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# The modified Cauchy transformation with applications to generalized Taylor expansions

by

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**Abstract.** We generalize to the case of several variables the classical theorems on the holomorphic extension of the Cauchy transforms. The Cauchy transformation is considered in the setting of tempered distributions and the Cauchy kernel is modified to a rapidly decreasing function. The results are applied to the study of "continuous" Taylor expansions and to singular partial differential equations.

This paper can be regarded as an extension and complement of the results of [16]. In [16] the reader will find the motivation for the developments conducted here as well as the definitions and notation which is used. We have tried, however, to make the paper self-contained by providing short explanations of the notions applied here. In particular, the first two sections on the modified Cauchy transformation in one and several variables can be read independently of [14]. We think that they present an independent interest. The next two sections are devoted to applications of the modified Cauchy transformation to the study of distributions having "continuous" Taylor expansions: we prove the local character of the notion of the type of the expansion ($A$-meromorphy) and study the behaviour of the "continuous" Taylor expansions under reticular changes of coordinates.

In the final Section 5, devoted to the Mellin analysis on manifolds, we introduce the notion of an $A$-spectral support of a distribution $u$ in a pyramid $\triangle$, which is an important invariant describing the asymptotic behaviour of $u$ in $\triangle$ when approaching the vertex of $\triangle$. It is a generalization to several dimensions of the invariant introduced in [13] and [15].

Application of the techniques developed here and in [16] to singular elliptic partial differential equations is given in [17]–[19] and [11], [12] (see also [8]–[10]).

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**1. Modified Cauchy and Hilbert transformations in dimension 1.** We start by recalling certain well-known facts (cf. [4]) about the classical Cauchy and Hilbert transformations in $L^2(\mathbb{R})$ in a slightly changed setting; namely, the Cauchy transformation is not considered relative to the real axis $\mathbb{R}$ but relative to a fixed purely imaginary line $\mathring{\alpha} + i\mathbb{R}$ for some $\mathring{\alpha} \in \mathbb{R}$.

Let $T \in L^2(\mathbb{R})$ and fix $\mathring{\alpha} \in \mathbb{R}$. The *right* and the *left Cauchy transforms* of $T$ (relative to the line $\mathring{\alpha} + i\mathbb{R}$) are defined as

$$C_+T(z) = -\frac{1}{2\pi}\int_{\mathbb{R}} \frac{T(\gamma)}{z-\mathring{\alpha}-i\gamma}\,d\gamma \quad \text{for } \operatorname{Re} z > \mathring{\alpha},$$

$$C_-T(z) = -\frac{1}{2\pi}\int_{\mathbb{R}} \frac{T(\gamma)}{z-\mathring{\alpha}-i\gamma}\,d\gamma \quad \text{for } \operatorname{Re} z < \mathring{\alpha}.$$

The following is a classical fact in the theory of the Hilbert transformation:

THEOREM (see [4]). *If $T \in L^2(\mathbb{R})$ then the $L^2(\mathbb{R})$ limits*

$$H_\pm T(\beta) = \lim_{\alpha\to\mathring{\alpha}_\pm} C_\pm T(\alpha + i\beta)$$

*exist and $H_-T - H_+T = T$.*

The functions $H_+T$, $H_-T \in L^2(\mathbb{R})$ are called the *right* and the *left Hilbert transforms* of $T$.

We want to establish analogues of these facts in $S'$ instead of $L^2$. To this end we introduce modified Cauchy kernels which replace the classical Cauchy kernel $-1/z$. To justify the definition below observe that

$$-1/z = (\mathcal{M}\chi_{(0,1]})(z) \quad \text{for } z \neq 0,$$

where $\chi_{(0,1]}$ is the characteristic function of the interval $(0,1]$, and the Mellin transform of a (bounded) function $f$ on $\mathbb{R}_+$, of bounded support at $+\infty$, was defined in [15] as a holomorphic extension of the function

$$\mathcal{M}f(z) = \int_0^\infty f(x)x^{-z-1}\,dx \quad \text{for } \operatorname{Re} z < 0.$$

LEMMA 1. *Let $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ in a neighbourhood of zero. Then*

(i) $\mathcal{M}\chi \in \mathcal{O}(\mathbb{C}\setminus\{0\})$,

(ii) $\mathcal{M}\chi(z) = -1/z + \widetilde{G}(z)$ *with a* $\widetilde{G} \in \mathcal{O}(\mathbb{C})$,

(iii) *for every fixed $\alpha \in \mathbb{R}$ the function $\mathbb{R} \ni \beta \mapsto (\alpha + i\beta)\mathcal{M}\chi(\alpha+i\beta)$ is in $S(\mathbb{R})$.*

Proof. Define $\kappa(y) = \chi(e^{-y})$ for $y \in \mathbb{R}$, and $G(z) = \mathcal{M}\chi(z)$. Then $G$ may be written in the following explicit form:

$$G(z) = \begin{cases} \mathcal{L}\kappa(z) & \text{for } \operatorname{Re} z < 0, \\ \mathcal{L}(\kappa - 1)(z) & \text{for } \operatorname{Re} z > 0, \end{cases} \tag{1}$$

where $\mathcal{L}\kappa(z) = \int_{\mathbb{R}} \kappa(y)e^{zy}\,dy$ is the Laplace transform of $\kappa$. To see this observe that

$$z(\mathcal{L}\kappa)(z) = -\mathcal{L}\Big(\frac{d\kappa}{dy}\Big)(z) \quad \text{for } \operatorname{Re} z < 0,$$

$$z(\mathcal{L}(\kappa-1))(z) = -\mathcal{L}\Big(\frac{d\kappa}{dy}\Big)(z) \quad \text{for } \operatorname{Re} z > 0,$$

and since $d\kappa/dy$ is a compactly supported smooth function it follows that $zG(z)$ is an entire function and (iii) holds. To compute the residue of $G$ at zero note that

$$zG(z)|_{z=0} = \mathcal{M}\Big(x\frac{d\chi}{dx}\Big)(0) = \int_0^\infty \frac{d\chi}{dx}\,dx = -\chi(0) = -1.$$

DEFINITION 1. Let $\chi$ be a fixed function as in Lemma 1. The function

$$G(z) = \mathcal{M}\chi(z), \quad z \in \mathbb{C}\setminus\{0\},$$

is called a *modified Cauchy kernel* (determined by $\chi$).

DEFINITION 2. Fix $\mathring{\alpha} \in \mathbb{R}$ and let $T \in S'(\mathbb{R})$. The functions

$$\mathcal{C}_\pm T(z) = \frac{1}{2\pi}T[G(z - \mathring{\alpha} - i\gamma)] \quad \text{for } \pm\operatorname{Re} z > \pm\mathring{\alpha}$$

are called the *right* and the *left modified Cauchy transforms* of $T$. In the sequel we often omit the word *modified* since the function $G$ and the point $\mathring{\alpha}$ will remain fixed.

THEOREM 1. *Let $T \in S'(\mathbb{R})$ and fix $\mathring{\alpha} \in \mathbb{R}$. Then the $S'(\mathbb{R})$ limits*

$$\mathcal{H}_\pm T = \lim_{\alpha\to\mathring{\alpha}_\pm} \mathcal{C}_\pm T(\alpha + i\cdot)$$

*exist and*

$$\mathcal{H}_-T - \mathcal{H}_+T = T. \tag{2}$$

We call the distributions $\mathcal{H}_\pm T \in S'(\mathbb{R})$ the *right* and the *left (modified) Hilbert transforms* of $T \in S'(\mathbb{R})$.

Proof. By translation in $z$ we may assume that $\mathring{\alpha} = 0$. By (1) we have

$$G(\alpha + i\beta) = \begin{cases} 2\pi\mathcal{F}^{-1}(e^{\alpha y}\kappa(y))(\beta) & \text{for } \alpha < 0, \\ 2\pi\mathcal{F}^{-1}(e^{\alpha y}(\kappa(y) - 1))(\beta) & \text{for } \alpha > 0, \end{cases}$$

where the Fourier transformation $\mathcal{F}$ and the inverse Fourier transformation $\mathcal{F}^{-1}$ are defined by

$$\mathcal{F}\sigma(y) = \int_{\mathbb{R}} e^{-iy\beta}\sigma(\beta)\,d\beta, \quad \mathcal{F}^{-1}\psi(\beta) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iy\beta}\psi(y)\,dy, \quad \sigma,\psi \in S(\mathbb{R}).$$

Now, by the exchange formula for the Fourier transformation

$$\mathcal{C}_-T(\alpha+i\cdot) = \mathcal{F}^{-1}(e^{\alpha y}\kappa(y)\mathcal{F}T) \quad \text{for } \alpha<0,$$
$$\mathcal{C}_+T(\alpha+i\cdot) = \mathcal{F}^{-1}(e^{\alpha y}(\kappa(y)-1)\mathcal{F}T) \quad \text{for } \alpha>0.$$

Since $\mathcal{F}$ and $\mathcal{F}^{-1}$ are topological isomorphisms of $S'$ onto $S'$, in order to prove the first part of the theorem it is enough to show that for every $\sigma \in S(\mathbb{R})$

$$e^{\alpha y}\kappa(y)\sigma(y) \to \kappa(y)\sigma(y) \quad \text{in } S(\mathbb{R}) \text{ as } \alpha \to 0_-,$$
$$e^{\alpha y}(\kappa(y)-1)\sigma(y) \to (\kappa(y)-1)\sigma(y) \quad \text{in } S(\mathbb{R}) \text{ as } \alpha \to 0_+.$$

But this is straightforward since $\operatorname{supp}\kappa$ is contained in the half line $y \ge -a$, and $\operatorname{supp}(\kappa-1)$ in $y \le a$ for some $a \in \mathbb{R}_+$, $\kappa$ is bounded together with all its derivatives, and $e^{\alpha y} \to 1$ as $\alpha \to 0$ uniformly on bounded sets.

From the above we get

$$\mathcal{H}_-T = \mathcal{F}^{-1}(\kappa\cdot\mathcal{F}T), \quad \mathcal{H}_+T = \mathcal{F}^{-1}((\kappa-1)\cdot\mathcal{F}T),$$

which gives (2).

Remark 1. In his book [1] Bremermann extends the Theorem to certain spaces of distributions which admit evaluations on functions behaving like $1/|\beta|$ at infinity. In Theorem 1 we avoid such spaces by modifying the Cauchy kernel so that it becomes a Schwartz class function, without affecting the essential properties of the corresponding Cauchy transformation.

Remark 2. In the case of compactly supported distributions (more generally, of compactly supported hyperfunctions) Theorem 1 (or its hyperfunction version) is immediate in view of Lemma 1(ii) and amounts to the statement that the modified Cauchy transform of $T$ is a nonstandard defining function of $T$ regarded as a distribution (hyperfunction) on the line $\mathring{\alpha} + i\mathbb{R}$.

We end this section by an application of Theorem 1 to the study of holomorphic extensions of the left Cauchy transform, which is the leading theme of the paper.

Let $F$ be holomorphic in an open set $U \subset \mathbb{C}$. Let $\mathring{\alpha} \in \mathbb{R}$ and suppose that the function $\beta \mapsto F(\mathring{\alpha}+i\beta)$, defined for $\beta \in \mathbb{R}$ such that $\mathring{\alpha}+i\beta \in U$, extends to a distribution in $S'(\mathbb{R})$ which we denote by $F_{\mathring{\alpha}}$.



Corollary 1. *Under the assumptions on $F$ given above the function*

$$\psi(z) = \begin{cases} \mathcal{C}_-F_{\mathring{\alpha}}(z) & \text{for } \operatorname{Re} z < \mathring{\alpha}, \\ \mathcal{C}_+F_{\mathring{\alpha}}(z) + F(z) & \text{for } z \in \{\operatorname{Re} z > \mathring{\alpha}\} \cap U, \end{cases}$$

*extends to a holomorphic function on* $\{\operatorname{Re} z < \mathring{\alpha}\} \cup U$.

Proof. This follows immediately from (2) in view of the distributional Painlevé theorem [3] (= one-dimensional edge of the wedge theorem [6]).

**2. Modified Cauchy transformation in several dimensions.** This section is devoted to a generalization of Corollary 1 to several dimensions. For that purpose we must define not only an $n$-dimensional Cauchy transformation but also partial Cauchy transformations with respect to a part of variables. Below we introduce the necessary notation and definitions.

Let $I = \{i_1,\ldots,i_k\} \subset \{1,\ldots,n\}$, $i_1 < \ldots < i_k$, and let $J = \{j_1,\ldots,j_{n-k}\}$, $j_1 < \ldots < j_{n-k}$, be the complement of $I$ in $\{1,\ldots,n\}$. For $z \in \mathbb{C}^n$ we write $z = (z_I, z_J)$, where $z_I = (z_{i_1},\ldots,z_{i_k})$, $z_J = (z_{j_1},\ldots,z_{j_{n-k}})$. Corresponding to this notation we write $U^I \times U^J = \{z = (z_I,z_J) : z_I \in U^I, z_J \in U^J\}$ for any sets $U^I$ and $U^J$ of points $z_I$ and $z_J$, respectively.

Obviously $\mathbb{C}^I = \{z_I : z_{i_l} \in \mathbb{C} \text{ for } l = 1,\ldots,k\}$ and analogously for $\mathbb{R}^I$, $\mathbb{R}^J$.

Definition 3. Let $F$ be a function holomorphic in an open set $U \subset \mathbb{C}^n$. Let $\mathring{\alpha} \in \mathbb{R}^n$ and let $I \ne \emptyset$ be a proper subset of $\{1,\ldots,n\}$. By a *holomorphic family of partial regularizations* of $F$ at $\mathring{\alpha}$ we mean any $S'(\mathbb{R}^J)$-valued holomorphic function

$$U^I \ni z_I \mapsto F_{z_I,\mathring{\alpha}_J} \in S'(\mathbb{R}^J)$$

defined on an open subset $U^I \subset \mathbb{C}^I$ such that on the set $\{(z_I,\beta_J) \in U^I \times \mathbb{R}^J : (z_I, \mathring{\alpha}_J + i\beta_J) \in U\}$, $F_{z_I,\mathring{\alpha}_J}$ is a function (of $\beta_J$) and $F_{z_I,\mathring{\alpha}_J}(\beta_J) = F(z_I, \mathring{\alpha}_J + i\beta_J)$.

If $I = \emptyset$ then $U^I = \emptyset$ and we have a single distribution $F_{\mathring{\alpha}} \in S'(\mathbb{R}^n)$. If $I = \{1,\ldots,n\}$ then there is no regularization, $U^I = U$ and $F_z = F(z)$.

Definition 4. Let $\{F_{z_I,\mathring{\alpha}_J} \in \mathcal{O}(U^I, S'(\mathbb{R}^J))$ for $I \subset \{1,\ldots,n\}\}$ be a set of families of partial regularizations of $F$ at $\mathring{\alpha}$. We say that the set is *compatible* if for any pair $I' \subset I$ the following is true:

For every $z_{I'} \in U^I \cap U^{I'} \overset{\text{def}}{=} \pi^{I'}(U^I) \cap U^{I'}$ where $\pi^{I'}$ is the natural projection of $\mathbb{R}^I$ onto $\mathbb{R}^{I'}$, the function

$$\gamma_{I\setminus I'} \mapsto F_{z'_I, \mathring{\alpha}_{I\setminus I'} + i\gamma_{I\setminus I'}, \mathring{\alpha}_J} \in S'(\mathbb{R}^J)$$

admits a regularization $\widetilde{F}_{z_{I'},\mathring{\alpha}_{I\setminus I'},\mathring{\alpha}_J}$ such that

$$\widetilde{F}_{z_{I'},\mathring{\alpha}_{I\setminus I'},\mathring{\alpha}_J}\in\mathcal{O}(U^I\cap U^{I'},S'(\mathbb{R}^{I\setminus I'},S'(\mathbb{R}^J)))$$

and

$$\widetilde{F}_{z_{I'},\mathring{\alpha}_{I\setminus I'},\mathring{\alpha}_J}=F_{z_{I'},\mathring{\alpha}_{J'}}$$

(under a natural identification $S'(\mathbb{R}^{I\setminus I'},S'(\mathbb{R}^J))\simeq S'(\mathbb{R}^{J'})$, since $(I\setminus I')\cup J=J'$; this is an $S'$ version of the Schwartz kernel theorem (see [12])).

Definition 5. Let $F_{z_I,\mathring{\alpha}_J}\in\mathcal{O}(U^I,S'(\mathbb{R}^J))$ be a holomorphic family of partial regularizations of $F$. By a *modified partial Cauchy transform* of $F$ (with respect to the variables $z_J$) we mean the function $\mathcal{C}^I(z)=\mathcal{C}^IF_{z_I,\mathring{\alpha}_J}(z_J)$ defined by

$$\mathcal{C}^I(z)=\frac{1}{(2\pi)^{|J|}}F_{z_I,\mathring{\alpha}_J}[G^J(z_J-\mathring{\alpha}_J-i\gamma_J)]$$

for $z=(z_I,z_J)$ where $z_I\in U^I$, $\operatorname{Re}z_{j_l}\neq\mathring{\alpha}_{j_l}$ for $l=1,\ldots,n-k$, $|J|=\operatorname{card}J$ and

$$G^J(z_J)=G(z_{j_1})\ldots G(z_{j_{n-k}}).$$

In the case $I\doteq\emptyset$ we shall write

$$\mathcal{C}(z)=\mathcal{C}^{\emptyset}(z)\quad\text{ for }\operatorname{Re}z_j\neq\mathring{\alpha}_j,\ j=1,\ldots,n.$$

We also put by definition

$$\mathcal{C}^{\{1,\ldots,n\}}(z)=F(z)\quad\text{ for }z\in U=U^{\{1,\ldots,n\}}.$$

Before stating the main result we introduce a final piece of notation: For $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_n)$ with $\varepsilon_j=\pm$ for $j=1,\ldots,n$ we define $I_\varepsilon=\{k:\varepsilon_k=+\}$, $\mathbb{C}^n_{\mathring{\alpha}_\varepsilon}=\{z\in\mathbb{C}^n:\varepsilon_j\operatorname{Re}z_j>\varepsilon_j\mathring{\alpha}_j,\ j=1,\ldots,n\}$ and

$$\mathbb{C}^n\#(\mathring{\alpha}+i\mathbb{R}^n)=\bigcup_{\varepsilon\in\{+,-\}^n}\mathbb{C}^n_{\mathring{\alpha}_\varepsilon}.$$

Theorem 2. *Let $F$, $\mathring{\alpha}$ be as above and suppose $F$ admits a compatible set of partial regularizations at $\mathring{\alpha}$. For $\varepsilon\in\{+,-\}^n$ let the function*

$$\psi_\varepsilon(z)=\sum_{I\subset I_\varepsilon}\mathcal{C}^I(z)$$

*be defined for $z\in U_\varepsilon\subset\mathbb{C}^n_{\mathring{\alpha}_\varepsilon}$. Let $U_\#=\bigcup_\varepsilon U_\varepsilon\subset\mathbb{C}^n\#(\mathring{\alpha}+i\mathbb{R}^n)$ and denote by $U_{\text{ext}}$ the set of those points $z$ in the closure $\overline{U}_\#$ of $U_\#$ in $\mathbb{C}^n$ such that there is an open neighbourhood $W\subset\mathbb{C}^n$ of $z$ with the property that $W\cap(\mathbb{C}^n\#(\mathring{\alpha}+i\mathbb{R}^n))\subset U_\#$. Then there exists a function $\psi\in\mathcal{O}(U_{\text{ext}})$ such that $\psi(z)=\psi_\varepsilon(z)$ for $z\in U_\varepsilon$.*



*Thus $\psi$ is a holomorphic extension of the function*

$$\psi_{(-,\ldots,-)}(z)=\mathcal{C}(z)\quad\textit{for }\operatorname{Re}z<\mathring{\alpha}.$$

Proof. Let $Z_j$, $j=0,\ldots,n-1$, be the stratification of the analytic set $\mathbb{C}^n\setminus\mathbb{C}^n\#(\mathring{\alpha}+i\mathbb{R}^n)$ into piecewise linear submanifolds $Z_j$ of real dimension $\dim Z_j=j+n$. For points $z\in U_{\text{ext}}\cap Z_{n-1}$ extendability follows easily from a parameter version of Corollary 1. For the remaining $j=n-2,\ldots,0$ the proof goes by induction in view of the following theorem which is an easy consequence of the edge of the wedge theorem [5].

Theorem. *Let $\mathring{z}\in\mathbb{C}^n$ and $\widetilde{\Phi}\in\mathcal{O}(W)$ where $W=V+iG$, $V$ is a connected local cone with vertex at $\operatorname{Re}\mathring{z}$ and $G$ is an open neighbourhood of $\operatorname{Im}\mathring{z}$ in $\mathbb{R}^n$. Then there exists an open neighbourhood $\Omega$ of $\mathring{z}$ in $\mathbb{C}^n$ such that $\widetilde{\Phi}$ extends to a function $\Phi$ holomorphic on $(\operatorname{conv}(V)+iG)\cap\Omega\cup W$, where* conv *denotes convex hull.*

*In particular, if $\operatorname{conv}(V)$ is an open neighbourhood of $\operatorname{Re}\mathring{z}$ in $\mathbb{R}^n$, then $\Phi$ is holomorphic in a neighbourhood of $\mathring{z}$ in $\mathbb{C}^n$.*

Example 1. We shall write down the functions $\psi_\varepsilon$ of Theorem 2 explicitly in the case of $n=2$. We assume that all integrals below are absolutely convergent; then the compatibility condition of Definition 4 follows from the Fubini theorem.

$$\begin{aligned}\psi_{-,-}(z_1,z_2)=&\int\!\!\int F(\mathring{\alpha}_1+i\gamma_1,\mathring{\alpha}_2+i\gamma_2)\\&\qquad\times G(z_1-\mathring{\alpha}_1-i\gamma_1)G(z_2-\mathring{\alpha}_2-i\gamma_2)\,d\gamma_1\,d\gamma_2\\&\qquad\qquad\text{for }\operatorname{Re}z_1<\mathring{\alpha}_1,\ \operatorname{Re}z_2<\mathring{\alpha}_2,\\ \psi_{-,+}(z_1,z_2)=&\int\!\!\int F(\mathring{\alpha}_1+i\gamma_1,\mathring{\alpha}_2+i\gamma_2)\\&\qquad\times G(z_1-\mathring{\alpha}_1-i\gamma_1)G(z_2-\mathring{\alpha}_2-i\gamma_2)\,d\gamma_1\,d\gamma_2\\&\quad+\int F(\mathring{\alpha}_1+i\gamma_1,z_2)G(z_1-\mathring{\alpha}_1-i\gamma_1)\,d\gamma_1\\&\qquad\qquad\text{for }\operatorname{Re}z_1<\mathring{\alpha}_1,\ \operatorname{Re}z_2>\mathring{\alpha}_2,\\ \psi_{+,+}(z_1,z_2)=&\int\!\!\int F(\mathring{\alpha}_1+i\gamma_1,\mathring{\alpha}_2+i\gamma_2)\\&\qquad\times G(z_1-\mathring{\alpha}_1-i\gamma_1)G(z_2-\mathring{\alpha}_2-i\gamma_2)\,d\gamma_1\,d\gamma_2\\&\quad+\int F(z_1,\mathring{\alpha}_2+i\gamma_2)G(z_2-\mathring{\alpha}_2-i\gamma_2)\,d\gamma_2\\&\quad+\int F(\mathring{\alpha}_1+i\gamma_1,z_2)G(z_1-\mathring{\alpha}_1-i\gamma_1)\,d\gamma_1+F(z_1,z_2)\\&\qquad\qquad\text{for }\operatorname{Re}z_1>\mathring{\alpha}_1,\ \operatorname{Re}z_2>\mathring{\alpha}_2,\end{aligned}$$

$$\psi_{+,-}(z_1,z_2) = \int\int F(\mathring{\alpha}_1+i\gamma_1,\mathring{\alpha}_2+i\gamma_2) \times G(z_1-\mathring{\alpha}_1-i\gamma_1)G(z_2-\mathring{\alpha}_2-i\gamma_2)\,d\gamma_1\,d\gamma_2 + \int F(z_1,\mathring{\alpha}_2+i\gamma_2)G(z_2-\mathring{\alpha}_2-i\gamma_2)\,d\gamma_2$$
$$\text{for } \operatorname{Re} z_1 > \mathring{\alpha}_1,\ \operatorname{Re} z_2 < \mathring{\alpha}_2 .$$

The quadrants given above indicate the sets on which the respective functions should be considered. Their domains of definition may, however, be much smaller and depend on the behaviour of $F$.

EXAMPLE 2. We shall find the set $U_{\text{ext}}$ for $F(z_1,z_2) = 1/(z_1^2+z_2^2)$ and $\mathring{\alpha} \in \mathbb{R}^2$. Observe that in the case $\mathring{\alpha} \neq 0$, since the function $\mathbb{R}^2 \ni \gamma \mapsto F(\mathring{\alpha}+i\gamma)$ is integrable, we have no problem with regularizations and compatibility.

In the case $\mathring{\alpha}=0$ the function $\gamma \mapsto -1/(\gamma_1^2+\gamma_2^2)$ is no longer integrable on $\mathbb{R}^2$ so we have to regularize it so that the condition of compatibility hold. In our case this amounts to checking that for any $\varphi \in S(\mathbb{R}^2)$

$$\operatorname{Pf}_{\gamma_2=0}\int\Big(\int \frac{\varphi(\gamma_1,\gamma_2)}{\gamma_1^2+\gamma_2^2}\,d\gamma_1\Big)\,d\gamma_2 = \operatorname{Pf}_{\gamma_1=0}\int\Big(\int \frac{\varphi(\gamma_1,\gamma_2)}{\gamma_1^2+\gamma_2^2}\,d\gamma_2\Big)\,d\gamma_1 .$$

Since integrals of the form

$$\int_{\mathbb{R}^2} \frac{\gamma_1\widetilde{\varphi}(\gamma_1,\gamma_2)}{\gamma_1^2+\gamma_2^2}\,d\gamma, \quad \int_{\mathbb{R}^2} \frac{\gamma_2\widetilde{\varphi}(\gamma_1,\gamma_2)}{\gamma_1^2+\gamma_2^2}\,d\gamma \quad \text{for } \widetilde{\varphi}\in S(\mathbb{R}^2)$$

are absolutely convergent, it is enough to prove that for some $\chi \in C_0^\infty(\mathbb{R})$ with $\chi \equiv 1$ in a neighbourhood of zero we have

$$\operatorname{Pf}_{\gamma_2=0}\int \chi(\gamma_2)\Big(\int \frac{\chi(\gamma_1)}{\gamma_1^2+\gamma_2^2}\,d\gamma_1\Big)\,d\gamma_2 = \operatorname{Pf}_{\gamma_1=0}\int \chi(\gamma_1)\Big(\int \frac{\chi(\gamma_2)}{\gamma_1^2+\gamma_2^2}\,d\gamma_2\Big)\,d\gamma_1 .$$

But this is obvious since the expressions are symmetric with respect to $\gamma_1,\gamma_2$.

We have to find sets $U^{\{1\}}$ and $U^{\{2\}}$ such that the functions

$$U^{\{1\}} \ni z_1 \mapsto F(z_1,\mathring{\alpha}_2+i\cdot) \in S'(\mathbb{R}),$$
$$U^{\{2\}} \ni z_2 \mapsto F(\mathring{\alpha}_1+i\cdot,z_2) \in S'(\mathbb{R})$$

are holomorphic. For instance, we may take

$$U^{\{1\}} = \bigcap_{\gamma_2\in\mathbb{R}} \{z_1 : (z_1,\mathring{\alpha}_2+\gamma_2)\in U\},$$
$$U^{\{2\}} = \bigcap_{\gamma_1\in\mathbb{R}} \{z_2 : (\mathring{\alpha}_1+i\gamma_1,z_2)\in U\}$$



where $U = \{(z_1,z_2)\in\mathbb{C}^2 : z_1 \neq \pm iz_2\}$ is the domain of holomorphy of $F$. In our case

$$U^{\{1\}} = \mathbb{C}\setminus\{\mathbb{R}\pm i\mathring{\alpha}_2\}, \quad U^{\{2\}} = \mathbb{C}\setminus\{\mathbb{R}\pm i\mathring{\alpha}_1\}.$$

Thus we have

$$U_{-,-} = \{\operatorname{Re} z_1 < \mathring{\alpha}_1\}\times\{\operatorname{Re} z_2 < \mathring{\alpha}_2\},$$
$$U_{-,+} = \{\operatorname{Re} z_1 < \mathring{\alpha}_1\}\times(\{\operatorname{Re} z_2 > \mathring{\alpha}_2\}\setminus\{\mathbb{R}\pm i\mathring{\alpha}_1\}),$$
$$U_{+,+} = (\{\operatorname{Re} z_1 > \mathring{\alpha}_1\}\setminus\{\mathbb{R}\pm i\mathring{\alpha}_2\}) \times (\{\operatorname{Re} z_2 > \mathring{\alpha}_2\}\setminus\{\mathbb{R}\pm i\mathring{\alpha}_1\})\cap\{z_1\neq\pm iz_2\},$$
$$U_{+,-} = (\{\operatorname{Re} z_1 > \mathring{\alpha}_1\}\setminus\{\mathbb{R}\pm i\mathring{\alpha}_2\})\times\{\operatorname{Re} z_2 < \mathring{\alpha}_2\},$$

and consequently $U_{\text{ext}} = \mathbb{C}^2\setminus U_{\text{sing}}$ where

$$U_{\text{sing}} = \{\alpha_1\pm i\mathring{\alpha}_2 : \alpha_1 \geq \mathring{\alpha}_1\}\times\mathbb{C} \cup \mathbb{C}\times\{\alpha_2\pm i\mathring{\alpha}_1 : \alpha_2\geq\mathring{\alpha}_2\}\cup\{\operatorname{Re} z \geq \mathring{\alpha},\ z_1\neq\pm iz_2\}.$$

EXAMPLE 3. We consider the function

$$F(z_1,z_2,z_3) = \frac{1}{z_1^2+z_2^2+z_3^2}$$

and $\mathring{\alpha}=0$. Now there are more functions in play. Namely,

$$(3)\qquad \begin{aligned} U^{\{1,2\}} &\ni (z_1,z_2)\mapsto F(z_1,z_2,i\cdot)\in S'(\mathbb{R}),\\ U^{\{1,3\}} &\ni (z_1,z_3)\mapsto F(z_1,i\cdot,z_3)\in S'(\mathbb{R}),\\ U^{\{2,3\}} &\ni (z_2,z_3)\mapsto F(i\cdot,z_2,z_3)\in S'(\mathbb{R}),\end{aligned}$$

and

$$(4)\qquad \begin{aligned} U^{\{1\}} &\ni z_1\mapsto F(z_1,i\cdot,i\cdot)\in S'(\mathbb{R}^2),\\ U^{\{2\}} &\ni z_2\mapsto F(i\cdot,z_2,i\cdot)\in S'(\mathbb{R}^2),\\ U^{\{3\}} &\ni z_3\mapsto F(i\cdot,i\cdot,z_3)\in S'(\mathbb{R}^2).\end{aligned}$$

We want to find the sets $U^I$ as large as possible so that the functions above are holomorphic.

Consider the first function in (3). Write $z_j = \alpha_j + i\beta_j$, $j=1,2$. Then if $z_1^2+z_2^2-\gamma_3^2=0$ for some $\gamma_3\in\mathbb{R}$, we must have $\alpha_2\beta_2 = -\alpha_1\beta_1$, so we may take

$$U^{\{1,2\}} = \{(z_1,z_2)\in\mathbb{C}^2 : \alpha_1\beta_1+\alpha_2\beta_2\neq 0\}.$$

By symmetry,

$$U^{\{1,3\}} = \{(z_1,z_3)\in\mathbb{C}^2 : \alpha_1\beta_1+\alpha_3\beta_3\neq 0\},$$
$$U^{\{2,3\}} = \{(z_2,z_3)\in\mathbb{C}^2 : \alpha_2\beta_2+\alpha_3\beta_3\neq 0\}.$$

Now, consider the first function in (4). We see that if $z_1^2-\gamma_2^2-\gamma_3^2=0$ for some $\gamma_2,\gamma_3\in\mathbb{R}$ and $z_1=\alpha_1+i\beta_1$, then we must have $\beta_1=0$ and the zeros lie on the circle $\gamma_2^2+\gamma_3^2=\alpha_1^2$. Consequently, $U^{\{1\}}=\{z_1\in\mathbb{C}:\beta_1\neq 0\}$ and by symmetry,

$$U^{\{2\}}=\{z_2\in\mathbb{C}:\beta_2\neq 0\},\qquad U^{\{3\}}=\{z_3\in\mathbb{C}:\beta_3\neq 0\}.$$

We proceed to study the case of $F\in\mathcal{O}(U)$ such that some of the sets

$$\bigcap_{\gamma_J\in\mathbb{R}^J}\{z_I\in\mathbb{C}^I:(z_I,\mathring{\alpha}_J+i\gamma_J)\in U\}$$

may be empty (unlike the examples above) but the sets of "partial singularities"

$$\{\gamma_J\in\mathbb{R}^J:(z_I,\mathring{\alpha}_J+i\gamma_J)\notin U\}$$

are independent of $z_I$.

Let $W$ be an open set in $\mathbb{C}^n$ and $W\#\mathbb{R}^n\overset{\text{def}}{=}\{z=x+iy\in W:\forall j\ \operatorname{Im}z_j\neq 0\}$. Recall (see [16]) that a function $F$ is called Id-*meromorphic* on $W$ if $F\in\mathcal{O}(W\#\mathbb{R}^n)$.

DEFINITION 6. Let $F$ be an Id-meromorphic function on a set $W=V+i\mathbb{R}^n$. We say that $F$ is *of polynomial growth along imaginary planes* if for every $\mathring{\alpha}\in V$ there exists a neighbourhood $V_{\mathring{\alpha}}$ of $\mathring{\alpha}$ in $\mathbb{R}^n$, a number $p\geq 0$ and a positive function $K(z)$ on $W_{\mathring{\alpha}}=V_{\mathring{\alpha}}+i\mathbb{R}^n$ such that

$$|F(z)|\leq K(z)/\operatorname{dist}(z,Z)^p\quad\text{ for } z\in W\#\mathbb{R}^n\cap W_{\mathring{\alpha}},$$

where $Z=W\setminus W\#\mathbb{R}^n$, and $K(z)\|z\|^{-p}$ is bounded on $W_{\mathring{\alpha}}$.

PROPOSITION 1. *Suppose $F$ is* Id-*meromorphic on an open set* $W=V+i\mathbb{R}^n\subset\mathbb{C}^n$, *and of polynomial growth along imaginary planes. Then for every* $\mathring{\alpha}\in V$ *there exists a compatible set of partial regularizations of* $F$

$$F_{z_I,\mathring{\alpha}_J}\in\mathcal{O}(U^I,S'(\mathbb{R}^J)),\qquad I\subset\{1,\ldots,n\},$$

*where*

(5) $$U^I=\{z_I:(z_I,\mathring{\alpha}_I+i\gamma_I)\in W\#\mathbb{R}^n \textit{ for some } \gamma_I\in\mathbb{R}^n\}.$$

Proof. Consider the function $z_1\mapsto F(z_1,z_2,\ldots,z_n)$ for $z_2,\ldots,z_n$ fixed, with $\operatorname{Im}z_j\neq 0$ for $j=2,\ldots,n$. It has singularities on the real line $\operatorname{Im}z_1=0$. Fix $\mathring{z}_1$ with $\operatorname{Im}\mathring{z}_1\neq 0$ so that $(\mathring{z}_1,z_2,\ldots,z_n)\in W\#\mathbb{R}^n$. Suppose for instance that $\operatorname{Im}\mathring{z}_1>0$. For every $z_1$ with $\operatorname{Im}z_1>0$ define

$$\mathcal{J}_+^1F(z_1,\ldots,z_n)=\int_{L_{z_1}}F(\zeta,z_2,\ldots,z_n)\,d\zeta$$

where $L_{z_1}$ is a curve joining $z_1$ to $\mathring{z}_1$ consisting of the line segment from $z_1$ to $\operatorname{Re}z_1+i\varepsilon_0$ for some fixed $\varepsilon_0>0$ (if $\operatorname{Im}z_1<\varepsilon_0$) and another line segment from $\operatorname{Re}z_1+i\varepsilon_0$ to $\mathring{z}_1$. If $\operatorname{Im}z_1\geq\varepsilon_0$, $L_{z_1}$ is the line segment from $z_1$ to $\mathring{z}_1$.



In a similar way we define $\mathcal{J}_-^1$ corresponding to $\operatorname{Im}\mathring{z}_1<0$, and $\mathcal{J}_\pm^k$ for $k=2,\ldots,n$ corresponding to the variables $z_2,\ldots,z_n$.

If $F$ is of polynomial growth then by standard reasoning we infer that there exists a multiindex $\varrho\in\mathbb{N}_0^n$ such that the function

$$G(z)=\mathcal{J}_\varepsilon^\varrho F(z)\quad\text{ for } z\in W_\varepsilon=\{z\in W:\varepsilon_j\operatorname{Re}z_j>0,\ j=1,\ldots,n\}$$

and $\varepsilon\in\{+,-\}^n$ satisfies the estimate

$$|G(z)|\leq C(1+\|z\|)^q\quad\text{ for } z\in W\#\mathbb{R}^n$$

for some positive constants $C$ and $q$ (independent of $z$) and the same is true for the derivatives $D_{z_j}G$, $j=1,\ldots,n$.

Fix $\mathring{\alpha}\in V$, and for every subset $I\subset\{1,\ldots,n\}$ let $U^I$ be the set defined by (5). For $z_I\in U^I$ we define the tempered distribution

$$F_{z_I,\mathring{\alpha}_J}[\sigma]=(-1)^{|\varrho_J|}D_{z_I}^{\varrho_I}\int_{\mathbb{R}^J}G(z_I,\mathring{\alpha}_J+i\gamma_J)D_{\gamma_J}^{\varrho_J}\sigma(\gamma_J)\,d\gamma_J,\qquad\sigma\in S(\mathbb{R}^J).$$

It is clear that for $\sigma\in S(\mathbb{R}^J)$ the function

$$U^I\ni z_I\mapsto G_{z_I,\mathring{\alpha}_J}=(-1)^{|\varrho_J|}\int_{\mathbb{R}^J}G(z_I,\mathring{\alpha}_J+i\gamma_J)D_{\gamma_J}^{\varrho_J}\sigma(\gamma_J)\,d\gamma_J$$

is holomorphic on $U^I$ and the set $\{G_{z_I,\mathring{\alpha}_J}\in\mathcal{O}(U^I,S'(\mathbb{R}^I))\}$ for $I\subset\{1,\ldots,n\}$ is a compatible family of regularizations of $G$ at $\mathring{\alpha}$. Hence the same is true for $F$, which was to be proven.

Remark 3. Observe that in the definition of a modified partial Cauchy transformation $C^I$ we evaluate $F_{z_I,\mathring{\alpha}_J}$ only on a family of analytic functions

$$\mathbb{R}^J\ni\gamma_J\mapsto G^J(z_J-\mathring{\alpha}_J-i\gamma_J).$$

Thus the assumption that $F_{z_I,\mathring{\alpha}_J}$ is a distribution in $S'(\mathbb{R}^J)$ may be dropped. Instead one can introduce a special class of hyperfunctions on $\overline{\mathbb{R}^J}$, analogous to Fourier hyperfunctions (see [2]), for which such evaluations make sense. However, a suitable compatibility condition should be retained.

In particular, in the hyperfunction formulation we could drop the assumption on the growth order of $F$ in Proposition 1.

**3. Mellin distributions with $A$-meromorphic Mellin transforms.** The spaces $M'_{(\omega)}((0,\mathbf{t}])$, $\omega\in(\mathbb{R}\cup\{\infty\})^n$, of Mellin distributions on a poly-interval $(0,\mathbf{t}]$, $\mathbf{t}\in\mathbb{R}_+^n$ were defined in [7, 12, 14]. We do not recall here the definition but only note that a Mellin distribution $u\in M'_{(\omega)}$ admits evaluation on functions $x\mapsto x^{-z-1}$ for $\operatorname{Re}z<\omega$. The holomorphic function

$$\mathcal{M}u(z)=u[x^{-z-1}],\quad\operatorname{Re}z<\omega,$$

is called the *Mellin transform* of $u$. That $\mathcal{M}u$ is $A$-meromorphic on $\Omega \subset \mathbb{C}^n$ (where $A \in \mathrm{GL}(n,\mathbb{R})$ with nonnegative entries) means that $(\mathcal{M}u) \circ A^{-1}$ is Id-meromorphic on $A(\Omega)$ (see [16] for other equivalent definitions).

We have the following variant of Th. 3 of [16] (see also [7]):

THEOREM 3. *A function $F$ is holomorphic on a set $\{\mathrm{Re}\, Az < A\omega\}$ (for some $\omega \in (\mathbb{R} \cup \{\infty\})^n$) and for every $b \in \mathbb{R}^n$ with $b < \omega$ satisfies the estimate*

$$|F(z)| \le p(|z|)\mathbf{t}^{-\mathrm{Re}\, Az}, \quad \mathrm{Re}\, Az \le Ab,$$

*for some polynomial $p$ and $\mathbf{t} \in \mathbb{R}^n_+$ if and only if there exists a Mellin distribution $u \in M'_{(\omega)}$ such that*

$$\mathrm{supp}\, u \subset Z^A_{\mathbf{t}} = \exp(A^{\mathrm{tr}}(\ln(\{0 < y \le \mathbf{t}\})))$$

*and $F(z) = \mathcal{M}u(z)$ for $\mathrm{Re}\, Az < A\omega$.*

Proof. Let $F$ satisfy the assumptions of the theorem. Define $H(z) = F \circ A^{-1}(z)$. Then $H \in \mathcal{O}(\{\mathrm{Re}\, z < A\omega\})$. Let $Q$ be a polynomial in $\mathbb{C}^n$ such that $H(z) = Q(z)G(z)$ where $G \in \mathcal{O}(\{\mathrm{Re}\, z < A\omega\})$ and for every $b < A\omega$

$$|G(z)| \le \frac{c}{(|z_1|^2+1)\ldots(|z_n|^2+1)}\mathbf{t}^{-\mathrm{Re}\, z} \quad \text{for } \mathrm{Re}\, z \le b.$$

Now, from Lemma 2 in [7] we know that the function

$$g(x) = \frac{1}{(2\pi i)^n}\int_{b+i\mathbb{R}^n} G(z)x^z\, dz \quad \text{for } x \in \mathbb{R}^n_+$$

is continuous, $\mathrm{supp}\, g \subset (0,\mathbf{t}]$ and $x^{-b}g(x)$ is bounded. Further, $g \in M'_{(A\omega)}$ and

$$g(z) = \mathcal{M}g(z) \quad \text{for } \mathrm{Re}\, z < A\omega.$$

As in the proof of Lemma 1 in [16] we have

$$\begin{aligned}\mathcal{M}g(Az) &= \int_{0<x\le\mathbf{t}} g(x)x^{-Az-1}\, dx \\ &= \int_{0<x\le\mathbf{t}} g(x)(\exp(A^{\mathrm{tr}}\ln x))^{-z}x^{-1}\, dx \\ &= \frac{1}{|\det A|}\int_{Z^A_{\mathbf{t}}} g(\exp((A^{\mathrm{tr}})^{-1}\ln y))y^{-z-1}\, dy.\end{aligned}$$

Note that the set $Z^A_{\mathbf{t}}$ is bounded since $A$ has nonnegative entries. Denote by $\chi^A_{\mathbf{t}}$ the characteristic function of $Z^A_{\mathbf{t}}$. We shall prove that the function

$$h(y) = \frac{1}{|\det A|}\chi^A_{\mathbf{t}}(y)g(\exp((A^{\mathrm{tr}})^{-1}\ln y)) \quad \text{is in } M'_{(\omega)}.$$



Indeed, it follows from the properties of $g$ that for every $b < A\omega$ the function

$$(\exp((A^{\mathrm{tr}})^{-1}\ln y))^{-b}h(y) = y^{-A^{-1}b}h(y)$$

is bounded, which is just what we want since $\{b < \omega\} \subset \{Ab < A\omega\}$.

Finally, we put

$$u = Q \circ A\left(y\frac{d}{dy}\right)h.$$

Then $u \in M'_{(\omega)}$, $\mathrm{supp}\, u \subset Z^A_{\mathbf{t}}$ and

$$\mathcal{M}u(z) = Q(Az)\mathcal{M}h(z) = Q(Az)\mathcal{M}g(Az) = H(Az) = F(z)$$

as desired. The converse implication follows from Theorem 4 in [11].

Before stating the main result of this section we need the following generalization of Lemma 1.

LEMMA 2. *Let $\varphi \in C^\infty_0(\mathbb{R}^n)$. Choose $\chi \in C^\infty_0(\mathbb{R})$ with $\chi \equiv 1$ in a neighbourhood of zero and such that $\chi(x_1)\ldots\chi(x_n) \equiv 1$ on $\mathrm{supp}\,\varphi$. Define $G = \mathcal{M}\chi$. Then for every $\mathbf{r} > 0$,*

$$\mathcal{M}\varphi(z) = \sum_{I\in\mathcal{P}_n}\ \sum_{\nu\in\mathbb{N}^J_0,\, \nu<\mathbf{r}_J} H_{I,\nu}(z_I)G^J(z_J - \nu_J)$$

*where $\mathcal{P}_n$ is the set of all subsets $I \subset \{1,\ldots,n\}$, $J$ is the complementary subset to $I$ and*

$$H_{I,\nu} \in \mathcal{O}(\{z_I \in \mathbb{C}^I : \mathrm{Re}\, z_I < \mathbf{r}_I\}),$$
$$H_{I,\nu}(\alpha_I + i\cdot) \in S(\mathbb{R}^I) \quad \textit{for every fixed } \alpha_I < \mathbf{r}_I$$

*(we assume that $H_{\emptyset,\nu} = c_\nu$ are constants).*

Proof. For simplicity of notation we assume $n = 2$, $\mathbf{r} = (r,r)$, $r > 0$. From the Taylor formula we have (see Lemma 1 in [16])

$$\begin{aligned}\chi(x_1)\chi(x_2)\varphi(x_1,x_2) &= \sum_{\nu_1,\nu_2=0}^{r-1} c_\nu\chi(x_1)x_1^{\nu_1}\chi(x_2)x_2^{\nu_2} \\ &\quad + x_1^r\sum_{j=0}^{r-1}\chi(x_1)s_j(x_1)\chi(x_2)x_2^j \\ &\quad + x_2^r\sum_{j=0}^{r-1}\chi(x_1)x_1^j\chi(x_2)n_j(x_2) \\ &\quad + \chi(x_1)\chi(x_2)x_1^r x_2^r p(x_1,x_2),\end{aligned}$$

where $c_\nu = (1/\nu!)D^\nu\varphi(0)$ for $0 \le \nu < \mathbf{r}$, and $n_j$, $s_j$ for $j = 0,\ldots,r-1$ and $p$ are smooth functions.

Computing the Mellin transform of the above we find

$$H_{\emptyset,\nu}=c_\nu,\quad 0\le\nu<\mathbf{r},$$
$$H_{\{1\},j}(z_1)=\mathcal{M}(x_1^r s_j(x_1)\chi_1(x_1))(z_1),\quad j=0,\ldots,r-1,$$
$$H_{\{2\},j}(z_2)=\mathcal{M}(x_2^r n_j(x_2)\chi_2(x_2))(z_2),\quad j=0,\ldots,r-1,$$
$$H_{\{1,2\}}(z_1,z_2)=\mathcal{M}(\chi(x_1)\chi(x_2)x_1^r x_2^r p(x_1,x_2))(z_1,z_2).$$

To see that the functions above are rapidly decreasing along the imaginary axis we note that if $\psi\in C_0^\infty$ then for $\operatorname{Re} z<0$ and any positive indices $\gamma$ and $\delta$,

$$D_\beta^\gamma((\alpha+i\beta)^\delta\mathcal{M}\psi)(\alpha+i\beta)=(-i)^{|\gamma|}\mathcal{M}((\ln x)^\gamma(xD)^\delta\psi(x))(\alpha+i\beta),$$

and for every $\varepsilon>0$, $x^\varepsilon(\ln x)^\gamma(xD)^\delta\psi(x)$ is bounded.

Now we can prove the following important result which shows that the notion of $A$-meromorphy of the Mellin transform of a Mellin distribution is "local".

THEOREM 4. *Let $A$ be an $n\times n$ real nonsingular matrix with nonnegative rational entries. Let $u\in M'_{(\omega)}$, $\operatorname{supp} u\subset Z_{\mathbf{t}}^A=\exp(A^{\mathrm{tr}}(\ln(\{0<y\le\mathbf{t}\})))$ for some $\mathbf{t}\in\mathbb{R}_+^n$ and suppose $\mathcal{M}u$ is $A$-meromorphic on $\Omega_A=\{z\in\mathbb{C}^n:\operatorname{Re} Az<\mathring{\omega}\}$ for some $\mathring{\omega}\in(\mathbb{R}\cup\{\infty\})^n$ and of polynomial growth along imaginary planes. Let $\varphi\in C_0^\infty(\mathbb{R}^n)$. Then $\varphi u\in M'_{(\omega)}$ and $\mathcal{M}(\varphi u)$ is $A$-meromorphic on $\Omega_A$.*

*Moreover, for every $\mathbf{r}\in\mathbb{R}_+^n$ with $\mathbf{r}\le\mathring{\omega}$,*

$$b^A(\mathcal{M}(\varphi u))(\alpha)=\sum_{0\le\nu\le\mathbf{r},\,\nu\in\mathbb{N}_0^n}\frac{D^\nu\varphi(0)}{\nu!}b^A(\mathcal{M}u)(\alpha-\nu)\tag{6}$$

*as a distribution on $\{\alpha\in\mathbb{R}^n:A\alpha<\mathbf{r}\}$, where $b^A(F)$ denotes the oriented boundary value of $F$, i.e. the hyperfunction defined by $F$* (see [16], Section 3, for details).

Proof. It follows from the proof of Theorem 3 that

$$\widetilde{u}=|\det A|\cdot u\circ\exp\circ A^{\mathrm{tr}}\circ\ln\in M'_{(A\omega)}((0,\mathbf{t}])$$

and $\mathcal{M}\widetilde{u}(z)=\mathcal{M}u\circ A^{-1}(z)$, i.e. $\mathcal{M}\widetilde{u}$ is Id-meromorphic on $\Omega=\{\operatorname{Re} z<\mathring{\omega}\}$. Set

$$\widetilde{\varphi}(x)=\varphi(\exp(A^{\mathrm{tr}}\ln x)),\quad x\in\mathbb{R}_+^n.$$

Multiplying $A$ by a suitable constant (which does not affect $A$-meromorphy) we may assume that $A$ has nonnegative integer entries. We shall now check that $\widetilde{\varphi}$ extends to a smooth function on $\mathbb{R}^n$. Writing $A^{\mathrm{tr}}=(a_{jl})_{j,l=1,\ldots,n}$ with $a_{jl}\in\mathbb{N}_0$ we have

$$\widetilde{\varphi}(x)=\varphi\Big(\prod_{j=1}^n x_j^{a_{1j}},\ldots,\prod_{j=1}^n x_j^{a_{nj}}\Big)\tag{7}$$

and since all $a_{jl}\in\mathbb{N}_0$, $\widetilde{\varphi}$ has the claimed property. To make it compactly supported we take a $\psi\in C_0^\infty(\mathbb{R}^n)$ with $\psi\equiv1$ on $(0,\mathbf{t}]$ and put

$$\widetilde{\widetilde{\varphi}}=\psi\cdot\widetilde{\varphi}.$$

Now, it is enough to prove the theorem in the case of a distribution $\widetilde{u}\in M'_{(A\omega)}((0,\mathbf{t}])$ whose Mellin transform is Id-meromorphic on $\Omega$ and of a smooth function $\widetilde{\widetilde{\varphi}}\in C_0^\infty(\mathbb{R}^n)$.

Fix $\overset{*}{\alpha}<A\omega$. Then (see [10], Proposition 2)

$$\mathcal{M}(\widetilde{\widetilde{\varphi}}\widetilde{u})(z)=\frac{1}{(2\pi)^n}T_{\overset{*}{\alpha}}[\mathcal{M}\widetilde{\widetilde{\varphi}}(z-\overset{*}{\alpha}-i\gamma)]\quad\text{for }\operatorname{Re} z<\overset{*}{\alpha}\tag{8}$$

where $T_{\overset{*}{\alpha}}(\beta)=\mathcal{M}\widetilde{u}(\overset{*}{\alpha}+i\beta)$ is of polynomial growth at infinity and the function $\gamma\mapsto\mathcal{M}\widetilde{\widetilde{\varphi}}(z-\overset{*}{\alpha}-i\gamma)$ is rapidly decreasing if $\operatorname{Re} z<\overset{*}{\alpha}$. Let $\mathbf{r}\in\mathbb{R}_+^n$. Then from Lemma 2,

$$\mathcal{M}\widetilde{\widetilde{\varphi}}(z)=\sum_{I\in\mathcal{P}_n}\sum_{\nu\in\mathbb{N}_0^J,\,\nu<\mathbf{r}_J}H_{I,\nu}(z_I)G^J(z_J-\nu_J)\tag{9}$$

and we need only prove that for every $I\in\mathcal{P}_n$ and $\nu\in\mathbb{N}_0^J$ with $\nu<\mathbf{r}_J$, the functions

$$\Psi_{I,\nu}(z)=\frac{1}{(2\pi)^n}T_{\overset{*}{\alpha}}[H_{I,\nu}(z_I-\overset{*}{\alpha}_I-i\gamma_I)G^J(z_J-\nu-\overset{*}{\alpha}_J-i\gamma_J)]\tag{10}$$

defined for $\operatorname{Re} z<\overset{*}{\alpha}$ are Id-meromorphic on $\Omega\cap\{\operatorname{Re} z<\mathbf{r}\}$. For $z_I<\mathbf{r}_I$ define

$$S_{\overset{*}{\alpha}_J+\nu_J}(z_I)=\frac{1}{(2\pi)^{|I|}}T_{\overset{*}{\alpha}}[H_{I,\nu}(z_I-\overset{*}{\alpha}_I-i\gamma_I)]\in S'(\mathbb{R}^J).$$

Then with $\mathring{\alpha}_J=\overset{*}{\alpha}_J+\nu_J$, formula (10) can be written as

$$\Psi_{I,\nu}(z_I,z_J)=\mathcal{C}^I(S_{\mathring{\alpha}_J}(z_I))(z_J)\quad\text{for }\operatorname{Re} z<\mathring{\alpha}$$

where $\mathcal{C}^I$ is the Cauchy transformation in the variables $z_J$. We want to apply Theorem 2 with $\{1,\ldots,n\}$ replaced by $J$. First, for every fixed $z_I$ with $\operatorname{Re} z_I<\mathbf{r}_I$ we want to holomorphically extend $S_{\mathring{\alpha}_J}(z_I)(\beta_J)$ regarded as a distribution in the variable $\beta_I$ to the plane $\mathring{\alpha}_J+i\mathbb{R}^J$.

Applying Proposition 1 to the Id-meromorphic function $T(z)=\mathcal{M}\widetilde{u}(z)$ we get the existence of a compatible set $T_{z_I,\overset{*}{\alpha}_J}\in\mathcal{O}(\widetilde{U}^I,S'(\mathbb{R}^J))$ where $\widetilde{U}^I=\{z_I:(z_I,\mathring{\alpha}_J+i\gamma_J)\in\Omega\#\mathbb{R}^n\text{ for some }\gamma_J\}$. Clearly $T_{\overset{*}{\alpha}}$ is the same as before since no regularization is required for it.

Now fix $J$ and $z_I$ with $\operatorname{Re} z_I < \mathbf{r}_I$. Then as $I'$ runs through the subsets of $J$ we get a compatible set of holomorphic distribution-valued functions

$$\widetilde{U}^{I'} \ni z_{I'} \mapsto S_{z_{I'},\mathring{\alpha}_{J'}}(z_I) \in S'(\mathbb{R}^{J'}),$$

where

$$S_{z_{I'},\mathring{\alpha}_{J'}}(z_I)[\sigma] = \frac{1}{(2\pi)^{|I|}} T_{z_{I'},\mathring{\alpha}_{J'\cup I}}[H_{I,\nu}(z_I - \overset{*}{\alpha}_I - i\gamma_I)\sigma(\gamma_{J'})]$$

for $\sigma \in S(\mathbb{R}^{J'})$. Clearly for every $I' \subset J$ and every fixed $z_{I'} \in \widetilde{U}^{I'}$ and $\sigma \in S(\mathbb{R}^{J'})$,

$$\{\operatorname{Re} z_I < \mathbf{r}_I\} \ni z_I \mapsto S_{z_{I'},\mathring{\alpha}_{J'}}(z_I)[\sigma]$$

is a holomorphic function.

Thus we can apply a "parameter version" of Theorem 2, with a holomorphic parameter $z_I \in \{\operatorname{Re} z_I < \mathbf{r}_I\}$, and with the set $\{1,\dots,n\}$ replaced by $J$. We conclude that

$$\Psi_{I,\nu} \in \mathcal{O}(\{\operatorname{Re} z_I < \mathbf{r}_I\} \times (\widetilde{U}^J_{\text{ext}} + \nu_J)). \tag{11}$$

Since $U^J = \{z_J < \mathring{\omega}_J\}$ is a product we observe that $U^J_{\text{ext}} = \widetilde{U}^J \# \mathbb{R}^J$. Summing the functions $\Psi_{I,\nu}$ over $I \in \mathcal{P}_n$ and $\mathbb{N}_0^J \ni \nu < \mathbf{r}_J$ we see by (8)–(11) that

$$\mathcal{M}(\widetilde{\varphi}\,\widetilde{u}) = \mathcal{M}(\widetilde{\widetilde{\varphi}}\,\widetilde{u}) \in \mathcal{O}(\Omega \# \mathbb{R}^n \cap \{\operatorname{Re} z < \mathbf{r}\}).$$

Since $\mathbf{r}$ was arbitrary this proves that $\mathcal{M}(\widetilde{\varphi}\,\widetilde{u}) \in \mathcal{O}(\Omega \# \mathbb{R}^n)$, which means that $\mathcal{M}(\widetilde{\varphi}\,\widetilde{u})$ is Id-meromorphic on $\Omega$.

It now remains to compute the boundary values. Suppose first that $\varphi$ is flat at zero of order $\mathbf{r} \in \mathbb{R}^n_+$, i.e.

$$\varphi(x) = \sum_{j=1}^n x_j^{r_j} R_j(x), \qquad R_j \text{ smooth at zero}.$$

Then from (7) we see that $\widetilde{\varphi}$, and consequently $\widetilde{\widetilde{\varphi}}$, is flat at zero of order at least $\mathbf{r}$. Thus (9) becomes

$$\mathcal{M}\widetilde{\widetilde{\varphi}}(z) = \sum_{\emptyset \neq I \in \mathcal{P}_n} \sum_{\nu \in \mathbb{N}_0^J,\, \nu < \mathbf{r}_J} H_{I,\nu}(z_I) G^J(z_J - \nu_J).$$

In view of (11) the functions $\Psi_{I,\nu}$ defined by (10) are holomorphic on

$$\{z = (z_I, z_J) : \operatorname{Re} z_I < \mathbf{r}_I,\ \operatorname{Re} z_J < \mathring{\omega}_J + \nu_J,\ z_j \neq 0\ \forall j \in J\}.$$

Thus

$$\Psi_{I,\nu} \circ A \in \mathcal{O}(\{z = (z_I, z_J) : \operatorname{Re} Az_I < \mathbf{r}_I,\ \operatorname{Re} Az_J < \mathring{\omega}_J + \nu,\ (Az)_j \neq 0\ \forall j \in J\}). \tag{12}$$



Since

$$\mathcal{M}(\varphi u) = \sum_{\emptyset \neq I \in \mathcal{P}_n} \sum_{\nu \in \mathbb{N}_0^J,\, \nu < \mathbf{r}_J} \Psi_{I,\nu} \circ A$$

it follows from (12) that

$$b^A(\mathcal{M}(\varphi u)) = 0 \quad \text{on } \{\alpha \in \mathbb{R}^n : A\alpha < \mathbf{r}\}.$$

Now for every $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have

$$\varphi(x) = \sum_{\nu \in \mathbb{N}_0^n,\, \nu < \mathbf{r}} \frac{D^\nu \varphi(0)}{\nu!} \chi(x) x^\nu + \varphi_{\mathbf{r}}(x)$$

where $\chi \equiv 1$ on $\operatorname{supp} u$ and $\varphi_{\mathbf{r}}$ is flat of order $\mathbf{r}$. Since $\mathcal{M}(\chi x^\nu u)(z) = \mathcal{M}(\chi u)(z - \nu)$ the desired result follows.

Remark 4. The assumption in Theorem 4 that $A$ has rational entries was necessary only to apply Lemma 2. It may be dropped by extending Lemma 2 to certain singular functions. Also the assumption that $\mathcal{M}u$ is of polynomial growth is not essential (see Remark 3).

## 4. Change of coordinates in the Mellin transformation

DEFINITION 7. Let $A$ be a real $n \times n$ nonsingular matrix with positive entries. By an *A-set at zero* we mean any set $Z \subset \mathbb{R}^n_+$ such that there exists an open set $U \ni 0$ and $\mathbf{t} \in \mathbb{R}^n_+$ satisfying

$$Z \cap U \subset Z^A_{\mathbf{t}} = \exp(A^{\mathrm{tr}}(\ln(\{0 < x \leq \mathbf{t}\}))).$$

DEFINITION 8. We say that a distribution $u$ on an open set $U_+ \subset \mathbb{R}^n_+$ with support in an $A$-set is *locally in* $M'_{(\omega)}$ and its local Mellin transform is *A-meromorphic on a set* $\Omega \subset \mathbb{C}^n$ if there exists an open set $U \subset \mathbb{R}^n$ with $0 \in U$ such that for every $\varphi \in C_0^\infty(U)$ with $\varphi \equiv 1$ in a neighbourhood of zero, $\varphi u \in M'_{(\omega)}$ and $\mathcal{M}(\varphi u)$ is $A$-meromorphic on $\Omega$.

Remark 5. It follows from the modified version of Theorem 4 (suggested by Remark 4) that it is enough to take in Definition 8 only one $\varphi$ having the required properties. Actually, one may only require that $\varphi(0) \neq 0$.

DEFINITION 9. By a *reticular coordinate system* at zero we mean a local diffeomorphism $f : U \to V$, $U, V$ open in $\mathbb{R}^n$ with $0 \in U, V$, which preserves the coordinate surfaces, i.e. for $j = 1, \dots, n$,

$$f(\{x_j = 0\} \cap U) \subset \{y_j = 0\}.$$

For $l \in \mathbb{N}_0$ and a distribution $u \in D'(\mathbb{R}^n)$ with support in $\overline{\mathbb{R}^n_+}$, $u^{*l}$ denotes the convolution power $u^{*l} = u * \dots * u$ ($l$ times) if $l \geq 1$, and $u^{*0} = \delta_0$.

THEOREM 5. *Let $A$ be an $n \times n$ real nonsingular matrix with nonnegative rational entries. Let $u \in M'_{(\omega)}$ with $\operatorname{supp} u \subset Z^A_{\mathbf{t}}$ for some $\omega \in \mathbb{R}^n$, $\mathbf{t} \in \mathbb{R}^n_+$*

*and suppose $\mathcal{M}u$ is $A$-meromorphic on $\Omega_A = \{Az < \mathring{\omega}\}$ for some $\mathring{\omega} \in (\mathbb{R} \cup \{\infty\})^n$ and is of polynomial growth along imaginary planes. Let $f : U \to V$ be a reticular coordinate system at zero such that $f(U \cap \mathbb{R}^n_+) \subset V \cap \mathbb{R}^n_+$. Then $u \circ f$ is supported by an $A$-set, is locally in $M'_{(\omega)}$ and its local Mellin transform is $A$-meromorphic on $\Omega_A$.*

*Moreover, for every $\mathbf{r} \in \mathbb{R}^n_+$ with $\mathbf{r} \le \mathring{\omega}$,*

$$b^A(\mathcal{M}(u \circ f)) = \mathbf{S}_{\mathbf{r}} * \Big( \sum_{\nu \in \mathbb{N}_0^n,\, \nu < \mathbf{r}} \frac{1}{\nu!} D^\nu (Jf^{-1})(0) \delta_\nu \Big) * b^A(\mathcal{M}u) \tag{13}$$

*on $\{\alpha \in \mathbb{R}^n : A\alpha < \mathbf{r}\}$, where $b^A(\mathcal{M}(u \circ f)) \overset{\text{def}}{=} b^A(\mathcal{M}(\varphi \cdot (u \circ f)))$ for any $\varphi \in C_0^\infty(\mathbb{R}^n)$ with sufficiently small support and with $\varphi \equiv 1$ about zero, and*

$$\mathbf{S}_{\mathbf{r}} = \mathbf{S}^1_{r_1} * \ldots * \mathbf{S}^n_{r_n} \in D'_{\overline{\mathbb{R}^n_+}}(\mathbb{R}^n)$$

*with*

$$\mathbf{S}^j_{r_j} = \sum_{\gamma \in \mathbb{N}_0^{[r_j]},\, \gamma \le \mathbf{r}} \binom{-\alpha_j - 1}{|\gamma|} \frac{|\gamma|!}{\gamma!} (D_j f_j^{-1}(0))^{-\alpha_j - 1 - |\gamma|} L_j^{*\gamma} \in D'_{\overline{\mathbb{R}^n_+}}(\mathbb{R}^n), \quad j = 1, \ldots, n,$$

*and for $j = 1, \ldots, n$ and $\gamma = (\gamma_1, \ldots, \gamma_{[r_j]})$,*

$$L_j^{*\gamma} = L_{j,1}^{*\gamma_1} * \ldots * L_{j,[r_j]}^{*\gamma_{[r_j]}}$$

*where for $j = 1, \ldots, n$, $k = 1, \ldots, [r_j]$, $e_j = (0, \ldots, 1, \ldots, 0)$ ($1$ in the $j$-th place),*

$$L_{j,k} = \sum_{\kappa \in \mathbb{N}^n,\, |\kappa| = k} \frac{1}{\kappa!} D^{\kappa + e_j} f_j^{-1}(0) \delta_\kappa .$$

Proof. Take a function $\varphi \in C_0^\infty(V)$ with $\varphi \equiv 1$ close to zero. Later on we require that $\operatorname{supp}\varphi$ is sufficiently small. By Theorem 4, $\varphi u \in M'_{(\omega)}$ and $\mathcal{M}(\varphi u)$ is $A$-meromorphic on $\Omega_A$. Since $\varphi u \in M'_{(\omega)}$ and $f$ is a diffeomorphism, clearly $(\varphi u) \circ f \in M'_{(\omega)}$. Further, $(\varphi u) \circ f$ is supported by an $A$-set because $f^{-1}$ preserves the order of tangency to the coordinate planes. By definition we have

$$\mathcal{M}((\varphi u) \circ f)(z) = \varphi u[Jf^{-1}(y) \cdot (f^{-1}(y))^{-z-1}] \quad \text{for } \operatorname{Re} Az < \omega ,$$

where $Jf^{-1}$ denotes the Jacobian of $f^{-1}$. Define

$$f^{-1}(y) = H(y) = (H_1(y), \ldots, H_n(y)) .$$

Since $H$ is reticular it follows that $H_j(y_1, \ldots, y_{j-1}, 0, y_{j+1}, \ldots, y_n) = 0$ for $j = 1, \ldots, n$ and $|y|$ small. By the Taylor formula then $H_j(y) = y_j \widetilde{H}_j(y)$ for some smooth $\widetilde{H}_j$. Next, since $H$ is a local diffeomorphism $D_j H_j(0) \ne 0$, and since it preserves the positive octant we must have $D_j H_j(0) > 0$, $j = 1, \ldots, n$. Fix $\mathbf{r} \in \mathbb{R}^n_+$ and for $j = 1, \ldots, n$ expand $\widetilde{H}_j$ in a Taylor series at zero up to order $[r_j]$. For $y \in V$ we have



$$H_j(y) = y_j \Big( \sum_{k=0}^{[r_j]} L_{j,k}(y) + |y|^{[r_j]+1} R_{j,r_j}(y) \Big)$$

where $R_{j,r_j}$ is bounded on $V$ and

$$L_{j,k}(y) = \sum_{\kappa \in \mathbb{N}^n,\, |\kappa| = k} \frac{1}{\kappa!} D^\kappa \widetilde{H}_j(0) y^\kappa \quad \text{for } j = 1, \ldots, n,\ k = 1, \ldots, [r_j] .$$

We shall need the Taylor formula for the function $y^a$, $a \in \mathbb{C}$, at a point $c \ne 0$ with remainder in the integral form:

$$(c + d)^a = \sum_{l=0}^{s} \binom{a}{l} c^{a-l} d^l + d^{s+1} \binom{a}{s+1} \int_0^1 (1 - \theta)^s (c + \theta d)^{a-s-1}\, d\theta ,$$

and the binomial expansion

$$(\omega_1 + \ldots + \omega_s + e)^l = \sum_{\gamma \in \mathbb{N}_0^s,\, |\gamma| = l} \frac{l!}{\gamma!} \omega^\gamma + e \cdot P_l(\omega)$$

where $\omega = (\omega_1, \ldots, \omega_s)$ and $P_l$ is a polynomial in $\omega$.

We put for $j = 1, \ldots, n$

$$c = L_{j,0}(0) = D_j H_j(0), \qquad d = \sum_{k=1}^{[r_j]+1} \omega_{j,k}$$

where

$$\omega_{j,k} = L_{j,k}(y) \quad \text{for } k = 1, \ldots, [r_j] , \qquad L_{j,[r_j]+1} = e = |y|^{[r_j]+1} R_{j,r_j}(y) .$$

Then for $L_j(y) = (L_{j,1}(y), \ldots, L_{j,[r_j]}(y))$ we have

$$(H(y))^{-z-1} = y^{-z-1} \prod_{j=1}^{n} \Big( \sum_{\gamma^j \in \mathbb{N}_0^{[r_j]},\, |\gamma^j| \le [r_j]} \binom{-z_j - 1}{|\gamma^j|} \times \frac{|\gamma^j|!}{\gamma^j!} (D_j H_j(0))^{-z_j - 1 - |\gamma^j|} (L_j(y))^{\gamma^j} \Big) + y^{-z-1} F(y, z)$$

where $F(y, z) = \prod_{j=1}^n F_j(y_j, z_j)$ and

$$F_j(y, z_j) = F_j(y) + \binom{-z_j - 1}{[r_j] + 1} \Big( \sum_{k=1}^{[r_j]+1} L_{j,k}(y) \Big)^{[r_j]+1} \psi_j(y, z_j) \tag{14}$$

with

$$F_j(y) = \sum_{l=1}^{[r_j]} (L_{j,[r_j]+1})^l P_l(y)$$

with $P_l(y)$, $l=1,\ldots,[r_j]$, a polynomial in $y$,

$$L_{j,[r_j]+1}=|y|^{[r_j]+1}R_{j,r_j}(y)\,,$$

$$\psi_j(y,z_j)=\binom{-z_j-1}{[r_j]+1}\int\limits_0^1(1-\theta)^{[r_j]}\times\Big(D_jH_j(0)+\theta\Big(\sum_{k=1}^{[r_j]+1}L_{j,k}(y)\Big)\Big)^{-z_j-[r_j]-2}\,d\theta\,.$$

It follows from Theorem 4 that the Mellin transform of

$$T=\varphi\cdot JG^{-1}\cdot u$$

is $A$-meromorphic on $\Omega_A$ and for every $0<\mathbf{r}\le\mathring{\omega}$

$$b^A(\mathcal{M}T)(\alpha)=\sum_{0\le\nu\le\mathbf{r}}\frac{1}{\nu!}D^\nu(JG^{-1})(0)b^A(\mathcal{M}u)(\alpha-\nu)$$

as a distribution on $\{A\alpha<\mathbf{r}\}$. Again it follows from the same theorem that the Mellin transform of $F\cdot T$, where $F(y)=\prod_{j=1}^n F_j(y)$, is $A$-meromorphic on $\{A\alpha<\mathbf{r}\}$ and

$$b^A(\mathcal{M}(F\cdot T))=0\quad\text{ on }\{A\alpha<\mathbf{r}\}$$

because $D^\nu F(0)=0$ for $\nu\in\mathbb{N}_0^n$, $\nu\le\mathbf{r}$.

Next observe that the function $\psi_j(y,z_j)$ is an entire function of $z_j$ if $y$ is small enough that

$$D_jH_j(0)+\theta\Big(\sum_{k=1}^{[r_j]+1}L_{j,k}(y)\Big)>0\quad\text{ for }\theta\in(0,1),\ j=1,\ldots,n\,.$$

To ensure this we assume that the support of $\varphi$ is sufficiently small.

To study the second summand in (14) we need the following generalization of Theorem 4.

THEOREM 4′. *Let $v$ be a Mellin distribution supported by an $A$-set such that $\mathcal{M}v$ is $A$-meromorphic on $\{Az<\mathbf{r}\}$ and is of polynomial growth along imaginary planes. Let $\varphi(y,z)$ be a smooth function of $y$ and an entire function of $z$. Then $\mathcal{M}(\varphi(y,z)v)$ is $A$-meromorphic on $\{Az<\mathbf{r}\}$.*

Proof. It follows from Theorem 4 that for every fixed $\zeta\in\mathbb{C}^n$, $\mathcal{M}(\varphi(y,\zeta)v)(z)$ is $A$-meromorphic on $\{Az<\mathbf{r}\}$, and from the version of the theorem with a complex parameter $\zeta$ we see that for every fixed $z\in\{Az\le\mathbf{r}\}\#\mathbb{R}^n$ the function $\mathcal{M}(\varphi(y,\zeta)v)(z)$ is an entire function of $\zeta$. On putting $\zeta=z$ the assertion follows in view of the Hartogs theorem.



Returning to the proof of the theorem we define

$$S_j^{[r_j]+1}(y)=\Big(\sum_{k=1}^{[r_j]+1}L_{j,k}(y)\Big)^{[r_j]+1},\qquad S^{[\mathbf{r}]+\mathbf{1}}=\prod_{j=1}^n S_j^{[r_j]+1}\,,$$

$$\psi(y,z)=\prod_{j=1}^n\psi_j(y,z_j)\,.$$

From Theorem 4′ we conclude that $\mathcal{M}(S^{[\mathbf{r}]+\mathbf{1}}(y)\psi(y,z)T)(z)$ is $A$-meromorphic on $\{Az\le\mathbf{r}\}$ with zero boundary value since $D^\nu S^{[\mathbf{r}]+\mathbf{1}}(0)=0$ for $\nu\le\mathbf{r}$.

More generally, for $I\subset\{1,\ldots,n\}$, $J=\{1,\ldots,n\}\setminus I$ set

$$F_I(y)=\prod_{j\in I}F_j(y),\qquad E_J(y,z_j)=\prod_{j\in J}S_j^{[r_j]+1}(y)\psi_j(y,z_j)\,.$$

Then it follows from Theorem 4′ that $\mathcal{M}(F_I(y)E_J(y,z_J)T)(z)$ is $A$-meromorphic on $\{Az\le\mathbf{r}\}$ and in view of the flatness of $F_I$ and $E_J(\cdot,z_J)$ its boundary value is zero there. This ends the proof of the theorem.

Remark 6. Theorem 4 will remain valid even if $f$ is allowed to have certain singularities at zero starting with a discrete term $y^{\alpha_0}$ with $\alpha_0>0$.

COROLLARY 2 (Generalized Taylor formula for composite functions). *Under the assumptions of Theorem 4 for every $\mathbf{r}\le\mathring{\omega}$ there exists a Mellin distribution $R\in M'_{(\mathring{\omega}-\mathbf{r})}$ which is $(\mathring{\omega},\mathbf{r})$-flat at zero and*

$$u\circ f=T_\mathbf{r}^A[x^\alpha\chi^A(x)]+R$$

*where $\chi^A$ is the characteristic function of the set*

$$Z_\mathbf{1}^A=\exp(A^{\mathrm{tr}}(\ln\{0<x\le\mathbf{1}\}))$$

*and $T_\mathbf{r}^A$ is a distribution (in the variable $\alpha$) equal to*

$$T_\mathbf{r}^A=\mathbf{S}_\mathbf{r}*\Big(\sum_{\nu\in\mathbb{N}_0^n,\,\nu<\mathbf{r}}\frac{1}{\nu!}D^\nu(Jf^{-1})(0)\delta_\nu\Big)*K_\mathbf{r}^A\in E'(\mathbb{R}^n)$$

*with*

$$K_\mathbf{r}^A=\begin{cases}\frac{1}{(2\pi i)^n}b^A(\mathcal{M}u) & \text{on }\Omega_\mathbf{r}\,,\\ 0 & \text{on }\mathbb{R}^n\setminus\{\alpha\in\mathbb{R}^n:|\alpha_j-\mathring{\omega}_j|<r_j,\ j=1,\ldots,n\}\,,\end{cases}$$

*where*

$$\Omega_\mathbf{r}=\{\alpha\in\mathbb{R}^n:\alpha<\mathring{\omega}\}\cup\{\alpha\in\mathbb{R}^n:|\alpha_j-\mathring{\omega}_j|<r_j,\ j=1,\ldots,n\}\,,$$

(Recall (see Def. 7 in [14]) that a Mellin distribution $R$ is $(\mathring{\omega},\mathbf{r})$-*flat* if $\mathcal{M}R$ is $A$-meromorphic and $b^A(\mathcal{M}R)=0$ on $\Omega_\mathbf{r}$.)

Proof. This is immediate from Theorem 4 and Theorem 5 in [16].

**5. Mellin analysis on manifolds.** To perform the Mellin analysis on a smooth $n$-dimensional manifold $N$ we replace the positive octant $\mathbb{R}^n_+$ by its image under a local diffeomorphism $H : \widetilde{U} \to N$, $0 \in \widetilde{U}$ an open set in $\mathbb{R}^n$. The image $\triangle = H(\widetilde{U} \cap \mathbb{R}^n_+)$ is called a (local) *pyramid* at $m = H(0)$. We shall identify pyramids at $m \in N$ identical in a neighbourhood of $m$. The space of germs of pyramids at $m$ is denoted by $P_m(N)$. Below we do not distinguish between pyramids and their germs.

DEFINITION 10. We say that $K : U \to \mathbb{R}^n$ is a *chart* for $\triangle \in P_m(N)$ if $m \in U \subset M$, $K$ is a chart for $N$ and $K(\triangle) = \widetilde{U} \cap \mathbb{R}^n_+$ where $\widetilde{U}$ is an open neighbourhood of zero in $\mathbb{R}^n$.

A chart $K$ defines an ordering of the walls of the pyramid $\triangle$ (= the set $K^{-1}(\{x_i = 0\})$ for $i = 1, \ldots, n$). We call such a pyramid *ordered*. In the sequel all pyramids are assumed to be ordered and we consider only order preserving charts. Observe that if $K_1$, $K_2$ are two such charts then the transition mapping $K_1 \circ K_2^{-1}$ is reticular and preserves $\mathbb{R}^n_+$.

Let $\triangle \in P_m(N)$. We define the space $\widetilde{M}'_{(\omega)}(\triangle)$ of (germs) of Mellin distributions on $\triangle$ as the space of distributions $u \in D'(\triangle)$ such that for some chart $K$ for $\triangle$ and $\varphi \in C_0^\infty(N)$ with $\varphi(m) \neq 0$, $(\varphi u) \circ K$ extends to an element of $M'_{(\omega)}$. Observe that by Theorem 5, $\omega$ is independent of the choice of $K$. It is also independent of $\varphi$ since $\varphi$ is a Mellin multiplier (see [7]).

DEFINITION 11. Let $\triangle \in P_m(N)$ and $u \in D'(U)$, $m \in U \subset N$. By the *Mellin restriction* of $u$ to $\triangle$ we understand a Mellin distribution $u_\triangle \in \widetilde{M}'_{(\omega)}(\triangle)$, for some $\omega \in \mathbb{R}^n$, which coincides with $u$ on $\triangle$.

Since $D$ is dense in $M_{(\omega)}$ (see Section 1 in [16]) it follows that $u_\triangle$ is unique for every $\omega$ for which it makes sense.

Let $A$ be a real $n \times n$ nonsingular matrix with positive entries. Let $\triangle \in P_m(N)$. By an *A-subset* of $\triangle$ we mean any subset $Z \subset \triangle$ such that $K(Z)$ is an $A$-set in $\mathbb{R}^n_+$, where $K$ is a chart for $\triangle$. As noted in the proof of Theorem 5 the image of an $A$-set under a reticular diffeomorphism is again an $A$-set, which means that the above definition is independent of the choice of $K$.

DEFINITION 12. Let $u \in D'(U)$, $U \subset N$, $m \in U$. Let $\triangle \in P_m(N)$. Suppose $u_\triangle \in M'_{(\omega)}(\triangle)$ is supported by an $A$-subset. We say that the local Mellin transform of $u_\triangle$ is *A-meromorphic* on $\mathbb{C}^n$ if there exists a function $\varphi \in C_0^\infty(U)$ with $\varphi(m) \neq 0$ such that for some chart $K$ for $\triangle$, $\mathcal{M}((\varphi u_\triangle) \circ K^{-1})$ is $A$-meromorphic on $\mathbb{C}^n$.

Independence of all the choices made follows from Theorems 4 and 5 supplemented by Remarks 4 and 5.



Now we introduce an invariant which describes the asymptotic behaviour of $u \in D'(U)$ at the point $m$ in a pyramid $\triangle \in P_m(N)$.

DEFINITION 13. Let $u \in D'(U)$, $m \in U \subset N$, $\triangle \in P_m(N)$ and suppose $u_\triangle$ supported by an $A$-set has $A$-meromorphic local Mellin transform. We define the *A-spectral support* of $u$ in $\triangle$ as the set

$$\mathrm{ss}^A_\triangle\, u = \bigcup_{\nu \in \mathbb{N}_0^n} \{\mathrm{supp}\, b^A(\mathcal{M}(\varphi u_\triangle) \circ K^{-1})\} + \nu$$

where $K$ is a chart for $\triangle$, $\varphi$ is a cut-off function at $m$ (the sum $\theta + \nu$ denotes the translation of the set $\theta$ by the vector $\nu$) and supp denotes the support of the hyperfunction on $\mathbb{R}^n$ with defining function $\mathcal{M}((\varphi u_\triangle) \circ K^{-1})(z)$.

That the set $\mathrm{ss}^A_\triangle\, u$ is independent of the choice of $\varphi$ and $K$ follows from formulas (6) and (13) which show that the change of $\varphi$ and $K$ results in the translation of the support of $b^A(\mathcal{M}(\varphi u_\triangle) \circ K^{-1})$ in the directions $\nu \in \mathbb{N}_0^n$.

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# Characterization of Mellin distributions supported by certain noncompact sets

by

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**Abstract.** A class of distributions supported by certain noncompact regular sets $K$ are identified with continuous linear functionals on $C_0^\infty(K)$. The proof is based on a parameter version of the Seeley extension theorem.

The paper is devoted to establishing theorems characterizing Mellin distributions supported by sets $Z_t^A$ (see Section 4). They can be regarded as the extension to certain noncompact sets of the following theorem characterizing compactly supported distributions (cf. [1]):

THEOREM 1. *Let $u \in D'_K(\mathbb{R}^n)$, where $K$ is a connected compact set in $\mathbb{R}^n$ such that any two points $x, y \in K$ can be joined by a rectifiable curve in $K$ of length $\le \widetilde{C}|x-y|$. Then there exists a constant $C < \infty$ and $k \in \mathbb{N}_0$ such that*

$$|u[\psi]| \le C \sum_{|\alpha|\le k} \sup_{y\in K} \left|\left(\frac{\partial}{\partial y}\right)^\alpha \psi(y)\right|, \quad \text{for } \psi \in C^k(\mathbb{R}^n).$$

## 1. Notation and necessary facts of the theory of Mellin distributions.

Any set in $\mathbb{R}^n$ of the form

$$\{(x_1,\ldots,x_n) : a_i < x_i < b_i \text{ for } i = 1,\ldots,n\},$$

where $a_1,\ldots,a_n, b_1,\ldots,b_n$ are given real numbers or $\pm\infty$ with $a_i < b_i$ for $i = 1,\ldots,n$, is called an *open polyinterval* in $\mathbb{R}^n$. Any set of the form

$$\{(x_1,\ldots,x_n) : a_i < x_i \le b_i < +\infty \text{ for } i = 1,\ldots,n\}$$

is called a *right-closed polyinterval.* $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0$ is the set of nonnegative integers. For $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}_0^n$ we write $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1! \ldots \alpha_n!$.

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