$$\leq \frac{1}{\Omega_n \varepsilon^n} \int_{|y|\leq\varepsilon} \sum_{j=1}^{K} \|\tau_{\theta_j y} f_j - f_j\|_{L^{p_j}} \prod_{k\neq j} \|f_j\|_{L^{p_k}}\, dy \to 0$$

as $|y| \to 0$ since the last integrand is a continuous function of y which vanishes at the origin. The last inequality above follows by adding and subtracting $2K-2$ suitable terms and applying Hölder's inequality K times.

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DEPARTMENT OF MATHEMATICS
YALE UNIVERSITY
BOX 2155, YALE STATION
NEW HAVEN, CONNECTICUT 06520
U.S.A.

Received May 24, 1991 (2804)

STUDIA MATHEMATICA 102 (1) (1992)

Orthogonal polynomials and middle Hankel operators on Bergman spaces

by

LIZHONG PENG† (Beijing), RICHARD ROCHBERG‡ (St. Louis, Mo.) and ZHIJIAN WU (Tuscaloosa, Ala.)

Abstract. We introduce a sequence of Hankel style operators H^k, $k = 1,2,3,\ldots$, which act on the Bergman space of the unit disk. These operators are intermediate between the classical big and small Hankel operators. We study the boundedness and Schatten–von Neumann properties of the H^k and show, among other things, that H^k are cut-off at $1/k$. Recall that the big Hankel operator is cut-off at 1 and the small Hankel operator at 0.

Introduction and background. Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$ and let $dA(z) = (1/\pi)dxdy$ be normalized Lebesgue measure. $L^2 = L^2(dA)$ is the Hilbert space of functions u for which the norm

$$\|u\| = \Big(\int_{\mathbb{D}} |u(z)|^2\, dA(z) \Big)^{1/2}$$

is finite. The Bergman space, A, is the subspace of all analytic functions in L^2.

The *big* and *small Hankel operators* on A with symbol b are defined by

$$H_b(f) = (I-P)(\overline{b}f), \qquad h_b(f) = Q(\overline{b}f).$$

P and Q are orthogonal projections from L^2 onto A and $\bar{A}_0 = \{f \in L^2 : \bar{f} \in A \text{ and } f(0) = 0\}$ respectively.

For $0 < p \leq \infty$ denote the Schatten–von Neumann ideal by S_p (S_∞ is the class of bounded operators) and the analytic Besov space in $\mathbb{D}$ by B_p (B_∞ is the Bloch space). The main S_p results for the big and small Hankel operators with analytic symbols can be summarized as follows (see [A], [Pel1, 2], [R1, 2], [S], [AFP] and [J1]).

1991 *Mathematics Subject Classification*: 30C40, 47B35.

† Lizhong Peng supported in part by the NNSF of China.

‡ Richard Rochberg supported in part by NSF grant DSM 9007491.

THEOREM A. *Suppose b is analytic.*

(1) *For $0 < p \leq \infty$, $h_b \in S_p$ if and only if $b \in B_p$.*

(2) *For $1 < p \leq \infty$, $H_b \in S_p$ if and only if $b \in B_p$; for $0 < p \leq 1$, $H_b \in S_p$ if and only if b is constant.*

The change in behavior at $p = 1$ is sometimes referred to as a "cut-off" (see [JR]). For convenience we say that the small Hankel operator has a cut-off at 0. Here we consider the problem raised in [JR] of finding a "middle" Hankel operator which is between h_b and H_b and has a cut-off between 0 and 1.

In general, we introduce the partial order $\prec$ between operators from one Hilbert space to another by

$$R \prec S \quad \text{if and only if} \quad R^*R \leq S^*S .$$

Clearly $S \in S_p$ and $R \prec S$ implies $R \in S_p$. It is easy to check that $h_b \prec H_b$. We are looking for a "Hankel" operator H with $h_b \prec H \prec H_b$.

Suppose X and Y are closed subspaces of L^2 and P_Y is the orthogonal projection from L^2 onto Y. If $A \subset X$ and $\bar{A} \subset Y$ then we can construct Hankel style operators intermediate between h_b and H_b by

$$H_b^{XY} : X \to Y, \quad g \mapsto P_Y(\overline{b}g) .$$

S. Janson and R. Rochberg, in [JR], considered the operator H_b^{JR} with $J = A$ and $R = \overline{\text{span}}\{z^n \overline{z}^m : m > n\}$. They proved that, for analytic symbols b, H_b^{JR} is between h_b and H_b and has cut-off 0. L. Peng and G. Zhang later, in [PZ], found a strict middle Hankel operator with cut-off 1/2 (see also [J1], [M] and [Z]).

In this paper, we give a sequence of middle Hankel operators $\{H_b^k\}$, $k = 0, 1, 2, \ldots$, which links H_b and H_b^{JR} in the following sense:

$$H_b = H_b^0 \succ H_b^1 \succ \ldots \succ H_b^k \succ H_b^{k+1} \succ \ldots \succ \lim_{k\to\infty} H_b^k = H_b^{JR} ,$$

and each H_b^k has cut-off $1/(k+1)$. This is Theorem 5 in Section 2.

The full story is based on a decomposition of L^2. To describe the decomposition we need the differential operator $\overline{D} = \overline{z}\partial/\partial\overline{z}$ acting on L^2.

$$L^2 = \sum_{k=0}^{\infty}\{\ker(\overline{D}^{k+1}) + \overline{\ker(\overline{D}^{k+1})}\} = \bigoplus_{k=0}^{\infty}(A^k + \bar{A}^k) .$$

Here $A^0 = A$. (Recall a similar decomposition $L^2(\partial\mathbb{D}) = H^2 + \overline{H}{}^2_0$, where H^2 is the Hardy space.) The A^k are copies of A^0 in a certain sense (related to Laguerre polynomials) and satisfy

$$A^k \oplus A^{k-1} \oplus \ldots \oplus A^1 \oplus A^0 = \ker(\overline{D}^{k+1}) .$$

Let P_k be the orthogonal projection from L^2 onto $\ker(\overline{D}^{k+1})$. For analytic symbols b our operators H_b^k, $k = 0, 1, 2, \ldots$, are defined by

$$H_b^k : A \to (\ker(\overline{D}^{k+1}))^\perp, \quad g \mapsto \overline{b}g - P_k(\overline{b}g) = (I - P_k)(\overline{b}g) .$$

In Section 1 we recall some basic results. In Section 2 we introduce the decomposition of L^2 and construct our middle Hankel operators. Sections 3 and 4 contain the proof of the main theorem together with other facts about the A^k. Section 5 includes a look at related operators and some results for nonanalytic b. Finally, in Section 6 we collect some remarks and problems. In particular, we indicate that our results extend to weighted Bergman spaces.

In this paper, the letters j, k, m, n, s and t will denote integers. C means a positive constant which may be different at each occurrence. The notation "$\approx$" means comparable.

1. Preliminaries and some notation. For $\alpha > -1$, Laguerre polynomials (see [Sz]) $\{L_n^\alpha(x)\}$ are defined by the following conditions of orthogonality and normalization:

$$(1.1) \quad \int_0^\infty L_k^\alpha(x)L_j^\alpha(x)e^{-x}x^\alpha\,dx = \Gamma(\alpha+1)\binom{k+\alpha}{k}\delta_{k,j}, \quad k, j = 0, 1, 2, \ldots$$

Here $\Gamma(\cdot)$ is the classical Gamma function and $\binom{k+\alpha}{k}$ is the binomial coefficient. We require that the coefficient of x^k in $L_k^\alpha(x)$ has the sign $(-1)^k$. The explicit formula is

$$L_k^\alpha(x) = \sum_{m=0}^{k}\binom{k+\alpha}{k-m}\frac{(-x)^m}{m!} .$$

The following result can be found in [Sz].

LEMMA B. *Fix $\alpha > -1$. The system $e^{-x/2}x^{\alpha/2}x^k$, $k = 0, 1, 2, \ldots$, is complete in $L^2(0, \infty)$.*

The Besov space B_p has several equivalent characterizations. We will use the following one (see [P]).

LEMMA C. *Let b be an analytic function on $\mathbb{D}$. Then $b \in B_p$, $p > 0$, if and only if*

$$\{2^{j/p}\|b * \psi_j\|_{L^p(\partial\mathbb{D})}\}_{j\geq 0} \in l^p .$$

Moreover, if $b \in B_p$, then

$$\|b\|_p \approx \|\{2^{j/p}\|b * \psi_j\|^p_{L^p(\partial\mathbb{D})}\}_{j\geq 0}\|_{l^p} .$$

Here $\psi_0(z)=1+\bar z+z$ *and* $\psi_j(z)$, $j\ge 1$, *is a trigonometric polynomial such that* $\widehat{\psi}_j(2^j)=1$, $\widehat{\psi}_j=0$ *outside* $(2^{j-1},2^{j+1})$ *and* $\widehat{\psi}_j(k)$ *is linear on* $(2^{j-1},2^j)$ *and* $(2^j,2^{j+1})$.

Remark. In Lemma C, the function $b*\psi_j(z)$ is a polynomial of degree not greater than 2^{2+j}.

The next two results can be found in [Pel2].

LEMMA D. *If* $b(z)=\sum_{j=0}^N b_j z^j$, *then*

$$b_j=\frac{1}{2N}\sum_{k=0}^{2N-1} b(\eta_k)\overline{\eta}_k^j\,.$$

Here $\eta_k=\exp(2\pi ki/(2N))$, $k=0,1,\ldots,2N-1$.

LEMMA E. *For the same* b *as in Lemma* D, *we have*

$$\|b\|^p_{L^p(\partial\mathbb{D})}=\int_0^{2\pi}|b(e^{i\theta})|^p\,d\theta\approx\frac{1}{2N}\sum_{j=0}^{2N-1}|b(\eta_j)|^p\,.$$

LEMMA F. *Let* $\xi_j,\eta_j\in\mathbb{C}$ *and* $|\xi_j|=|\eta_j|=1$, $j=0,1,\ldots$ *Then for* $p>0$

$$\|(\xi_m A_{m,n}\eta_n)_{m,n\ge0}\|_p=\|(A_{m,n})_{m,n\ge0}\|_p\,.$$

Here $(A_{m,n})_{m,n\ge0}$ *is a matrix and* $\|\cdot\|_p$ *is the* S_p *(quasi) norm.*

Proof. Obvious. ∎

LEMMA G. *Suppose* $\beta=\{\beta_j\}$ *and* $\gamma=\{\gamma_j\}$ *are in* l^2. *Then* $(\beta_m\gamma_n)_{m,n\ge0}$ *is a rank one matrix and*

$$\|(\beta_m\gamma_n)_{m,n\ge0}\|_p=\|\beta\|_{l^2}\|\gamma\|_{l^2}\,.$$

Proof. Obvious. ∎

2. Decomposition of L^2 and middle Hankel operators.

In this section we construct the middle Hankel operators and state our main result.

For $r\in[0,1)$ and $n,k=0,1,2,\ldots$, let

$$P_{k,n}(r)=\sqrt{n+1}L_k^0(\log(r^{-n-1}))=\sqrt{n+1}\sum_{m=0}^{k}\binom{k}{k-m}\frac{(n+1)^m}{m!}(\log r)^m\,,$$

$$E_{k,n}(z)=P_{k,n}(|z|^2)z^n,\qquad E_{k,-n}(z)=\overline{E_{k,n}(z)}\,.$$

In particular, for $n\ge0$

$$E_{0,n}(z)=\sqrt{n+1}\,z^n,\qquad E_{1,n}(z)=\sqrt{n+1}\,((n+1)\log|z|^2+1)z^n\,.$$

Clearly $P_{k,n}(r)$ is a polynomial in $\log r$ of degree k. Changing variables in (1.1), we get

$$\int_0^1 P_{j,n}(r)P_{k,n}(r)r^n\,dr=\delta_{jk}\,. \tag{2.1}$$

Hence we have the following version of Lemma B on $L^2([0,1))$.

LEMMA B′. *For any* $n\ge0$, *the system* $P_{k,n}(r)r^{n/2}$, $k=0,1,2,\ldots$, *is an orthonormal basis of* $L^2([0,1))$.

THEOREM 1. *The system*

$$E_{k,n}(z),\qquad k=0,1,2,\ldots;\ n=0,\pm1,\pm2,\ldots,$$

is an orthonormal basis of L^2.

Proof. Every function f in L^2 can be written as $f(z)=f(re^{i\theta})=\sum_{n=-\infty}^{\infty}f_n(r)e^{in\theta}$ with $\|f\|^2=\sum_{n=-\infty}^{\infty}\int_0^1|f_n(r)|^2 2r\,dr$. Notice that Lemma B′ also says the system $P_{k,|n|}(r^2)r^{|n|}$, $k=0,1,2,\ldots$, is an orthonormal basis for $L^2([0,1),2rdr)$. Thus

$$f_n(r)=\sum_{k=0}^{\infty}c_{k,n}P_{k,|n|}(r^2)r^{|n|}\,.$$

Hence

$$f(z)=f(re^{i\theta})=\sum_{n=-\infty}^{\infty}f_n(r)e^{in\theta}=\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}c_{k,n}P_{k,|n|}(r^2)r^{|n|}e^{in\theta}=\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}c_{k,n}E_{k,n}(z)\,.\ ∎$$

Let

$$A^k=\operatorname{span}\{E_{k,n}\}_{n\ge0},\qquad \bar A^k=\operatorname{span}\{E_{k,n}\}_{n\le0}\,.$$

By Theorem 1, we have $L^2=\bigoplus_{k=0}^{\infty}(A^k+\bar A^k)$. For $k=0,1,2,\ldots$, let J_k be the linear map defined by

$$J_k(E_{0,n})=E_{k,n},\qquad n=0,1,2,\ldots$$

If we extend J_k to A by linearity, we get a unitary map from A to A^k. This is why we say that the A^k are copies of A.

For $k=0,1,2,\ldots$, let $A_k=\ker(\overline{D}^{k+1})$.

THEOREM 2. $A_0=A^0=A$ *and* $A_k=A^k\oplus A^{k-1}\oplus\ldots\oplus A^0$, $k=1,2,\ldots$

Proof. For $n \geq 0$ and $k \geq 1$, writing $\log|z|^2 = \log z + \log \overline{z}$ we find

$$\overline{D}(z^n) = 0, \quad \overline{D}((\log|z|^2)^k z^n) = k(\log|z|^2)^{k-1} z^n. \tag{2.2}$$

These imply $A_k \supset A^k \oplus A^{k-1} \oplus \ldots \oplus A^0$.

We show the other direction by induction. Clearly $A_0 \subset A^0$. Suppose $A_k \subset A^k \oplus A^{k-1} \oplus \ldots \oplus A^0$. If $f \in A_{k+1}$, then $\overline{D}^{k+2} f = 0$. Hence $\overline{D}^{k+1} f \in A^0 = A$, i.e.,

$$\overline{D}^{k+1} f(z) = \sum_{m=0}^{\infty} a_m E_{0,m}(z).$$

By (2.2) we have $\overline{D}^j E_{j,m}(z) = (m+1)^j E_{0,m}(z)$, hence

$$f - \sum_{m=0}^{\infty} \frac{a_m}{(m+1)^{k+1}} E_{k+1,m} \in A_k.$$

Finally, by induction, we get $f \in A^{k+1} \oplus A_k \subset A^{k+1} \oplus A^k \oplus \ldots \oplus A^0$. ∎

Denote by P^k and P_k the orthogonal projections from L^2 onto A^k and A_k respectively. Clearly

$$P_k = \bigoplus_{j=0}^{k} P^j. \tag{2.3}$$

With this notation the *generalized Hankel operator* with symbol b is

$$H_b^k : A \to A_k^{\perp} = \Big(\bigoplus_{j=0}^{k} A^j\Big)^{\perp}, \quad g \mapsto (I - P_k)(\overline{b}g) = \overline{b}g - \sum_{j=0}^{k} P^j(\overline{b}g).$$

LEMMA 3. *Let $b(z) = z^s$. Then*

$$P^j(\overline{b}E_{0,m}) = \begin{cases} 0 & \text{if } m < s, \\ \sqrt{\dfrac{m-s+1}{m+1}}\Big(\dfrac{s}{m+1}\Big)^j E_{j,m-s}(z) & \text{if } m \geq s. \end{cases}$$

Proof. Since

$$P^j(\overline{b}E_{0,m})(z) = \sum_{n=0}^{\infty} \langle E_{0,m}, bE_{j,n}\rangle E_{j,n}(z),$$

the result for $m < s$ is obvious. Suppose $m \geq s$. Then

$$P^j(\overline{b}E_{0,m})(z) = \langle E_{0,m}, bE_{j,m-s}\rangle E_{j,m-s}(z).$$

Direct computation yields

$$\begin{aligned}\langle E_{0,m}, bE_{j,m-s}\rangle &= \int_{\mathbb{D}} \sqrt{m+1}\, z^m \overline{z}^{\,s} P_{j,m-s}(|z|^2)\overline{z}^{\,m-s}\, dA(z) \\ &= \sqrt{m+1}\sqrt{m-s+1}\sum_{k=0}^{j}\binom{j}{j-k}\frac{(m-s+1)^k}{k!}\int_0^1 r^m(\log r)^k\, dr \\ &= \sqrt{m+1}\sqrt{m-s+1}\sum_{k=0}^{j}\binom{j}{j-k}(-1)^k\frac{(m-s+1)^k}{(m+1)^{k+1}} \\ &= \sqrt{m+1}\sqrt{m-s+1}\frac{1}{m+1}\Big(\frac{s}{m+1}\Big)^j. \ \blacksquare\end{aligned}$$

LEMMA 4. *Suppose b is analytic and $k = 0, 1, 2, \ldots$ We have*

(1) $H_b^k \succ H_b^{k+1} \succ H_b^{JR}$,

(2) $\lim_{k\to\infty} H_b^k = H_b^{JR}$ *(in the weak operator topology).*

Proof. (1) follows from the definitions. For (2) we only need to check the monomials. Let $b(z) = z^m$ and $g(z) = z^n$. If $m > n$ we have clearly $P_k(\overline{b}g) = 0$. Hence $H_b^k(g) = H_b^{JR}(g) = \overline{b}g$. If $m \leq n$, since $\|H_b^{JR}(g)\| = 0$ we only need to show

$$\lim_{k\to\infty}\|H_b^k(g)\| = 0.$$

By definition we have

$$\begin{aligned}\|H_b^k(g)\|^2 &= \|\overline{b}g\|^2 - \|P_k(\overline{b}g)\|^2 = \|bg\|^2 - \sum_{j=0}^{k}\|P^j(\overline{b}g)\|^2 \\ &= \int_{\mathbb{D}} |z^{m+n}|^2\, dA(z) \\ &\quad - \sum_{j=0}^{k}\frac{(n-m+1)m^{2j}}{(n+1)^{2j+2}}\int_{\mathbb{D}} |E_{j,n-m}(z)|^2\, dA(z) \\ &= \frac{1}{n+m+1} - \sum_{j=0}^{k}\frac{(n-m+1)m^{2j}}{(n+1)^{2j+2}},\end{aligned}$$

hence

$$\begin{aligned}\lim_{k\to\infty}\|H_b^k(g)\|^2 &= \frac{1}{n+m+1} - \sum_{j=0}^{\infty}\frac{(n-m+1)m^{2j}}{(n+1)^{2j+2}} \\ &= \frac{1}{n+m+1} - \frac{n-m+1}{(n+1)^2}\Big(1 - \frac{m^2}{(n+1)^2}\Big)^{-1} = 0. \ \blacksquare\end{aligned}$$

Our main theorem is

THEOREM 5. *Suppose b is analytic and $k=0,1,2,\ldots$ Then*

$$H_b = H_b^0 \succ H_b^1 \succ \ldots \succ H_b^k \succ H_b^{k+1} \succ \ldots \succ \lim_{k\to\infty} H_b^k = H_b^{JR}.$$

Moreover,

(1) *if $0<p\le 1/(k+1)$, then $H_b^k \in S_p$ if and only if b is constant;*
(2) *if $1/(k+1)<p\le\infty$, then $H_b^k\in S_p$ if and only if $b\in B_p$.*

Using Theorem A and Lemma 4 we see that we are reduced to proving that if b is analytic and $k=0,1,2,\ldots$, then

(cut-off) if $0<p\le 1/(k+1)$, then $H_b^k\in S_p \Rightarrow b$ is constant;
(S_p estimates) if $1/(k+1)<p\le 1$, then $b\in B_p \Rightarrow H_b^k\in S_p$.

We prove these in the next two sections.

3. The cut-off. We now compute the matrix elements of the operator $(H_b^k)^* H_b^k$ on A with respect to the standard basis.

LEMMA 6. *Let $b(z)=\sum_{s=0}^{\infty} b_s z^s$. Then*

$$\begin{aligned}\langle H_b^k(E_{0,m}), H_b^k(E_{0,n})\rangle &= \sum_{t<0} \overline{b}_{m-t} b_{n-t} \frac{\sqrt{m+1}\sqrt{n+1}}{m+n-t+1}\\ &\quad + \sum_{0\le t\le \min(m,n)} \overline{b}_{m-t} b_{n-t} \frac{\sqrt{m+1}\sqrt{n+1}}{m+n-t+1}\left(\frac{(m-t)(n-t)}{(m+1)(n+1)}\right)^{k+1}.\end{aligned}$$

Proof. By the definition and (2.3), we get

$$\begin{aligned}\langle H_b^k(E_{0,m}), H_b^k(E_{0,n})\rangle &= \sum_{s,t=0}^{\infty} \overline{b}_s b_t \Big\langle \overline{z}^{\,s} E_{0,m} - \sum_{j=0}^{k} P^j(\overline{z}^{\,s} E_{0,m}), \overline{z}^{\,t} E_{0,n} - \sum_{j=0}^{k} P^j(\overline{z}^{\,t} E_{0,n})\Big\rangle\\ &= \sum_{s,t=0}^{\infty} \overline{b}_s b_t \Big(\langle \overline{z}^{\,s} E_{0,m}, \overline{z}^{\,t} E_{0,n}\rangle - \Big\langle \overline{z}^{\,s}E_{0,m}, \sum_{j=0}^{k} P^j(\overline{z}^{\,t}E_{0,n})\Big\rangle\\ &\quad - \Big\langle \sum_{j=0}^{k} P^j(\overline{z}^{\,s}E_{0,m}), \overline{z}^{\,t}E_{0,n}\Big\rangle + \Big\langle \sum_{j=0}^{k} P^j(\overline{z}^{\,s}E_{0,m}), \sum_{j=0}^{k} P^j(\overline{z}^{\,t}E_{0,n})\Big\rangle\Big)\end{aligned}$$

(replace s by $m-t$ and t by $n-t$)

$$\begin{aligned}&= \sum_{t=-\infty}^{\min(m,n)} \overline{b}_{m-t} b_{n-t} \Big(\langle \overline{z}^{\,m-t} E_{0,m}, \overline{z}^{\,n-t} E_{0,n}\rangle\\ &\quad - \sum_{j=0}^{k} \langle P^j(\overline{z}^{\,m-t}E_{0,m}), P^j(\overline{z}^{\,n-t}E_{0,n})\rangle\Big) = \sum_{t<0} + \sum_{0\le t\le\min(m,n)}\end{aligned}$$

For $t\le\min(m,n)$, a direct computation yields

$$\langle \overline{z}^{\,m-t} E_{0,m}, \overline{z}^{\,n-t} E_{0,n}\rangle = \frac{\sqrt{m+1}\sqrt{n+1}}{m+n-t+1}.$$

If $t<0$, by Lemma 3 we have clearly

$$\langle P^j(\overline{z}^{\,m-t}E_{0,m}), P^j(\overline{z}^{\,n-t}E_{0,n})\rangle = 0.$$

If $\min(m,n)\ge t\ge 0$, again by Lemma 3 we get

$$\begin{aligned}&\sum_{j=0}^{k} \langle P^j(\overline{z}^{\,m-t}E_{0,m}), P^j(\overline{z}^{\,n-t}E_{0,n})\rangle\\ &= \sum_{j=0}^{k} \frac{t+1}{\sqrt{m+1}\sqrt{n+1}} \left(\frac{m-t}{m+1}\right)^j \left(\frac{n-t}{n+1}\right)^j\\ &= \frac{t+1}{\sqrt{m+1}\sqrt{n+1}} \left(1-\left(\frac{(m-t)(n-t)}{(m+1)(n+1)}\right)^{k+1}\right)\left(1-\frac{(m-t)(n-t)}{(m+1)(n+1)}\right)^{-1}.\end{aligned}$$

Hence

$$\sum_{t<0} = \sum_{t<0} \overline{b}_{m-t} b_{n-t} \frac{\sqrt{m+1}\sqrt{n+1}}{m+n-t+1}$$

and

$$\begin{aligned}\sum_{0\le t\le\min(m,n)} &= \sum_{0\le t\le\min(m,n)} \overline{b}_{m-t} b_{n-t} \sqrt{m+1}\sqrt{n+1}\\ &\quad\times\left(\frac{1}{m+n-t+1} - \frac{t+1}{(m+1)(n+1)}\left(1-\left(\frac{(m-t)(n-t)}{(m+1)(n+1)}\right)^{k+1}\right)\right.\\ &\quad\times\left.\left(1-\frac{(m-t)(n-t)}{(m+1)(n+1)}\right)^{-1}\right)\\ &= \sum_{0\le t\le\min(m,n)} \overline{b}_{m-t} b_{n-t} \frac{\sqrt{m+1}\sqrt{n+1}}{m+n-t+1}\left(\frac{(m-t)(n-t)}{(m+1)(n+1)}\right)^{k+1}. \ \blacksquare\end{aligned}$$

LEMMA 7. *Let $b(z)=z^s$, $s\ge 1$. Then $H_b^k\in S_p$ if and only if $p>1/(k+1)$.*

Proof. We can compute the singular values of H_b^k explicitly. In fact, by Lemma 6 we know the matrix $(\langle H_b^k(E_{0,m}), H_b^k(E_{0,n})\rangle)_{m,n\ge 0}$ is a diagonal matrix with the (m,m)th entry

$$\langle H_b^k(E_{0,m}), H_b^k(E_{0,m})\rangle = \begin{cases} \dfrac{m+1}{m+s+1} & \text{if } m<s,\\ \dfrac{s^{2k+2}}{(m+s+1)(m+1)^{2k+1}} & \text{if } m\ge s;\end{cases}$$
$$\approx \begin{cases} 1 & \text{if } m<s,\\ (m+1)^{-2k-2} & \text{if } m\ge s.\end{cases}$$

Hence for this choice of b we have

$$\|H_b^k\|_p^p = \|(H_b^k)^* H_b^k\|_{p/2}^{p/2} \approx \sum_{m=0}^{\infty}(m+1)^{-(k+1)p}$$

and the lemma follows. ■

Proof of the cut-off. Suppose b is an analytic function in $\mathbb{D}$ such that $H_b^k\in S_p$ for $0<p\le 1/(k+1)$. We prove b is a constant by showing its jth derivative $b^{(j)}(0)$ is 0 for $j=1,2,\ldots$ The use of the maximum principle for S_p, $p\le 1$, which plays an important role in the following comes from [PZ] and has its root in [J1].

For $|\zeta|\le 1$, let

$$b_\zeta^{(j)}(z) = \begin{cases} b(\zeta z) - \displaystyle\sum_{s=0}^{j-1}\frac{b^{(s)}(0)}{s!}(\zeta z)^s\zeta^{-j} & \text{if } \zeta\ne 0,\\ \dfrac{b^{(j)}(0)}{j!}z^j & \text{if } \zeta = 0.\end{cases}$$

Clearly the map $\zeta\mapsto b_\zeta^{(j)}$ is analytic and the map $\zeta\mapsto H^k_{b_\zeta^{(j)}}$ is anti-analytic in $\mathbb{D}$. If $|\zeta|=1$, by the rotation invariant property of H_b^k (this is clear),

$$\|H^k_{b_\zeta^{(1)}}\|_p = \|H_b^k\|_p < \infty\,.$$

The maximum principle yields that $H^k_{b_\zeta^{(1)}}\in S_p$ for all $\zeta\in\mathbb{D}$. In particular, this implies $H^k_{b_0^{(1)}}\in S_p$. Hence by Lemma 7, $b'(0)=0$.

Assume we have already proved that $b^{(s)}(0)=0$ for $s=1,\ldots,j$. Repeating the above argument yields

$$\|H^k_{b_\zeta^{(j+1)}}\|_p = \|H_b^k\|_p < \infty \quad \text{for } |\zeta|=1.$$

Thus $H^k_{b_0^{(j+1)}}\in S_p$ and this implies $b^{(j+1)}(0)=0$. ■

4. S_p norm estimates. Notice that the second part of Lemma 4 says

$$\overline{b}g = \sum_{j=0}^{\infty}P^j(\overline{b}g) + H_b^{JR}(g).$$

Define T_b^j by $T_b^j(g) = P^j(\overline{b}g)$. Another way to think of H_b^k is to write

$$H_b^k(g) = \overline{b}g - \sum_{j=0}^{k}P^j(\overline{b}g) = \sum_{j=k+1}^{\infty}T_b^j(g) + H_b^{JR}(g)\,.$$

Hence to finish the proof it is enough to show $\sum_{j=k+1}^{\infty}T_b^j\in S_p$ if $b\in B_p$ for $1/(k+1)<p\le 1$.

For convenience, let $T_k=\sum_{j=k+1}^{\infty}T_b^j$. By Lemma C and the remark that follows it we only need to show the following theorem.

THEOREM 8. *Suppose b is an N-th degree polynomial and $1/(k+1)<p\le 1$. Then*

$$\|T_k\|_p^p \le CN\|b\|_{L^p(\partial\mathbb{D})}^p\,. \tag{4.1}$$

The idea of the proof is the following. Consider the partition for A: $A=\bigoplus_{t=0}^{\infty}U_t$, with $U_t=\operatorname{span}\{E_{0,n}: n\in I_t\}$, where $I_t=\{j: tN\le j<(t+1)N\}$. Clearly $T_k=\sum_{t=0}^{\infty}T_k|_{U_t}$. Because $p\le 1$, we have

$$\|T_k\|_p^p \le \sum_{t=0}^{\infty}\|T_k|_{U_t}\|_p^p\,. \tag{4.2}$$

Different techniques will be used to estimate $\|T_k|_{U_t}\|_p^p$ for small t and for large t. When t is small we have

LEMMA 9. *Suppose $b(z)=\sum_{j=0}^{N}b_jz^j$ and $t=0,1,2,\ldots$ Then for $p\le 1$*

$$\|T_k|_{U_t}\|_p^p \le C(t+1)^{p/2}N\|b\|_{L^p(\partial\mathbb{D})}^p.$$

When t is large, using $T_k|_{U_t}=\sum_{j=k+1}^{\infty}T_b^j|_{U_t}$ and $p\le 1$ we get

$$\|T_k|_{U_t}\|_p^p \le \sum_{j=k+1}^{\infty}\|T_b^j|_{U_t}\|_p^p\,. \tag{4.3}$$

The following lemma gives the desired estimate for the individual terms.

LEMMA 10. *Suppose $b(z)=\sum_{s=0}^{N-1}b_sz^s$, $j\ge k+1$, $t\ge 1$ and $p\le 1$. Then*

$$\|T_b^j|_{U_t}\|_p^p \le C\left(\frac{2}{t^p}\right)^j N\|b\|_{L^p(\partial\mathbb{D})}^p\,.$$

Using these lemmas, here is how to finish.

Proof of Theorem 8. Choose an integer M such that $M^p \geq 4$. Then (4.2), (4.3), Lemmas 9 and 10 yield

$$
\begin{aligned}
\|T_k\|_p^p &\leq \sum_{t=0}^{\infty} \|T_k|_{U_t}\|_p^p = \sum_{t=0}^{M-1} + \sum_{t=M}^{\infty} \\
&\leq \sum_{t=0}^{M-1} C(t+1)^{p/2} N \|b\|_{L^p(\partial\mathbb{D})}^p + \sum_{t=M}^{\infty} \Big\| \sum_{j=k+1}^{\infty} T_b^j|_{U_t} \Big\|_p^p \\
&\leq CM^{1+p/2} N \|b\|_{L^p(\partial\mathbb{D})}^p + \sum_{t=M}^{\infty} \sum_{j=k+1}^{\infty} C\Big(\frac{2}{t^p}\Big)^j N \|b\|_{L^p(\partial\mathbb{D})}^p \\
&\leq C\Big(M^{1+p/2} + \sum_{t=M}^{\infty} \Big(\frac{2}{t^p}\Big)^{k+1}\Big) N \|b\|_{L^p(\partial\mathbb{D})}^p \leq CN \|b\|_{L^p(\partial\mathbb{D})}^p . \quad \blacksquare
\end{aligned}
$$

Proof of Lemma 9. Because $T_k(\text{constant}) = 0$, we can assume $b(0) = b_0 = 0$. It is clear that $T_k \prec M_{\overline{b}}$. We only need to show

$$
\|M_{\overline{b}}|_{U_t}\|_p^p \leq C(t+1)^{p/2} N \|b\|_{L^p(\partial\mathbb{D})}^p .
$$

By Lemma D, for $\eta_s = \exp(2\pi si/(2N))$, we have

$$
M_{\overline{b}} = \sum_{j=1}^{N} \overline{b}_j M_{\overline{z}^j} = \sum_{j=1}^{N} \frac{1}{2N} \sum_{s=0}^{2N-1} \overline{b(\eta_s)} \eta_s^j M_{\overline{z}^j} ,
$$

hence

$$
\|M_{\overline{b}}|_{U_t}\|_p^p \leq \Big(\frac{1}{2N}\Big)^p \sum_{s=0}^{2N-1} |b(\eta_s)|^p \Big\| \sum_{j=1}^{N} \eta_s^j M_{\overline{z}^j}|_{U_t} \Big\|_p^p .
$$

By Lemma E, we only need to show

$$
\Big\| \sum_{j=1}^{N} \eta_s^j M_{\overline{z}^j}|_{U_t} \Big\|_p^p \leq C(t+1)^{p/2} N^p . \tag{4.4}
$$

Since

$$
\begin{aligned}
&\Big(\Big\langle \sum_{j=1}^{N} \eta_s^j M_{\overline{z}^j}|_{U_t}(E_{0,m}), \sum_{j=1}^{N} \eta_s^j M_{\overline{z}^j}|_{U_t}(E_{0,n}) \Big\rangle\Big)_{m,n\geq 0} \\
&\quad = \Big(\Big\langle \sum_{j=1}^{N} \eta_s^j M_{\overline{z}^j}(E_{0,m}), \sum_{j=1}^{N} \eta_s^j M_{\overline{z}^j}(E_{0,n}) \Big\rangle\Big)_{m,n\in I_t} \\
&\quad = \Big(\sum_{j=1}^{N} \eta_s^{m-n} \frac{\sqrt{m+1}\sqrt{n+1}}{j+n+1} \Big)_{m,n\in I_t} \\
&\quad = \Big(\eta_s^{m-n} \sum_{j=1}^{N} \frac{\sqrt{m+1}\sqrt{n+1}}{j+n+1} \Big)_{m,n\in I_t}
\end{aligned}
$$

and the matrix

$$
\Big(\sum_{j=1}^{N} \frac{\sqrt{m+1}\sqrt{n+1}}{j+n+1} \Big)_{m,n\in I_t} = \Big(\sqrt{m+1} \sum_{j=1}^{N} \frac{\sqrt{n+1}}{j+n+1} \Big)_{m,n\in I_t}
$$

is a rank one matrix, we have

$$
\begin{aligned}
&\Big\| \sum_{j=1}^{N} \eta_s^j M_{\overline{z}^j}|_{U_t} \Big\|_p^p \\
&= \Big\| \Big(\sum_{j=1}^{N} \eta_s^j M_{\overline{z}^j}|_{U_t} \Big)^* \Big(\sum_{j=1}^{N} \eta_s^j M_{\overline{z}^j}|_{U_t} \Big) \Big\|_{p/2}^{p/2} \\
&= \Big\| \Big(\Big\langle \sum_{j=1}^{N} \eta_s^j M_{\overline{z}^j}|_{U_t}(E_{0,m}), \sum_{j=1}^{N} \eta_s^j M_{\overline{z}^j}|_{U_t}(E_{0,n}) \Big\rangle \Big) \Big\|_{p/2}^{p/2} \\
&= \Big\| \Big(\eta_s^{m-n} \sqrt{m+1} \sum_{j=1}^{N} \frac{\sqrt{n+1}}{j+n+1} \Big)_{m,n\in I_t} \Big\|_{p/2}^{p/2} \\
&= \Big\| \Big(\sqrt{m+1} \sum_{j=1}^{N} \frac{\sqrt{n+1}}{j+n+1} \Big)_{m,n\in I_t} \Big\|_{p/2}^{p/2} \quad \text{(by Lemma F)} \\
&\leq \Big(\Big(\sum_{m\in I_t} (m+1) \Big)^{1/2} \Big(\sum_{n\in I_t} \Big(\sum_{j=1}^{N} \frac{\sqrt{n+1}}{j+n+1} \Big)^2 \Big)^{1/2} \Big)^{p/2} \quad \text{(by Lemma G)} \\
&\leq \Big(\sqrt{t+1} N \sqrt{(t+1)N} \Big(\sum_{n\in I_t} \Big(\log \frac{N+n+1}{n+1} \Big)^2 \Big)^{1/2} \Big)^{p/2} .
\end{aligned}
$$

To complete the proof of (4.4), and hence of the lemma, we need only estimate the sum

$$
\begin{aligned}
\sum_{n\in I_t} \Big(\log \frac{N+n+1}{n+1} \Big)^2 &\approx \int_{tN}^{(t+1)N} \Big(\log \frac{N+x+1}{x+1} \Big)^2 dx \\
&= N \int_{t}^{t+1} \Big(\log \frac{r+2}{r} \Big)^2 dr \leq N \int_{0}^{\infty} \Big(\log \frac{r+2}{r} \Big)^2 dr = CN . \quad \blacksquare
\end{aligned}
$$

Proof of Lemma 10. By Lemma 3 we have

$$
\langle T_b^j(E_{0,m}), E_{j,n} \rangle = \overline{b}_{m-n} \sqrt{\frac{n+1}{m+1}} \Big(\frac{m-n}{m+1} \Big)^j . \tag{4.5}
$$

Notice that if $m \in I_t$ then the right hand side of (4.5) is 0 unless $n \in \widetilde{I}_t = I_{t-1} \cup I_t \cup I_{t+1}$.

Let $b_s = 0$ for $s \in [-2N, -1] \cup [N+1, 2N]$, and

$$\widetilde{b}(z) = b(z) z^{2N} = \sum_{s=-2N}^{2N} b_s z^{s+2N}.$$

Then $\widetilde{b}$ is a $4N$th degree polynomial. By Lemmas D and E, for $\xi_k = \exp(2\pi k i/(8N))$, $k = 0, 1, \ldots, 8N-1$, we have

$$b_j = \frac{1}{8N} \sum_{k=0}^{8N-1} \widetilde{b}(\xi_k) \overline{\xi}_k^{j+2N} = \frac{1}{8N} \sum_{k=0}^{8N-1} b(\xi_k) \overline{\xi}_k^{j}, \qquad j = -2N, -2N+1, \ldots, 2N,$$

and

$$\int_0^{2\pi} |b(e^{i\theta})|^p \, d\theta = \int_0^{2\pi} |\widetilde{b}(e^{i\theta})|^p \, d\theta \approx \frac{1}{8N} \sum_{k=0}^{8N-1} |\widetilde{b}(\xi_k)|^p = \frac{1}{8N} \sum_{k=0}^{8N-1} |b(\xi_k)|^p.$$

Now for $m \in I_t$ and $n \in \widetilde{I}_t$ we have

$$\langle T_b^j (E_{0,m}), E_{j,n} \rangle = \frac{1}{8N} \sum_{k=0}^{8N-1} \overline{b(\xi_k)} \xi_k^{m-n} \sqrt{\frac{n+1}{m+1}} \left(\frac{m-n}{m+1} \right)^j = \frac{1}{8N} \sum_{k=0}^{8N-1} \overline{b(\xi_k)} \left(\xi_k^{m-n} \sqrt{\frac{n+1}{m+1}} \left(\frac{m-n}{m+1} \right)^j \right).$$

Hence

$$\|T_b^j|_{U_t}\|_p^p = \|(\langle T_b^j(E_{0,m}), E_{j,n}\rangle)_{m \in I_t, n \in \tilde{I}_t}\|_p^p \le \left(\frac{1}{8N}\right)^p \sum_{k=0}^{8N-1} |b(\xi_k)|^p \left\| \left(\xi_k^{m-n} \sqrt{\frac{n+1}{m+1}} \left(\frac{m-n}{m+1} \right)^j \right)_{m \in I_t, n \in \tilde{I}_t} \right\|_p^p.$$

By Lemma F we have

$$\left\| \left(\xi_k^{m-n} \sqrt{\frac{n+1}{m+1}} \left(\frac{m-n}{m+1} \right)^j \right)_{m \in I_t, n \in \tilde{I}_t} \right\|_p^p = \left\| \left(\sqrt{\frac{n+1}{m+1}} \left(\frac{m-n}{m+1} \right)^j \right)_{m \in I_t, n \in \tilde{I}_t} \right\|_p^p = \left\| \left(\sqrt{\frac{n+1}{m+1}} \sum_{s=0}^{j} \binom{j}{s} \frac{(m-tN)^s}{(m+1)^j} (tN-n)^{j-s} \right)_{m \in I_t, n \in I_t} \right\|_p^p$$

and using the p-triangle inequality and Lemma G we can continue with

$$\le \sum_{s=0}^{j} \binom{j}{s}^p \Big(\sum_{m \in I_t} \frac{(m-tN)^{2s}}{(m+1)^{2j+1}} \Big)^{p/2} \Big(\sum_{n \in \tilde{I}_t} (n+1)(tN-n)^{2j-2s} \Big)^{p/2} \le \sum_{s=0}^{j} \binom{j}{s}^p \left(\frac{N}{t^j} \right)^p \le 2^j \left(\frac{N}{t^j} \right)^p.$$

Hence

$$\|T_b^j|_{U_t}\|_p^p \le \left(\frac{1}{8N} \right)^p \sum_{k=0}^{8N-1} |b(\xi_k)|^p 2^j \left(\frac{N}{t^j} \right)^p \le C \left(\frac{2}{t^p} \right)^j N \|b\|_{L^p(\partial \mathbb{D})}^p. \ \blacksquare$$

5. Other operators. In this section we look at two other operators related to the decomposition we have been using.

Recall that P^k is the orthogonal projection from L^2 onto A^k. Similarly, let $\overline{P}^k$ be the orthogonal projection from L^2 onto $\overline{A}^k$. Consider the operators

$$h_b^{(k,j)} : A^j \to \overline{A}^k, \quad g \mapsto \overline{P}^k(\overline{b} g),$$
$$T_b^{(k,j)} : A^j \to A^k, \quad g \mapsto P^k(\overline{b} g).$$

$h_b^{(k,j)}$ and $T_b^{(k,j)}$ can be viewed as generalized small Hankel operators and Toeplitz operators. In fact,

$$h_b^{(0,0)} = h_b \quad \text{and} \quad T_b^{(0,0)} = T_b.$$

Let $b(z) = \sum_{s=0}^{\infty} b_s E_{t,s}(z) \in A^t$. Then

$$\langle h_b^{(k,j)}(E_{j,m}), \overline{E}_{k,n} \rangle = \langle \overline{P}^k(\overline{b} E_{j,m}), \overline{E}_{k,n} \rangle = \langle \overline{b} E_{j,m}, \overline{E}_{k,n} \rangle = \sum_{s=0}^{\infty} \overline{b}_s \langle E_{j,m} E_{k,n}, E_{t,s} \rangle = \overline{b}_{m+n} \langle E_{j,m} E_{k,n}, E_{t,m+n} \rangle.$$

LEMMA 11. *If $k + j = t$, then*

$$\langle E_{j,m} E_{k,n}, E_{t,m+n} \rangle = \frac{t!}{k!j!} \frac{(m+1)^{j+1/2}(n+1)^{k+1/2}}{(m+n+1)^{t+1/2}}.$$

Proof. Using the definitions we find

$$\langle E_{j,m} E_{k,n}, E_{t,m+n} \rangle = \int_0^1 P_{j,m}(r) P_{k,n}(r) P_{t,m+n}(r) r^{m+n} \, dr.$$

Because $P_{j,m}(r) P_{k,n}(r)$ is a polynomial in $\log r$ of degree $j + k = t$, by Lemma B′, we have

$$P_{j,m}(r) P_{k,n}(r) = \sum_{s=0}^{t} c_s P_{s,m+n}(r)$$

with appropriate constants c_s. The tth order coefficients of $P_{t,m+n}$ and $P_{j,m}P_{k,n}$ are

$$\frac{(m+n+1)^{t+1/2}}{t!} \quad \text{and} \quad \frac{(m+1)^{j+1/2}(n+1)^{k+1/2}}{j!k!}$$

respectively. We have

$$c_t = \frac{t!}{k!j!}\frac{(m+1)^{j+1/2}(n+1)^{k+1/2}}{(m+n+1)^{t+1/2}}.$$

Hence using Lemma B′ we get

$$\langle E_{j,m}E_{k,n}, E_{t,m+n}\rangle = \int_0^1 \sum_{s=0}^t c_s P_{s,m+n}(r)P_{t,m+n}(r)r^{m+n}\,dr = c_t. \ ■$$

LEMMA 12.

$$\langle E_{0,m}E_{k,n}, E_{0,m+n}\rangle = \frac{m^k\sqrt{m+1}\sqrt{n+1}}{(m+n+1)^{k+1/2}}.$$

Proof. Straightforward. ■

Recall that J_t is the unitary map of $A^0 = A$ to A^t.

THEOREM 13. *Suppose* $0 < p \le \infty$. *Then*

(1) *for* $b \in A^t$ *and* $k + j = t$, $h_b^{(k,j)} \in S_p$ *if and only if* $J_t^{-1}(b) \in B_p$;

(2) *for* $b \in A$, $h_b^{(k,0)} \in S_p$ *if and only if* $b \in B_p$.

Proof. These are consequences of Lemmas 11, 12 and the results in [Pel1] and [S] about operators which have matrix entries, a_{mn}, of the form which shows up in Lemmas 11 and 12. ■

THEOREM 14. *If* b *is analytic and* $k \ge 1$ *then*

(1) *for* $1/k < p \le \infty$, $T_b^{(k,0)} \in S_p$ *if and only if* $b \in B_p$;

(2) *for* $0 < p \le 1/k$, $T_b^{(k,0)} \in S_p$ *if and only if* $b = \text{constant}$.

Proof. Suppose $b(z) = \sum_{n=0}^\infty b_n z^n$. Then

$$(5.1)\quad (\langle T_b^{(k,0)}(E_{0,m}), E_{k,n}\rangle)_{m,n\ge0} = \left(\overline{b}_{m-n}\sqrt{\frac{n+1}{m+1}}\left(\frac{m-n}{m+1}\right)^k\right)_{m,n\ge0}.$$

For (1), Section 4 gives the proof of "if", we now prove "only if".

Denote the matrix of (5.1) by M_k. Let $\{e_n\}_{n\ge0}$ be an orthogonal basis of l^2. For $j \ge 0$ and fixed $\zeta \in \mathbb{D}$, consider

$$\phi_j(\zeta) = (1-|\zeta|^2)^j \sum_{n=0}^\infty (n+1)^{j+1/2}\overline{\zeta}^n e_n, \quad j \ge 0.$$

It is easy to check $\|\phi_j(\zeta)\|_{l^2} \approx 1$. Let $\{\zeta_t\}$ be a δ-lattice in $\mathbb{D}$. By a result in [R2] we know that, for δ small enough, $\{\phi_j(\zeta_t)\}_{t\ge0}$ is WO in l^2 (see [R2] for the definition of WO and related properties).

Computation shows that for $t \ge 0$

$$M_k(\phi_k(\zeta_t)) = (1-|\zeta_t|^2)^k\overline{D}^k\overline{b}(\overline{\zeta}_t)\phi_0(\zeta_t),$$

and thus

$$\langle M_k(\phi_k(\zeta_t)), \phi_0(\zeta_t)\rangle \approx (1-|\zeta_t|^2)^k\overline{D}^k\overline{b}(\overline{\zeta}_t).$$

Hence for $p > 1/k$, by a result in [R2] if $p > 1$ and by Semmes' method in [S] if $p \le 1$, we have

$$\sum_{t=0}^\infty |\overline{D}^k\overline{b}(\overline{\zeta}_t)|^p(1-|\zeta_t|^2)^{kp} \le C\|M_k\|_p^p = C\|T_b^{(k,0)}\|_p^p.$$

This is equivalent to

$$\int_{\mathbb{D}} |D^k b(\zeta)|^p(1-|\zeta|^2)^{pk-2}\,dA(\zeta) \le C\|T_b^{(k,0)}\|_p^p,$$

i.e. $\|b\|_p \le C\|T_b^{(k,0)}\|_p$.

For part (2), as in Section 3, it is enough to prove it for monomial symbols (see Lemma 7).

Let $b(z) = z^s$. Then nonzero entries in the matrix (5.1), which are entries on the subdiagonal $(n+s, n)$, are

$$\sqrt{\frac{n+1}{n+s+1}}\left(\frac{s}{n+s+1}\right)^k, \quad n = 0, 1, 2, \ldots$$

These numbers are also all the singular values of $T_b^{(k,0)}$. Hence

$$\|T_b^{(k,0)}\|_p^p = \sum_{n=0}^\infty \left(\sqrt{\frac{n+1}{n+s+1}}\left(\frac{s}{n+s+1}\right)^k\right)^p \approx \sum_{n=0}^\infty (n+1)^{-kp}.$$

This implies $T_b^{(k,0)} \in S_p$ if and only if $kp > 1$. ■

THEOREM 15. *If* b *is in* A^t *and* $t + k = j$, *then*

(1) *for* $1/t < p \le \infty$, $T_b^{(k,j)} \in S_p$ *if and only if* $J_t^{-1}(b) \in B_p$;

(2) *for* $0 < p \le 1/t$, $T_b^{(k,0)} \in S_p$ *if and only if* $b = \text{constant}$.

Proof. Suppose $b(z) = \sum_{n=0}^\infty b_n E_{t,n}(z)$. By Lemma 12 we have

$$(\langle T_b^{(k,j)}(E_{j,m}), E_{k,n}\rangle)_{m,n\ge0} = \frac{t!k!}{j!}\left(\overline{b}_{m-n}\frac{(m-n)^{t+1/2}(n+1)^{k+1/2}}{(m+1)^{j+1/2}}\right)_{m,n\ge0}.$$

The proof is now similar to that of Theorem 14. ■

Remarks. (1) Similar results can be obtained for the weighted space $L^2(dA_\alpha^{(j)})$ with $dA_\alpha^{(1)}(z) = \Gamma(\alpha+1)^{-1}(-\log|z|^2)^\alpha dA(z)$ or $dA_\alpha^{(2)}(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$, $\alpha > -1$. For $dA_\alpha^{(1)}$ the results follow by straightforward extension of what we have done, using the Laguerre polynomials L_n^α. On the other hand, $dA_\alpha^{(2)}$ is the measure generally considered when studying "weighted Bergman spaces". Although the Laguerre formalism does not work well in that case, the results involving $dA_\alpha^{(2)}$ can be derived from those with $dA_\alpha^{(1)}$ by showing that the two associated operators differ by an operator which can be estimated well. This general theme, that some of Hankel and Toeplitz operator theory is rather stable under mild change of measure, is developed in [J2]. In particular, see Theorem 2 and Example 4 of [J2]. Alternatively, $dA_\alpha^{(2)}$ can be studied by using Jacobi polynomials (see [PX]).

(2) We conjecture that if b is nice then H_b^k is in the weak Schatten–Lorentz space $S_{1/(k+1),\infty}$. The case $k=0$ is in [N].

(3) Similar spaces can be studied on the half plane. Some results are in [JP].

(4) A_k is a closed subspace of L^2 characterized as a solution to a simple PDE. It would be interesting to know about its function theory, in particular how the theory varies *as a function of k*. As a simple example, how does $\sup_{g\in A_k,\|g\|=1}|\nabla g(0)|$ vary with k?

(5) $T_b^{(k,j)}$ with b in A^t can be viewed as generalized Toeplitz operators. For $t+k<j$, it is easy to check that $T_b^{(k,j)}$ is a zero operator. Our results are for $t+k=j$. For $t+k>j>0$, the situation seems more complicated. One can also study the family of generalized Hankel type operators $H_b^{(k,j)} : A^j \to A_k^\perp$ defined by

$$H_b^{(k,j)}(g) = (I-P_k)(\overline{b}g)\,.$$

$H_b^{(k,0)}$ is just our H_b^k.

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DEPARTMENT OF MATHEMATICS
PEKING UNIVERSITY
BEIJING 100871, P.R. CHINA

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALABAMA
TUSCALOOSA, ALABAMA 35487
U.S.A.
E-mail: ZWU@UA1VM.UA.EDU

DEPARTMENT OF MATHEMATICS
WASHINGTON UNIVERSITY
ST. LOUIS, MISSOURI 63130
U.S.A.
E-mail: RR@JEZEBEL.WUSTL.EDU

Received September 10, 1991 (2805)