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Law equivalence of solutions of some linear stochastic equations in Hilbert spaces

by

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Abstract. Sufficient and necessary conditions for equivalence of the distributions of the solutions of some linear stochastic equations in Hilbert spaces are given. Some facts in the theory of perturbations of semigroup generators and Zabczyk's results on law equivalence are used.

0. Introduction. Let X_1 and X_2 be the solutions of the following stochastic linear equations on a real separable Hilbert space H:

$$dX_1 = A_1 X_1 dt + dW, \quad X_1(0) = x \in H, \tag{0.1}$$

$$dX_2 = A_2 X_2 dt + dW, \quad X_2(0) = x \in H. \tag{0.2}$$

In (0.1) and (0.2), W is a cylindrical Wiener process on H, A_1 and A_2 stand for the infinitesimal generators of C_0-semigroups S_1 and S_2 from a class to be specified later. By the solution of (0.1) and (0.2) we understand the so-called mild solution.

Let $\mathcal{L}(X_1(\ ,x))$ and $\mathcal{L}(X_2(\ ,x))$ be the laws (distributions) in $L^2(0,T;H)$ of the solutions of (0.1) and (0.2). This paper is concerned with the study of necessary and sufficient conditions for equivalence of $\mathcal{L}(X_1(\ ,x))$ and $\mathcal{L}(X_2(\ ,x))$ (i.e. law equivalence of X_1 and X_2). It will be shown that if $D(A_1) = D(A_2)$ and

$$\int_0^1 \|(A_2 - A_1)S_1(t)\|_2^2\, dt < \infty,$$

then for all $T > 0$ and $x \in H$ the laws $\mathcal{L}(X_1(\ ,x))$ and $\mathcal{L}(X_2(\ ,x))$ are equivalent in $L^2(0,T;H)$, and in many important cases these conditions are also necessary.

Conditions for law equivalence of solutions of general linear stochastic differential equations were given by Zabczyk [10]. In our paper Zabczyk's

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conditions are formulated in terms of generators and semigroups. The problem of law equivalence of the processes (0.1) and (0.2) was also considered by Kozlov [6] and [7] for elliptic generators and by Koski and Loges [5] for self-adjoint and commuting operators. Using a completely different method Peszat [9] obtained sufficient conditions for law equivalence. The cases of general self-adjoint and elliptic generators are considered in the last section concerning particular cases. Note that the results obtained in [7] and [5] follow from the main theorems of the present paper.

1. Notations and preliminaries. In this paper, the spaces of bounded, Hilbert–Schmidt and trace (i.e. nuclear) operators on H are denoted by $L(H)$, $L_2(H)$ and $L_1(H)$, respectively. $\| \|$, $\| \|_2$ and $\| \|_1$ stand for the operator, the Hilbert–Schmidt and the trace norm. The space $L^2(0,T;H)$ is denoted by $\mathcal{H}_T$. By C_0 we denote the collection of all generators of C_0-semigroups acting on H. In our considerations an important role is played by the class $\mathcal{U}$ of generators $A \in C_0$ such that the semigroup S generated by A satisfies the condition

$$\int_0^1 \|S(t)\|_2^2\, dt < \infty .$$

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space with a right-continuous increasing family $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$ each containing $\mathbf{P}$-null sets. Let $\{e_m\}$ be an orthonormal basis in H and let $\{W_m\}$ be a sequence of independent, real-valued $\mathbf{F}$-Wiener processes. By a *cylindrical Wiener process* on H we mean the series

$$W(t) = \sum_{m=1}^{\infty} W_m(t) e_m .$$

This series does not converge in H but in an arbitrary Hilbert space $\widetilde{H}$ containing H with a Hilbert–Schmidt embedding.

Assume that K is a separable Hilbert space and denote by $\mathcal{M}_2(K)$ the space of all $\mathcal{B}([0,+\infty)) \times \mathcal{F}$-measurable processes ϕ taking values in K such that:

(i) $\phi(t)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$,
(ii) $E\int_0^t |\phi(s)|_K^2\, ds < \infty$, $t \geq 0$.

THEOREM 1.1 ([8], Theorem 5.1). *There exists a linear operator $\mathcal{J}$ acting from $\mathcal{M}_2(L_2(H,K))$ into $\mathcal{M}_2(K)$ such that:*

(a) *$\mathcal{J}\phi$ has continous sample paths,*
(b) *$\mathcal{J}\phi$ is a martingale,*
(c) $E\mathcal{J}\phi(t) = 0$, $E|\mathcal{J}\phi(t)|_K^2 = E\int_0^t \|\phi(s)\|_{L_2(H,K)}^2\, ds$, $t \geq 0$.

Remark 1.1. $\mathcal{J}\phi(t)$, usually denoted by $\int_0^t \phi(s)\, dW(s)$, is called the *Itô integral* and can be defined as the $L^2(\Omega,\mathcal{F},\mathbf{P})$ limit of the series

$$\sum_{m=1}^{\infty} \int_0^t \phi(s) e_m\, dW_m .$$

For more information on Wiener processes, stochastic Itô integrals and measures on Hilbert spaces the reader is referred to [8] or [2].

In this paper we study stochastic equations of the form

$$dX = AX\,dt + dW, \quad X(0) = x \in H, \tag{1.3}$$

where $A \in \mathcal{U}$ is the infinitesimal generator of a semigroup S. By Theorem 1.1 the process

$$X(t) = S(t)x + \int_0^t S(t-s)\, dW(s), \quad t \geq 0, \tag{1.4}$$

takes values in H. The process X is called the *mild solution* of equation (1.3).

Let $\mathbf{L} : \mathcal{H}_T \to \mathcal{H}_T$ and $\mathbf{m} \in \mathcal{H}_T$ be defined by the formulas

$$(\mathbf{L}u)(t) = \int_0^t S(t-s)u(s)\, ds, \quad \mathbf{m}(t) = S(t)x ,$$

for $u \in \mathcal{H}_T$, $t \in [0,T]$. Note that the operator $\mathbf{L}$ is injective. The following well known theorem (see [2], Chapter 5) presents properties of the process X which will be used in the sequel.

THEOREM 1.2. *Let $A \in \mathcal{U}$, $x \in H$ and let X be given by* (1.4). *Then:*

(i) *for each $T > 0$, $X(\ ,\omega) \in \mathcal{H}_T$ $\mathbf{P}$-almost surely and $X : \Omega \to \mathcal{H}_T$ is a measurable mapping,*
(ii) *the law of X, $\mathcal{L}(X)(B) := \mathbf{P}(X^{-1}(B))$, $B \in \mathcal{B}(\mathcal{H}_T)$, is Gaussian with mean $\mathbf{m}$ and covariance operator $\mathbf{Q} = \mathbf{L}\mathbf{L}^*$.*

Recall ([8], p. 28) that a probability measure ν on a Hilbert space $\mathcal{H}$ is *Gaussian* if and only if for every $h \in \mathcal{H}$

$$\nu_h(B) := \nu(k \in \mathcal{H} : \langle k, h\rangle_{\mathcal{H}} \in B), \quad B \in \mathcal{B}(\mathbf{R}),$$

is a Gaussian measure on $\mathbf{R}$ with mean $\mathbf{m}_h$ and covariance $\mathbf{Q}_h$. The (unique) element $\mathbf{m} \in \mathcal{H}$ and self-adjoint operator $\mathbf{Q}$ on $\mathcal{H}$ such that $\langle \mathbf{m}, h\rangle_{\mathcal{H}} = \mathbf{m}_h$ and $\langle \mathbf{Q}h, h\rangle_{\mathcal{H}} = \mathbf{Q}_h$ are called the *mean* and the *covariance operator* of ν.

Suppose that $A_1, A_2 \in \mathcal{U}$, $T > 0$ and $x \in H$. Let S_1 and S_2 be the semigroups generated by A_1 and A_2, respectively. The processes

$$X_1(t,x) = S_1(t)x + \int_0^t S_1(t-s)\, dW(s), \tag{1.5}$$

$$X_2(t,x) = S_2(t)x + \int_0^t S_2(t-s)\,dW(s) \tag{1.6}$$

are the mild solutions of equations (0.1) and (0.2). We consider X_1 and X_2 as random elements in the space $\mathcal{H}_T = L^2(0,T;H)$ (i.e. measurable mappings from Ω into $\mathcal{H}_T$).

Let $\mathbf{L}_1, \mathbf{L}_2 : \mathcal{H}_T \to \mathcal{H}_T$, and $\mathbf{m}_1, \mathbf{m}_2 \in \mathcal{H}_T$ be defined by the formulas

$$(\mathbf{L}_1 u)(t) = \int_0^t S_1(t-s)u(s)\,ds, \qquad (\mathbf{L}_2 u)(t) = \int_0^t S_2(t-s)u(s)\,ds,$$

for $u \in \mathcal{H}_T$, $t \in [0,T]$, and

$$\mathbf{m}_1(t) = S_1(t)x, \quad \mathbf{m}_2(t) = S_2(t)x, \quad \text{for } t \in [0,T].$$

As a consequence of Theorem 1.2, we have

Remark 1.2. The laws $\mathcal{L}(X_1(\ ,x))$ and $\mathcal{L}(X_2(\ ,x))$ are Gaussian; $\mathbf{m}_1$ and $\mathbf{m}_2$, $\mathbf{Q}_1 = \mathbf{L}_1\mathbf{L}_1^*$ and $\mathbf{Q}_2 = \mathbf{L}_2\mathbf{L}_2^*$ are the means and the covariance operators of $\mathcal{L}(X_1(\ ,x))$ and $\mathcal{L}(X_2(\ ,x))$.

We say that X_1 and X_2 are *law equivalent* if the laws $\mathcal{L}(X_1)$ and $\mathcal{L}(X_2)$ are equivalent (i.e. mutually absolutely continuous). This means that for every $B \in \mathcal{B}(\mathcal{H}_T)$, $\mathcal{L}(X_1)(B) = 0$ if and only if $\mathcal{L}(X_2)(B) = 0$.

Recall [8] that two Gaussian measures are either singular or equivalent. For further reference we present Zabczyk's theorem on equivalence (see [10]). Its proof is based on the Feldman–Hájek theorem (see [8]) and is omitted here.

THEOREM 1.3. *The laws $\mathcal{L}(X_1(\ ,x))$ and $\mathcal{L}(X_2(\ ,x))$ are equivalent if and only if the following conditions hold*:

(i) $\operatorname{Im}\mathbf{L}_1 = \operatorname{Im}\mathbf{L}_2$,
(ii) *the operator* $(\mathbf{L}_2^{-1}\mathbf{L}_1)(\mathbf{L}_2^{-1}\mathbf{L}_1)^* - \mathcal{I}$ *is Hilbert–Schmidt*,
(iii) $\mathbf{m}_1 - \mathbf{m}_2 \in \operatorname{Im}\mathbf{L}_1$.

Using elementary arguments one can prove that (i) implies that the domains $D(A_1)$ and $D(A_2)$ are equal (more generally, $\operatorname{Im}\mathbf{L}_1 \subseteq \operatorname{Im}\mathbf{L}_2$ implies that $D(A_1) \subseteq D(A_2)$). Moreover, the operator $\mathcal{K} = \mathbf{L}_2^{-1}\mathbf{L}_1 - \mathcal{I}$ has on $C^1(0,T;H)$ the following expression:

$$(\mathcal{K}u)(t) = (A_1 - A_2)\int_0^t S_1(t-s)u(s)\,ds. \tag{1.7}$$

Therefore, we may rephrase condition (ii) in a more convenient but equivalent form. The operator $\mathcal{K}$ defined on $C^1(0,T;H)$ by (1.7) has a bounded extension to $\mathcal{H}_T$ and the operator $\mathcal{M} = \mathcal{K} + \mathcal{K}^* + \mathcal{K}\mathcal{K}^*$ is Hilbert–Schmidt. Of course, if $\mathcal{K}$ is Hilbert–Schmidt then so is $\mathcal{M}$. Difficulties in application

of Theorem 1.3 lie in the fact that the form of the operator $\mathcal{K}$ is known only on $C^1(0,T;H)$ and a priori it is possible that $\mathcal{M}$ is Hilbert–Schmidt even if $\mathcal{K}$ is not. In fact, the main theorems of the present paper show that the latter situation does not occur for a variety of cases.

Before proceeding further we need a few definitions. In what follows $1 \le p < \infty$, $A \in C_0$ and S stands for the semigroup generated by A.

DEFINITION 1.1. We say that a linear operator $(K, D(K))$ belongs to $\mathcal{P}_p(A)$ if it has the following properties:

(i) $D(A) = D(K)$,
(ii) $KR(\lambda, A)$ is bounded for some λ from the resolvent set of A,
(iii) for $t > 0$, the operators $KS(t)$ defined on $D(A)$ have bounded extensions to H,
(iv) $\int_0^1 \|KS(t)\|^p\,dt < \infty$.

Now, let $A \in \mathcal{U}$.

DEFINITION 1.2. We say that a linear operator $(K, D(K))$ belongs to $\mathcal{R}(A)$ iff $K \in \mathcal{P}_1(A)$ and

$$\int_0^1 \|KS(t)\|_2^2\,dt < \infty.$$

The class $\mathcal{P}_1(A)$ was introduced by E. Hille and R. Phillips (see [4]). The classes $\mathcal{P}_2(A)$ and $\mathcal{R}(A)$ were introduced by Peszat in [9] (with different notation: $\mathcal{P}_1(A)$, $\mathcal{P}_2(A)$ and $\mathcal{R}(A)$ were denoted by $\mathcal{P}(A)$, $\mathcal{P}_1(A)$ and $\mathcal{P}_2(A)$, respectively).

DEFINITION 1.3. A generator $A \in C_0$ belongs to $\mathcal{S}$ if there exist a self-adjoint strictly negative operator $(\overline{A}, D(\overline{A}))$ and $1 < p < \infty$ such that:

(i) $D(A) = D(\overline{A}) = D(A^*)$,
(ii) $A - \overline{A} \in \mathcal{P}_p(\overline{A})$,
(iii) $A^* - \overline{A} \in \mathcal{P}_p(\overline{A})$.

$\mathcal{S}$ is a subspace of the space of all analytic generators, containing the self-adjoint generators. One can show that each uniformly elliptic operator, with smooth coefficients and 0-boundary conditions on a bounded region, belongs to $\mathcal{S}$ (for more details see the last section).

2. Formulation of main results. Now, we are ready to formulate the main results. Recall that $A_1, A_2 \in \mathcal{U}$ and X_1, X_2 are given by (1.3) and (1.4). The main results of this paper present relations between the following conditions of probabilistic nature:

(C.1) *For some $T > 0$, $\mathcal{L}(X_1(\ ,0))$ and $\mathcal{L}(X_2(\ ,0))$ are equivalent in $\mathcal{H}_T$*, (C.2) *For all* $T > 0$, $x \in H$, $\mathcal{L}(X_1(\,, x))$ *and* $\mathcal{L}(X_2(\,, x))$ *are equivalent in* $\mathcal{H}_T$,

and the following analytic conditions:

(C.3) $A_2 - A_1 \in \mathcal{R}(A_1)$,

(C.4) $A_1 - A_2 \in \mathcal{R}(A_2)$.

It will be proved (see the next section) that (C.3) and (C.4) are equivalent and (see the proof of Theorem 2.1, or [10]) that the operator $\mathcal{K}$ is Hilbert–Schmidt if and only if (C.3) holds. The first theorem contains sufficient conditions for law equivalence of the processes X_1 and X_2.

THEOREM 2.1. (C.3) (*or equivalently* (C.4)) *implies* (C.2).

Two theorems below contain sufficient and necessary conditions under the additional assumption that the difference $A_2 - A_1$ is "small".

THEOREM 2.2. *Suppose that* $A_2 - A_1 \in \mathcal{P}_2(A_1)$. *Then* (C.1)–(C.4) *are equivalent.*

THEOREM 2.3. *Suppose that* $D(A_1) \cap D(A_1^*) \cap D(A_2^*)$ *is dense in* H *and* $A_2 - A_1 \in \mathcal{P}_1(A_1)$. *Then* (C.1)–(C.4) *are equivalent.*

Theorem 2.4 contains necessary and sufficient conditions for law equivalence, under the assumption that the generators differ only slightly from self-adjoint operators.

THEOREM 2.4. *Suppose that* $A_1, A_2 \in \mathcal{S}$. *Then* (C.1)–(C.4) *are equivalent.*

COROLLARY 2.1. *Suppose that* A_1 *and* A_2 *are self-adjoint. Then* (C.1)–(C.4) *are equivalent.*

The proofs, based on the theory of perturbations of semigroup generators, are given in Section 4.

3. Perturbations of generators. The following theorem, whose proof is omitted here, contains facts obtained by E. Hille and R. Phillips [4] (Theorem 3.4.1, p. 399, Corollary 1, p. 400, and Theorem 13.5.3, p. 410; see also [9] and [3]).

THEOREM 3.1. *Let* $K \in \mathcal{P}_1(A)$. *Then*:

(i) *The operator* $A + K$, *with domain* $D(A)$, *belongs to* C_0.
(ii) $\mathcal{P}_1(A) = \mathcal{P}_1(A + K)$.
(iii) *If* U *is the semigroup generated by* $A + K$ *then*

$$U(t) = S(t) + \int_0^t S(t-s)KU(s)\,ds = S(t) + \int_0^t U(t-s)KS(s)\,ds. \tag{3.8}$$

Before proceeding further we will prove the following easy lemma.

LEMMA 3.1. *The conditions*:

(i) $A \in \mathcal{U}$,
(ii) *for each* $t > 0$, $S(t) \in L_1(H)$ *and* $\int_0^1 \|S(t)\|_1 < \infty$,

are equivalent.

Proof. (i)⇒(ii) follows from the inequality

$$\|S(t)\|_1 = \|S(t/2)S(t/2)\|_1 \le \|S(t/2)\|_2 \|S(t/2)\|_2 = \|S(t/2)\|_2^2.$$

(ii)⇒(i) follows from the inequality

$$\|S(t)\|_2^2 = \|S(t)S^*(t)\|_2 \le \|S(t)\| \|S(t)\|_1.$$

The following theorem gives some more information about the class of $\mathcal{R}(A)$ perturbations and it will be used in the proofs of the main theorems.

THEOREM 3.2. *Let* $A \in \mathcal{U}$, $K \in \mathcal{P}_1(A)$. *Then*:

(i) $A + K \in \mathcal{U}$,
(ii) $\mathcal{R}(A + K) = \mathcal{R}(A)$.

Proof. Let U be the semigroup generated by $A + K$. We will show that U satisfies condition (ii) from Lemma 3.1. By (3.8)

$$\|U(t)\|_1 \le \|S(t)\|_1 + \int_0^t \|S(t-s)\|_1 \|KU(s)\|\,ds.$$

Since $K \in \mathcal{P}_1(A + K)$, we have

$$\int_0^1 \|U(t)\|_1\,dt \le \int_0^1 \|S(t)\|_1\,dt + \int_0^1 \|S(t)\|_1\,dt \int_0^1 \|KU(s)\|\,ds < \infty.$$

Now, we prove (ii). By Theorem 3.1, $-K \in \mathcal{P}_1(A+K)$ and it is enough to prove that $\mathcal{R}(A) \subseteq \mathcal{R}(A+K)$. Let $F \in \mathcal{R}(A)$; then $F \in \mathcal{P}_1(A+K) = \mathcal{P}_1(A)$. Since

$$\int_0^1 \int_0^s \|FS(s-r)\| \|KU(r)\|\,dr\,ds \le \int_0^1 \|FS(s)\|\,ds \int_0^1 \|KU(r)\|\,dr < \infty,$$

we have (see [4], Lemma 13.3.2, p. 393)

$$FU(s) = FS(s) + \int_0^s FS(s-r)KU(r)\,dr$$

and for each $t \in [0,1]$

$$\int_0^t \|FU(s)\|_2\,ds \le \int_0^t \|FS(s)\|_2\,ds + \int_0^t \|FS(s)\|_2\,ds \int_0^1 \|KU(r)\|\,dr$$
$$= \Big(1+\int_0^1 \|KU(r)\|\,dr\Big)\int_0^t \|FS(s)\|_2\,ds = M\int_0^t \|FS(s)\|_2\,ds\,.$$

Note that there are only two possibilities:

1° There exists $0<\varepsilon\le 1$ such that for all $0<s\le\varepsilon$

$$\|FU(s)\|_2 \le M\|FS(s)\|_2\,.$$

2° There exists a decreasing sequence $\{t_n\}$ converging to 0 such that for all $n\in\mathbf{N}$, $0<t_n\le 1$ and

$$\|FU(t_n)\|_2 = M\|FS(t_n)\|_2\,.$$

In the first case

$$\int_0^1 \|FU(s)\|_2^2\,ds = \int_0^\varepsilon \|FU(s)\|_2^2\,ds + \int_\varepsilon^1 \|FU(s)\|_2^2\,ds$$
$$\le M^2\int_0^\varepsilon \|FS(s)\|_2^2\,ds + \|FU(\varepsilon)\|^2\int_0^{1-\varepsilon}\|U(s)\|_2^2\,ds < \infty\,.$$

In the second case

$$\int_0^1 \|FU(s)\|_2^2\,ds = \sum_{n=1}^\infty \int_{t_{n+1}}^{t_n} \|FU(s)\|_2^2\,ds + \int_{t_1}^1 \|FU(s)\|_2^2\,ds$$
$$\le \sum_{n=1}^\infty \|FU(t_{n+1})\|_2^2 \sup_{t_{n+1}\le s\le t_n}\|U(s-t_{n+1})\|^2(t_n-t_{n+1})$$
$$+\int_{t_1}^1 \|FU(s)\|_2^2\,ds$$
$$\le M^2\sup_{0\le s\le 1}\|U(s)\|^2\sum_{n=1}^\infty \|FS(t_{n+1})\|_2^2(t_n-t_{n+1})$$
$$+\|FU(t_1)\|^2\int_0^{1-t_1}\|U(s)\|_2^2\,ds<\infty\,.$$

Thus the proof is complete.

4. Proofs of Theorems 2.1–2.4. Although Theorem 2.1 was essentially proved in [10] (and in [9] using a completely different method) we

present a proof. We show that condition (C.3) implies that the operator $\mathcal{K}$ (given by (1.7)) has a Hilbert–Schmidt extension. Let $\{e_n\}$ be an orthonormal basis in H and let $\{f_m\}$ be an orthonormal basis in $L^2(0,T;\mathbf{R})$. We may assume that for each n, $e_n\in D(A_1)=D(A_2)$. Moreover, we assume that for each m, $f_m\in C^1$. The functions $\{f_me_n;\ m,n\in\mathbf{R}\}$ form an orthonormal basis in $\mathcal{H}_T$ and

$$\mathcal{K}f_me_n(t) = (A_1-A_2)\int_0^t S_1(t-s)f_m(s)e_n\,ds$$
$$= \int_0^t (A_1-A_2)S_1(t-s)f_m(s)e_n\,ds\,.$$

Hence (see [8], Example 3, p. 6)

$$\sum_{n,l}\sum_{m,d}\langle\mathcal{K}f_me_n, f_de_l\rangle_{\mathcal{H}_T}^2$$
$$=\sum_{n,l}\sum_{m,d}\Big(\int_0^T\int_0^t \langle(A_1-A_2)S_1(t-s)e_n,e_l\rangle_H f_m(s)f_d(t)\,ds\,dt\Big)^2$$
$$=\sum_{n,l}\int_0^T\int_0^t \langle(A_1-A_2)S_1(t-s)e_n,e_l\rangle_H^2\,ds\,dt$$
$$=\int_0^T\int_0^t \|(A_1-A_2)S_1(t-s)\|_2^2\,ds\,dt$$
$$\le T\int_0^T \|(A_1-A_2)S_1(t)\|_2^2\,dt<\infty\,.$$

This means that $\mathcal{K}$ has a Hilbert–Schmidt extension to $\mathcal{H}_T$. As $\mathbf{L}_2(\mathcal{K}+\mathcal{I}) = \mathbf{L}_1$, we have $\operatorname{Im}\mathbf{L}_1\subseteq\operatorname{Im}\mathbf{L}_2$. Since $\mathcal{R}(A_1)=\mathcal{R}(A_2)$, the inclusion $\operatorname{Im}\mathbf{L}_2\subseteq\operatorname{Im}\mathbf{L}_1$ can be proved by replacement of A_1 with A_2. Now, we show condition (iii) in Theorem 1.3. By (3.8)

$$\mathbf{m}_1(t)-\mathbf{m}_2(t) = S_1(t)x - S_2(t)x = -\int_0^t S_1(t-s)(A_2-A_1)S_2(s)x\,ds\,.$$

Let $u(s) = -(A_2-A_1)S_2(s)x$. Since

$$\int_0^T |(A_2-A_1)S_2(s)x|_H^2\,ds \le |x|_H^2\int_0^T \|(A_2-A_1)S_2(s)\|^2\,ds<\infty\,,$$

we have $u \in \mathcal{H}_T$ and $\mathbf{m}_1 - \mathbf{m}_2 = \mathbf{L}_1 u \in \operatorname{Im} \mathbf{L}_1$. Thus the proof of Theorem 2.1 is complete.

Now, we prove Theorems 2.2–2.4. Note that (C.3)⇔(C.4) follows from Theorem 3.2, (C.3)⇒(C.2) follows from Theorem 2.1 and (C.2)⇒(C.1) is obvious. Therefore we only have to prove (C.1)⇒(C.3). Assume that (C.1) holds. Let $K = A_1 - A_2$, let $\{e_n\}$ be an orthonormal basis in H and $\{f_m\}$ an orthonormal basis in $L^2(0,T;\mathbf{R})$. As in the proof of Theorem 2.1 we assume that for each m, $f_m \in C^1$. In the cases of Theorems 2.3 and 2.4 we assume that $e_n \in D(A_1) \cap D(A_1^*) \cap D(A_2^*)$. These assumptions guarantee that for all n we have

$$\int_0^T (|KS_1(t)e_n|_H^2 + |(KS_1(t))^* e_n|_H^2)\, dt < \infty .$$

Let

$$q^1_{n,l}(t,s) = \mathbf{1}_{\{s<t\}} \langle KS_1(t-s)e_n, e_l\rangle_H ,$$
$$q^2_{n,l}(t,s) = \mathbf{1}_{\{t<s\}} \langle (KS_1(s-t))^* e_n, e_l\rangle_H ,$$
$$q^3_{n,l}(t,s) = \int_0^{t\wedge s} \langle (KS_1(s-r))^* e_n, (KS_1(t-r))^* e_l\rangle_H \, dr .$$

After simple calculations we have

$$\langle (\mathcal{K} + \mathcal{K}^* + \mathcal{K}\mathcal{K}^*) f_m e_n(t), f_d(t) e_l\rangle_H = \int_0^T \{q^1_{n,l}(t,s) + q^2_{n,l}(t,s) + q^3_{n,l}(t,s)\} f_m(s) f_d(t)\, ds .$$

Using arguments similar to the proof of Theorem 2.1, we have

$$\sum_{m,d} \langle (\mathcal{K} + \mathcal{K}^* + \mathcal{K}\mathcal{K}^*) f_m e_n, f_d e_l\rangle^2_{\mathcal{H}_T} = \int_0^T \int_0^T \{q^1_{n,l}(t,s) + q^2_{n,l}(t,s) + q^3_{n,l}(t,s)\}^2\, ds\, dt .$$

Let $0 < T_0 \le T$, to be specified later. Assumption (C.1) and Theorem 1.3 imply that $\mathcal{M} = \mathcal{K} + \mathcal{K}^* + \mathcal{K}\mathcal{K}^*$ is Hilbert–Schmidt. Hence

$$\sum_{n,l} \int_0^{T_0} \int_0^t \{q^1_{n,l}(t,s) + q^2_{n,l}(t,s) + q^3_{n,l}(t,s)\}^2\, ds\, dt \le \|\mathcal{M}\|_2^2 < \infty .$$

Since for all $0 < s < t$, $q^2_{n,l}(t,s) = 0$, we have

$$\sum_{n,l} \int_0^{T_0} \int_0^t \{q^1_{n,l}(t,s) + q^3_{n,l}(t,s)\}^2\, ds\, dt < \infty .$$

This means that

$$\sum_{n,l} \int_0^{T_0} \int_0^t \Big\{ \langle KS_1(t-s)e_n, e_l\rangle_H + \int_0^s \langle (KS_1(s-r))^* e_n, (KS_1(t-r))^* e_l\rangle_H\, dr \Big\}^2 ds\, dt \tag{4.9}$$
$$= \sum_{n,l} \int_0^{T_0} \int_0^t \Big\langle KS_1(t-s)\Big(e_n + \int_0^s S_1(s-r)(KS_1(s-r))^* e_n\, dr\Big), e_l\Big\rangle_H^2 ds\, dt < \infty .$$

Suppose that $A_2 - A_1 = -K \in \mathcal{P}_1(A_1)$ (we consider the cases of Theorems 2.2–2.3). The operators

$$Z(s) = \int_0^s S_1(s-r)(KS_1(s-r))^*\, dr, \qquad s \ge 0 ,$$

are bounded and

$$\|Z(s)\| \le \int_0^s \|S_1(s-r)\|\|KS_1(s-r)\|\, dr \le \sup_{0\le r\le s} \|S_1(r)\| \int_0^s \|KS_1(r)\|\, dr .$$

Hence we may choose $0 < T_0 \le T$ such that

$$\sup\{\|Z(s)\| : 0 \le s \le T_0\} \le 1/2 . \tag{4.10}$$

From (4.9)

$$\int_0^{T_0} \int_0^t \|KS_1(t-s)(\mathcal{I} + Z(s))\|_2^2\, ds\, dt < \infty .$$

From (4.10) the operators $\mathcal{I} + Z(s)$, $s \in [0, T_0]$, are invertible and

$$\sup\{\|(\mathcal{I} + Z(s))^{-1}\| : 0 \le s \le T_0\} \le 2 .$$

Therefore

$$\int_0^{T_0} \int_0^t \|KS_1(t-s)\|_2^2\, ds\, dt = \int_0^{T_0} \int_0^t \|KS_1(t-s)(\mathcal{I} + Z(s))(\mathcal{I} + Z(s))^{-1}\|_2^2\, ds\, dt$$
$$\le 4 \int_0^{T_0} \int_0^t \|KS_1(t-s)(\mathcal{I} + Z(s))\|_2^2\, ds\, dt < \infty$$

and finally $\int_0^{T_0} \|KS_1(t)\|_2^2\,dt < \infty$. This proves that $A_2 - A_1 \in \mathcal{R}(A_1)$. Thus Theorems 2.2 and 2.3 are proved.

The case of Theorem 2.4 is more complicated. We need the following lemma.

LEMMA 4.1. *Suppose that $A \in \mathcal{S}$. Then for all $s > 0$ the operator*

$$Z_0(s) = \int_0^s S(s-r)(AS(s-r))^* dr = \int_0^s S(r)(AS(r))^* dr,$$

$$D(Z_0(s)) = D(A^*),$$

has a bounded extension (denoted also by $Z_0(s)$) to the whole space H. Moreover, if $A \in \mathcal{S} \cap \mathcal{U}$ then for each $\varepsilon > 0$ there exists $0 < T_0 \le T$ such that

$$\sup\{\|Z_0(s)\| : 0 \le s \le T_0\} \le 1/2 + \varepsilon.$$

Proof.

$$Z_0(s) = \int_0^s S^*(r)(AS(r))^* dr + \int_0^s (S(r) - S^*(r))(AS(r))^* dr$$
$$= Z_{01}(s) + Z_{02}(s).$$

Obviously $Z_{01}(s) = \frac{1}{2}(S^*(2s) - \mathcal{I})$. Suppose that $\overline{A}$ is a self-adjoint strictly negative definite operator, $p \in (1, \infty)$ and $A - \overline{A} \in \mathcal{P}_p(\overline{A})$, $A^* - \overline{A} \in \mathcal{P}_p(\overline{A})$. From (3.8) (here $\overline{S}$ stands for the semigroup generated by $\overline{A}$)

$$S(r) = \overline{S}(r) + \int_0^r S(r-u)(A - \overline{A})\overline{S}(u)\,du, \tag{4.11}$$

$$S^*(r) = \overline{S}(r) + \int_0^r S^*(r-u)(A^* - \overline{A})\overline{S}(u)\,du. \tag{4.12}$$

Hence, by the Hölder Inequality (q is such that $1/p + 1/q = 1$)

$$\|S(r) - S^*(r)\| \le \int_0^r \|S(r-u)\|\|(A - \overline{A})\overline{S}(u)\|\,du$$
$$+ \int_0^r \|S^*(r-u)\|\|(A^* - \overline{A})\overline{S}(u)\|\,du$$
$$\le \sup_{0 \le v \le r} \|S(v)\| \Big\{ \Big(\int_0^r \|(A - \overline{A})\overline{S}(u)\|^p\,du \Big)^{1/p}$$
$$+ \Big(\int_0^r \|(A^* - \overline{A})\overline{S}(u)\|^p\,du \Big)^{1/p} \Big\} r^{1/q} = M(r) r^{1/q}.$$

Since S is an analytic semigroup, there exists M such that

$$\|Z_{02}(s)\| \le \int_0^s M(r) r^{1/q} \|(AS(r))^*\|\,dr \le \int_0^s M(r) r^{1/q} M r^{-1}\,dr$$
$$\le \sup_{0 \le r \le s} M(r) M \int_0^s r^{1/q - 1}\,dr = C_1(s) s^{1/q}.$$

Hence $Z_0(s)$ is bounded and

$$\|Z_0(s)\| \le \tfrac{1}{2}\|S^*(2s) - \mathcal{I}\| + C_1(s) s^{1/q}. \tag{4.13}$$

If $A \in \mathcal{U}$ then $\overline{A} \in \mathcal{U}$ as well. Since $\overline{A}$ is strictly negative, $\|\overline{S}(2s) - \mathcal{I}\| = 1$, for $s > 0$. From (4.12)

$$\|S^*(2s) - \mathcal{I}\| \le \|\overline{S}(2s) - \mathcal{I}\| + \int_0^{2s} \|S^*(2s-u)\|\|(A^* - \overline{A})\overline{S}(u)\|\,du = 1 + C_2(2s).$$

From (4.13)

$$\|Z_0(s)\| \le \tfrac{1}{2} + \tfrac{1}{2} C_2(2s) + C_1(s) s^{1/q} = \tfrac{1}{2} + C(s).$$

Let $\varepsilon > 0$. Since $C(s)$ converges to 0 as s converges to 0, there exists $T_0 > 0$ such that $\sup\{C(s) : 0 \le s \le T_0\} \le \varepsilon$. This completes the proof of the lemma.

According to Theorems 2.1 and 3.2 we may replace A_1 by $A_1 + \lambda\mathcal{I}$ and A_2 by $A_2 + \beta\mathcal{I}$. Therefore, without any loss of generality we can assume that A_1 is invertible and $\|\mathcal{I} - A_2 A_1^{-1}\| < 2$. In this situation

$$K := A_1 - A_2 = (\mathcal{I} - A_2 A_1^{-1}) A_1 = B A_1,$$

where $\|B\| < 2$. By Lemma 4.1 the operators

$$Z(s) = \int_0^s S_1(s-r)(KS_1(s-r))^* dr = Z_0(s) B^*, \qquad s \in [0, T],$$

are bounded and there exist $0 < \eta < 1$ and $0 < T_0 \le T$ such that

$$\sup\{\|Z(s)\| : 0 \le s \le T_0\} \le 1 - \eta.$$

Using the same arguments as in the proofs of Theorems 2.2 and 2.3 we can conclude from (4.9) that

$$\int_0^{T_0} \int_0^t \|KS_1(t-s)\|_2^2\,ds\,dt \le \eta^{-2} \int_0^{T_0} \int_0^t \|KS_1(t-s)(\mathcal{I} + Z(s))\|_2^2\,ds\,dt < \infty$$

and finally $\int_0^{T_0} \|KS_1(t)\|_2^2\,dt < \infty$. This completes the proof of the theorem.

5. Some special cases. In this section we consider particular classes of generators. First we deal with self-adjoint generators and then with elliptic generators.

Let $(A, D(A))$ be a self-adjoint, negative definite generator of the semigroup S acting on H. Note that $A \in \mathcal{U}$ iff A^{-1} is nuclear. Therefore we assume that $A^{-1} \in L_1(H)$. Let $\{-\lambda_k, e_k\}$ be the sequence of all eigenvalues and the corresponding normalized eigenvectors of A. We assume that $0 < \lambda_1 \le \lambda_2 \le \ldots$ The semigroup S has the form

$$S(t)x = \sum_{k=1}^{\infty} \exp(-\lambda_k t)\langle x, e_k\rangle_H e_k .$$

First we solve the problem when an operator $(F, D(F))$ belongs to $\mathcal{R}(A)$.

LEMMA 5.1. *$F \in \mathcal{R}(A)$ iff $D(F) = D(A)$ and the operator $F(-A)^{-1/2}$ defined on $D(A)$ has a Hilbert–Schmidt extension to the whole space H.*

Proof. For $k \in \mathbb{N}$ and $0 \le t$, we have $FS(t)e_k = \exp(-\lambda_k t)Fe_k$. Hence

$$\begin{aligned}\int_0^1 \|FS(t)\|_2^2\,dt &= \sum_{k=1}^{\infty}\int_0^1 \exp(-2\lambda_k t)\,dt|Fe_k|_H^2\\ &= \frac{1}{2}\sum_{k=1}^{\infty}(1-\exp(-2\lambda_k))\lambda_k^{-1}|Fe_k|_H^2\\ &= \frac{1}{2}\sum_{k=1}^{\infty}(1-\exp(-2\lambda_k))|F(-A)^{-1/2}e_k|_H^2\\ &\le \frac{1}{2}\|F(-A)^{-1/2}\|_2^2 \le (1-\exp(-2\lambda_1))^{-1}\int_0^1 \|FS(t)\|_2^2\,dt .\end{aligned}$$

This completes the proof.

Corollary 2.1 and Lemma 5.1 lead directly to the result obtained earlier by Koski and Loges ([5], Proposition 1).

THEOREM 5.1. *Let A_1, A_2 be self-adjoint operators having the form*

$$A_1 e_n = -\lambda_n e_n, \quad A_2 e_n = -\beta_n e_n, \quad n \in \mathbb{N},$$

where $\{e_n\}$ is an orthonormal basis in H and $\{\lambda_n\}$, $\{\beta_n\}$ are sequences of positive numbers such that

$$\sum_{n=1}^{\infty} \lambda_n^{-1} < \infty, \quad \sum_{n=1}^{\infty} \beta_n^{-1} < \infty .$$

Then for each $T > 0$ and $x \in H$ the laws $\mathcal{L}(X_1(\ ,x))$ and $\mathcal{L}(X_2(\ ,x))$ are equivalent in $\mathcal{H}_T$ if and only if

$$\sum_{n=1}^{\infty} (\beta_n - \lambda_n)^2/\lambda_n < \infty . \tag{5.14}$$

Proof. (5.14) holds if and only if $D(A_1) = D(A_2)$ and $(A_2 - A_1) \circ (-A_1)^{-1/2}$ is a Hilbert–Schmidt operator on H. This completes the proof.

Now, suppose that A_1 and A_2 are uniformly elliptic differential operators of orders $2m_1$ and $2m_2$ on a bounded region $\mathcal{G} \subset \mathbf{R}^n$. We assume that the operators have infinitely differentiable coefficients in $\overline{\mathcal{G}}$, $\mathcal{G}$ is of class C^∞ (see [1], p. 128) and that their domains are given by

$$D(A_1) = H^{2m_1}(\mathcal{G}) \cap H_0^{m_1}(\mathcal{G}), \quad D(A_2) = H^{2m_2}(\mathcal{G}) \cap H_0^{m_2}(\mathcal{G}) .$$

LEMMA 5.2. (i) *$D(A_1) = D(A_2)$ iff $m_1 = m = m_2$.*

(ii) *The operators A_i, $i = 1, 2$, belong to $\mathcal{S}$.*

(iii) *For $i = 1, 2$ the operator A_i belongs to $\mathcal{U}$ iff $2m_i > n$.*

Proof. (i) is obvious, (ii) follows from the fact that for every elliptic operator $\mathcal{A}$ of order $2m$ one can choose an elliptic self-adjoint operator $\overline{\mathcal{A}}$ of order $2m$ such that the order of $\mathcal{A} - \overline{\mathcal{A}}$ is less than $2m - 1$ (see [1]) and it is easy to verify that there exists $p > 1$ such that $\mathcal{A} - \overline{\mathcal{A}} \in \mathcal{P}_p(\overline{\mathcal{A}})$. Part (iii) follows from: Theorem 3.2, statement (ii) and the fact that each self-adjoint elliptic differential operator of order $2m$ on a $\mathcal{G} \subset \mathbf{R}^n$ has a pure point spectrum $\{-\lambda_k, k \in \mathbb{N}\}$ such that $\lambda_k \approx ck^{2m/n}$ as k converges to ∞ (see [1], Sec. 14).

The following theorem is a special case of Theorem 2.4.

THEOREM 5.2. *Suppose that the orders $2m_1$ and $2m_2$ of the operators A_1 and A_2 are greater than n. Then for all $T > 0$ and for all initial values $x \in L^2(\mathcal{G})$ the laws of solutions of the stochastic partial differential equations*

$$dX_1 = A_1 X_1 dt + dW, \quad X_1(0) = x ,$$
$$dX_2 = A_2 X_2 dt + dW, \quad X_2(0) = x ,$$

are equivalent in $L^2(0, T; L^2(\mathcal{G}))$ if and only if $A_2 - A_1$ belongs to $\mathcal{R}(A_1)$.

The remark below is a consequence of Theorem 3.2, Lemmas 5.1 and 5.2 and Theorem 13.5 from [1].

Remark 5.1. (i) If the order of $A_2 - A_1$ is less than $m - n/2$ then $A_2 - A_1$ belongs to $\mathcal{R}(A_1)$.

(ii) If $A_2 - A_1$ belongs to $\mathcal{R}(A_1)$, then the order of $A_2 - A_1$ is less than m.

Kozlov ([7], Theorem 4, p. 161) proved that if A_1 and A_2 are elliptic and self-adjoint operators on a smooth compact manifold $\mathcal{M}$ without boundary, then the laws $\mathcal{L}(X_1)$ and $\mathcal{L}(X_2)$ are equivalent if and only if the order of

$A_2 - A_1$ is less than $m - n/2$, where $2m$ is the order of A_1 and A_2 and n is the dimension of $\mathcal{M}$. Using Theorem 2.4, Theorem 3.2(ii) and the same arguments as in the proof of Lemma 5.2(ii) one can generalize this result to the class of all elliptic operators on $\mathcal{M}$. However, for arbitrary elliptic operators the problem whether $A_2 - A_1 \in \mathcal{R}(A_1)$ if and only if the order of $A_2 - A_1$ is less than $m - n/2$ is open.

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The Littlewood–Paley function and φ-transform characterizations of a new Hardy space HK_2 associated with the Herz space

by

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Abstract. We give a Littlewood–Paley function characterization of a new Hardy space HK_2 and its φ-transform characterizations in M. Frazier & B. Jawerth's sense.

§ 0. Introduction. In [8] we have introduced some new Hardy spaces HK_p associated with the Herz spaces K_p, where $1 < p < \infty$. More importantly, we have established the atomic and molecular structural theorems for HK_p, $1 < p < \infty$. In §1 of this paper, we present a Littlewood–Paley function characterization of HK_2. In §2, using the atomic and molecular character of HK_2 and the characterization of a special "tent space" TK_2 introduced in [8], we give the φ-transform characterization of HK_2 in Frazier & Jawerth's sense (see [3] or [4]).

§ 1. The Littlewood–Paley function characterization of HK_2. Let $Q_k = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n : |x_i| \le 2^k\}$, $C_k = Q_k \setminus Q_{k-1}$, and $\chi_k = \chi_{C_k}$, $k \in \mathbf{Z}$. The following definitions are given in [8] and [6].

DEFINITION 1.1. Suppose $1 < p < \infty$, $1/p + 1/p' = 1$. The *Herz space* K_p consists of those functions $f \in L^p_{\mathrm{loc}}(\mathbf{R}^n \setminus \{0\})$ for which

$$\|f\|_{K_p} := \sum_{k=-\infty}^{\infty} 2^{(k+1)n/p'} \|f\chi_k\|_p < \infty.$$

DEFINITION 1.2. Let $1 < p < \infty$. A function $a(x)$ defined on $\mathbf{R}^n$ is said to be a *central symmetry* $(1,p)$-*atom* if

(1) $\operatorname{supp} a \subset Q$, where Q is a cube centered at the origin,
(2) $\int a(x)\,dx = 0$,

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