$A_2 - A_1$ is less than $m - n/2$, where $2m$ is the order of A_1 and A_2 and n is the dimension of $\mathcal{M}$. Using Theorem 2.4, Theorem 3.2(ii) and the same arguments as in the proof of Lemma 5.2(ii) one can generalize this result to the class of all elliptic operators on $\mathcal{M}$. However, for arbitrary elliptic operators the problem whether $A_2 - A_1 \in \mathcal{R}(A_1)$ if and only if the order of $A_2 - A_1$ is less than $m - n/2$ is open.

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The Littlewood–Paley function and φ-transform characterizations of a new Hardy space HK_2 associated with the Herz space

by

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Abstract. We give a Littlewood–Paley function characterization of a new Hardy space HK_2 and its φ-transform characterizations in M. Frazier & B. Jawerth's sense.

§ 0. Introduction. In [8] we have introduced some new Hardy spaces HK_p associated with the Herz spaces K_p, where $1 < p < \infty$. More importantly, we have established the atomic and molecular structural theorems for HK_p, $1 < p < \infty$. In §1 of this paper, we present a Littlewood–Paley function characterization of HK_2. In §2, using the atomic and molecular character of HK_2 and the characterization of a special "tent space" TK_2 introduced in [8], we give the φ-transform characterization of HK_2 in Frazier & Jawerth's sense (see [3] or [4]).

§ 1. The Littlewood–Paley function characterization of HK_2. Let $Q_k = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n : |x_i| \le 2^k\}$, $C_k = Q_k \setminus Q_{k-1}$, and $\chi_k = \chi_{C_k}$, $k \in \mathbf{Z}$. The following definitions are given in [8] and [6].

DEFINITION 1.1. Suppose $1 < p < \infty$, $1/p + 1/p' = 1$. The *Herz space* K_p consists of those functions $f \in L^p_{\mathrm{loc}}(\mathbf{R}^n \setminus \{0\})$ for which

$$\|f\|_{K_p} := \sum_{k=-\infty}^{\infty} 2^{(k+1)n/p'} \|f\chi_k\|_p < \infty.$$

DEFINITION 1.2. Let $1 < p < \infty$. A function $a(x)$ defined on $\mathbf{R}^n$ is said to be a *central symmetry* $(1,p)$*-atom* if

(1) $\operatorname{supp} a \subset Q$, where Q is a cube centered at the origin,
(2) $\int a(x)\,dx = 0$,

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(3) $(|Q|^{-1}\int_Q |a(x)|^p\,dx)^{1/p} \le |Q|^{-1}$.

In [8], we define some new Hardy spaces HK_p as follows.

DEFINITION 1.3. Let $1 < p < \infty$. Define $HK_p = \{f = \sum \lambda_i a_i$, where each a_i is a central symmetry $(1,p)$-atom and $\sum|\lambda_i| < \infty\}$. We write $\|f\|_{HK_p} = \inf\{\sum|\lambda_i|\}$, where inf is taken over all decompositions of f as above.

For HK_2, we have the following theorem.

THEOREM 1.1. *Suppose $f \in L^1(\mathbf{R}^n)$, $v(x,t) = t(\partial/\partial t)(f * p_t)(x)$, where $p_t(x)$ is the Poisson kernel, and set*

$$S^1_{2\sqrt{n}}(v)(x) = \Big(\int\limits_{|x-y|<2\sqrt{n}t} |v(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\Big)^{1/2} \in K_2\,.$$

Then there is a sequence of central symmetry $(1,2)$-atoms $\{a_j\}$ and a sequence of numbers $\{\lambda_j\}$ satisfying:

(i) $f = \sum \lambda_j a_j$.

(ii) $\sum|\lambda_j| \le C\|S^1_{2\sqrt{n}}(v)\|_{K_2}$, *where C is independent of f.*

Proof. Let $\Omega_k = \{x \in \mathbf{R}^n : S^1_{2\sqrt{n}}(v)(x) > 2^k\}$, $k \in \mathbf{Z}$; $Q_{j,K} = \{x \in \mathbf{R}^n : 2^j x - K \in [0,1)^n\}$, $\mathfrak{D} = \{Q_{j,K} : j \in \mathbf{Z}, K \in \mathbf{Z}^n\}$ and $\mathfrak{D}^l_k = \{Q \in \mathfrak{D} : |Q \cap \Omega_k \cap C_l| > |Q|/2^{n+1}$, $|Q \cap \Omega_{k+1} \cap C_l| \le |Q|/2^{n+1}$, and $Q \subset Q_l$, but $Q \not\subset Q_{l-1}\}$, where $l, k \in \mathbf{Z}$.

Write $\widetilde{Q} = \{(y,t) \in \mathbf{R}^{n+1}_+ : y \in Q,\ l(Q) < t \le 2l(Q)\}$, where $l(Q)$ is the side length of Q. Obviously $\mathbf{R}^{n+1}_+ = \bigcup_{l=-\infty}^{\infty}\bigcup_{k=-\infty}^{\infty}\bigcup_{Q\in\mathfrak{D}^l_k}\widetilde{Q}$ is a disjoint partition of $\mathbf{R}^{n+1}_+$.

By the Calderón's representation lemma (see [9]), there exists a function $\Psi \in C_0^\infty(\mathbf{R}^n)$ satisfying:

(i) $\operatorname{supp}\Psi \subset B(0,1)$, Ψ is radial and real-valued,

(ii) $\widehat{\Psi}(0) = 0$,

(iii) $\int_0^\infty e^{-t}\widehat{\Psi}(t)\,dt = -1$,

and such that

$$f(x) = \int\limits_0^\infty \int\limits_{\mathbf{R}^n} v(y,t)\Psi_t(x-y)\frac{dy\,dt}{t}$$

$$= \sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{Q\in\mathfrak{D}^l_k}\int\limits_{\widetilde{Q}} v(y,t)\Psi_t(x-y)\frac{dy\,dt}{t} = \sum_{l=-\infty}^{\infty}\lambda_l a_l(x)$$

where

$$a_l(x) = \lambda_l^{-1}\sum_{k=-\infty}^{\infty}\sum_{Q\in\mathfrak{D}^l_k}\int\limits_{\widetilde{Q}} v(y,t)\Psi_t(x-y)\frac{dy\,dt}{t}\,,$$

and λ_l is a constant to be determined.

First of all, we have $\operatorname{supp} a_l \subset Q_{l+2}$, $l \in \mathbf{Z}$. In fact, let $x \in \operatorname{supp} a_l$. Since $\operatorname{supp}\Psi \subset B(0,1)$, we may assume $|x-y| \le t$. For $(y,t) \in \widetilde{Q}$, we have $t \le 2l(Q) \le l(Q_l) = 2^{l+1}$ and $y \in Q_l$, and so

$$|x| \le |x-y| + |y| \le 2^{l+1} + 2^l < 2^{l+2}\,,$$

that is, $\operatorname{supp} a_l \subset Q_{l+2}$.

By the condition (ii) on Ψ, we know $\int a_l(x)\,dx = 0$. Moreover, setting $U = \bigcup_{Q\in\mathfrak{D}^l_k}\widetilde{Q}$, we obtain

$$\|a_l\|_2 = \sup_{\|\eta\|_2\le 1}\Big|\int\limits_{\mathbf{R}^n} a_l(x)\eta(x)\,dx\Big|$$

$$= \lambda_l^{-1}\sup_{\|\eta\|_2\le 1}\Big|\sum_{k=-\infty}^{\infty}\sum_{Q\in\mathfrak{D}^l_k}\int\limits_{\widetilde{Q}} v(y,t)(\Psi_t * \eta)(y)\frac{dy\,dt}{t}\Big|$$

$$\le \lambda_l^{-1}\sup_{\|\eta\|_2\le 1}\Big(\sum_{k=-\infty}^{\infty}\int\limits_U |v(y,t)|^2\frac{dy\,dt}{t}\Big)^{1/2}$$

$$\times\Big(\sum_{k=-\infty}^{\infty}\int\limits_U |(\Psi_t * \eta)(y)|^2\frac{dy\,dt}{t}\Big)^{1/2}$$

$$\le \lambda_l^{-1}\Big(\sum_{k=-\infty}^{\infty}\int\limits_U |v(y,t)|^2\frac{dy\,dt}{t}\Big)^{1/2}\sup_{\|\eta\|_2\le 1} C_0\|\eta\|_2$$

$$\le C_0\lambda_l^{-1}\Big(\sum_{k=-\infty}^{\infty}\int\limits_U |v(y,t)|^2\frac{dy\,dt}{t}\Big)^{1/2} = |Q_{l+2}|^{-1/2}\,,$$

where we take

$$\lambda_l = C_0\Big(\sum_{k=-\infty}^{\infty}\int\limits_U |v(y,t)|^2\frac{dy\,dt}{t}\Big)^{1/2}|Q_{l+2}|^{1/2}\,.$$

Consequently, if we take λ_l as above, then a_l is a central symmetry $(1,2)$-atom and $f = \sum_{l=-\infty}^{\infty}\lambda_l a_l$. It remains to estimate $\sum_{l=-\infty}^{\infty}|\lambda_l|$.

First, we show that

$$\sum_{Q\in\mathfrak{D}^l_k}\int\limits_{\widetilde{Q}} |v(y,t)|^2\frac{dy\,dt}{t} \le C2^{2k}|\Omega_k \cap C_l|\,.$$

If we write $\widetilde{\Omega}_k = \{x \in \mathbf{R}^n : M(\chi_{\Omega_k \cap C_l})(x) > 1/2^{n+2}\}$, where M is the Hardy–Littlewood maximal function, then we have

$$\int_{(\widetilde{\Omega}_k \setminus \Omega_{k+1}) \cap C_l} \{S^1_{2\sqrt{n}}(v)(x)\}^2 dx \le 2^{2(k+1)} |(\widetilde{\Omega}_k \setminus \Omega_{k+1}) \cap C_l| \le C 2^{2k} |\Omega_k \cap C_l|$$

and

$$\int_{(\widetilde{\Omega}_k \setminus \Omega_{k+1}) \cap C_l} \{S^1_{2\sqrt{n}}(v)(x)\}^2 dx$$
$$= \int_{(\widetilde{\Omega}_k \setminus \Omega_{k+1}) \cap C_l} dx \int_{|x-y|<2\sqrt{n}t} |v(y,t)|^2 \frac{dy\,dt}{t^{n+1}}$$
$$= \int_0^\infty \int_{\mathbf{R}^n} |v(y,t)|^2 |\{x \in (\widetilde{\Omega}_k \setminus \Omega_{k+1}) \cap C_l : |x-y| < 2\sqrt{n}t\}| \frac{dy\,dt}{t^{n+1}}$$
$$\ge \sum_{Q \in \mathfrak{D}^l_k} \int_{\widetilde{Q}} |v(y,t)|^2 |\{x \in (\widetilde{\Omega}_k \setminus \Omega_{k+1}) \cap C_l : |x-y| < 2\sqrt{n}t\}| \frac{dy\,dt}{t^{n+1}}.$$

If $x \in Q$, then $|x-y| < \sqrt{n} l(Q) \le 2\sqrt{n}t$ for all $(y,t) \in \widetilde{Q}$. Therefore, if $(y,t) \in \widetilde{Q}$, we have

$$|\{x \in (\widetilde{\Omega}_k \setminus \Omega_{k+1}) \cap C_l : |x-y| < 2\sqrt{n}t\}| \ge |Q \cap (\widetilde{\Omega}_k \setminus \Omega_{k+1}) \cap C_l| = |Q \cap \widetilde{\Omega}_k \cap C_l| - |Q \cap \Omega_{k+1} \cap C_l|.$$

If $Q \in \mathfrak{D}^l_k$, we have the following two cases:

Case I: $Q \cap Q_{l-1} = \emptyset$. Then $Q \subset C_l$. Since $|Q \cap C_l \cap \Omega_k| > |Q|/2^{n+1}$, we have $Q \subset \widetilde{\Omega}_k$. Furthermore, $Q \subset \widetilde{\Omega}_k \cap C_l$. Note that $|Q \cap C_l \cap \Omega_{k+1}| = |Q \cap \Omega_{k+1}| \le |Q|/2^{n+1}$. Hence

$$|Q \cap (\widetilde{\Omega}_k \setminus \Omega_{k+1}) \cap C_l| \ge |Q| - |Q|/2^{n+1} = Ct^n$$

and thereby

$$\sum_{Q \in \mathfrak{D}^l_k, Q \subset C_l} \int_{\widetilde{Q}} |v(y,t)|^2 \frac{dy\,dt}{t} \le C 2^{2k} |\Omega_k \cap C_l|.$$

Case II: $Q \cap Q_{l-1} \ne \emptyset$. Note that $l(Q) < t \le 2l(Q) = 2^{l+1}$ and if $y \in Q$ and $x \in C_l, |x-y| < 2\sqrt{n}t$, we have

$$|\{x \in (\widetilde{\Omega}_k \setminus \Omega_{k+1}) \cap C_l : |x-y| < 2\sqrt{n}t\}| \ge |(\widetilde{\Omega}_k \setminus \Omega_{k+1}) \cap C_l|.$$

We can suppose $|(\widetilde{\Omega}_k \setminus \Omega_{k+1}) \cap C_l| \ne \emptyset$. Then

$$\sum_{Q \cap Q_{l-1} \ne \emptyset, Q \in \mathfrak{D}^l_k} \int_{\widetilde{Q}} |v(y,t)|^2 \frac{dy\,dt}{t}$$
$$\le C 2^{ln} \sum_{Q \in \mathfrak{D}^l_k} \int_{\widetilde{Q}} \frac{|v(y,t)|^2 |\{x \in (\widetilde{\Omega}_k \setminus \Omega_{k+1}) \cap C_l : |x-y| < 2\sqrt{n}t\}|}{|(\widetilde{\Omega}_k \setminus \Omega_{k+1}) \cap C_l|} \frac{dy\,dt}{t^{n+1}}$$
$$\le C 2^{ln} 2^{2k} \le C 2^{2k} |\Omega_k \cap Q \cap C_l| \le C 2^{2k} |\Omega_k \cap C_l|.$$

On the other hand,

$$\int_{C_l} \{S^1_{2\sqrt{n}}(v)(x)\}^2 dx = \sum_{k=-\infty}^{\infty} \int_{(C_l \cap \Omega_k) \setminus (C_l \cap \Omega_{k+1})} \{S^1_{2\sqrt{n}}(v)(x)\}^2 dx$$
$$\ge \sum_{k=-\infty}^{\infty} 2^{2k} (|C_l \cap \Omega_k| - |C_l \cap \Omega_{k+1}|)$$
$$\ge C \sum_{k=-\infty}^{\infty} 2^{2k} |C_l \cap \Omega_k|.$$

So, $\lambda_l \le C 2^{(l+1)n/2} \|S^1_{2\sqrt{n}}(v) \chi_l\|_2$, and therefore

$$\sum_{l=-\infty}^{\infty} |\lambda_l| \le C \|S^1_{2\sqrt{n}}(v)\|_{K_2}.$$

This finishes the proof of Theorem 1.1.

Remark 1.1. If we replace $S^1_{2\sqrt{n}}(v)$ by

$$S^1_{\alpha,p}(v)(x) = \Big(\int_{|x-y|<\alpha t} |v(y,t)|^p \frac{dy\,dt}{t^{n+1}} \Big)^{1/p}, \quad \alpha \ge 2\sqrt{n},$$

then Theorem 1.1 still holds for $HK_p, 1 < p \le 2$.

Next, let $g^*_\lambda(f)$, $S_\alpha(f)$ and $g(f)$ denote respectively the Littlewood–Paley g^*_λ-function, the Lusin area integral and the Littlewood–Paley g-function (see [9]).

THEOREM 1.2. *If* $\alpha > 0$, $\lambda > (3n+1)/n$, $1 < p < \infty$ *then* $\|g^*_\lambda(f)\|_{K_p} \le C(\lambda)\|f\|_{HK_p}$, $\|S_\alpha(f)\|_{K_p} \le C(\alpha)\|f\|_{HK_p}$, $\|g(f)\|_{K_p} \le C\|f\|_{HK_p}$.

Proof. We need only prove that if f is a central symmetry $(1,p)$-atom, then $\|g^*_\lambda(f)\|_{K_p} \le C$, where C is independent of f. Suppose $\operatorname{supp} f \subset Q_0$, where Q_0 is a cube centered at the origin. Furthermore, suppose $Q_{l_0-1} \subset 4\sqrt{n} Q_0 \subset Q_{l_0}$, $l_0 \in \mathbf{Z}$. Using the procedure of Theorem 3 in [10], we can prove that

$$g_\lambda^*(f)(x) \le C\frac{|Q_0|^{1/2n}}{|x|^{n+1/2}}, \quad \text{for } x \in \complement Q_{l_0}.$$

Then, using this fact and the L^p-boundedness of $g_\lambda^*(f)$, we have

$$\begin{aligned}\|g_\lambda^*(f)\|_{K_p} &= \sum_{k=-\infty}^{\infty} 2^{(k+1)n/p'}\|g_\lambda^*(f)\chi_k\|_p \\ &\le C\sum_{k=-\infty}^{l_0} 2^{(k+1)n/p'}\|f\|_p + \sum_{k=l_0+1}^{\infty} 2^{(k+1)n/p'}\|g_\lambda^*(f)\chi_k\|_p \\ &\le C\sum_{k=-\infty}^{l_0} 2^{-(l_0-k)n/p'} + C\sum_{k=l_0+1}^{\infty} 2^{(k+1)n/p'}2^{l_0/2}2^{-k(n+1/2-n/p)} \\ &\le C\sum_{l=0}^{\infty} 2^{-ln/p'} + C\sum_{l=-1}^{\infty} 2^{-l/2} \le C.\end{aligned}$$

This completes the proof of Theorem 1.2.

We point out that it is possible to prove (see [2])

$$\|S_\alpha(f)\|_{K_p} \le C\|g(f)\|_{K_p}, \quad 1<p<\infty.$$

This yields

THEOREM 1.3. *Let $f \in L^1(\mathbf{R}^n)$. Then*

(i) $f \in HK_2(\mathbf{R}^n)$ *if and only if* $g(f) \in K_2$.
(ii) $f \in HK_2(\mathbf{R}^n)$ *if and only if* $S_\alpha(f) \in K_2$, *where* $\alpha \ge 2\sqrt{n}$.
(iii) $f \in HK_2(\mathbf{R}^n)$ *if and only if* $g_\lambda^*(f) \in K_2$, *where* $\lambda > (3n+1)/n$.

The spaces $\mathrm{CSMO}_p(\mathbf{R}^n)$ are introduced by the authors in [8]. For convenience sake, we restate the definition as follows:

DEFINITION 1.4. Suppose $1 < p < \infty$. Then $f \in L^p_{\mathrm{loc}}(\mathbf{R}^n)$ is said to belong to CSMO_p, the space of functions of central symmetry mean oscillation, if and only if for every $R > 0$, there is a constant C_R such that

$$\sup_{R>0}\Big(|B(0,R)|^{-1}\int_{B(0,R)}|f(x)-C_R|^p\,dx\Big)^{1/p} < \infty.$$

It is easy to verify that C_R can be taken as

$$C_R = m_R(f) = |B(0,R)|^{-1}\int_{B(0,R)} f(x)\,dx.$$

We shall write $\|f\|_{\mathrm{CSMO}_p}$ for the above supremum with $C_R = m_R(f)$.

THEOREM 1.4. *Let $f \in \mathrm{CSMO}_p$, $1<p<\infty$, $\lambda > 1$ and $\alpha > 0$. Then either $g(f) = \infty$, $S_\alpha(f) = \infty$, $g_\lambda^*(f) = \infty$ almost everywhere or $g(f) < \infty$, $S_\alpha(f) < \infty$, $g_\lambda^*(f) < \infty$ almost everywhere and*

$$\|g(f)\|_{\mathrm{CSMO}_p} \le C\|f\|_{\mathrm{CSMO}_p},$$
$$\|S_\alpha(f)\|_{\mathrm{CSMO}_p} \le C(\alpha)\|f\|_{\mathrm{CSMO}_p},$$
$$\|g_\lambda^*(f)\|_{\mathrm{CSMO}_p} \le C(\lambda)\|f\|_{\mathrm{CSMO}_p}.$$

This is easy to prove by the procedure in [7]. We omit the details.

Remark 1.2. The Poisson kernels of Theorems 1.3 and 1.4 can be replaced by the general Littlewood–Paley functions (see [9]).

§ 2. The φ-transform characterizations of HK_2.

Let φ be a Schwartz function, supp $\widehat{\varphi} \subseteq \{\xi \in \mathbf{R}^n : 1/2 \le |\xi| \le 2\}$ and $|\widehat{\varphi}(\xi)| \ge C > 0$ if $3/5 \le |\xi| \le 5/3$. Moreover, $\sum_{v\in\mathbf{Z}}|\widehat{\varphi}(2^v\xi)|^2 = 1$ if $\xi \ne 0$.

Write $\varphi_v(x) = 2^{vn}\varphi(2^v x), v \in \mathbf{Z}$, and for $Q = Q_{v,K} = \{x = (x_1,\ldots,x_n) \in \mathbf{R}^n : 2^v x - K \in [0,1)^n\}$, $K \in \mathbf{Z}^n$, set $x_Q = 2^{-v}K$ and $\varphi_Q(x) = |Q|^{-1/2}\varphi(2^v x - K) = |Q|^{1/2}\varphi_v(x - x_Q)$.

Let $\widetilde{\chi}_Q = |Q|^{-1/2}\chi_Q$ and $s_Q = (f, \varphi_Q)$ for a given $f \in L^1(\mathbf{R}^n)$.

THEOREM 2.1. *Let φ be as above. The following are equivalent.*

(i) $f \in HK_2$.

(ii) *There is a constant $r > 0$ such that for each $Q \in \mathfrak{D}$, there is a dyadic cube $R \subset Q$ satisfying $|R| \ge r|Q|$ and*

$$\sigma(f) := \Big(\sum_{Q\in\mathfrak{D}}|s_Q|^2|Q|^{-1}\chi_R(x)\Big)^{1/2} \in K_2.$$

(iii) $$s(f)(x) := \Big(\sum_{Q\in\mathfrak{D}}(|s_Q|\widetilde{\chi}_Q(x))^2\Big)^{1/2} \in K_2.$$

The equivalence of (ii) and (iii) is contained in Theorem 3.2 of [8]. In fact, (iii) implies that the sequence $s = \{s_Q\}_{Q\in\mathfrak{D}}$ of complex numbers is in TK_2, which is a special "tent space" introduced in [8]. To be exact, we have

DEFINITION 2.1. We say a sequence $\alpha = \{\alpha(Q)\}_{Q\in\mathfrak{D}}$ of complex numbers is in TK_2 if

$$s(\alpha)(x) := \Big(\sum_{x\in Q}|\alpha(Q)|^2|Q|^{-1}\Big)^{1/2} \in K_2.$$

Let $\|\alpha\|_{TK_2} := \|s(\alpha)\|_{K_2}$.

For a sequence $\alpha = \{\alpha(Q)\}_{Q\in\mathfrak{D}}$ of complex numbers, define

$$\operatorname{supp}\ \alpha = \bigcup_{Q\in\mathfrak{D},\alpha(Q)\ne 0} Q.$$

DEFINITION 2.2. If there exists a cube R centered at the origin such that $R \supset \operatorname{supp} \alpha$ and $\sum_{Q\in\mathfrak{D}} |\alpha(Q)|^2 \le 1/|R|$, then $\alpha = \{\alpha(Q)\}_{Q\in\mathfrak{D}}$ is called a *central symmetry atom-sequence*.

In [8], we have given the following characterization of the space TK_2.

LEMMA 2.1. *For a sequence $\alpha = \{\alpha(Q)\}_{Q\in\mathfrak{D}}$ of complex numbers, the following are equivalent.*

(1) *There exists a constant $C_0 \in (0,1]$ such that for any $Q \in \mathfrak{D}$, there is a dyadic cube $R \subset Q$ satisfying $|R| \ge C_0|Q|$ and*

$$\sigma(x) := \Big(\sum_{Q\in\mathfrak{D}} |\alpha(Q)|^2 |Q|^{-1}\chi_R(x)\Big)^{1/2} \in K_2 .$$

(2) $\alpha \in TK_2$.

(3) *There is a sequence $\{\alpha_j\}_{j=-\infty}^{\infty}$ of central symmetry atom-sequences and a sequence $\{\lambda_j\}_{j=-\infty}^{\infty}$ of numbers such that*

$$\alpha = \sum_{j=-\infty}^{\infty} \lambda_j \alpha_j \quad \text{and} \quad \sum_{j=-\infty}^{\infty} |\lambda_j| < \infty .$$

In fact, we can have $\operatorname{supp} \alpha_j \subset C_1 Q_j$, where C_1 only depends on C_0, and $Q_j = \{x = (x_1, \cdots, x_n) \in \mathbf{R}^n : |x_i| \le 2^j\}$, $j \in \mathbf{Z}$.

To prove Theorem 2.1, we need only prove the equivalence of (i) and (iii). To do this, suppose $s(f) \in K_2$. We show that $f(x) = \sum_{Q\in\mathfrak{D}} s_Q \varphi_Q(x)$ (see [3]) has a decomposition into central symmetry $(1,2,\varepsilon)$-molecules by the molecular theory of HK_2 presented in [8], where $\varepsilon > 0$.

First of all, let $\varepsilon > 0$, $b = 1/2 + \varepsilon$. A function $M(x) \in L^2(\mathbf{R}^n)$ is called a *central symmetry* $(1,2,\varepsilon)$-*molecule* if

(i) $|x|^{nb} M(x) \in L^2(\mathbf{R}^n)$,
(ii) $\Re_p(M) := \|M\|_2^{\varepsilon/b} \| M(x)|x|^{nb}\|_2^{1-\varepsilon/b} < \infty$,
(iii) $\int M(x)\,dx = 0$.

By Lemma 2.1, there exists a sequence $\{s_j\}_{j=-\infty}^{\infty}$ of central symmetry atom-sequences and a sequence $\{\lambda_j\}_{j=-\infty}^{\infty}$ of numbers such that

$$s = \sum_{j=-\infty}^{\infty} \lambda_j s_j \quad \text{and} \quad \sum_{j=-\infty}^{\infty} |\lambda_j| < \infty .$$

Moreover, $R_j = \operatorname{supp} s_j \subset C_r Q_j$ and therefore

$$f(x) = \sum_{Q\in\mathfrak{D}} s_Q \varphi_Q(x) = \sum_{Q\in\mathfrak{D}} \sum_{j=-\infty}^{\infty} \lambda_j s_j(Q) \varphi_Q(x)$$

$$= \sum_{j=-\infty}^{\infty} \lambda_j \sum_{Q\in\mathfrak{D}} s_j(Q)\varphi_Q(x) .$$

Write $s_j(x) := \sum_{Q\in\mathfrak{D}} s_j(Q)\varphi_Q(x)$. We want to prove $s_j(x)$ is a central symmetry $(1,2,\varepsilon)$-molecule. Obviously, we need only verify $\Re_2(s_j) \le C$, where C is independent of j.

By the results in [3] (see also [4]), we have

$$\|s_j\|_2 \le C\Big(\int \sum_{Q\in\mathfrak{D}} |s_j(Q)|^2 \widetilde{\chi}_Q^2(x)\,dx\Big)^{1/2}$$
$$\le C\Big(\sum_{Q\in\mathfrak{D}} |s_j(Q)|^2\Big)^{1/2} \le C|Q_j|^{-1/2} .$$

Again, we write $b = 1/2 + \varepsilon$, $\varepsilon > 0$, $|\mu| = nb$. Then

$$\|s_j(x)x^\mu\|_2 = \|D^\mu \widehat{s}_j(x)\|_2 = \Big[\int \Big(\sum_{Q\in\mathfrak{D}} s_j(Q) D^\mu \widehat{\varphi}_Q(x)\Big)^2 dx\Big]^{1/2}$$
$$\le \Big[\int \Big(\sum_{Q\in\mathfrak{D}} |s_j(Q)| 2^{-vn/2} 2^{-vnb} |(D^\mu \widehat{\varphi})(2^{-v}x)|\Big)^2 dx\Big]^{1/2} .$$

Note $R_j = \operatorname{supp} s_j \subset C_r Q_j$; we can suppose that

$$2^{k_0} \le C_r < 2^{k_0+1}, \quad k_0 \in \mathbf{Z}.$$

If $s_j(Q) \ne 0$, then $Q \subset R_j \subset C_r Q_j$, and therefore $2^{-v} \le 2^{j+k_0+1}$, that is, $v \ge -(j+k_0+1)$. Thus

$$\|s_j(x)x^\mu\|_2$$
$$\le \sum_{v=-(j+k_0+1)}^{\infty} \Big[\int \Big(\sum_{l(Q)=2^{-v}} |s_j(Q)| 2^{-vn/2} 2^{-vnb} |(D^\mu\widehat{\varphi})(2^{-v}x)|\Big)^2 dx\Big]^{1/2}$$
$$\le C \sum_{v=-(j+k_0+1)}^{\infty} \sum_{l(Q)=2^{-v}} |s_j(Q)| 2^{-vnb}$$
$$\le C \sum_{v=-(j+k_0+1)}^{\infty} \Big(\sum_{l(Q)=2^{-v}} |s_j(Q)|^2\Big)^{1/2} \Big(\frac{2^{(j+k_0+1)n}}{2^{-vn}}\Big)^{1/2} 2^{-vnb}$$
$$\le C \frac{2^{jn/2}}{|Q_j|^{1/2}} \sum_{v=-(j+k_0+1)}^{\infty} 2^{(n/2-bn)v} \le C|Q_j|^{b-1/2} .$$

From the above discussion, we have

$$\Re_2(s_j) = \|s_j\|_2^{\varepsilon/b} \|s_j(x)|x|^{nb}\|_2^{1-\varepsilon/2}$$
$$\le C|Q_j|^{-\varepsilon/(2b)} |Q_j|^{(b-1/2)(1-\varepsilon/b)} \le C .$$

We have proved that (iii) implies (i). Now, we prove that if $f \in HK_2$, then $s(f) \in K_2$. In fact, we can establish the following general proposition.

PROPOSITION 2.1. *If* $f(x) = \sum_{Q\in\mathfrak{D}} s_Q\varphi_Q(x) \in HK_p$, $1 < p < \infty$, *then*

$$s(f)(x) := \Big[\sum_{Q\in\mathfrak{D}} (|s_Q|\widetilde{\chi}_Q(x))^2\Big]^{1/2} \in K_p\,.$$

For this purpose, we need only show that for any central symmetry $(1,p)$-atom f, $\|s(f)\|_{K_p} \le C$, where C is independent of f. In fact, we can suppose $\operatorname{supp} f \subset R$, where R is a cube centered at the origin. Let $l(R)$ satisfy $2^{k_0} \le l(R) < 2^{k_0+1}$, $k_0 \in \mathbf{Z}$, and let $1/p + 1/p' = 1$. We have

$$\|s(f)\|_{K_p} = \sum_{k=-\infty}^{\infty} 2^{(k+1)n/p'}\Big(\int_{C_k} s(f)^p(x)dx\Big)^{1/p} = \sum_{k=-\infty}^{k_0+2}\ldots + \sum_{k=k_0+3}^{\infty}\ldots =: I_1 + I_2\,.$$

By the results of [3] (see also [4]), we have

$$I_1 \le C\sum_{k=-\infty}^{k_0+2} 2^{(k+1)n/p'}\|f\|_p \le C\sum_{k=-\infty}^{k_0+2} 2^{(k-k_0)n/p'} \le C\sum_{k=-2}^{\infty} 2^{-kn/p'} = C < \infty\,.$$

To estimate I_2, we consider two cases:

Case I. If $p/2 \ge 1$, then

$$\begin{aligned} I_2 &= \sum_{k=k_0+3}^{\infty} 2^{(k+1)n/p'}\Big\{\int_{C_k}\Big(\sum_{Q\in\mathfrak{D}} |s_Q|^2|Q|^{-1}\chi_Q(x)\Big)^{p/2}dx\Big\}^{1/p} \\ &\le \sum_{k=k_0+3}^{\infty} 2^{(k+1)n/p'}\Big(\sum_{Q\in\mathfrak{D}} |s_Q|^2|Q|^{-1}|Q\cap C_k|^{2/p}\Big)^{1/2} \\ &\le \sum_{k=k_0+3}^{\infty} 2^{(k+1)n/p'}\Big\{\Big(\sum_{Q\subset C_k} |s_Q|^2|Q|^{-1+2/p}\Big)^{1/2} \\ &\quad + \Big(\sum_{Q\not\subset C_k,\, Q\cap C_k\ne\emptyset} |s_Q|^2|Q|^{-1}|C_k|^{2/p}\Big)^{1/2}\Big\} \\ &=: I_{2,1} + I_{2,2} \end{aligned}$$

and

$$\begin{aligned} |s_Q| &= \Big|\int f(x)\varphi_Q(x)\,dx\Big| = \Big|\int f(x)(\varphi_Q(x)-\varphi_Q(0))\,dx\Big| \\ &\le \int_{\mathbf{R}^n} |f(x)||\nabla\varphi_Q(\xi x)|\,|x|\,dx, \quad \xi\in(0,1), \\ &\le C_L|Q|^{-1/2-1/n}\int_{\mathbf{R}^n} |f(x)||x|\frac{1}{(1+l(Q)^{-1}|\xi x - x_Q|)^{L+1}}\,dx \end{aligned}$$

where L will be determined later.

If $Q \subset C_k$, $k \ge k_0+3$, then $|\xi x - x_Q| \ge 2^{k-1} - 2^{k_0+1} \ge 2^{k-2}$ and therefore

$$\begin{aligned} |s_Q| &\le C_L|Q|^{-1/2-1/n}\frac{1}{(1+l(Q)^{-1}2^{k-2})^{L+1}}\int_{\mathbf{R}^n} |f(x)||x|\,dx \\ &\le C_L|Q|^{-1/2-1/n}|R|^{1/n}\frac{1}{(1+l(Q)^{-1}2^{k-2})^{L+1}}\,. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{Q\subset C_k} |s_Q|^2|Q|^{-1+2/p} \\ &\quad\le C_L|R|^{2/n}\sum_{Q\subset C_k} |Q|^{-1-2/n-1+2/p}\frac{1}{(1+l(Q)^{-1}2^{k-2})^{2L+2}} \\ &\quad\le C_L|R|^{2/n}\sum_{v=-\infty}^{k-1} 2^{-2nv-2v+2nv/p}\frac{2^{(k-1-v)n}}{(1+2^{k-v-2})^{2L+2}} \\ &\quad\le C_L|R|^{2/n}2^{k(n-2L-2)}\sum_{v=-\infty}^{k-1} 2^{v(-3n+2L+2n/p)} \quad (\text{choose } L > (\tfrac{3}{2}-\tfrac{1}{p})n) \\ &\quad\le C_L|R|^{2/n}2^{k(n-2L-2)}2^{k(-3n+2L+2n/p)} \\ &\quad\le C_L|R|^{2/n}2^{k(-2n-2+2n/p)}\,, \end{aligned}$$

and hence

$$I_{2,1} \le C\sum_{k=k_0+3}^{\infty} 2^{(k+1)n/p'}|R|^{1/n}2^{k(n/p-n-1)} \le C|R|^{1/n}\sum_{k=k_0+3}^{\infty} 2^{-k} \le C\,.$$

If $Q\cap C_k \ne \emptyset$ and $Q \not\subset C_k$, then $Q \in \{Q_{-l,K} : l \ge k,\ K \in \{-1,0\}^n\}$. We then have

$$|s_Q| \le C_L |Q|^{-1/2-1/n} \int_{\mathbf{R}^n} |f(x)|\,|x|\,dx$$

$$= C_L |Q_l|^{-1/2-1/n} |R|^{1/n} \quad \text{if } Q \subset Q_l \text{ and } Q \not\subset Q_{l-1},\ l \ge k,$$

and therefore

$$\Big(\sum_{Q\not\subset C_k, Q\cap C_k\ne\emptyset} |s_Q|^2 |Q|^{-1} |C_k|^{2/p}\Big)^{1/2}$$

$$\le C_L |R|^{1/n} |C_k|^{1/p} \Big(\sum_{l=k}^{\infty} |Q_l|^{-2-2/n}\Big)^{1/2}$$

$$\le C_L |R|^{1/n} |C_k|^{1/p} 2^{kn(-1-1/n)} = C_L |R|^{1/n} 2^{kn/p-kn(1+1/n)}.$$

Thus

$$I_{2,2} \le C \sum_{k=k_0+3}^{\infty} 2^{(k+1)n/p'} |R|^{1/n} 2^{kn/p-kn(1+1/n)}$$

$$= C|R|^{1/n} \sum_{k=k_0+3}^{\infty} 2^{-k} \le C.$$

To sum up, we have $I_2 \le I_{2,1} + I_{2,2} \le C$.

Case II. If $p/2 \le 1$, then

$$I_2 \le \sum_{k=k_0+3}^{\infty} 2^{(k+1)n/p'} \Big(\sum_{Q\in\mathfrak{D}} \int_{C_k} |s_Q|^p |Q|^{-p/2} \chi_Q(x)\,dx\Big)^{1/p}$$

$$= \sum_{k=k_0+3}^{\infty} 2^{(k+1)n/p'} \Big(\sum_{Q\in\mathfrak{D}} |s_Q|^p |Q|^{-p/2} |Q\cap C_k|\Big)^{1/p}.$$

Similarly to Case I, we have

$$\sum_{Q\in\mathfrak{D}} |s_Q|^p |Q|^{-p/2} |Q\cap C_k|$$

$$= \sum_{Q\subset C_k} |s_Q|^p |Q|^{-p/2+1} + \sum_{\substack{Q\not\subset C_k\\ Q\cap C_k\ne\emptyset}} |s_Q|^p |Q|^{-p/2} |C_k|$$

$$\le C \sum_{Q\subset C_k} |Q|^{-p/2-p/n} |R|^{p/n} |Q|^{1-p/2} \frac{1}{(1+l(Q)^{-1}2^{k-2})^{pL+p}}$$

$$+ C\sum_{l=k}^{\infty} |Q_l|^{-p/2-p/n} |R|^{p/n} |Q_l|^{-p/2} |C_k|$$

$$\le C|R|^{p/n} \sum_{v=-\infty}^{k-1} 2^{-vnp-vp+vn} 2^{(k-v)n} \frac{1}{2^{(k-v)(pL+p)}}$$

$$+ C|R|^{p/n} |C_k| \sum_{l=k}^{\infty} 2^{-lnp-lp} \quad (\text{choose } L > n)$$

$$= C|R|^{p/n} 2^{-k(pn+p-n)},$$

and therefore

$$I_2 \le C \sum_{k=k_0+3}^{\infty} |R|^{1/n} 2^{kn/p'} 2^{-k(n+1-n/p)} = C|R|^{1/n} \sum_{k=k_0+3}^{\infty} 2^{-k} \le C.$$

This finishes the proof of Proposition 2.1, and therefore the proof of Theorem 2.1.

Remark. It should be pointed out that the space HK_p is a homogeneous version of the space HA^p which is introduced in [1] and [5]. In fact, we have the following relation:

$$HK_p(\mathbf{R}^n) \cap L^p_{\mathrm{loc}}(\mathbf{R}^n) = HA^p(\mathbf{R}^n).$$

Proof. Suppose $f \in HA^p(\mathbf{R}^n)$. By Theorem 3.1 of [5], we have

$$HA^p(\mathbf{R}^n) \subset A^p(\mathbf{R}^n) \subset L^p_{\mathrm{loc}}(\mathbf{R}^n), \qquad HA^p(\mathbf{R}^n) \subset HK_p(\mathbf{R}^n),$$

where $A^p(\mathbf{R}^n)$ are the so-called Beurling algebras (see [5]). Thus

$$HA^p(\mathbf{R}^n) \subset HK_p(\mathbf{R}^n) \cap L^p_{\mathrm{loc}}(\mathbf{R}^n).$$

Conversely, let $f \in HK_p(\mathbf{R}^n) \cap L^p_{\mathrm{loc}}(\mathbf{R}^n)$. By Definition 1.3, we have

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x), \qquad \sum_{j=1}^{\infty} |\lambda_j| < \infty,$$

where each $a_j(x)$ is a central symmetry $(1,p)$-atom supported on $B(0,R_j)$. We write

$$I_1 = \{j : R_j \ge 1\}, \qquad I_2 = \{j : R_j < 1\}.$$

Then $f(x) = (\sum_{j\in I_1} + \sum_{j\in I_2})\lambda_j a_j(x) =: f_1(x) + f_2(x)$. By Theorem 3.1 of [5], obviously, we have $f_1 \in HA^p(\mathbf{R}^n)$ and

$$\|f_1\|_p \le \sum_{j\in I_1} |\lambda_j| \|a_j\|_p \le \sum_{j\in I_1} |\lambda_j| |B(0,R_j)|^{-1+1/p} \le C \sum_{j=1}^{\infty} |\lambda_j| < \infty.$$

Thus, for given $f \in L^p_{\mathrm{loc}}(\mathbf{R}^n)$, we have $f_2(x) = f(x) - f_1(x) \in L^p_{\mathrm{loc}}(\mathbf{R}^n)$, $\operatorname{supp} f_2 \subset B(0,1)$ and $\int f_2(x)\,dx = \sum_{j\in I_2} \lambda_j \int a_j(x)\,dx = 0$, and therefore

$$\|f_2\|_p^{-1} |B(0,1)|^{1/p-1} f_2(x) = b(x)$$

is a central $(1, p)$-atom (see [5]) of the space $HA^p(\mathbf{R}^n)$. So, by Theorem 3.1 of [5], it follows that

$$f_2(x) = \|f_2\|_p |B(0,1)|^{1-1/p} b(x) \in HA^p(\mathbf{R}^n).$$

Hence $f(x) = f_1(x) + f_2(x) \in HA^p(\mathbf{R}^n)$, that is, $HK_p(\mathbf{R}^n) \cap L^p_{\text{loc}}(\mathbf{R}^n) \subset HA^p(\mathbf{R}^n)$.

This finishes the proof of Remark.

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Sur les espaces de Fréchet ne contenant pas c_0

par

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Sommaire. Soit E un espace de Fréchet séparable ne contenant pas c_0; soit de plus (X_n) une suite symétrique de vecteurs aléatoires à valeurs dans E. Alors si la série de Fourier aléatoire $\sum X_n \exp(i\langle\lambda_n, t\rangle)$, $t \in \mathbf{R}^d$, a p.s. ses sommes partielles localement uniformément bornées dans E, nécessairement elle converge p.s. uniformément sur tout compact de $\mathbf{R}^d$ vers une fonction aléatoire à valeurs dans E et à trajectoires continues.

1. Introduction, notations, énoncé

1.1. Dans deux papiers récents, M. Talagrand et X. Fernique étudient respectivement les séries de Fourier gaussiennes à valeurs dans un espace de Banach séparable E ([8]) et les fonctions aléatoires gaussiennes stationnaires sur $\mathbf{R}^d$ à valeurs dans un espace lusinien quasi-complet E ([3], [3']). L'un et l'autre montrent que si les éléments aléatoires qu'ils étudient sont p.s. localement bornés dans E et si E ne contient pas c_0, alors ces éléments ont des propriétés de continuité. Dans ces études qui prolongent les plus anciens résultats de Hoffmann-Jørgensen ([4]) et de Kwapień ([6]) sur les espaces de Banach ne contenant pas c_0, le caractère gaussien semble important. On se propose de montrer ici qu'en fait l'analyse des énoncés de [8] et des méthodes d'étude de [3] permet de supprimer toute hypothèse gaussienne et même tout calcul gaussien, au prix d'une restriction sur les espaces considérés par rapport à [3], [3'].

Dans tout ce travail, E désignera un espace de Fréchet séparable (complexe); on notera (λ_n) une suite d'éléments de $\mathbf{R}^d$ et (X_n) une suite symétrique de vecteurs aléatoires à valeurs dans E; on supposera que l'espace d'épreuves est complet et on posera $S_0 = \{\mathbf{Q} \cap [0,1]\}^d$, $S = \mathbf{Q}^d$. On dira que E *contient* c_0 si :

(1.1.1) Il existe une suite (x_n) contenue dans E telle que pour toute suite

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