, 1, 2, \ldots$

Define

$$g_k = f - \sum_{j<k} h_j \quad \text{for } k = 1, 2, \ldots$$

The proof of the lemma will be finished if we show that the sequence $\{g_k\}$ tends to zero in the Mackey topology $\mu = \mu_n$ of $LN_*(\mathbb{U}^n)$.

We know that $\|h_j\|_\infty \le 1$ for each $j \in \mathbb{Z}_+$ and that $\|f\|_\infty \le 1$, hence $\|g_k\|_\infty \le k + 1$, $k = 1, 2, \ldots$ Define

$$\gamma_k(z', z_n) = \sum_{\beta \in \mathbb{Z}_+^{n-1}} \sum_{j \ge k} \widehat{f}(\beta, j) z'^\beta z_n^{j-k}$$

for $z' \in \mathbb{U}^{n-1}$, $z_n \in \mathbb{U}$, $k = 1, 2, \ldots$ Then $g_k(z', z_n) = \gamma_k(z', z_n) z_n^k$, so, by the maximum modulus principle, $\|\gamma_k\|_\infty \le k + 1$, $k = 1, 2, \ldots$ Set $\Gamma_k = \gamma_k/(k+1)$ and $\mathcal{G} = \{\Gamma_k : k = 1, 2, \ldots\}$. $\mathcal{G}$ being bounded in $H^\infty(\mathbb{U}^n)$ is bounded in $LN_*(\mathbb{U}^n)$. Therefore, the family $\mathcal{M}$ of operators $M_k : h \to h\Gamma_k$ is μ_n-equicontinuous (see Lemma 6.2(b)). Moreover, by Lemma 6.2(c), $(k+1)\zeta_k \to 0$ (μ_n), so $g_k = M_k((k+1)\zeta_k) \to 0$ in the Mackey topology of $LN_*(\mathbb{U}^n)$. The proof is complete.

Proof of Theorem 6.1. (a) Fix $f \in LN_*(\mathbb{U}^n)$. By Theorem 3.1, f can be written in the form $f = he^g$, where $h \in H^\infty(\mathbb{U}^n)$, $g \in H(\mathbb{U}^n)$, $u = \operatorname{Re} g > 1$, $e^g \in LN_*(\mathbb{U}^n)$. Let G be the space of all $\gamma \in LN_*(\mathbb{U}^n)$ such that μ-$\lim_{r \to 1-} \gamma_r = \gamma$ (as usual, $\gamma_r(z) = \gamma(rz)$). G is μ-closed. Indeed, it suffices to note that the base $\mathcal{V}$ of neighbourhoods of zero for μ consisting of the sets $U_\varepsilon = \operatorname{conv} V_\varepsilon$, $\varepsilon > 0$, where V_ε is the ε-ball in $LN_*(\mathbb{U}^n)$, has the following property: if $\gamma \in U_\varepsilon$ and $r \in (0,1)$, then $\gamma_r \in U_\varepsilon$. It is obvious that $A(\mathbb{U}^n) \subset G$, so Lemma 6.3 tells us that $H^\infty(\mathbb{U}^n) \subset G$. Therefore, using Lemma 6.2(a) and Lemma 6.3 once again we see that for the proof of our assertion it is enough to show that $e^g \in G$.

Let $\Gamma = e^{-g}$ and $\Delta_r = e^{g + g_r}$, for $r \in (0,1)$. Then the family $\mathcal{G} = \{\Delta_r : r \in (0,1)\}$ is ν-bounded in $LN_*(\mathbb{U}^n)$ (see Proposition 2.2(a) and (b)). By Lemma 6.2(b) the family of multiplication operators defined by $\mathcal{G}$ is μ-equicontinuous. Moreover, Γ is holomorphic and $|\Gamma| = e^{-u} \le e^{-1}$, so

$\Gamma \in H^{\infty}(\mathbb{U}^n) \subset G$. Finally,

$$e^g - (e^g)_r = \Delta_r(\Gamma_r - \Gamma) \to 0 \ (\mu) \quad \text{as } r \to 1-,$$

and so $e^g \in G$. The proof of (a) is finished.

(b) and (c) follow immediately from (a) and Theorem 5.3.

COROLLARY 6.4. *$LN_0(\mathbb{U}^n)$ is a proper closed weakly dense Möbius invariant subalgebra of $LN_*(\mathbb{U}^n)$.*

7. Applications. In this section we present a few consequences of Theorem 6.1.

As was observed in Section 5, $LF_*(\mathbb{U}^n)$ is isomorphic to the nuclear power series space $E_n = \Lambda_1(j^{1/2n} : j \in \mathbb{Z}_+)$. Using [3], Proposition 3, we see that E_n is not isomorphic to E_m if $n \neq m$. The Fréchet envelopes of isomorphic spaces are isomorphic, so we obtain the following result.

PROPOSITION 7.1. *If $n \neq m$ then $LN_*(\mathbb{U}^n)$ is not isomorphic to $LN_*(\mathbb{U}^m)$.*

It is easy to prove that the sequence $\{z^\alpha\}_{\alpha \in \mathbb{Z}_+^n}$ is a Schauder basis of $LF_*(\mathbb{U}^n)$. This, Theorem 6.1, and Proposition 4.1 imply the following pathological property of $LN_*(\mathbb{U}^n)$.

PROPOSITION 7.2. *The Taylor series $\sum \widehat{f}(\alpha) z^\alpha$ of each function $f \in LN_*(\mathbb{U}^n)$ is μ-convergent. Consequently, the sequence $\{z^\alpha\}_{\alpha \in \mathbb{Z}_+^n}$ is a Schauder basis of $(LN_*(\mathbb{U}^n), \mu)$, and so $LN_*(\mathbb{U}^n)$ is weakly separable but nonseparable in its own topology ν.*

PROPOSITION 7.3. *No infinite-dimensional locally bounded subspace of $LN_*(\mathbb{U}^n)$ is complemented.*

Proof. Suppose that X is a locally bounded complemented subspace of $LN_*(\mathbb{U}^n)$. Then the Mackey topology of X coincides with the topology induced on it by the Mackey topology of the entire space $LN_*(\mathbb{U}^n)$. By Theorem 6.1 the closure of X in $LF_*(\mathbb{U}^n)$ is a locally bounded subspace of $LF_*(\mathbb{U}^n)$. However, every locally bounded nuclear space is finite-dimensional.

We recall that a topological vector space X has the *Orlicz–Pettis property* if every weakly subseries convergent series in X (i.e., a series $\sum x_n$ in X such that weak-$\lim_{n\to\infty} \sum_{j=1}^{n} x_{k_j}$ exists for each increasing sequence $\{k_j\}$ of positive integers) is convergent in X.

PROPOSITION 7.4. *If $n > 1$ then $LN_*(\mathbb{U}^n)$ does not have the Orlicz–Pettis property.*

Proof. Let $\{f_k\}$ be a sequence of homogeneous polynomials constructed in Proposition 4.1. For every sequence $\{\varepsilon_k\} \subset \{0,1\}$ the series $(*)$ $\sum \varepsilon_k f_k$ is κ-convergent to a function $f_\varepsilon \in LN_*(\mathbb{U}^n)$. The series $(*)$ is just a block series of the Taylor series of f_ε, so it is μ-convergent (see Proposition 7.2). Finally, the series $\sum f_k$ is both Mackey and weakly subseries convergent but, by Proposition 4.1(b), it is not convergent in the original topology of $LN_*(\mathbb{U}^n)$. The proof is complete.

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Received August 2, 1991 (2827)

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