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Cluster sets of analytic multivalued functions

by

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Abstract. Classical theorems about the cluster sets of holomorphic functions on the unit disc are extended to the more general setting of analytic multivalued functions, and examples are given to show that these extensions cannot be improved.

1. Introduction. In this paper we give analogues for a.m.v. functions of some classical theorems on the cluster sets of analytic and subharmonic functions.

Let $\kappa(\mathbf{C})$ denote the non-empty compact sets in $\mathbf{C}$. A function $K : U \to \kappa(\mathbf{C})$ on a domain $U \subset \mathbf{C}$ is said to be *analytic multivalued* (*a.m.v.*) if it is upper semicontinuous and if, for every open set $U_1 \subset U$ and every function φ which is plurisubharmonic on a neighbourhood of the graph of $K|_{U_1}$, we have that φ is subharmonic on U_1, where φ is defined by

$$\varphi(\lambda) = \sup\{\psi(\lambda, z) : z \in K(\lambda)\}.$$

In the case where $\psi(\lambda, z) = \log|z|$, we shall denote the subharmonic function $\exp\varphi$ by ϱ, so

$$\varrho(\lambda) = \sup\{|z| : z \in K(\lambda)\}.$$

The two simplest examples of a.m.v. functions are

$$K(\lambda) = \{f(\lambda)\},$$

where f is a single-valued holomorphic function on U, and, writing $\Delta(\omega, r)$ for the open disc with centre ω and radius r,

$$K(\lambda) = \overline{\Delta}(0, \exp u(\lambda)),$$

where u is a subharmonic function on U. This latter assertion is proved in Proposition 4.5 of [9].

Other examples include the spectrum of a holomorphic function $f : U \to A$, where A is a Banach algebra. This was shown to be a.m.v. by Słodkowski in [11], a paper that gave rise to the modern interest in a.m.v. functions.

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Słodkowski also applied results for a.m.v. functions to the theory of uniform algebras.

In the construction of various counter-examples we shall draw upon the fact that an u.s.c. function K is a.m.v. if it has local holomorphic selections. This means that for every $\lambda_0 \in U$ and every $z_0 \in \partial K(\lambda_0)$ there exist an open neighbourhood U_1 of λ_0 in U and a holomorphic function f on U_1 such that $f(\lambda_0) = z_0$ and $f(\lambda) \in K(\lambda)$ for all $\lambda \in U_1$. We shall also construct, from a.m.v. functions K and L on U, new functions $K \cup L$ and $K + L$, where $K + L$ is defined by

$$(K+L)(\lambda) = \{z + \omega : z \in K(\lambda),\ \omega \in L(\lambda)\}.$$

The analyticity of the former function is clear from the definition, and the latter function is a.m.v. according to a theorem of Słodkowski in [11].

Many theorems of complex analysis, for example the Open Mapping Theorem, the Picard theorems and Rouché's theorem, have analogues for a.m.v. functions (see [9] and [10]); applying these versions to singleton-valued a.m.v. functions brings us back to the original theorems, and this will also be the case with several of the theorems proved here.

NOTATION. We abbreviate $\Delta(0,1)$ to Δ, and write $\mathbf{C}^\infty$ for $\mathbf{C} \cup \{\infty\}$. Let $K : \Delta \to \kappa(\mathbf{C})$, and suppose that $G \subset \Delta$ and $e^{i\theta} \in \overline{G}$.

The *cluster set* of K at $e^{i\theta}$ with respect to G, which we shall write as $K^G(e^{i\theta})$, is then the set of $z \in \mathbf{C}^\infty$ such that there exist a sequence (λ_n) in G with $\lambda_n \to e^{i\theta}$ and a sequence (z_n) such that $z_n \in K(\lambda_n)$ for each n and $z_n \to z$. Equivalently,

$$K^G(e^{i\theta}) = \bigcap_{r>0} \overline{\bigcup_{|\lambda' - e^{i\theta}|<r,\, \lambda' \in G} K(\lambda')}.$$

When K is a single-valued function, say $K(\lambda) = \{f(\lambda)\}$ for $\lambda \in \Delta$, we write $f^G(e^{i\theta})$ for the set $K^G(e^{i\theta})$.

We also define $K_G(e^{i\theta})$ to be the set of $z \in \mathbf{C}^\infty$ such that for every sequence (λ_n) in G with $\lambda_n \to e^{i\theta}$ there exists a sequence (z_n) such that $z_n \in K(\lambda_n)$ for each n and $z_n \to z$.

Thus, if $K(\lambda) = \{f(\lambda)\}$ is a single-valued function then the set $K_G(e^{i\theta})$, which we shall write as $f_G(e^{i\theta})$, is non-empty if and only if

$$\lim_{\lambda' \to e^{i\theta},\, \lambda' \in G} f(\lambda')$$

exists, and if this happens then $f_G(e^{i\theta}) = f^G(e^{i\theta})$.

When $G = \Delta$ we write $K^*(e^{i\theta})$ and $K_*(e^{i\theta})$ for $K^G(e^{i\theta})$ and $K_G(e^{i\theta})$, and when G is a curve γ ending at $e^{i\theta}$ we shall write $K^\gamma(e^{i\theta})$ and $K_\gamma(e^{i\theta})$. In particular, we write $K^{\mathrm{r}}(e^{i\theta})$ and $K_{\mathrm{r}}(e^{i\theta})$ for the *radial limit sets*, when γ is the radius of Δ ending at $e^{i\theta}$.

Fatou's theorem (see [5]) implies that a function $f \in H^\infty$ has radial limits almost everywhere. We begin this paper with an analogous result for a.m.v. functions, although we shall see that it is necessarily stated in terms of the polynomial hulls of the limit sets. Using this theorem and with some extra hypotheses, we proceed to prove a form of maximum principle for a.m.v. functions which links the values inside the disc with the radial limits of the function. We then look at two other famous theorems in this area, due to Lindelöf, and in Theorem 3 and the following examples we show to what extent these can be generalized to the set-valued case. Finally, we consider Bagemihl's theorem on the ambiguous points of a function defined on the unit disc, and we show that the same theorem holds true for a set-valued function.

2. Radial limit sets of an a.m.v. function. Littlewood in [7] proved that certain subharmonic functions on the unit disc resemble bounded analytic functions in having radial limits almost everywhere. The condition corresponding to boundedness becomes

$$\lim_{r\to 1} \int_0^{2\pi} u(re^{i\theta})\, d\theta < \infty$$

for the subharmonic function u.

Our first theorem is proved by applying this result to certain subharmonic functions obtained from K.

THEOREM 1. *Let $K : \Delta \to \kappa(\mathbf{C})$ be an a.m.v. function for which*

$$\lim_{r\to 1} \int_0^{2\pi} \log^+ \varrho(e^{i\theta})\, d\theta < \infty. \tag{1}$$

Then there is a set $A \subset \mathbf{T}$ of measure 2π such that, for all $e^{i\theta} \in A$,

$$[K^{\mathrm{r}}(e^{i\theta})]^\wedge = [K_{\mathrm{r}}(e^{i\theta})]^\wedge,$$

where the hats denote polynomially convex hulls.

Proof. Suppose that $p(z) = \sum_{m=1}^M a_m z^m$ is a polynomial, and for $\lambda \in \Delta$ define

$$u(\lambda) = \sup\{\log^+ |p(z)| : z \in K(\lambda)\}.$$

Then u is subharmonic on Δ, since K is a.m.v. there. Furthermore,

$$\log^+ |p(z)| \le \sum_{m=1}^M (\log^+ |a_m| + m \log^+ |z|) + \log M$$

$$= \log^+ |z| \sum_{m=1}^M m + \text{const},$$

so, for all $\lambda \in \Delta$,

$$u(\lambda) \le b(\sup_{z\in K(\lambda)} \log^+ |z|) + c \quad \text{for some constants } b \text{ and } c$$
$$= b\log^+ \varrho(\lambda) + c.$$

Hence

$$\lim_{r\to 1} \int_0^{2\pi} u(re^{i\theta})\, d\theta \le \lim_{r\to 1} \int_0^{2\pi} (b\log^+ \varrho(re^{i\theta}) + c)\, d\theta < \infty$$

by (1), and so, by Littlewood's theorem, u has radial limits a.e. on $\mathbb{T}$.

Let $\{p_n : n \in \mathbb{N}\}$ be the set of all polynomials with coefficients whose real and imaginary parts are rational, and for each n define the subharmonic function u_n as above. Put

$$A_n = \{e^{i\theta} \in \mathbb{T} : \lim_{r\to 1} u_n(re^{i\theta}) \text{ exists}\},$$

and define $A = \bigcap_{n\ge 1} A_n$, so A has measure 2π. Fix $e^{i\theta} \in A$. Then, in particular,

$$\lim_{r\to 1} \sup\{\log^+ |z| : z \in K(re^{i\theta})\}$$

exists, so we can find $d > 0$ and $R < 1$ such that $K(re^{i\theta}) \subset \Delta(0,d)$ for $r \ge R$.

Step 1. *Let F be a polynomially convex set such that there exists a sequence $(r_m) \to 1$ with $K(r_m e^{i\theta}) \subset F$ for all m. Then $K^{\mathrm{r}}(e^{i\theta}) \subset F$.*

Proof of Step 1. Suppose that, on the contrary, there exists $\zeta \in K^{\mathrm{r}}(e^{i\theta}) \setminus F$. In this case we can find a polynomial p which is one of the p_n and is such that $|p(\zeta)| > |p|_F > 1$. Let

$$\eta = |p(\zeta)| - |p|_F,$$

so $\eta > 0$. Since $\zeta \in K^{\mathrm{r}}(e^{i\theta})$, by definition there exist $s_j \to 1$ and a sequence (z_j) converging to ζ such that $z_j \in K(s_j e^{i\theta})$ for each j. We have $|p(z_j)| > |p(\zeta)| - \frac{1}{2}\eta = |p|_F + \frac{1}{2}\eta$ and so

$$u(s_j e^{i\theta}) = \sup\{\log^+ |p(z)| : z \in K(s_j e^{i\theta})\} > \log^+(|p|_F + \tfrac{1}{2}\eta)$$

for j sufficiently large. Hence

$$\limsup_{r\to 1} u(re^{i\theta}) \ge \log^+(|p|_F + \tfrac{1}{2}\eta). \tag{2}$$

Now, for all m, we have $K(r_m e^{i\theta}) \subset F$, so $|p(z)| \le |p|_F$ for all $z \in K(r_m e^{i\theta})$, and so $u(r_m e^{i\theta}) \le \log^+ |p|_F$. Thus,

$$\liminf_{r\to 1} u(re^{i\theta}) \le \log^+ |p|_F. \tag{3}$$

Inequalities (2) and (3) together imply that u has no radial limit at $e^{i\theta}$. But $e^{i\theta} \in A$ and p is p_n for some n, so u has a radial limit at $e^{i\theta}$; this gives the necessary contradiction.

Step 2. *Let U be an open set such that $K^{\mathrm{r}}(e^{i\theta}) \subset U$. Then there exists $R_1 < 1$ such that $K(re^{i\theta}) \subset U$ for all $r \ge R_1$.*

Proof of Step 2. Suppose, if possible, that there exist $r_j \to 1$ and a sequence (z_j) such that for each j we have $z_j \in K(r_j e^{i\theta})$ and $z_j \notin U$. For $r \ge R$ we have $K(re^{i\theta}) \subset \Delta(0,d)$, so, for j sufficiently large, $z_j \in \Delta(0,d) \setminus U$. Therefore (z_j) has a convergent subsequence, say $z_{j_k} \to z$. But then $z \in K^{\mathrm{r}}(e^{i\theta})$, and $z \notin U$, which is a contradiction.

Now suppose that $\omega \in \partial([K^{\mathrm{r}}(e^{i\theta})]^\wedge)$. We shall prove that $\omega \in K_{\mathrm{r}}(e^{i\theta})$ by showing that, given a sequence (r_j) converging to 1 and some $\varepsilon > 0$, we can find J such that for $j \ge J$ there exist $z_j \in K(r_j e^{i\theta})$ with $|z_j - \omega| < \varepsilon$. This will be achieved by constructing sets F and U with $U \setminus F \subseteq \Delta(\omega, \varepsilon)$, and applying Steps 1 and 2 to show that, for r sufficiently large,

$$K(re^{i\theta}) \not\subset F \quad \text{but} \quad K(re^{i\theta}) \subset U,$$

and so, since $r_j \to 1$, that there is some J such that

$$K(r_j e^{i\theta}) \not\subset F \quad \text{but} \quad K(r_j e^{i\theta}) \subset U$$

for $j \ge J$. This last statement implies that for $j \ge J$ there is some $z_j \in K(r_j e^{i\theta})$ with $z_j \in \Delta(\omega,\varepsilon)$ as required.

So suppose that $r_j \to 1$ and $\varepsilon > 0$ are given. Since $\omega \in \partial([K^{\mathrm{r}}(e^{i\theta})]^\wedge)$, we can, taking a smaller value of ε if necessary, find $a \notin [K^{\mathrm{r}}(e^{i\theta})]^\wedge$ with $|a - \omega| = \varepsilon$. The set $\mathbb{C} \setminus [K^{\mathrm{r}}(e^{i\theta})]^\wedge$ is connected, so there is a curve α mapping the interval $[0,1]$ into it such that $\alpha(0) = a$, $|\alpha(1)| = d$. As $\mathbb{C} \setminus [K^{\mathrm{r}}(e^{i\theta})]^\wedge$ is open, there is a connected open neighbourhood V of $\alpha([0,1])$ such that $\overline{V}$ does not intersect $[K^{\mathrm{r}}(e^{i\theta})]^\wedge$. Put

$$F = \overline{\Delta}(0,d) \setminus (V \cup \Delta(\omega,\varepsilon)), \quad U = \Delta(0,d) \setminus \overline{V}.$$

Then $\mathbb{C} \setminus F$ is connected (since $z \notin F \Rightarrow z \in V \cup \Delta(\omega,\varepsilon)$ or $|z| > d$), so F is polynomially convex and U is open. Also,

$$z \in U \setminus F \;\Rightarrow\; |z| < d \text{ and } z \notin \overline{V} \text{ and } z \in V \cup \Delta(\omega,\varepsilon)$$
$$\Rightarrow\; z \in \Delta(\omega,\varepsilon),$$

so $U \setminus F \subseteq \Delta(\omega,\varepsilon)$ as required.

Now $\omega \notin F$, so $[K^{\mathrm{r}}(e^{i\theta})]^\wedge \not\subset F$, and hence $K^{\mathrm{r}}(e^{i\theta}) \not\subset F$. Therefore, by Step 1, there exists $R_0 < 1$ such that $K(re^{i\theta}) \not\subset F$ for all $r \ge R_0$. Also, $z \in K^{\mathrm{r}}(e^{i\theta})$ implies that $|z| < d$ and $z \notin \overline{V}$, so $K^{\mathrm{r}}(e^{i\theta}) \subset U$. Thus, by Step 2, there is an $R_1 < 1$ such that if $r \ge R_1$ then $K(re^{i\theta}) \subset U$. So if $r \ge \max(R_0, R_1)$ then $K(re^{i\theta}) \not\subset F$ and $K(re^{i\theta}) \subset U$, and by our previous remarks we deduce that $\omega \in K_{\mathrm{r}}(e^{i\theta})$.

We have thus shown that $\partial([K^{\mathrm{r}}(e^{i\theta})]^\wedge) \subset K_{\mathrm{r}}(e^{i\theta})$, and combining this with the obvious inclusion $K_{\mathrm{r}}(e^{i\theta}) \subset K^{\mathrm{r}}(e^{i\theta})$ gives $[K^{\mathrm{r}}(e^{i\theta})]^\wedge = [K_{\mathrm{r}}(e^{i\theta})]^\wedge$ for $e^{i\theta} \in A$, which proves the theorem. ∎

We showed in this proof that $\partial([K^{\mathrm{r}}(e^{i\theta})]^\wedge) \subset K_{\mathrm{r}}(e^{i\theta})$ except possibly on a set of measure zero; however, it need not be the case that $\partial K^{\mathrm{r}}(e^{i\theta}) \subset K_{\mathrm{r}}(e^{i\theta})$ almost everywhere, as we shall see in the following example.

EXAMPLE. Bagemihl and Seidel in [2] construct a holomorphic function g on Δ whose radial cluster set $g^{\mathrm{r}}(e^{i\theta})$ is the unit circle $\mathbf{T}$ for almost every $e^{i\theta} \in \mathbf{T}$. Let $K : \Delta \to \kappa(\mathbf{C})$ be defined by

$$K(\lambda) = \begin{cases} 2\mathbf{T} \cup \{g(\lambda)\} & \text{for } |g(\lambda)| < 2, \\ 2\mathbf{T} & \text{for } |g(\lambda)| \geq 2. \end{cases}$$

Then K is a.m.v. on Δ, since it is u.s.c. and has local holomorphic selections, and it is bounded. However, for almost every $e^{i\theta} \in \mathbf{T}$,

$$K^{\mathrm{r}}(e^{i\theta}) = 2\mathbf{T} \cup \mathbf{T}, \quad \text{but} \quad K_{\mathrm{r}}(e^{i\theta}) = 2\mathbf{T}.$$

We next prove our maximum principle for a.m.v. functions on the unit disc, and we shall see that, in fact, the essential range of K_{r} is sufficient to determine the values of $K(\lambda)$ for $\lambda \in \Delta$.

We begin by defining the *essential range* of a function $L : \mathbf{T} \to \kappa(\mathbf{C})$:

$$\operatorname{ess\,range}(L) = \mathbf{C}^\infty \setminus \bigcup \{\text{open } V : m(\{\lambda : L(\lambda) \cap V \neq \emptyset\}) = 0\},$$

where m is Lebesgue measure on $\mathbf{T}$.

In the proof of Theorem 2 we shall want to apply the inequality

$$u(\lambda) \leq P[\limsup_{r\to 1} u(re^{i\theta})](\lambda) \quad \text{for all } |\lambda| < 1 \tag{4}$$

when u is a subharmonic function of the type used in the proof of Theorem 1, and this will mean that we have to strengthen the hypothesis on $\log^+ \varrho$ of Theorem 1 (here, for a function f such that $f^+ \in L^1(\mathbf{T})$, we denote the Poisson integral of f by $P[f]$). Dahlberg in [4] proves that the following three conditions on a subharmonic function u defined in Δ are sufficient for inequality (4) to hold:

$$\limsup_{r\to 1} u(re^{i\theta}) < \infty \quad \text{for all } e^{i\theta} \in \mathbf{T}; \tag{5}$$

(6) there is a $g \in L^1(\mathbf{T})$ such that $\liminf_{r\to 1} u(re^{i\theta}) \leq g(e^{i\theta})$ a.e. on $\mathbf{T}$;

$$\max\{u(re^{i\theta}) : \theta \in [0, 2\pi)\} = o[(1-r)^{-2}] \quad \text{as } r \to 1. \tag{7}$$

To simplify the notation of this theorem, the a.m.v. function K, defined on the open unit disc, is extended to be u.s.c. on the closure of the disc by writing $K^*(e^{i\theta})$ as $K(e^{i\theta})$ for $e^{i\theta} \in \mathbf{T}$.

THEOREM 2. *Suppose that $K : \Delta \to \kappa(\mathbf{C})$ is a.m.v., and that $\log^+ \varrho$ satisfies conditions* (5)–(7) *above. Then*

$$[K(\overline{\Delta})]^\wedge = [\operatorname{ess\,range}(K_{\mathrm{r}})]^\wedge.$$

Proof. We need only prove that if $A \subset \mathbf{T}$ has measure 2π then

$$[K(\overline{\Delta})]^\wedge \subseteq \Big[\bigcup_{e^{i\theta}\in A} K_{\mathrm{r}}(e^{i\theta})\Big]^\wedge,$$

since clearly $[K(\overline{\Delta})]^\wedge \supset [\operatorname{ess\,range}(K_{\mathrm{r}})]^\wedge$. Accordingly, suppose that $A \supset \mathbf{T}$ has measure 2π. Then

$$\begin{aligned}\Big[\bigcup_{e^{i\theta}\in A} K_{\mathrm{r}}(e^{i\theta})\Big]^\wedge &= \Big[\bigcup_{e^{i\theta}\in A} [K_{\mathrm{r}}(e^{i\theta})]^\wedge\Big]^\wedge \\ &= \Big[\bigcup_{e^{i\theta}\in A'} [K^{\mathrm{r}}(e^{i\theta})]^\wedge\Big]^\wedge \\ &= \widehat{F}, \quad \text{say,}\end{aligned}$$

where $A' \subset A$ also has measure 2π, by Theorem 1.

Suppose that $\zeta \notin \widehat{F}$, so there is some polynomial p such that

$$|p(\zeta)| > |p|_F > 1.$$

For any $e^{i\theta} \in A'$ and any open set $U \supset \widehat{F}$, there exists $r_0 < 1$ such that $r \geq r_0$ implies that $K(re^{i\theta}) \subset U$. So, given $\varepsilon > 0$ such that $\varepsilon < |p(\zeta)| - |p|_F$ and $e^{i\theta} \in A'$, we can find r_0 such that for every $r \geq r_0$ we have $|p(z)| < |p(\zeta)| - \varepsilon$ for all $z \in K(re^{i\theta})$. Define the subharmonic function u by

$$u(\lambda) = \sup\{\log^+ |p(z)| : z \in K(\lambda)\}.$$

Then, for $r \geq r_0$, $u(re^{i\theta}) < \log^+(|p(\zeta)| - \varepsilon)$, so

$$\limsup_{r\to 1} u(re^{i\theta}) \leq \log^+(|p(\zeta)| - \varepsilon).$$

Note that, since $\log^+ \varrho$ fulfils conditions (5)–(7) of Dahlberg's theorem, so does u, by virtue of the relationship $u(\lambda) \leq b \log^+ \varrho(\lambda) + c$ for some constants b and c, which we obtained in Theorem 1. Thus for any $se^{i\varphi} \in \Delta$, writing $P(se^{i\varphi}, e^{i\theta})$ for the Poisson kernel, we have

$$\begin{aligned}u(se^{i\varphi}) &\leq \int_{\mathbf{T}} P(se^{i\varphi}, e^{i\theta}) \limsup_{r\to 1} u(re^{i\theta}) \frac{d\theta}{2\pi} \\ &= \int_{A'} P(se^{i\varphi}, e^{i\theta}) \limsup_{r\to 1} u(re^{i\theta}) \frac{d\theta}{2\pi} \leq \log^+(|p(\zeta)| - \varepsilon).\end{aligned}$$

Therefore, for all $e^{i\theta} \in \mathbf{T}$,

$$\limsup_{\lambda\to e^{i\theta}} u(\lambda) < \log^+ |p(\zeta)|,$$

so $|p(z)| < |p(\zeta)|$ for all $z \in K(\mathrm{T})$, and thus, by the maximum principle applied to u, $|p(z)| < |p(\zeta)|$ for all $z \in K(\overline{\Delta})$. We conclude that $\zeta \notin [K(\overline{\Delta})]^\wedge$, and the theorem is proved. ∎

3. Analogues of Lindelöf's theorems. Our objective in this section is to prove extensions of Lindelöf's theorems (see [6]) to bounded a.m.v. functions, using a method suggested by Nevanlinna's proof in [8]. We suppose that a domain G in Δ is given and is such that $\overline{G} \cap \mathrm{T} = \{e^{i\theta}\}$ for some $e^{i\theta} \in \mathrm{T}$, and that $\partial G = \alpha_1 \cup \alpha_2$, where α_1 and α_2 are Jordan arcs meeting at $e^{i\theta}$. Theorems L1 and L2 are both due to Lindelöf.

THEOREM L1. *Suppose f is bounded and analytic on Δ, that the domain G is as above, and that*

$$\lim_{\lambda \to e^{i\theta}, \lambda \in \alpha_1} f(\lambda) = a \quad \text{and} \quad \lim_{\lambda \to e^{i\theta}, \lambda \in \alpha_2} f(\lambda) = b .$$

Then $a = b$ and $\lim_{\lambda \to e^{i\theta}, \lambda \in G} f(\lambda) = a$.

THEOREM L2. *Suppose that f is a bounded analytic function on Δ, that G is as above, and that $\lim_{\lambda \to e^{i\theta}, \lambda \in \alpha_1} f(\lambda)$ exists. Then $\lim_{r \to 1} f(re^{i\theta})$ exists and takes the same value.*

In Theorem 3 and in the examples following it, we abuse our previous notation by writing K^S for $K^S(e^{i\theta})$, where S is any set in Δ such that $e^{i\theta} \in \overline{S}$; this will cause no confusion, since $e^{i\theta}$ is fixed.

THEOREM 3. *Let $K : \Delta \to \kappa(\mathbf{C})$ be a.m.v. and bounded on a neighbourhood N of G in Δ, and let $A = \widehat{K^{\alpha_1}} \cup \widehat{K^{\alpha_2}}$. Then:*

(i) *for each component C of $\widehat{A}$ we have $C \cap \widehat{K^{\alpha_j}} \neq \emptyset$ for $j = 1, 2$;*

(ii) *the inclusion $K^G \subset \widehat{A}$ holds;*

(iii) *for each component B of $\widehat{A} \setminus A$, either $B \subset K^G$ or $B \cap K^G = \emptyset$.*

Proof. Since K is bounded on N, there is a bounded open set U such that $\widehat{K(N)}$ is relatively compact in U. We begin the proof by supposing that V is some open, simply connected set such that $\widehat{A} \subset V \Subset U$, and we show how a subharmonic function φ, non-positive on G and depending on V, can be constructed. We then use such a function φ to prove (i) and (ii) separately, and subsequently apply (ii) to prove (iii).

So suppose that V is a set as described above. Then there is an $r < 1$ such that $0 < |e^{i\theta} - \zeta| \le r$ and $\zeta \in \partial G$ imply that $K(\zeta) \subset V$. For $\lambda \in G$, put

$$\theta(\lambda) = \omega_G(\partial G \cap \{|e^{i\theta} - \zeta| < r\}, \lambda) ,$$

the value at λ of the harmonic function on G taking the values 1 on $\partial G \cap \{|e^{i\theta} - \zeta| < r\}$ and 0 on $\partial G \cap \{|e^{i\theta} - \zeta| > r\}$. Define $\nu : U \to [-\infty, \infty)$ by

$$\nu(z) = \begin{cases} -\omega_{U \setminus \overline{V}}(\partial V, z) & \text{for } z \in U \setminus \overline{V}, \\ -1 & \text{for } z \in \overline{V}, \end{cases}$$

so ν is subharmonic on U. Finally, define φ by

$$\varphi(\lambda) = \sup\{\theta(\lambda) + \nu(z) : z \in K(\lambda)\} ,$$

so φ is a subharmonic function on G.

Suppose that $\zeta \in \partial G \setminus \{e^{i\theta}\}$ and $|\zeta - e^{i\theta}| \le r$. Then $K(\zeta) \subset V$, so $\nu \equiv -1$ in $K(\zeta)$, and so, since $\theta(\lambda) \le 1$ for all λ,

$$\limsup_{\lambda \to \zeta, \lambda \in G} \varphi(\lambda) \le 0 .$$

If $\zeta \in \partial G$ and $|\zeta - e^{i\theta}| > r$ then

$$\limsup_{\lambda \to \zeta, \lambda \in G} \theta(\lambda) = 0 ,$$

and so, since $\nu \le 0$, we again have

$$\limsup_{\lambda \to \zeta, \lambda \in G} \varphi(\lambda) \le 0 .$$

Therefore, by the (extended) maximum principle, $\varphi \le 0$ on G.

Proof of (i). Suppose for a contradiction that there is some component C of $\widehat{A}$ such that $C \cap \widehat{K^{\alpha_1}} = \emptyset$. Then we can find an open set V as above, and a set $W = W_1 \cup W_2$ such that W is simply connected, $V \Subset W \subset U$, the sets W_1 and W_2 are open and disjoint, $\widehat{K^{\alpha_1}}$ and W_1 are disjoint, and $C \subset W_1$. Let φ be the function constructed as above with this V. For any arc l in G, define

$$m(l) = \inf\{\theta(\lambda) : \lambda \in l\} , \qquad p(l) = \inf\{-\nu(z) : z \in K(l)\} .$$

Then, as $\theta(\lambda) + \nu(z) \le 0$ for all $z \in K(\lambda)$ and all $\lambda \in l$, we have $m(l) + \nu(z) \le 0$ for all $z \in K(l)$, so

$$m(l) \le p(l) \le 1 . \tag{8}$$

Now $\widehat{K^{\alpha_1}} \cap C = \emptyset$, and $C \subset \widehat{A}$, so $\widehat{K^{\alpha_2}} \cap C \neq \emptyset$, and hence, as C is polynomially convex and cannot be a proper subset of a component of $\widehat{K^{\alpha_2}}$, the set $K^{\alpha_2} \cap C$ is non-empty. Therefore there exist $\lambda_n \to e^{i\theta}$, with each $\lambda_n \in \alpha_2$, such that for all n the sets $K(\lambda_n)$ and W_1 have non-empty intersection. Choose $\mu_n \to e^{i\theta}$ in α_1 such that for each n there is an arc l_n joining λ_n to μ_n inside G, and such that $m(l_n) \to 1$. Then $p(l_n) \to 1$ by (8).

There exists $\varepsilon > 0$ such that $\nu(z) < -1 + \varepsilon$ implies that $z \in W$; for n sufficiently large, $p(l_n) > 1 - \varepsilon$ so $\nu(z) < -1 + \varepsilon$ for all $z \in K(l_n)$, and thus $K(l_n) \subset W$. By upper semicontinuity there are open connected sets

$H_n \subset G$ with $l_n \subset H_n$ and $K(H_n) \subset W$. Since $\widehat{K^{\alpha_1}} \cap W_1 = \emptyset$, we see that $K(\mu_n) \subset W_2$ for n sufficiently large. But, by Theorem 1.6 of [9],

$$K(\mu_n) \subset W_2 \Rightarrow K(\lambda) \subset W_2 \text{ for all } \lambda \in H_n \Rightarrow K(\lambda_n) \subset W_2,$$

which contradicts the assertion that $K(\lambda_n) \cap W_1 \neq \emptyset$ for all n.

Proof of (ii). Suppose that we are given $V \subset U$ as above, with φ the corresponding subharmonic function, and that W is an open simply connected subset of U such that $V \Subset W$. We can find $\varepsilon > 0$ such that $\nu(z) < -1 + \varepsilon$ implies that $z \in W$. Since

$$\lim_{\lambda \to e^{i\theta}, \lambda \in G} \theta(\lambda) = 1,$$

we can find a corresponding $\delta > 0$ such that $|\lambda - e^{i\theta}| < \delta$ and $\lambda \in G$ imply that $\theta(\lambda) > 1 - \varepsilon$. But $\theta(\lambda) \leq -\nu(z)$ for all $z \in K(\lambda)$ (since $\varphi(\lambda) \leq 0$), so

$$\theta(\lambda) > 1 - \varepsilon \Rightarrow \nu(z) < -1 + \varepsilon \Rightarrow z \in W \text{ for all } z \in K(\lambda).$$

Thus we have found δ such that $|\lambda - e^{i\theta}| < \delta$ and $\lambda \in G$ imply that $K(\lambda) \subset W$. We can do this for every W such that $V \Subset W$, and so we deduce that $K^G \subset V$. This holds for all V such that $\widehat{A} \Subset U$, so we deduce that $K^G \subset \widehat{A}$.

Proof of (iii). Let B be a component of $\widehat{A} \setminus A$, and suppose that $a \in B$ and $a \notin K^G$. Then there is some neighbourhood of a which does not intersect $A \cup K^G$, and so the function L defined by

$$L(\lambda) = \{1/(z-a) : z \in K(\lambda)\}$$

is a.m.v. and bounded on a neighbourhood of G in Δ. We apply (ii) to the function L. Since B is mapped to the unbounded component of $\widehat{L^{\alpha_1}} \cup \widehat{L^{\alpha_2}}$, we deduce that the intersection of B and K^G is empty. ∎

Theorem 3 includes Lindelöf's first theorem as a special case. Suppose that K is a single-valued function, say $K(\lambda) = \{f(\lambda)\}$ for $\lambda \in \Delta$, and that the hypotheses of Theorem L1 are satisfied. Then $A = \{a\} \cup \{b\}$, and (i) implies that $a = b$, which in turn implies the second conclusion of Lindelöf's theorem, by (ii). However, Theorem L2 cannot be deduced from Theorem 3. We shall see in Example 3 below that Theorem L2 is not generally true for a.m.v. functions. The question of whether $\widehat{A}$ is ever strictly larger than A remains open, but we now give some examples to show that it is not possible to improve certain other aspects of Theorem 3.

Example 1. Our first example tells us that the conclusion of part (i) of Theorem 3 cannot be extended from intersections to inclusions. We describe a case in which A is polynomially convex and connected, but neither $A \subset \widehat{K^{\alpha_1}}$ nor $A \subset \widehat{K^{\alpha_2}}$ holds.

Let $G \subset \Delta$ be the triangular domain bounded by

$$\alpha_1(t) = \tfrac{1}{2}(1 + i + (1-i)t), \quad t \in [0,1],$$
$$\alpha_2(t) = \begin{cases} \tfrac{1}{2}(1+i)(1-t), & t \in [0,1], \\ t - 1, & t \in [1,2], \end{cases}$$

so $e^{i\theta} = 1$.

Let $f_1 : \mathbf{T} \to \mathbf{C}$ be defined by $f_1(e^{i\beta}) = \beta/\pi$, $\beta \in [0, 2\pi)$. Then f_1 is discontinuous at 1: $\lim_{\beta \to 0+} f_1(e^{i\beta}) = 0$, while $\lim_{\beta \to 2\pi-} f_1(e^{i\beta}) = 2$. Let h_1 be the harmonic extension of f_1 to Δ, so

$$\lim_{\lambda \to 1, \lambda \in \alpha_1} h_1(\lambda) = 1/2, \quad \lim_{r \to 1} h_1(r) = 1.$$

Similarly, defining $f_2 : \mathbf{T} \to \mathbf{C}$ by $f_2(e^{i\beta}) = (2\pi - \beta)/\pi$, and h_2 to be the harmonic extension of f_2 to Δ, we obtain

$$\lim_{\lambda \to 1, \lambda \in \alpha_1} h_2(\lambda) = 3/2, \quad \lim_{r \to 1} h_2(r) = 1.$$

Now set $K(\lambda) = \overline{\Delta}(-e^{1/2}, e^{h_1(\lambda)}) \cup \overline{\Delta}(-e^{3/2}, e^{h_2(\lambda)})$, so K is a.m.v. We have

$$K^{\alpha_1} = \overline{\Delta}(-e^{1/2}, e^{1/2}) \cup \overline{\Delta}(-e^{3/2}, e^{3/2}),$$

while

$$K^{\alpha_2} = K^{\mathrm{r}} = \overline{\Delta}(-e^{1/2}, e^1) \cup \overline{\Delta}(-e^{3/2}, e^1).$$

Example 2. We show that part (i) of Theorem 3 does not hold for K_{α_1} and K_{α_2} by constructing an example in which K_{α_1} and K_{α_2} are non-empty, but $\widehat{K_{\alpha_1}} \cap \widehat{K_{\alpha_2}} = \emptyset$.

We let G be the semicircular domain bounded by

$$\alpha_1 = \{|\lambda - 1/2| = 1/2, \operatorname{Im} \lambda < 0\}, \quad \alpha_2(t) = t, \quad t \in [0,1],$$

so again $e^{i\theta} = 1$ and $K_{\alpha_2} = K_{\mathrm{r}}$. The function K will be the union of the a.m.v. functions L and H defined below.

Choose M with $64 < M < 80$, and define k by $\pi k = \log(M/4)$. Then $e^{\pi k} < M/2$, and $2M > e^{3\pi k/2} = M(M/64)^{1/2} > M$, as is easily checked. Put

$$L(\lambda) = \left\{ \exp\left(\frac{\lambda+1}{\lambda-1}\right) + 6M \right\}.$$

Then

$$L^{\alpha_1} = \{|z - 6M| = e^{-1}\}, \quad L_{\alpha_1} = \emptyset, \quad L_{\mathrm{r}} = L^{\mathrm{r}} = \{6M\}.$$

Now choose $g \in H^\infty$ such that $\liminf_{r \to 1} |g(r)| = 0$, $\limsup_{r \to 1} |g(r)| = M$, and $|g| < M$ on Δ (see e.g. [3, p. 24] for a construction of such a function).

Define f on $\mathbf{T}$ by $f(e^{i\beta}) = \beta k$, and let h be the harmonic extension of f to the disc. Then $h(\lambda) = k(\pi + 2\arg(1-\lambda))$; for $\lambda \in \alpha_1$ and λ near enough to 1, $\arg(1-\lambda) > \pi/4$, so $h(\lambda) > 3\pi k/2$, while on the other hand $\lim_{r\to 1} h(r) = \pi k$.

Define the a.m.v. function $H : \Delta \to \kappa(\mathbf{C})$ by

$$H(\lambda) = \overline{\Delta}(g(\lambda), e^{h(\lambda)}).$$

Then we claim that (a) $H_{\mathfrak{r}} = \emptyset$, but (b) $0 \in H_{\alpha_1}$.

Proof of (a). Choose $r_n \to 1$ such that $|g(r_n)| \to 0$. Then $h(r_n) \to \pi k$ so, for n large enough, $|z| < M/2$ for all $z \in H(r_n)$. Choose $s_j \to 1$ such that $|g(s_j)| \to M$. Then, if j is large enough, $|z| > M/2$ for all $z \in H(s_j)$.

Thus we have $H(r_n) \cap H(s_j) = \emptyset$ for n, j large enough, and so (a) holds.

Proof of (b). Suppose $\lambda_n \to 1$ with $(\lambda_n) \subset \alpha_1$. Then, for n sufficiently large, $\overline{\Delta}(g(\lambda_n), e^{3\pi k/2}) \subset H(\lambda_n)$. Now $|g(\lambda_n)| < M$ for all n, and $e^{3\pi k/2} > M$, so $0 \in H(\lambda_n)$ for n sufficiently large, and thus $0 \in H_{\alpha_1}$.

We can now put $K(\lambda) = L(\lambda) \cup H(\lambda)$ for $\lambda \in \Delta$. When r is sufficiently large we have $H(r) \subset \Delta(0, 2M)$ and

$$\left|\exp\left(\frac{r+1}{r-1}\right) + 6M\right| > 5M,$$

so the sets $H(r)$ and $L(r)$ are disjoint and thus $K_{\mathfrak{r}} = L_{\mathfrak{r}} = \{6M\}$. Also, $0 \in K_{\alpha_1}$, since $0 \in H_{\alpha_1}$, so K_{α_1} is non-empty. Lastly, $|\omega| < 6M$ for all $\omega \in K_{\alpha_1}$, since L_{α_1} is empty and $|z| < M + 5M$ for all $z \in H(\Delta)$. So neither $K_{\mathfrak{r}}$ nor K_{α_1} is empty, but $\widehat{K_{\alpha_1}} \cap \widehat{K_{\mathfrak{r}}} = \emptyset$.

Example 3. We end this section by proving that Lindelöf's second theorem does not hold for subharmonic functions; the counter-example we obtain can be immediately extended to give an a.m.v. function K for which $\widehat{K^{\alpha_1}} = \widehat{K_{\alpha_1}}$, but $\widehat{K^{\mathfrak{r}}} \neq \widehat{K_{\mathfrak{r}}}$.

We first construct a subharmonic function u for which Theorem L2 fails, taking G to be the same domain as in Example 2.

For $\lambda \in \Delta$ put $u(\lambda) = \log|g(\lambda)| \vee \log|f(\lambda)| \vee (-M)$, where $M > 0$, and $g, f \in H^\infty$ are such that

$$g(\lambda) = k\exp\left(\frac{\lambda+1}{\lambda-1}\right) \quad \text{for some } k > e,$$

and

$$|f| < t \text{ on } \Delta, \quad \liminf_{r\to 1} |f(r)| = 0, \quad \limsup_{r\to 1} |f(r)| = t$$

for some t such that

$$-M < \log t < \log(k/e).$$

On α_1 we have $|g| = ke^{-1}$, so $\log|g| > (\log|f| \vee -M)$, so $u \equiv \log(k/e)$.

Radially, $|g| \to 0$, so, for r sufficiently large, $\log|g(r)| < -M$. For all x in the interval $[0, t]$, there exist $r_n \to 1$ such that $|f(r_n)| \to x$. If $x > e^{-M}$,

then, for n sufficiently large, $\log|f(r_n)| > -M$, and so $u(r_n) = \log|f(r_n)|$. Thus for each $x \in (e^{-M}, t]$ there is some sequence $r_n \to 1$ such that $u(r_n) \to \log x$, and so we have

$$\liminf_{r\to 1} u(r) = -M, \quad \limsup_{r\to 1} u(r) = \log t.$$

Defining $K : \Delta \to \kappa(\mathbf{C})$ by $K(\lambda) = \overline{\Delta}(0, e^{u(\lambda)})$ gives the promised counter-example, since

$$\widehat{K^{\alpha_1}} = \widehat{K_{\alpha_1}} = \overline{\Delta}(0, k/e),$$

but

$$K^{\mathfrak{r}} = \overline{\Delta}(0, t), \quad K_{\mathfrak{r}} = \overline{\Delta}(0, e^{-M}).$$

Furthermore, if we change the domain G by letting α_2 be the reflection of α_1 in the imaginary axis, and let K be the a.m.v. function which was defined above, then we have $\widehat{K^{\alpha_1}} = \widehat{K_{\alpha_1}} = \widehat{K^{\alpha_2}} = \widehat{K_{\alpha_2}}$, but $\widehat{K^{\mathfrak{r}}} \neq \widehat{K_{\mathfrak{r}}}$.

4. Ambiguous points of set-valued functions.

Having seen that even for a bounded a.m.v. function we might not get very good behaviour when a point on the boundary of the disc is approached along two different curves, we now turn to Bagemihl's theorem and discover that all set-valued functions are subject to some constraints on their cluster sets. We first extend a definition made by Bagemihl for a single-valued function to the set-valued case.

Definition. A point $e^{i\theta} \in \mathbf{T}$ is an *ambiguous point* of the set-valued function $K : \Delta \to \kappa(\mathbf{C})$ if there are curves γ_1, γ_2 ending at $e^{i\theta}$ for which

$$K^{\gamma_1}(e^{i\theta}) \cap K^{\gamma_2}(e^{i\theta}) = \emptyset.$$

Bagemihl in [1] proved that the set of ambiguous points of any function $f : \Delta \to \mathbf{C}$ is at most countable, and we show in the first half of our next theorem, using his method, that this remains true for a set-valued function. We need the following lemma, which is also due to Bagemihl.

Lemma. *Let $S \subset \Delta$. Then there is a set $A \subset \mathbf{T}$, with $\mathbf{T} \setminus A$ at most countable, such that for every $e^{i\theta} \in A$, if γ_1, γ_2 are simple curves ending at $e^{i\theta}$ then either γ_1, γ_2 both intersect S or they both intersect $\Delta \setminus S$.*

Theorem 4. *Let $K : \Delta \to \kappa(\mathbf{C})$ be an arbitrary set-valued function. Then there is a set $A \subset \mathbf{T}$, with $\mathbf{T} \setminus A$ at most countable, such that if $e^{i\theta} \in A$ and γ_1, γ_2 are simple curves ending at $e^{i\theta}$ then*

(i) *the intersection $K^{\gamma_1}(e^{i\theta}) \cap K^{\gamma_2}(e^{i\theta})$ is non-empty;*
(ii) *for $j = 1, 2$ we have $K_{\gamma_j}(e^{i\theta}) \subset K^{\gamma_1}(e^{i\theta}) \cap K^{\gamma_2}(e^{i\theta})$.*

Proof. For each $m \geq 0$ and $k \geq 1$, put

$$V_{mk} = \begin{cases} \{z : |z| > k\} & \text{if } m = 0, \\ \Delta(\lambda_m, 1/k) & \text{if } m > 0, \end{cases}$$

where $\{\lambda_m : m > 0\}$ is the set of points with rational coordinates. Then let $\{U_n\}$ be the countable collection of sets which are unions of finitely many V_{mk}. Define

$$S_n = \{\lambda \in \Delta : K(\lambda) \cap U_n \neq \emptyset\}.$$

Then for each n we can find a set A_n corresponding to S_n as in the lemma, and if we let $A = \bigcap_{n \geq 1} A_n$ then $\mathrm{T} \setminus A$ is at most countable. Fix $e^{i\theta} \in A$, and let γ_1 and γ_2 be simple curves ending at $e^{i\theta}$.

Proof of (i). Suppose that $K^{\gamma_1}(e^{i\theta}) \cap K^{\gamma_2}(e^{i\theta}) = \emptyset$. Then, for some n, $K^{\gamma_1}(e^{i\theta}) \subset U_n$ and $K^{\gamma_2}(e^{i\theta}) \subset \mathbb{C}^\infty \setminus \overline{U}_n$. Hence there exist t_1, t_2 such that

$$K(\gamma_1(t)) \subset U_n \quad (t \geq t_1), \qquad K(\gamma_2(t)) \subset \mathbb{C}^\infty \setminus \overline{U}_n \quad (t \geq t_2).$$

But then $\{\gamma_1(t) : t \geq t_1\} \subset S_n$, and $\{\gamma_2(t) : t \geq t_2\} \subset \Delta \setminus S_n$, which contradicts the hypothesis that $e^{i\theta} \in A$.

Proof of (ii). Clearly $K_{\gamma_j}(e^{i\theta}) \subset K^{\gamma_j}(e^{i\theta})$ for $j = 1, 2$. Now suppose that $\omega \in K_{\gamma_1}(e^{i\theta})$; we shall prove that $\omega \in K^{\gamma_2}(e^{i\theta})$ by showing that given $\varepsilon > 0$ there exist sequences (s_m) with $s_m \to 1$ and (z_m) with $z_m \in K(\gamma_2(s_m))$ and $|z_m - \omega| < \varepsilon$ for each m.

Choose a subset $\{U_{n_m}\}$ of the $\{U_n\}$ such that $\omega \in U_{n_m} \subset \Delta(\omega, 1/m)$ for each m. Then, since $\omega \in K_{\gamma_1}(e^{i\theta})$, for each m we can find t_m such that if $t \geq t_m$ then

$$U_{n_m} \cap K(\gamma_1(t)) \neq \emptyset,$$

or equivalently $\{\gamma_1(t) : t \geq t_m\} \subset S_{n_m}$. Since $e^{i\theta} \in A$, for every m and every T with $0 < T < 1$ we have

$$\{\gamma_2(t) : t \geq T\} \cap S_{n_m} \neq \emptyset.$$

Thus for each m there is a sequence (t_{m_q}) converging to 1 such that for every q we can find

$$z_{m_q} \in K(\gamma_2(t_{m_q})) \cap \Delta(\omega, 1/m).$$

Now, for each m, choose $s_m = t_{m_Q}$ for some Q in such a way that the s_m form an increasing sequence, and choose M such that $1/M < \varepsilon$. Then for all $m \geq M$ we have $z_{m_Q} \in K(\gamma_2(s_m)) \cap \Delta(\omega, 1/m)$, and so $z_{m_Q} \in K(\gamma_2(s_m)) \cap \Delta(\omega, \varepsilon)$. ∎

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