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On multilinear fractional integrals

by

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Abstract. In $\mathbb{R}^n$, we prove $L^{p_1} \times \ldots \times L^{p_K}$ boundedness for the multilinear fractional integrals $I_\alpha(f_1, \ldots, f_K)(x) = \int f_1(x-\theta_1 y) \ldots f_K(x-\theta_K y)|y|^{\alpha-n}\,dy$ where the θ_j's are nonzero and distinct. We also prove multilinear versions of two inequalities for fractional integrals and a multilinear Lebesgue differentiation theorem.

1. Introduction. Although it is not known whether the bi(sub)linear maximal function

$$M(f,g)(x) = \sup_{N>0} \frac{1}{2N} \int_{-N}^{N} |f(x+t)g(x-t)|\,dt$$

or the bilinear Hilbert transform

$$H(f,g)(x) = \text{p.v.} \int f(x+t)g(x-t)\,\frac{dt}{t}$$

map $L^p(\mathbb{R}^1) \times L^{p'}(\mathbb{R}^1) \to L^1(\mathbb{R}^1)$ boundedness into L^1 for the corresponding multilinear fractional integrals can be obtained.

Throughout this note, K will denote an integer ≥ 2 and θ_j, $j = 1, \ldots, K$, will be fixed, distinct and nonzero real numbers. We are going to be working in $\mathbb{R}^n$ and α will be a fixed real number number strictly between 0 and n. We denote by $\mathbf{f}$ the K-tuple $(f_1, \ldots, f_K)$ and by I_α the K-linear fractional integral operator defined as follows:

$$I_\alpha(\mathbf{f})(x) = \int f_1(x-\theta_1 y) \ldots f_K(x-\theta_K y)|y|^{\alpha-n}\,dy\,.$$

When $K = 1$ the operators I_α are the usual fractional integrals as studied in [ST]. We also denote by $M(\mathbf{f})$ the K-sublinear maximal function

$$M(\mathbf{f})(x) = \sup_{N>0} (\Omega_n N^n)^{-1} \int |f_1(x-\theta_1 y)| \ldots |f_K(x-\theta_K y)|\,dy$$

where Ω_n is the volume of the unit ball in $\mathbb{R}^n$. It is trivial to check that for any positive $p_1, \ldots, p_K$ with harmonic mean $s > 1$, M maps $L^{p_1} \times \ldots \times L^{p_K}$ into L^s. If we denote by f^* the Hardy–Littlewood maximal function of f, then $M(\mathbf{f})$ is dominated by the product $C_{\theta_k}((f_1^{p_1/s})^*)^{s/p_1}\dots((f_1^{p_K/s})^*)^{s/p_K}$ and hence its boundedness follows from Hölder's inequality and the L^s boundedness of the Hardy–Littlewood maximal function. This argument breaks down when $s=1$ but a slight modification of it gives that M maps into weak L^1 in this endpoint case. It is conceivable, however, that M map into L^1 since it carries K-tuples of compactly supported functions into compactly supported functions. This problem remains unsolved. The $L^p\times L^q\to L^r$ boundedness of the bilinear Hilbert transform $H(f,g)$ is more subtle and it remains unsolved even in the case $r>1$.

In this note, we study the easier problem of the multilinear fractional integrals. Our first result concerns the $L^{p_1}\times\dots\times L^{p_K}\to L^r$ boundedness of I_α for $r\ge 1$.

THEOREM 1. *Let s be the harmonic mean of $p_1,\dots,p_K>1$ and let r be such that $1/r+\alpha/n=1/s$. Then I_α maps $L^{p_1}\times\dots\times L^{p_K}$ into L^r for $n/(n+\alpha)\le s<n/\alpha$ (equivalently $1\le r<\infty$).*

Note that in the case $K=1$, the corresponding range of s is the smaller interval $1<s<n/\alpha$ (equivalently $n/(n-\alpha)<r<\infty$).

When $K=1$, the following theorem has been proved by Hirschman [HI] for periodic functions and by Hedberg [HE] for positive functions.

THEOREM 2. *Let p_j be positive real numbers and let $s>1$ be their harmonic mean. Then for $q,r>1$ and $0<\theta<1$,*

$$\|I_{\alpha\theta}(\mathbf{f})\|_{L^r}\le C\|I_\alpha(\mathbf{f})\|_{L^q}^{\theta}\prod_k\|f_k\|_{L^{p_k}}^{1-\theta}\quad \textit{where}\quad \frac{1}{r}=\frac{\theta}{q}+\frac{1-\theta}{s}.$$

In the endpoint case $s=n/\alpha$, Trudinger [T] for $\alpha=1$, and Strichartz [STR] for other α proved exponential integrability of I_α when $K=1$. Hedberg [HE] gave a proof of Theorem 3 below when $K=1$.

By ω_{n-1} we denote the area of the unit sphere $\mathbf{S}^{n-1}$. The factor L in the exponent below is a normalizing factor and should be there by homogeneity.

THEOREM 3. *Let $s=n/\alpha$ be the harmonic mean of $p_1,\dots,p_K>1$. Let B be a ball of radius R in $\mathbb{R}^n$ and let $f_j\in L^{p_j}(B)$ be supported in B. Then for any $\gamma<1$, there exists a constant $C_0(\gamma)$ depending only on n, α, the θ_j's and on γ such that*

$$(1.1)\qquad \int_B \exp\left(\frac{n}{\omega_{n-1}}\gamma\left(\frac{LI_\alpha(f_1,\dots,f_K)}{\|f_1\|_{L^{p_1}}\dots\|f_K\|_{L^{p_K}}}\right)^{n/(n-\alpha)}\right)dx\le C_0(\gamma)R^n$$

$$\textit{where}\quad L=\prod_k|\theta_k|^{n/p_k}.$$

All the comments in this paragraph refer to the case $K=1$. Hempel *et al.* [HMT] (for $\alpha=1$) and later Adams [A] (for all α) showed that inequality (1.1) cannot hold if $\gamma>1$. Moser [M] showed exponential integrability of $n\omega_{n-1}^{1/(n-1)}(|\phi(x)|/\|\nabla\phi\|_{L^n})^{n/(n-1)}$ suggesting that Theorem 3 be true in the endpoint case $\gamma=1$. (Use formula (18), p. 125 in [ST] to show that Moser's result follows from an improved Theorem 3 with $\gamma=1$.) In fact, Adams [A] proved inequality (1.1) in the endpoint case $\gamma=1$ and also deduced the sharp constants for Moser's exponential inequality for higher order derivatives. Chang and Marshall [CM] proved a similar sharp exponential inequality concerning the Dirichlet integral.

When $\gamma>1$, Theorem 3 does not hold either, while the case $\gamma=1$ remains open when $K\ge 2$.

2. Proof of Theorem 1.

We denote by $|B|$ the measure of the set B and by χ_A the characteristic function of the set A. We also use the notation $s'=s/(s-1)$ for $s\ge 1$.

We consider first the case $s\ge 1$. We will show that I_α maps $L^{p_1}\times\dots\times L^{p_K}\to L^{r,\infty}$. The required result when $s>1$ is going to follow from an application of the Marcinkiewicz interpolation theorem. Without loss of generality we can assume that $f_j\ge 0$ and that $\|f_j\|_{L^{p_j}}=1$. Fix a $\lambda>0$ and define $\mu>0$ by

$$L^{-1}\left(\frac{\omega_{n-1}}{(\alpha-n)s'+n}\right)^{1/s'}\mu^{-n/r}=\frac{\lambda}{2}$$

where ω_{n-1} and L are as in Theorem 3. Hölder's inequality and our choice of μ give

$$(2.1)\qquad I_\alpha^\infty(\mathbf{f})(x)=\int_{|y|>\mu}f_1(x-\theta_1y)\dots f_K(x-\theta_Ky)|y|^{\alpha-n}\,dy$$
$$\le\Big\|\prod f_k(x-\theta_ky)\Big\|_{L^s(y)}\||y|^{\alpha-n}\chi_{|y|>\mu}\|_{L^{s'}}$$
$$\le\prod\|f_k(x-\theta_ky)\|_{L^{p_k}(y)}\left(\frac{\omega_{n-1}}{(\alpha-n)s'+n}\right)^{1/s'}\mu^{n(\alpha/n-1+1/s')}=\frac{\lambda}{2}.$$

Let $I_\alpha^0(\mathbf{f})(x)=\int_{|y|\le\mu}f_1(x-\theta_1y)\dots f_K(x-\theta_Ky)|y|^{\alpha-n}\,dy$. We compute its L^s norm:

$$(2.2)\qquad \|I_\alpha^0(\mathbf{f})\|_{L^s}\le\Big\|\Big(\int\Big(\prod f_k\Big)^s|y|^{\alpha-n}\chi_{|y|\le\mu}dy\Big)^{1/s}$$
$$\times\Big(\int 1^{s'}|y|^{\alpha-n}\chi_{|y|\le\mu}\,dy\Big)^{1/s'}\Big\|_{L^s}$$

$$\le C\mu^{\alpha/s'}\Big(\int\int\Big(\prod f_k\Big)^s |y|^{\alpha-n}\chi_{|y|\le\mu}\,dx\,dy\Big)^{1/s}$$
$$\le C\mu^{\alpha/s'}\Big(\prod\|f_k\|^s_{L^{p_k}}\int_{|y|\le\mu}|y|^{\alpha-n}\,dy\Big)^{1/s}$$
$$= C\mu^{\alpha/s'}\mu^{\alpha/s} = C\mu^{\alpha}\,.$$

By (2.1) the set $\{x : I_\alpha^\infty(\mathbf{f})(x) > \lambda/2\}$ is empty. This fact together with Chebyshev's inequality and (2.2) gives

$$|\{x : I_\alpha(\mathbf{f})(x) > \lambda\}| \le |\{x : I_\alpha^0(\mathbf{f})(x) > \lambda/2\}|$$
$$\le 2^s\lambda^{-s}\|I_\alpha^0\|^s_{L^s} \le C\lambda^{-s}\mu^{s\alpha} = C_{\theta_k}\lambda^{-r}\,,$$

which is the required weak type estimate for I_α.

We now do the case $n/(n+\alpha)\le s\le 1$. The corresponding range of r's is $1\le r\le n/(n-\alpha)$. Assume that $K=2$ and that $p_1\ge p_2>1$. Also assume that $r=1$ first. Since $s<n/\alpha$ we must have $p_2<n/\alpha$. For $r=1$ we get

$$\text{(2.3)}\quad \|I_\alpha(f_1,f_2)\|_{L^1} = \int\int f_1(x-\theta_1 y)f_2(x-\theta_2 y)|y|^{\alpha-n}\,dx\,dy$$
$$= \int f_1(x)\int f_2(x-(\theta_2-\theta_1)y)|y|^{\alpha-n}\,dy\,dx$$
$$= |\theta_2-\theta_1|^{-\alpha}\int f_1(x)I_\alpha(f_2)(x)\,dx$$
$$\le C_{\theta_1,\theta_2}\|f_1\|_{L^{p_1}}\|I_\alpha(f_2)\|_{L^{p_1'}}\,.$$

Note that $r=1$ implies $1/p_1'+\alpha/n=1/p_2$. Since $1<p_2<n/\alpha$, we can apply Theorem 1, Ch. V on fractional integrals in [ST] to bound (2.3) by $C_{\theta_1,\theta_2}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}$. The case of general $r>1$ follows by interpolating between the endpoint case $r=1$ and the case of r close to ∞. Suppose now that the theorem is true for $K-1$, $K\ge 3$. We will show that it is true for K. Again we first do the case $r=1$. We may assume without loss of generality that $p_1\ge\ldots\ge p_K>1$. Now,

$$\text{(2.4)}\quad \|I_\alpha(\mathbf{f})\|_{L^1} = \int\int f_1(x-\theta_1 y)\ldots f_K(x-\theta_K y)|y|^{\alpha-n}\,dx\,dy$$
$$= \int f_1(x)\int f_2(x-(\theta_2-\theta_1)y)\ldots f_K(x-(\theta_K-\theta_1)y)|y|^{\alpha-n}\,dy\,dx$$
$$= \prod_{k\ne 1}|\theta_k-\theta_1|^{-\alpha}\int f_1(x)I_\alpha(f_2,\ldots,f_K)(x)\,dx$$
$$\le C_{\theta_k}\|f_1\|_{L^{p_1}}\|I_\alpha(f_2,\ldots,f_K)\|_{L^{p_1'}}\,.$$

Define s_1 by $1/s_1=1/s-1/p_1$. Since $r=1$, we have $1/p_1'+\alpha/n=1/s_1$. We can apply the induction hypothesis only provided $n/(n+\alpha)\le s_1<n/\alpha$. This inequality follows from the identity $1+\alpha/n=1/s$, which relates s and $r=1$. From our induction hypothesis we deduce that (2.4) is bounded by $C_{\theta_j}\prod\|f_k\|_{L^{p_k}}$. The case $r\ge 1$ follows by interpolation.

3. Proof of Theorem 2. As in the proof of Theorem 1, fix $f_j\ge 0$ such that $\|f_j\|_{L^{p_j}}=1$. As in [HE], split

$$I_{\alpha\theta}(\mathbf{f})(x) = \Big(\int_{|y|<\delta}+\int_{|y|\ge\delta}\Big)\prod f_k(x-\theta_k y)|y|^{\alpha\theta-n}\,dy$$
$$\le \sum_{m=1}^{\infty}\int_{|y|\sim\delta 2^{-m}}\prod f_k(x-\theta_k y)|y|^{\alpha\theta-n}\,dy$$
$$+\int_{|y|\ge\delta}\prod f_k(x-\theta_k y)|y|^{\alpha-n}|y|^{(\theta-1)\alpha}\,dy$$
$$\le \sum_{m=1}^{\infty}(\delta 2^{-m})^{\alpha\theta}\int_{|y|\sim\delta 2^{-m}}\prod f_k(x-\theta_k y)|y|^{-n}\,dy$$
$$+\delta^{(\theta-1)\alpha}\int_{|y|\ge\delta}\prod f_k(x-\theta_k y)|y|^{\alpha-n}\,dy$$
$$\le C\delta^{\alpha(\theta-\varepsilon)}M(\mathbf{f})(x)+\delta^{\alpha(\theta-1)}I_\alpha(\mathbf{f})(x)\,.$$

Now choose $\delta=(I_\alpha(\mathbf{f})(x)/M(\mathbf{f})(x))^{1/\alpha}$ to get

$$I_{\alpha\theta}(\mathbf{f})(x)\le C(I_\alpha(\mathbf{f})(x))^\theta\,(M(\mathbf{f})(x))^{1-\theta}\,.$$

Hölder's inequality with exponents

$$1/r = 1\Big/\Big(\frac{s}{1-\theta}\Big)+1\Big/\Big(\frac{q}{\theta}\Big)$$

will give

$$\|I_{\alpha\theta}(\mathbf{f})\|_{L^r}\le C\|I_\alpha(\mathbf{f})^\theta\|_{L^{q/\theta}}\|M(\mathbf{f})\|_{L^{s/(1-\theta)}}$$
$$= C\|I_\alpha(\mathbf{f})\|^\theta_{L^q}\|M(\mathbf{f})\|^{1-\theta}_{L^s}\le C\|I_\alpha(\mathbf{f})\|^\theta_{L^q}$$

by the boundedness of the maximal function M on L^q. This concludes the proof of Theorem 2.

4. Proof of Theorem 3. A simple dilation argument shows that if we know Theorem 3 for a specific value of $R=R_0$ with a constant $C_0'(\gamma)$ on the right hand side of (1.1), then we also know it for all other values of R with constant $C_0'(\gamma)(R/R_0)^n$. We select $R_0=1/P$ where $P=2\min|\theta_k|^{-1}$ and we will assume that the radius of B is R_0. Furthermore, we can assume that the f_j's satisfy $f_j\ge 0$ and $\|f_j\|_{L^{p_j}}=1$. Now fix $x\in B$. The same argument as in Theorem 2 with $\theta=1$ gives

$$\text{(4.1)}\qquad I_\alpha(\mathbf{f})(x)\le C\delta^\alpha M(\mathbf{f})(x)+\int_{|y|\ge\delta}\prod f_k(x-\theta_k y)|y|^{\alpha-n}\,dy\,.$$

Since all f_k are supported in the ball B and $x \in B$ the integral in (4.1) is over the set $\{y : \delta \le |y| \le PR_0 = 1\}$. Hölder's inequality with exponents $p_1, \ldots, p_K$ and $n/(n-\alpha)$ gives

$$(4.2) \quad \int_{\delta\le|y|\le1} \prod f_k(x-\theta_k y)|y|^{\alpha-n}\,dy \le \prod \|f_k(x-\theta_k y)\|_{L^{p_k}(y)} \Big(\int_{\delta\le|y|\le1} |y|^{-n}\,dy\Big)^{(n-\alpha)/n} = L^{-1}\Big(\omega_{n-1}\ln\frac{1}{\delta}\Big)^{(n-\alpha)/n}.$$

Combining (4.1) and (4.2) we get

$$(4.3) \quad I_\alpha(\mathbf{f})(x) \le C\delta^\alpha M(\mathbf{f})(x) + L^{-1}\Big(\frac{\omega_{n-1}}{n}\ln\Big(\frac{1}{\delta}\Big)^n\Big)^{(n-\alpha)/n}.$$

The choice $\delta = 1$ gives $I_\alpha(\mathbf{f})(x) \le CM(\mathbf{f})(x)$ for all $x \in B$ and therefore the selection $\delta = \delta(x) = \varepsilon(I_\alpha(\mathbf{f})(x)(CM(\mathbf{f})(x))^{-1})^{1/\alpha}$ will satisfy $\delta \le 1$ for all $\varepsilon \le 1$. (4.3) now implies

$$I_\alpha(\mathbf{f})(x) \le \varepsilon^\alpha I_\alpha(\mathbf{f})(x) + L^{-1}\Big(\frac{\omega_{n-1}}{n}\ln\Big(\frac{(CM(\mathbf{f})(x))^{n/\alpha}}{\varepsilon^n I_\alpha(\mathbf{f})(x)^{n/\alpha}}\Big)\Big)^{(n-\alpha)/n}.$$

Algebraic manipulation of the above gives

$$(4.4) \quad \frac{n}{\omega_{n-1}}\gamma(LI_\alpha(\mathbf{f})(x))^{n/(n-\alpha)} \le \ln\Big(\frac{(CM(\mathbf{f})(x))^{n/\alpha}}{\varepsilon^n I_\alpha(\mathbf{f})(x)^{n/\alpha}}\Big)$$

where we set $\gamma = (1-\varepsilon^\alpha)^{n/(n-\alpha)}$. We exponentiate (4.4) and we integrate over the set $B_1 = \{x \in B : I_\alpha(\mathbf{f})(x) \ge 1\}$ to obtain

$$\int_{B_1} \exp\Big(\frac{n}{\omega_{n-1}}\gamma(LI_\alpha(\mathbf{f})(x))^{n/(n-\alpha)}\Big)\,dx \le \frac{1}{\varepsilon^n}\int_{B_1}\frac{(CM(\mathbf{f})(x))^{n/\alpha}}{I_\alpha(\mathbf{f})(x)^{n/\alpha}}\,dx \le \frac{C_1}{\varepsilon^n}\int M(\mathbf{f})(x)^{n/\alpha}\,dx \le \frac{C_2}{\varepsilon^n}.$$

The last inequality follows from the boundedness of the maximal function of $\mathbf{f}$ on $L^{n/\alpha}$. The integral of the same exponential over the set $B_2 = B - B_1$ is estimated trivially by

$$\int_{B_2} \exp\Big(\frac{n}{\omega_{n-1}}\gamma(LI_\alpha(\mathbf{f})(x))^{n/(n-\alpha)}\Big)\,dx \le \exp\Big(\frac{n}{\omega_{n-1}}L^{n/(n-\alpha)}\Big)|B_2| \le C_3\Omega_n R_0^n = C_4.$$

Adding the integrals above over B_1 and B_2 we obtain the required inequality with a constant $C_0'(\gamma) = \max(C_2, C_4)(1+(1-\gamma^{(n-\alpha)/n})^{-n/\alpha})$. The constant $C_0(\gamma)$ in the statement of Theorem 3 is then $C_0'(\gamma)R_0^{-n} = C_0'(\gamma)P^n$. We obtain the following

Corollary. *Let B, f_k, p_k, and s be as in Theorem 3. Then $I_\alpha(f_1,\ldots,f_K)$ is in $L^q(B)$ for every $q > 0$. In fact,*

$$\|I_\alpha(f_1,\ldots,f_K)\|_{L^q(B)} \le C\prod_k \|f_k\|_{L^{p_k}}$$

for some constant C depending only on q, n, α and the θ_j's.

The corollary follows since exponential integrability of I_α implies integrability to any power q. (Here γ is fixed < 1.)

5. A multilinear differentiation theorem. We end this note by proving the following multilinear Lebesgue differentiation theorem.

Let $f_j \in L^{p_j}(\mathbb{R}^n)$ and suppose that the harmonic mean of $p_1,\ldots,p_K$ is $s \ge 1$. Then

$$\lim_{\varepsilon\to0} T_\varepsilon(\mathbf{f})(x) = \lim_{\varepsilon\to0}\frac{1}{\Omega_n\varepsilon^n}\int_{|y|\le\varepsilon} f_1(x-\theta_1 y)\ldots f_K(x-\theta_K y)\,dy = f_1(x)\ldots f_K(x) \quad a.e.$$

The case $s = 1$ is a consequence of the weak type inequality

$$|\{x \in \mathbb{R}^n : M(\mathbf{f})(x) > \lambda\}| \le \frac{C}{\lambda}\|f_1\|_{L^{p_1}}\ldots\|f_1\|_{L^{p_K}},$$

which is easily obtained from

$$|\{x \in \mathbb{R}^n : M(\mathbf{f})(x) > \lambda\}| \le \sum_{j=1}^K |\{x \in \mathbb{R}^n : (f_j)^*(x) > (\varepsilon_{j-1}/\varepsilon_j)^{p_j}\}| \le C\sum_{j=1}^K(\varepsilon_{j-1}/\varepsilon_j)^{-p_j}\|f_j\|_{L^{p_j}}$$

after minimizing over all $\varepsilon_1,\ldots,\varepsilon_K > 0$. (Take $\varepsilon_0 = \lambda$.) The standard argument presented in [SWE], p. 61, will prove that the sequence $\{T_\varepsilon(\mathbf{f})(x)\}_{\varepsilon>0}$ is Cauchy for almost all x and therefore it converges. Since for continuous $f_1,\ldots,f_K$ it converges to the value of their product at the point $x \in \mathbb{R}^n$, to deduce the general case it will suffice to show that $\{T_\varepsilon(\mathbf{f})\}_{\varepsilon>0}$ converges to the product of the f_j's in the L^s norm as $\varepsilon \to 0$. (Then some subsequence will converge to the product a.e.) Setting $(\tau_y f)(x) = f(x-y)$, we get

$$\|T_\varepsilon(\mathbf{f}) - f_1\ldots f_K\|_{L^s} \le \frac{1}{\Omega_n\varepsilon^n}\int_{|y|\le\varepsilon}\Big\|\prod_j \tau_{\theta_j y}f_j - \prod_j f_j\Big\|_{L^s}\,dy$$

$$\leq \frac{1}{\Omega_n \varepsilon^n} \int_{|y|\leq\varepsilon} \sum_{j=1}^{K} \|\tau_{\theta_j y} f_j - f_j\|_{L^{p_j}} \prod_{k\neq j} \|f_j\|_{L^{p_k}}\, dy \to 0$$

as $|y| \to 0$ since the last integrand is a continuous function of y which vanishes at the origin. The last inequality above follows by adding and subtracting $2K-2$ suitable terms and applying Hölder's inequality K times.

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Orthogonal polynomials and middle Hankel operators on Bergman spaces

by

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Abstract. We introduce a sequence of Hankel style operators H^k, $k = 1, 2, 3, \ldots$, which act on the Bergman space of the unit disk. These operators are intermediate between the classical big and small Hankel operators. We study the boundedness and Schatten–von Neumann properties of the H^k and show, among other things, that H^k are cut-off at $1/k$. Recall that the big Hankel operator is cut-off at 1 and the small Hankel operator at 0.

Introduction and background. Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$ and let $dA(z) = (1/\pi)dxdy$ be normalized Lebesgue measure. $L^2 = L^2(dA)$ is the Hilbert space of functions u for which the norm

$$\|u\| = \Big(\int_{\mathbb{D}} |u(z)|^2 \, dA(z) \Big)^{1/2}$$

is finite. The Bergman space, A, is the subspace of all analytic functions in L^2.

The *big* and *small Hankel operators* on A with symbol b are defined by

$$H_b(f) = (I-P)(\overline{b}f), \quad h_b(f) = Q(\overline{b}f).$$

P and Q are orthogonal projections from L^2 onto A and $\bar{A}_0 = \{f \in L^2 : \bar{f} \in A \text{ and } f(0) = 0\}$ respectively.

For $0 < p \leq \infty$ denote the Schatten–von Neumann ideal by S_p (S_∞ is the class of bounded operators) and the analytic Besov space in $\mathbb{D}$ by B_p (B_∞ is the Bloch space). The main S_p results for the big and small Hankel operators with analytic symbols can be summarized as follows (see [A], [Pel1, 2], [R1, 2], [S], [AFP] and [J1]).

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