[13] B. Ziemian, *An analysis of microlocal singularities of functions and distributions on the real line*, Bull. Polish Acad. Sci. Math. 32 (1984), 157–164.

[14] —, *Taylor formula for distributions in several dimensions*, ibid. 34 (1986), 277–286.

[15] —, *Taylor formula for distributions*, Dissertationes Math. 264 (1988).

[16] —, *The Mellin transformation and multidimensional generalized Taylor expansions of singular functions*, J. Fac. Sci. Univ. Tokyo 36 (1989), 263–295.

[17] —, *Elliptic corner operators in spaces with continuous radial asymptotics I*, J. Differential Equations, to appear.

[18] —, *Elliptic corner operators in spaces with continuous radial asymptotics II*, in: Banach Center Publ. 27, to appear.

[19] —, *Continuous radial asymptotic for solutions to elliptic Fuchsian equations in 2 dimensions*, in: Proc. Sympos. Microlocal Analysis and its Applications, RIMS Kokyuroku 750, Kyoto Univ., 1991, 3–19.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
P.O. BOX 137
00-950 WARSZAWA, POLAND

Received October 12, 1990
Revised version April 16, 1991 (2726)

STUDIA MATHEMATICA 102 (1) (1992)

Characterization of Mellin distributions supported by certain noncompact sets

by

ZOFIA SZMYDT and BOGDAN ZIEMIAN (Warszawa)

Abstract. A class of distributions supported by certain noncompact regular sets K are identified with continuous linear functionals on $C_0^\infty(K)$. The proof is based on a parameter version of the Seeley extension theorem.

The paper is devoted to establishing theorems characterizing Mellin distributions supported by sets Z_t^A (see Section 4). They can be regarded as the extension to certain noncompact sets of the following theorem characterizing compactly supported distributions (cf. [1]):

THEOREM 1. *Let $u \in D'_K(\mathbb{R}^n)$, where K is a connected compact set in $\mathbb{R}^n$ such that any two points $x, y \in K$ can be joined by a rectifiable curve in K of length $\le \widetilde{C}|x-y|$. Then there exists a constant $C < \infty$ and $k \in \mathbb{N}_0$ such that*

$$|u[\psi]| \le C \sum_{|\alpha| \le k} \sup_{y \in K} \left| \left(\frac{\partial}{\partial y}\right)^\alpha \psi(y) \right|, \quad \textit{for } \psi \in C^k(\mathbb{R}^n).$$

1. Notation and necessary facts of the theory of Mellin distributions.

Any set in $\mathbb{R}^n$ of the form

$$\{(x_1, \ldots, x_n) : a_i < x_i < b_i \text{ for } i = 1, \ldots, n\},$$

where $a_1, \ldots, a_n, b_1, \ldots, b_n$ are given real numbers or $\pm\infty$ with $a_i < b_i$ for $i = 1, \ldots, n$, is called an *open polyinterval* in $\mathbb{R}^n$. Any set of the form

$$\{(x_1, \ldots, x_n) : a_i < x_i \le b_i < +\infty \text{ for } i = 1, \ldots, n\}$$

is called a *right-closed polyinterval.* $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0$ is the set of nonnegative integers. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ we write $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1! \ldots \alpha_n!$.

1991 *Mathematics Subject Classification*: Primary 46F10.

Throughout the paper we use the following vector notation: if $a, b \in \mathbb{R}^n$, $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ then $a < b$ ($a \le b$, resp.) means $a_j < b_j$ ($a_j \le b_j$, resp.) for $j = 1, \ldots, n$. We set $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : 0 < x\}$, $I = (0, t] = \{x \in \mathbb{R}^n_+ : x \le t\}$, where $t \in \mathbb{R}^n_+$. We write $\mathbf{r} = (r, \ldots, r)$, in particular $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$.

For $x \in \mathbb{R}^n_+$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we write

$$x^z = x_1^{z_1} \ldots x_n^{z_n} .$$

We set $e^{-y} = (e^{-y_1}, \ldots, e^{-y_n})$ for $y \in \mathbb{R}^n$ and similarly, if $x \in \mathbb{R}^n_+$, $\ln x = (\ln x_1, \ldots, \ln x_n)$. In particular, for $x \in \mathbb{R}^n_+$ and $\alpha \in \mathbb{N}^n_0$, $(\ln x)^\alpha = (\ln x_1)^{\alpha_1} \ldots (\ln x_n)^{\alpha_n}$. Vector notation is also used for differentiation:

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right), \qquad x\frac{\partial}{\partial x} = \left(x_1\frac{\partial}{\partial x_1}, \ldots, x_n\frac{\partial}{\partial x_n}\right),$$

and if $\nu \in \mathbb{N}^n_0$ then

$$\left(\frac{\partial}{\partial x}\right)^\nu = \frac{\partial^{\nu_1}}{\partial x_1^{\nu_1}} \cdots \frac{\partial^{\nu_n}}{\partial x_n^{\nu_n}}, \qquad \left(x\frac{\partial}{\partial x}\right)^\nu = \left(x_1\frac{\partial}{\partial x_1}\right)^{\nu_1} \cdots \left(x_n\frac{\partial}{\partial x_n}\right)^{\nu_n}.$$

Let $A \subset \Omega$, Ω open in $\mathbb{R}^n$. We denote by $C^\infty_A(\Omega)$ the set of smooth (i.e. C^∞) functions on Ω with supports in A. We write C^∞_A if $\Omega = \mathbb{R}^n$. Observe that the formula

$$|||\varphi|||_k = \sum_{|\alpha| \le k} \sup_{x \in K} \left|\left(\frac{\partial}{\partial x}\right)^\alpha \varphi(x)\right| \quad \text{for } \varphi \in C^\infty_K \ (k = 0, 1, \ldots)$$

defines an increasing sequence of norms on C^∞_K. The set C^∞_K equipped with the topology defined by this sequence of (semi)norms is denoted by D_K. The space D_K is complete (see e.g. [3] or Proposition 3 for a similar proof) and so is the dual space $(D_K)'$.

Let $u \in (D_K)'$. The value of u on a function $\varphi \in C^\infty_K$ is denoted by $u[\varphi]$.

Let Ω be an open subset of $\mathbb{R}^n$. We denote by $C^k_0(\Omega)$ ($k \in \mathbb{N}_0 \cup \infty$) the set of functions of class $C^k(\Omega)$ whose supports are compact subsets of Ω. If $A \subset \Omega$ is relatively closed in Ω we denote by $C^\infty_{(0)}(A)$ ($C^\infty(A)$, resp.) the space of restrictions to A of functions in $C^\infty_0(\Omega)$ ($C^\infty(\Omega)$, resp.). We equip $C^\infty_{(0)}(A)$ with the inductive limit topology as follows:

$$D(A) := \varinjlim_{K \subset \Omega} D_K|_A ,$$

where K ranges over all compact subsets of Ω and $D_K|_A$ denotes the space of restrictions to A of elements of D_K with the topology induced from D_K.

The dual space $D'(A)$ is called the space of distributions on A.

Note that if A is open then we take $A = \Omega$ and $D'(A)$ is the "usual" space of distributions on an open set.

In applications we take $A = I = (0, t] \subset \mathbb{R}^n_+$, or $A = [0, t] \subset \mathbb{R}^n$ with $t > 0$, $t \in \mathbb{R}^n$.

Let $a \in \mathbb{R}^n$, $t \in \mathbb{R}^n_+$. Note that the polyinterval

$$I = (0, t] = \{x \in \mathbb{R}^n : 0 < x \le t\}$$

is relatively closed in $\mathbb{R}^n_+$ and that $C^\infty(I)$ denotes the space of restrictions to I of smooth functions on $C^\infty(\mathbb{R}^n_+)$.

For $a \in \mathbb{R}^n$ we introduce the space

$$M_a = M_a(I) = \left\{\varphi \in C^\infty(I) : \varrho_{a,\alpha}(\varphi) := \sup_{x \in I} \left|x^{a+\alpha+1}\left(\frac{\partial}{\partial x}\right)^\alpha \varphi(x)\right| < \infty,\ \alpha \in \mathbb{N}^n_0\right\}$$

equipped with the topology defined by the sequence of the seminorms $\{\varrho_{a,\alpha}\}_{\alpha \in \mathbb{N}^n_0}$. An equivalent sequence of seminorms is

$$\widetilde{\varrho}_{a,\alpha}(\varphi) = \sup_{x \in I} \left|x^{a+1}\left(x\frac{\partial}{\partial x}\right)^\alpha \varphi(x)\right|, \qquad \alpha \in \mathbb{N}^n_0 .$$

We shall show in the next section that the space M_a is complete. For $\omega \in (\mathbb{R} \cup \{\infty\})^n$ we define the function space $M_{(\omega)}(I)$ as the inductive limit

$$M_{(\omega)}(I) = \varinjlim_{a < \omega} M_a(I) .$$

The following topological inclusion is clear:

$$D(I) \subset M_{(\omega)}(I) .$$

Moreover, it can be proved that $C^\infty_{(0)}(I)$ is dense in $M_{(\omega)}(I)$. Hence we derive easily that $M'_{(\omega)}(I)$ is a subspace of $D'(I)$, where $M'_{(\omega)}(I)$ denotes as usual the dual space of $M_{(\omega)}(I)$ endowed with the pointwise convergence topology. Therefore the elements of $M'_{(\omega)}$ are called *Mellin distributions*. The totality of Mellin distributions is denoted by $M'(I)$:

$$M'(I) = \bigcup_{\omega \in (\mathbb{R} \cup \{\infty\})^n} M'_{(\omega)}(I) = \bigcup_{\omega \in \mathbb{R}^n} M'_{(\omega)}(I) .$$

Note that the following operations of (pointwise) multiplication are continuous:

$$\text{(1)} \qquad \begin{gathered} x^\beta : M_a \to M_{a - \operatorname{Re}\beta}, \qquad x^\beta : M_{(\omega)} \to M_{(\omega - \operatorname{Re}\beta)}, \\ x^\beta : M'_{(\omega)} \to M'_{(\omega + \operatorname{Re}\beta)} . \end{gathered}$$

We shall yet introduce a space $M_{[a]}(I)$:

$$M_{[a]}(I) = \varprojlim_{b > a} M_b(I)$$

equipped with the projective limit topology: $\varphi_\nu \to 0$ in $M_{[a]}$ if and only if $\varphi_\nu \to 0$ in M_b for every $b > a$.

$M_{[-1]}$ coincides with the space of *Mellin multipliers*, i.e. functions $m \in C^\infty(I)$ such that multiplication by m is continuous $M_{(\omega)} \to M_{(\omega)}$ for every $\omega \in (\mathbb{R} \cup \{\infty\})^n$. We formulate two propositions leaving the proof of the first one to the reader.

PROPOSITION 1. *Let $u \in M'_{(\omega)}(I)$, $\omega \in (\mathbb{R} \cup \{\infty\})^n$, $m_0, m_j \in M_{[-1]}$ $(j = 1, 2, \ldots)$, $m_j \to m_0$ in the projective limit topology of $M_{[-1]}$. Then $m_j u \to m_0 u$ in $M'_\omega(I)$.*

In the next proposition we study the properties of cut-off functions $\lambda \in C^\infty(\mathbb{R})$ with $0 \le \lambda \le 1$, $\lambda(s) = 0$ for $s \le 1$, $\lambda = 1$ for $s \ge 2$. Let $r \in \mathbb{R}_+$. We put

$$\chi_r(x) = \lambda(x_1/r)\ldots\lambda(x_n/r) \quad \text{for } x \in \mathbb{R}^n. \tag{2}$$

It is easy to see that $\chi_r \in C^\infty(\mathbb{R}^n)$, $0 \le \chi_r \le 1$, $\chi_r(x) = 1$ for $x \ge 2\mathbf{r}$, $\operatorname{supp}\chi_r \subset \{x \in \mathbb{R}^n : x \ge \mathbf{r}\}$.

PROPOSITION 2. *Let χ_r be the function defined by (2). Then $\chi_r \in M_{[-1]}$ and for every $\alpha \in \mathbb{N}_0^n$ there exists a constant $C_\alpha < \infty$ (independent of r) such that*

$$\sup_{x\in I}\left|\left(x\frac{\partial}{\partial x}\right)^\alpha \chi_r(x)\right| \le C_\alpha$$

and $\chi_r \to 1$ in $M_{[-1]}(I)$ as $r \to 0$.

Proof. Clearly $\chi_r \in M_{-\mathbf{1}} \subset M_{[-\mathbf{1}]}$. The desired estimate can be proved by induction. Now take $\delta > 0$ and observe that

$$\widetilde{\varrho}_{-\mathbf{1}+\delta,\alpha}(\chi_r - 1) = \sup_{x\in I}\left|x^\delta\left(x\frac{\partial}{\partial x}\right)^\alpha(\chi_r - 1)\right| \le C_\alpha \sup_{x\in J_r}|x^\delta| \to 0$$

as $r \to 0$, where $J_r = I \setminus \{x \in \mathbb{R}^n : x > 2\mathbf{r}\}$.

For the use of Section 2 it is convenient to introduce the following subspaces of $C^\infty((0,\varepsilon))$, $\varepsilon \in \mathbb{R}_+$:

$$\widetilde{C}^\infty((0,\varepsilon]) = \left\{\varphi \in C^\infty((0,\varepsilon)) : \frac{d^j}{dx^j}\varphi \text{ extends continuously to } (0,\varepsilon]\ (j = 0,1,2,\ldots)\right\},$$

and $\widetilde{C}^\infty([0,\varepsilon))$ defined analogously. It is clear that $C^\infty((0,\varepsilon]) \subset \widetilde{C}^\infty((0,\varepsilon])$, $C^\infty([0,\varepsilon)) \subset \widetilde{C}^\infty([0,\varepsilon))$. By the Seeley extension theorem (see [2]) the converse inclusions are also true. In Section 2 we shall prove a parameter version of the Seeley extension theorem.

2. Seeley type linear extension mapping on a polyinterval. We shall construct a linear continuous extension mapping

$$\mathcal{E} : M_a((0,t]) \to M_a((0,\widetilde{t}]) \quad \text{for any } \widetilde{t} > t.$$

We begin with a lemma which is a parameter version of the Seeley extension theorem from a half-line to the real line.

Let $a \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}^n_+$ and write $a = (a_1, a')$, $x = (x_1, x')$, $t = (t_1, t')$ with $a' = (a_2, \ldots, a_n)$, $x' = (x_2, \ldots, x_n)$, $t' = (t_2, \ldots, t_n)$. Take $\varepsilon \in \mathbb{R}_+$ and set

$$\widetilde{C}^\infty([0,\varepsilon); M_{a'}((0,t'])) = \Big\{\varphi \in C^\infty((0,\varepsilon)\times(0,t']) : \sup_{(0,\varepsilon)\times(0,t']}\left|(x')^{a'+\mathbf{1}}\left(x'\frac{\partial}{\partial x'}\right)^{\alpha'}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\varphi(x_1,x')\right| < \infty$$
$$\text{for } \alpha = (\alpha_1,\alpha') \in \mathbb{N}_0^n,\ \varphi(\cdot,x') \in \widetilde{C}^\infty([0,\varepsilon) \text{ for every } x' \in (0,t']\Big\}.$$

In an analogous way we define $C^\infty((-\varepsilon,\varepsilon); M_{a'}((0,t']))$ and $C^\infty_{(0,\varepsilon/2]}((0,\infty); M_{a'}((0,t']))$.

LEMMA 1. *There exists a linear extension mapping*

$$\mathcal{E}^1 : \widetilde{C}^\infty([0,\varepsilon); M_{a'}((0,t'])) \to C^\infty((-\varepsilon,\varepsilon); M_{a'}((0,t']))$$

such that for every $\alpha \in \mathbb{N}_0^n$ there exists a constant C_{α_1} such that

$$\sup_{(-\varepsilon,\varepsilon)\times(0,t']}\left|(x')^{a'+\mathbf{1}}\left(x'\frac{\partial}{\partial x'}\right)^{\alpha'}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}(\mathcal{E}^1\varphi)(x_1,x')\right| \tag{3}$$
$$\le C_{\alpha_1}\sum_{p=0}^{\alpha_1}\sup_{(0,\varepsilon)\times(0,t']}\left|(x')^{a'+\mathbf{1}}\left(x'\frac{\partial}{\partial x'}\right)^{\alpha'}\left(\frac{\partial}{\partial x_1}\right)^{p}\varphi(x_1,x')\right|$$

for every $\varphi \in \widetilde{C}^\infty([0,\varepsilon); M_{a'}((0,t']))$.

Proof. Let $\chi \in C_0^\infty(\mathbb{R})$ be 1 in a neighbourhood of zero and $\chi(x_1) = 0$ for $|x_1| \ge \varepsilon/2$. Define

$$(\widetilde{\mathcal{E}}^1\varphi)(x) = \sum_{l=0}^{\infty} a_l\varphi(3^l x_1, x')\chi(3^l x_1) \tag{4}$$

for $\varphi \in \widetilde{C}^\infty([0,\varepsilon); M_{a'}((0,t']))$, where $\{a_l\}$ is a sequence of real numbers. Observe that for each $x_1 > 0$ only finitely many terms on the right hand side of (4) are nonzero and $(\widetilde{\mathcal{E}}^1\varphi)(x) = 0$ for $x_1 \ge \varepsilon/2$. Clearly

$$\widetilde{\mathcal{E}}^1 : \widetilde{C}^\infty([0,\varepsilon); M_{a'}((0,t'])) \to C^\infty_{(0,\varepsilon/2]}((0,\infty); M_{a'}((0,t']))$$

and this map is linear. We shall choose the sequence $\{a_l\}$ to satisfy

(5) $$\sum_{l=0}^{\infty}|a_l|3^{lp}<\infty \quad \text{for } p\in\mathbb{N},$$

(6) $$\sum_{l=0}^{\infty}a_l 3^{lp}=(-1)^p \quad \text{for } p\in\mathbb{N}.$$

Assuming this for a moment we find by differentiating (4)

(7) $$\Big(\frac{\partial}{\partial x_1}\Big)^p(\widetilde{\mathcal{E}}^1\varphi)(x_1,x') = \sum_{l=0}^{\infty}a_l 3^{lp}\Big(\frac{\partial}{\partial s}\Big)^p(\chi(s)\varphi(s,x'))\Big|_{s=3^l x_1} \quad \text{for } p\in\mathbb{N}.$$

Thus from the properties of χ and from (5) we get the estimate

(8) $$\sup_{(0,\varepsilon)\times(0,t']}\Big|(x')^{a'+1}\Big(x'\frac{\partial}{\partial x'}\Big)^{\alpha'}\Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1}(\widetilde{\mathcal{E}}^1\varphi)(x_1,x')\Big| \le C_{\alpha_1}\sum_{p=0}^{\alpha_1}\sup_{(0,\varepsilon/2]\times(0,t']}\Big|(x')^{a'+1}\Big(x'\frac{\partial}{\partial x'}\Big)^{\alpha'}\Big(\frac{\partial}{\partial x_1}\Big)^{p}\varphi(x_1,x')\Big|,$$

with some constant $C_{\alpha_1}<\infty$, and on the other hand we find

$$\lim_{x_1\to 0+}\Big(x'\frac{\partial}{\partial x'}\Big)^{\alpha'}\Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1}(\widetilde{\mathcal{E}}^1\varphi)(x_1,x') = \sum_{l=0}^{\infty}a_l 3^{l\alpha_1}\Big(x'\frac{\partial}{\partial x'}\Big)^{\alpha'}\Big(\Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1}\varphi\Big)(0,x'),$$

where

$$\Big(x'\frac{\partial}{\partial x'}\Big)^{\alpha'}\Big(\Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1}\varphi\Big)(0,x')=\lim_{x_1\to 0+}\Big(x'\frac{\partial}{\partial x'}\Big)^{\alpha'}\Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1}\varphi(x_1,x').$$

Now by (6)

$$\lim_{x_1\to 0+}\Big(x'\frac{\partial}{\partial x'}\Big)^{\alpha'}\Big(\Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1}(\widetilde{\mathcal{E}}^1\varphi)\Big)(x_1,x') = (-1)^{\alpha_1}\Big(x'\frac{\partial}{\partial x'}\Big)^{\alpha'}\Big(\Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1}\varphi\Big)(0,x')$$

for every $\alpha\in\mathbb{N}_0^n$; denote this expression by

$$\Big(\Big(x'\frac{\partial}{\partial x'}\Big)^{\alpha'}\Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1}(\widetilde{\mathcal{E}}^1\varphi)\Big)(0,x').$$

The desired extension is obtained by taking

$$(\mathcal{E}^1\varphi)(x_1,x')=\begin{cases}\varphi(x_1,x') & \text{for } 0<x_1<\varepsilon,\ 0<x'\le t',\\ (\widetilde{\mathcal{E}}^1\varphi)(-x_1,x') & \text{for } -\varepsilon<x_1\le 0,\ 0<x'\le t'.\end{cases}$$

Now (8) and the assumption that $\varphi\in\widetilde{C}^\infty([0,\varepsilon);M_{a'}((0,t']))$ yield (3).

Finally, to find a sequence $\{a_l\}$ satisfying (5) and (6) we note that the function

$$h(z)=\cos(\pi(3^z-1)/2)$$

is entire and $h(p)=(-1)^p$ for $p\in\mathbb{N}$. We take a_l $(l=0,1,\ldots)$ to be the coefficients of the power series expansion of the function $\mathbb{C}\ni w\mapsto\cos(\pi((w-1)/2))$, i.e. $h(z)=\sum_{l=0}^{\infty}a_l(3^z)^l$.

Remark 1. $(\mathcal{E}^1\varphi)(x_1,x')=0$ for $x_1\le-\varepsilon/2$ since $(\widetilde{\mathcal{E}}^1\varphi)(x)=0$ for $x_1\ge\varepsilon/2$.

THEOREM 2. *Let $a\in\mathbb{R}^n$, $0<t<\widetilde{t}\in\mathbb{R}_+^n$. Then for every $0<\varepsilon<\widetilde{t}-t$, $\varepsilon<t$ there exists a linear extension mapping*

$$\mathcal{E}_\varepsilon : M_a((0,t])\to M_a((0,\widetilde{t}])$$

continuous in the respective topologies and such that for every $\varphi\in M_a((0,t])$, $(\mathcal{E}_\varepsilon\varphi)(x)=0$ if $t_j+\varepsilon_j<x_j\le\widetilde{t}_j$ for some $1\le j\le n$.

Proof. Let $\varphi\in M_a((0,t])$, choose $0<\varepsilon<t$, $\varepsilon<\widetilde{t}-t$, and observe that the function

$$(0,\varepsilon_1)\times(0,t']\ni(x_1,x')\mapsto\widetilde{\varphi}(x_1,x')=\varphi(t_1-x_1,x')$$

belongs to $\widetilde{C}^\infty([0,\varepsilon_1);M_{a'}((0,t']))$. Thus by Lemma 1 and Remark 1

$$\mathcal{E}^1\widetilde{\varphi}\in C^\infty((-\varepsilon_1,\varepsilon_1);M_{a'}((0,t'])),\quad (\mathcal{E}^1\widetilde{\varphi})(x_1,x')=0 \quad \text{for } x_1<-\varepsilon_1/2,$$
$$(\mathcal{E}^1\widetilde{\varphi})(t_1-x_1,x')=\widetilde{\varphi}(t_1-x_1,x') \quad \text{for } t_1-\varepsilon_1<x_1<t_1,\ 0<x'\le t'.$$

Since $\widetilde{\varphi}(t_1-x_1,x')=\varphi(x_1,x')$ for $t_1-\varepsilon_1<x_1<t_1$, $0<x'\le t'$ we get

$$\varphi(x_1,x')=(\mathcal{E}^1\widetilde{\varphi})(t_1-x_1,x') \quad \text{for } t_1-\varepsilon_1<x_1<t_1,\ 0<x'\le t',$$

which yields the correctness of the following definition:

$$(\mathcal{E}_{\varepsilon_1}\varphi)(x_1,x')=\begin{cases}\varphi(x_1,x') & \text{for } 0<x_1<t_1,\ 0<x'\le t',\\ (\mathcal{E}^1\widetilde{\varphi})(t_1-x_1,x') & \text{for } t_1-\varepsilon_1<x_1<t_1+\varepsilon_1,\ 0<x'\le t'.\end{cases}$$

It is clear that $\mathcal{E}^1\widetilde{\varphi}$ is an extension of $\widetilde{\varphi}$ to $M_a((0,(t_1+\varepsilon_1,t')])$ and in fact to $M_a((0,(\widetilde{t}_1,t')])$ since $(\mathcal{E}^1\widetilde{\varphi})(x_1,x')=0$ for $x_1<-\varepsilon_1/2$. The continuity of $\mathcal{E}_{\varepsilon_1}$ follows from the continuity of $\mathcal{E}^1$.

If $n\ge 2$ we iterate the above procedure starting with $\mathcal{E}_{\varepsilon_1}\varphi$ defined above instead of φ and $(\widetilde{t}_1,t')$ instead of t.

PROPOSITION 3. *The space $M_a(I)$ is complete.*

Proof. Let $\{\varphi_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $M_a(I)$. Take $\widetilde{t} > t$, $0 < \varepsilon < \widetilde{t} - t$, $\varepsilon < t$ and a continuous extension map $\mathcal{E}_\varepsilon$ from Theorem 2. Let $\widetilde{\varphi}_j = \mathcal{E}_\varepsilon \varphi_j$ $(j = 1, 2, \ldots)$. By the continuity of $\mathcal{E}_\varepsilon$, $\{\widetilde{\varphi}_j\}_{j=1}^{\infty}$ is also a Cauchy sequence in $M_a((0,\widetilde{t}])$. This means that for every $\alpha \in \mathbb{N}_0^n$ the sequence $\{x^{a+\alpha+1}(\partial/\partial x)^\alpha \widetilde{\varphi}_j\}_{j=1}^{\infty}$ satisfies the Cauchy condition for the uniform convergence on $\widetilde{I} = (0, \widetilde{t}]$. Thus there exist functions h_α $(\alpha \in \mathbb{N}_0^n)$ continuous on $\widetilde{I}$ vanishing near the boundary $\widetilde{I} \setminus I$ and such that

$$\sup_{x\in\widetilde{I}} |x^{a+\alpha+1}(\partial/\partial x)^\alpha \widetilde{\varphi}_j - h_\alpha| \to 0 \quad \text{as } j \to \infty. \tag{9}$$

Hence, for every compact set $K \subset \widetilde{I}$,

$$\sup_{x\in\widetilde{I}} \left| x^{a+\alpha+1}\left(\left(\frac{\partial}{\partial x}\right)^\alpha \widetilde{\varphi}_j - \frac{h_\alpha}{x^{a+\alpha+1}}\right)\right| \ge C_{\alpha K} \max_{x\in K} \left|\left(\frac{\partial}{\partial x}\right)^\alpha \widetilde{\varphi}_j - \frac{h_\alpha}{x^{a+\alpha+1}}\right|,$$

where $C_{\alpha K} = \min_{x\in K} x^{a+\alpha+1} > 0$. This in view of (9) implies that

$$\lim_{j\to\infty} (\partial/\partial x)^\alpha \widetilde{\varphi}_j = h_\alpha / x^{a+\alpha+1} \quad \text{almost uniformly on } \widetilde{I}. \tag{10}$$

Set $\widetilde{\varphi}(x) = h_0/x^{a+1}$ for $x \in \operatorname{Int}\widetilde{I}$ and $\varphi = \widetilde{\varphi}\big|_I$. Then $\varphi \in C^\infty(I)$ and from (10), $(\partial/\partial x)^\alpha \widetilde{\varphi} = h_\alpha / x^{a+\alpha+1}$ for $x \in \operatorname{Int}\widetilde{I}$, $\alpha \in \mathbb{N}_0^n$ and consequently

$$\sup_{x\in I} |x^{a+\alpha+1}(\partial/\partial x)^\alpha \varphi_j - h_\alpha| \to 0 \quad \text{as } j \to \infty,$$

which proves that $\varphi_j \to \varphi$ in $M_a(I)$.

Denote by $M_{(\omega)}((0,t])$ the space of functions $\varphi \in M_{(\omega)}((0,t])$ vanishing with all derivatives $(\partial/\partial x)^\alpha \varphi$ on the set $(0,t] \setminus (0,t)$ with the topology induced by the topology of $M_{(\omega)}((0,t])$. Here as usual $\omega \in (\mathbb{R} \cup \{\infty\})^n$, $t \in \mathbb{R}_+^n$. By Theorem 2 we get

COROLLARY 1. *Let $0 < t < \widetilde{t} \in \mathbb{R}_+^n$. Then for every $0 < \varepsilon < \widetilde{t} - t$, $\varepsilon < t$, there exists a linear extension mapping*

$$\mathcal{E}_\varepsilon : M_{(\omega)}((0,t]) \to M_{(\omega)}((0,\widetilde{t}])$$

continuous in the respective topologies and such that $(\mathcal{E}_\varepsilon \varphi)(x) = 0$ if $t_j + \varepsilon_j < x_j < \widetilde{t}_j$ for some $1 \le j \le n$.

3. Characterization of Mellin distributions supported by a smaller polyinterval

THEOREM 3. *Let $u \in M'_{(\omega)}((0,\widetilde{t}])$ and $\operatorname{supp} u \subset (0,t]$ for some $t < \widetilde{t}$. Then for any $b < \omega$ there exist constants $C = C(b)$ and $k = k(b) \in \mathbb{N}_0$ such that*

$$|u[\varphi]| \le C \sum_{|\alpha|\le k} \sup_{x\in(0,t]} \left| x^{b+1}\left(x\frac{\partial}{\partial x}\right)^\alpha \varphi(x)\right| \quad \textit{for } \varphi \in M_b((0,\widetilde{t}]). \tag{11}$$

Hence the restriction mapping

$$M_{(\omega)}((0,\widetilde{t}]) \ni \varphi \mapsto \varphi|_{(0,t]} \in M_{(\omega)}((0,t])$$

induces a linear isomorphism

$$\{u \in M'_{(\omega)}((0,\widetilde{t}]) : \operatorname{supp} u \subset (0,t]\} \simeq M'_{(\omega)}((0,t]). \tag{12}$$

Proof. Let χ_r be the functions defined by (2). Let $\widetilde{I} = (0,\widetilde{t}]$. Then by Propositions 1 and 2

$$\lim_{r\to 0} u_r = u \quad \text{in } M'_{(\omega)}(\widetilde{I}), \quad \text{where} \quad u_r = \chi_r u \text{ for } r > 0. \tag{13}$$

Observe that $\operatorname{supp} u_r$ is a compact set $K_r \subset (0,t] \cap \{x \in \mathbb{R}^n : x \ge \mathbf{r}\}$. Hence by Theorem 1 there exist constants $C = C_r < \infty$, $k = k_r \in \mathbb{N}_0$ such that for every φ of class C^k in a neighbourhood of K_r,

$$|u_r[\varphi]| \le C \sum_{|\alpha|\le k} \sup_{x\in K_r} \left|\left(\frac{\partial}{\partial x}\right)^\alpha \varphi(x)\right|. \tag{14}$$

Take any $b < \omega$, a function $\gamma \in M_b((0,t])$ and its extension $\widetilde{\gamma} \in M_b((0,\widetilde{t}])$ (e.g. $\widetilde{\gamma} = \mathcal{E}_\varepsilon \gamma$, see Th. 2). From (14) we derive that

$$|u_r[\widetilde{\gamma}]| \le C_{rb} \sum_{|\alpha|\le k} \sup_{x\in K_r} \left| x^{b+1}\left(x\frac{\partial}{\partial x}\right)^\alpha \gamma(x)\right|. \tag{15}$$

Define

$$v_r[\gamma] = u_r[\widetilde{\gamma}] \quad \text{for } \gamma \in M_b((0,t]),\ r > 0.$$

The definition of the functionals v_r is correct (i.e. does not depend on the choice of the extension $\widetilde{\gamma}$ in view of (14)) and by (15), $v_r \in M'_b((0,t])$ for $r > 0$ since $K_r \subset (0,t]$. We can also write $v_r[\varphi|_{(0,t]}] = u_r[\varphi]$ for $\varphi \in M_{(\omega)}((0,\widetilde{t}])$ and putting $\gamma = \varphi|_{(0,t]}$ we get by (13)

$$v_r[\gamma] \to u[\varphi] \quad \text{as } r \to 0 \text{ for } \varphi \in M_{(\omega)}((0,\widetilde{t}]).$$

Thus for every $b < \omega$ there exist constants $C_b < \infty$ and $k \in \mathbb{N}_0$ such that

$$|v_r[\gamma]|, |u[\varphi]| \le C_b \sum_{|\alpha|\le k} \sup_{x\in(0,t]} \left| x^{b+1}\left(x\frac{\partial}{\partial x}\right)^\alpha \gamma(x)\right| \quad \text{for } \varphi \in M_b((0,\widetilde{t}])$$

and hence (11) holds since $\gamma(x) = \varphi(x)$ for $x \in (0,t]$.

To prove the isomorphism (12) take u satisfying the assumptions of the theorem. Define

$$\widetilde{u}[\varphi] = u[\mathcal{E}_\varepsilon \varphi] \quad \text{for } \varphi \in M_{(\omega)}((0,t]).$$

It follows from Corollary 1 that $\widetilde{u} \in M'_{(\omega)}((0,t])$ and from (11) we see that the definition of $\widetilde{u}$ is independent of the choice of the extension $\mathcal{E}_\varepsilon\varphi$. Conversely, given $\widetilde{u} \in M'_{(\omega)}((0,t])$, the formula

$$u[\varphi] = \widetilde{u}[\varphi|_{(0,t]}] \quad \text{for } \varphi \in M_{(\omega)}((0,\widetilde{t}])$$

defines $u \in M'_{(\omega)}((0,\widetilde{t}])$ with support in $(0,t]$. This ends the proof.

4. Seeley type linear extension mapping on the set Z_t^A. Let $A \in \mathrm{GL}(n;\mathbb{R})$ have nonnegative entries. Define

$$S : \mathbb{R}^n_+ \to \mathbb{R}^n_+, \qquad S(y) = \exp(A^{\mathrm{tr}} \ln y) \quad \text{for } y \in \mathbb{R}^n_+ . \tag{16}$$

Since the transpose A^{tr} of A also has nonnegative entries it follows that the set

$$Z_t^A = S((0,t]) \subset \mathbb{R}^n_+ \quad (t \in \mathbb{R}^n_+)$$

is bounded. Actually $S((0,t]) \subset (0, S(t)]$. Note that

$$S^{-1}(x) = \exp((A^{\mathrm{tr}})^{-1} \ln x) \quad \text{for } x \in \mathbb{R}^n_+$$

and the Jacobian of S^{-1} equals

$$JS^{-1}(x) = \frac{1}{\det A}(S^{-1}(x))^{\mathbf{1}} x^{-\mathbf{1}} .$$

For $a \in \mathbb{R}^n$ let

$$M_a(Z_t^A) = \{\varphi \in C^\infty(Z_t^A) : \varrho^A_{a,\alpha}(\varphi) < \infty \text{ for } \alpha \in \mathbb{N}^n_0\},$$

where

$$\varrho^A_{a,\alpha}(\varphi) = \sup_{x \in Z_t^A} |x^{a+\alpha+\mathbf{1}}(\partial/\partial x)^\alpha \varphi(x)| \quad \text{for } \alpha \in \mathbb{N}^n_0$$

and, as in the case of $C^\infty(I)$, $C^\infty(Z_t^A)$ denotes the space of restrictions to Z_t^A of functions in $C^\infty(\mathbb{R}^n_+)$. We also define, for $\omega \in (\mathbb{R} \cup \{\infty\})^n$,

$$M_{(\omega)}(Z_t^A) = \varinjlim_{a<\omega} M_a(Z_t^A) .$$

Let $t < \widetilde{t} \in \mathbb{R}^n_+$ and let $\mathcal{E}_\varepsilon$ be the linear extension mapping of Corollary 1:

$$\mathcal{E}_\varepsilon : M_{(A\omega)}((0,t]) \to \dot{M}_{(A\omega)}((0,\widetilde{t}]) .$$

Define

$$\mathcal{E}^A(\varphi) = JS^{-1} \cdot \mathcal{E}_\varepsilon(JS \cdot \varphi \circ S) \circ S^{-1} \quad \text{for } \varphi \in M_{(\omega)}(Z_t^A) .$$

PROPOSITION 4. *Let $h = S(t)$, $\widetilde{h} = S(\widetilde{t})$, $t < \widetilde{t}$. Then $\mathcal{E}^A$ is a continuous linear extension mapping*

$$\mathcal{E}^A : M_{(\omega)}(Z_t^A) \to \dot{M}_{(\omega)}((0,\widetilde{h}]) .$$

In the proof we shall need assertions (17) and (18) of the following

LEMMA 2. *Let $\omega \in (\mathbb{R} \cup \{\infty\})^n$, $t < \widetilde{t}$, $h = S(t)$, $\widetilde{h} = S(\widetilde{t})$. The following mappings are continuous in the respective topologies:*

$$M_{(\omega)}(Z_t^A) \ni \varphi \mapsto JS \cdot (\varphi \circ S) \in M_{(A\omega)}((0,t]) , \tag{17}$$

$$\dot{M}_{(A\omega)}((0,\widetilde{t}]) \ni \psi \mapsto JS^{-1} \cdot (\psi \circ S^{-1}) \in \dot{M}_{(\omega)}((0,\widetilde{h}]) , \tag{18}$$

$$M_{(A\omega)}((0,t]) \ni \psi \mapsto JS^{-1} \cdot (\psi \circ S^{-1}) \in M_{(\omega)}(Z_t^A) . \tag{19}$$

Proof. We consider first the mapping (18). Take any $\psi \in \dot{M}_{(A\omega)}((0,\widetilde{t}])$, choose $\delta \in \mathbb{R}^n_+$ such that $\psi \in \dot{M}_{A\omega-\delta}((0,\widetilde{t}])$ and extend ψ by zero to $\mathbb{R}^n_+$. Hence $\psi \circ S^{-1}(x) = 0$ on $\mathbb{R}^n_+ \setminus \operatorname{Int} S((0,\widetilde{t}])$. Select $\widetilde{\delta} \in \mathbb{R}^n_+$ such that $A\widetilde{\delta} < \delta$. We shall prove that $(S^{-1}(x))^{\mathbf{1}} x^{-\mathbf{1}} \cdot \psi \circ S^{-1}(x)$ is in $M_{\omega-\widetilde{\delta}}((0,\widetilde{h}])$ and thus in $M_{(\omega)}((0,\widetilde{h}])$. Since $(S^{-1}(x))^{\mathbf{1}} x^{-\mathbf{1}} = x^{A^{-1}\mathbf{1}-\mathbf{1}}$, by (1) it follows that $(S^{-1}(x))^{\mathbf{1}} x^{-\mathbf{1}} \cdot (\psi \circ S^{-1})(x) \in M_{\omega-\widetilde{\delta}}((0,\widetilde{h}])$ if and only if $\psi \circ S^{-1}(x) \in M_{\omega-\widetilde{\delta}+A^{-1}\mathbf{1}-\mathbf{1}}((0,\widetilde{h}])$. Hence it suffices to prove that $\psi \circ S^{-1}(x) \in M_{\omega-\widetilde{\delta}+A^{-1}\mathbf{1}-\mathbf{1}}((0,\widetilde{h}])$. Observe that under the notation $x\partial/\partial x = (x_1\partial/\partial x_1, \ldots, x_n\partial/\partial x_n)$ we have the following vector formula:

$$x\frac{\partial}{\partial x}(\psi(S^{-1}(x))) = \left(A^{-1}\left(y\frac{\partial}{\partial y}\right)\psi\right)(S^{-1}(x)) .$$

Then for any $\alpha \in \mathbb{N}^n_0$ we get

$$\begin{aligned}
\sup_{x \in (0,\widetilde{h}]} \left| x^{\omega-\widetilde{\delta}+A^{-1}\mathbf{1}} \left(x\frac{\partial}{\partial x}\right)^\alpha (\psi(S^{-1}(x))) \right|
&\le \sup_{y \in (0,\widetilde{t}]} \left| S(y)^{\omega-\widetilde{\delta}+A^{-1}\mathbf{1}} \left(A^{-1}\left(y\frac{\partial}{\partial y}\right)\right)^\alpha \psi(y) \right| \\
&= \sup_{y \in (0,\widetilde{t}]} \left| y^{A\omega+\mathbf{1}-A\widetilde{\delta}} \left(A^{-1}\left(y\frac{\partial}{\partial y}\right)\right)^\alpha \psi(y) \right| .
\end{aligned}$$

The last expression is finite since $M_{A\omega-\delta}((0,\widetilde{t}]) \subset M_{A\omega-A\widetilde{\delta}}((0,\widetilde{t}])$ by the choice of $\widetilde{\delta}$. Thus

$$(S^{-1}(x))^{\mathbf{1}} x^{-\mathbf{1}} (\psi \circ S^{-1})(x) \in M_{(\omega)}((0,\widetilde{h}]) .$$

The proof of (19) is analogous and the extendibility of $JS^{-1}(\psi \circ S^{-1})$ to an element of $M_{(\omega)}((0,\widetilde{h}])$ follows from (18).

To prove (17) let $\varphi \in M_b(Z_t^A)$ for some $b < \omega$. Define

$$\psi(y) = \det A \cdot \varphi(S(y))(S(y))^{\mathbf{1}} y^{-\mathbf{1}} \quad \text{for } y \in (0,t] .$$

Proceeding analogously to the proof of (18) we prove for any $k \in \mathbb{N}_0$ the estimates

$$\sum_{|\alpha|\le k} \sup_{y\in(0,t]} \left| y^{Ab+1}\left(y\frac{\partial}{\partial y}\right)^{\alpha}\psi(y)\right|$$
$$\le \widetilde{C}\sum_{|\alpha|\le k}\sup_{x\in Z_t^A}\left|(S^{-1}(x))^{Ab+1}x^{1}(S^{-1}(x))^{-1}\left(A\left(x\frac{\partial}{\partial x}\right)\right)^{\alpha}\varphi(x)\right|$$
$$= C\sum_{|\alpha|\le k}\sup_{x\in Z_t^A}\left|x^{b+1}\left(x\frac{\partial}{\partial x}\right)^{\alpha}\varphi(x)\right|,$$

which ends the proof of the lemma.

Proof of Proposition 4. Let $\varphi \in M_{(\omega)}(Z_t^A)$. Then by Lemma 2(17), $\psi = JS\cdot(\varphi\circ S) \in M_{(A\omega)}((0,t])$ and hence by Corollary 1, $\mathcal{E}_\varepsilon\psi \in M_{(A\omega)}((0,\widetilde{t}])$. Again by Lemma 2(18), $JS^{-1}\cdot((\mathcal{E}_\varepsilon\psi)\circ S^{-1})$ is in $M_{(\omega)}((0,\widetilde{h}])$. Clearly $\mathcal{E}^A$ is linear and continuous.

Now in the same way as in the proof of Proposition 3 we prove, by using the extension mapping $\mathcal{E}^A$,

PROPOSITION 5. *The space $M_a(Z_t^A)$ with the topology given by the seminorms $\varrho^A_{a,\alpha}$, $\alpha\in\mathbb{N}_0^n$, is complete.*

Remark 2. Proposition 4 generalizes Corollary 1 ($\mathcal{E}_\varepsilon = \mathcal{E}^{\mathrm{Id}}$) and hence also the Seeley extension theorem.

5. Characterization of Mellin distributions supported by the set Z_t^A. Proceeding analogously to the proof of Theorem 3 we get the following generalization of that theorem:

THEOREM 3′. *Let $u \in M'_{(\omega)}((0,\widetilde{h}])$ with $\operatorname{supp} u \subset Z_t^A$ for some $t < \widetilde{t} = S^{-1}(\widetilde{h})$. Then for any $b<\omega$ there exist constants $C = C(b)$ and $k = k(b)\in\mathbb{N}_0$ such that*

$$|u[\varphi]| \le C\sum_{|\alpha|\le k}\sup_{x\in Z_t^A}|x^{b+1}(x\partial/\partial x)^{\alpha}\varphi(x)| \quad \text{for } \varphi\in M_b((0,\widetilde{h}]).$$

Hence the restriction mapping

$$M_{(\omega)}((0,\widetilde{h}]) \ni \varphi \mapsto \varphi|_{Z_t^A} \in M_{(\omega)}(Z_t^A)$$

induces a linear isomorphism

$$\{u\in M'_{(\omega)}((0,\widetilde{h}]) : \operatorname{supp} u\subset Z_t^A\} \simeq M'_{(\omega)}(Z_t^A).$$

6. Application: substitution in a Mellin distribution; the Mellin transform of substitution. Let $u \in M'_{(\omega)}(I)$ and let S be defined by (16). Since $M'_{(\omega)}(I) \subset D'(I)$ and S^{-1} is a one-to-one C^∞ mapping of $\mathbb{R}^n_+$ onto $\mathbb{R}^n_+$ with nonvanishing Jacobian JS^{-1} we may define the substitution $u\circ S$ by the formula

$$u\circ S[\psi] = \frac{1}{|\det A|}u[\psi\circ S^{-1}(x)\cdot(S^{-1}(x))^{1}x^{-1}], \tag{20}$$

for $\psi\in C^\infty_{(0)}(S^{-1}(I))$. Then $u\circ S\in D'(S^{-1}(I))$, where $S^{-1}(I)$ is relatively closed in $S^{-1}(\mathbb{R}^n_+)$ but need not be bounded. By imposing the restriction $\operatorname{supp} u\subset Z_t^A$ on the support of the Mellin distribution u we establish in Theorem 4 below that $u\circ S$ itself is a Mellin distribution and we give a formula for the Mellin transform of this substitution. Note that we adopt the following definition of the Mellin transformation (see [4]):

DEFINITION. Let $u\in M'_{(\omega)}(I)$ for some $\omega\in(\mathbb{R}\cup\{\infty\})^n$. We define the *Mellin transform* of u by

$$\mathcal{M}u(z) = u[x^{-z-1}] \quad \text{for } \operatorname{Re} z<\omega.$$

It turns out that $\mathcal{M}u$ is a holomorphic function for $\operatorname{Re} z<\omega$.

Now, we are in a position to state the following

THEOREM 4. *Let $t\in\mathbb{R}^n_+$ and let $S(y) = \exp(A^{\mathrm{tr}}\ln y)$ for $y\in\mathbb{R}^n_+$ where $A\in\mathrm{GL}(n;\mathbb{R})$ has nonnegative entries. Let $u\in M'_{(\omega)}((0,h])$ with $h=S(t)$ and $\operatorname{supp} u\subset Z_t^A = S((0,t])$. For $\psi\in M_{(A\omega)}((0,t])$ define*

$$u\circ S[\psi] = u[\varphi], \tag{21}$$

where (see (20))

$$\varphi(x) = \frac{1}{|\det A|}\psi\circ S^{-1}(x)\cdot(S^{-1}(x))^{1}x^{-1} \quad \text{for } x\in Z_t^A$$

(here u is regarded as an element of $M'_{(\omega)}(Z_t^A)$). Then $u\circ S\in M'_{(A\omega)}((0,t])$ and

$$\mathcal{M}(u\circ S)(\zeta) = \frac{1}{|\det A|}(\mathcal{M}u)\circ A^{-1}(\zeta) \quad \text{for } \operatorname{Re}\zeta < A\omega.$$

Proof. Let $\widetilde{t}>t$, $\widetilde{h} = S(\widetilde{t})$ and observe that u can be considered as a functional in $M'_{(\omega)}((0,\widetilde{h}])$ (namely as $\widetilde{u}[\psi] = u[\psi|_{(0,h]}]$ for $\psi\in M_{(\omega)}((0,\widetilde{h}])$). Hence by Theorem 3′, u can also be considered as an element of $M'_{(\omega)}(Z_t^A)$ and by Lemma 2(19) formula (21) defines correctly the functional $u\circ S\in$

$M'_{(A\omega)}((0,t])$. From (21) we get

$$\mathcal{M}(u\circ S)(\zeta) = u\circ S[y^{-\zeta-1}] = u\Big[(S^{-1}(x))^{-\zeta-1}\frac{1}{|\det A|}(S^{-1}(x))^{1}x^{-1}\Big]$$
$$= \frac{1}{|\det A|}u[x^{-A^{-1}\zeta-1}] = \frac{1}{|\det A|}(\mathcal{M}u)\circ A^{-1}\zeta\,,$$

which ends the proof.

Remark 3. After the change of variables $\mathbb{R}^n \ni y \mapsto e^y \in \mathbb{R}^n_+$, Theorem 3′ (and hence Theorem 3) extends Theorem 1 to the case of the non-compact set $A^{\mathrm{tr}}(\ln(0,t])$.

References

[1] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer, 1985.
[2] R. Melrose, *Analysis on Manifolds with Corners*, lecture notes, preprint MIT, 1988.
[3] Z. Szmydt, *Fourier Transformation and Linear Differential Equations*, PWN, Warszawa, and Reidel, Dordrecht, 1977.
[4] B. Ziemian, *The Mellin transformation and multidimensional generalized Taylor expansions of singular functions*, J. Fac. Sci. Univ. Tokyo 36 (1989), 263–295.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
P.O. BOX 137
00-950 WARSZAWA, POLAND

Received March 28, 1991 (2793)

STUDIA MATHEMATICA 102 (1) (1992)

Weighted inequalities for square and maximal functions in the plane

by

JAVIER DUOANDIKOETXEA* and ADELA MOYUA (Bilbao)

Abstract. We prove weighted inequalities for square functions of Littlewood–Paley type defined from a decomposition of the plane into sectors of lacunary aperture and for the maximal function over a lacunary set of directions. Some applications to multiplier theorems are also given.

1. Introduction. Square functions are often used in Harmonic Analysis because their action on a function gives a new one with equivalent L^p-norm. They can be viewed in some sense as a substitute of Plancherel's theorem in L^p, $p \neq 2$.

In this paper we consider two such square functions associated with a decomposition of $\mathbb{R}^2$ into angles of lacunary aperture. Let us take the lines through the origin with slope $\pm 2^j$, $j \in \mathbb{Z}$, and consider the angular sectors they determine. More precisely, we set

$$\Delta_j = \{(x_1,x_2) \in \mathbb{R}^2 \mid 2^{-j} \le |x_2/x_1| < 2^{-j+1}\}$$

and define the multiplier operator S_j as $(S_j f)^\wedge = \chi_{\Delta_j}\widehat{f}$ (we denote by χ_A the characteristic function of A). Our first square function will be

$$g(f) = \Big(\sum_{j=-\infty}^{\infty} |S_j f|^2\Big)^{1/2}.$$

We shall also consider a smooth decomposition defined as follows: let φ_j be a homogeneous function of degree zero, supported on $\Delta_j \cup \Delta_{j+1}$ and such that the restriction to the unit circle S^1 (denoted again by φ_j) is C^∞ and satisfies

$$|D^\alpha \varphi_j(\theta)| \le C2^{-|j|\alpha} \quad (C \text{ independent of } j)\,.$$

1991 *Mathematics Subject Classification*: 42B25, 42B15.
* Supported in part by DGICYT, Project PB 86-108.