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Weighted norm inequalities on spaces of homogeneous type

by

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Abstract. We give a characterization of the weights (u,w) for which the Hardy–Littlewood maximal operator is bounded from the Orlicz space $L_\Phi(u)$ to $L_\Phi(w)$. We give a characterization of the weight functions w (respectively u) for which there exists a nontrivial u (respectively $w>0$ almost everywhere) such that the Hardy–Littlewood maximal operator is bounded from the Orlicz space $L_\Phi(u)$ to $L_\Phi(w)$.

1. Preliminaries and main results. The main objective of this paper is to study weight pairs (u,w) for which the Hardy–Littlewood maximal operator is bounded from the Orlicz space $L_\Phi(u)$ to $L_\Phi(w)$ in the context of spaces of homogeneous type. Some work in this direction was done in [1]–[3], [4]–[9], [11]–[15]. With this aim, we introduce some notations.

Let X be a set. A nonnegative symmetric function $d(x,y)$ defined on $X\times X$ will be called a *quasi-distance* if there exists an absolute constant D such that

$$d(x,y)\le D(d(x,z)+d(z,y))$$

for every $x,y,z\in X$, and $d(x,y)=0$ if and only if $x=y$. Let μ be a positive measure defined on a σ-algebra of subsets of X which contains balls $B(x,r)=\{y;d(x,y)<r\}$. Now we say that (X,d,μ) is a *space of homogeneous type* if X is a set endowed with a quasi-distance d and a positive measure μ such that:

(i) The family $\{B(x,r);x\in X,\ r>0\}$ is a basis of the topology of X;

(ii) There exists a natural number N such that for any $x\in X$ and $r>0$ the ball $B(x,r)$ contains at most N points x_i with $d(x_i,x_j)\ge\frac12 r$;

(iii) μ is a doubling Borel measure, i.e., there exists a constant D such that $0<\mu(B(x,2r))\le D\mu(B(x,r))$ for all $x\in X$ and $r>0$.

Hereafter, we shall suppose that the continuous functions with compact support are dense in $L^p(X,d\mu)$ for $1\le p<\infty$.

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For a weight function v, define the *Hardy–Littlewood maximal operator* M_v by

$$M_v f(x) = \sup v(B)^{-1} \int_B |f| v \, d\mu ,$$

where the supremum is taken over all balls B containing x and $v(E) = \int_E v \, d\mu$ for any measurable set E.

When $v = 1$ we write Mf for $M_1 f$.

Now, we present the basic definitions concerning N-functions and Orlicz spaces which will be used later (see [5], [10]).

An *N-function* is a continuous and convex function $\Phi : [0,\infty) \to \mathbf{R}$ such that $\Phi(s) > 0$ for $s > 0$, $\Phi(s)/s \to 0$ as $s \to 0$ and $\Phi(s)/s \to \infty$ as $s \to \infty$. An N-function Φ has the representation $\Phi(s) = \int_0^s \varphi(t)\, dt$, where $\varphi : [0,\infty) \to \mathbf{R}$ is continuous from the right, nondecreasing and such that $\varphi(s) > 0$ for $s > 0$, $\varphi(0) = 0$ and $\varphi(s) \to \infty$ as $s \to \infty$. Associated with φ we define the *generalized inverse* ϱ of φ by $\varrho(t) = \sup\{s ; \varphi(s) \le t\}$ which has the same aforementioned properties of φ. Now we define the *complementary N-function* Ψ of Φ by $\Psi(t) = \int_0^t \varrho(t)\, dt$.

An N-function Φ is said to satisfy the Δ_2 *condition* in $[0,\infty)$ if

$$\sup_{s>0} \Phi(2s)/\Phi(s) < \infty .$$

In this paper we shall always suppose that Φ and the complementary N-function Ψ satisfy the Δ_2 condition.

Define the *Orlicz space*

$$L_\Phi(v) = \left\{ f ; \int \Phi(|f|)\, dv < \infty \right\},$$

with the *Luxemburg norm* $\|f\|_{(\Phi,v)} = \inf\{t > 0 ; \int \Phi(t^{-1}|f|)\, dv \le 1\}$. Therefore we have the Hölder inequality,

$$\int |fg| \, dv \le C \|f\|_{(\Phi,v)} \|g\|_{(\Psi,v)} .$$

When $v = w\, d\mu$ for a nonnegative measurable function w on X we write $L_\Phi(w)$ for $L_\Phi(v)$ and $\|f\|_{(\Phi,w)}$ for $\|f\|_{(\Phi,v)}$.

In this paper, we shall prove the following results.

THEOREM 1. *Let u and w be two weight functions. Suppose there exists a weight function σ for which $\sigma\, d\mu$ is a doubling measure and*

$$\|\sigma M_\sigma(f)\|_{(\Phi,u)} \le C \|\sigma f\|_{(\Phi,u)} \tag{1}$$

for all $f\sigma \in L_\Phi(u)$. Then M is bounded from $L_\Phi(u)$ to $L_\Phi(w)$ if and only if

$$\int_B \Phi(M(\chi_B t\sigma)) w \, d\mu \le C \int_B \Phi(t\sigma) u \, d\mu \tag{2}$$

for every ball B, all positive t and a constant C independent of B and t, where χ_B is the characteristic function of the set B.

THEOREM 2. *Let u be a weight function. Suppose there exists a weight function σ for which (1) holds and $\sigma\, d\mu$ is a doubling measure. Then there exists a measurable function w, which is positive and finite almost everywhere, such that M is bounded from $L_\Phi(u)$ to $L_\Phi(w)$ if and only if for every ball B_1 there exists a covering $\{E_j\}$ of B_1 such that*

$$\sup_{t>0} \sup_{B\supset B_1} \Big(\int_B \Phi(t\sigma) u \, d\mu \Big)^{-1} \int_{E_j} \Phi(tM(\chi_B \sigma)) u \, d\mu < \infty \tag{3}$$

for all j, where the second supremum is taken over all balls B containing B_1.

THEOREM 3. *Let w be a weight function. Then there exists $u \ge 0$ finite almost everywhere such that M is bounded from $L_\Phi(u)$ to $L_\Phi(w)$ if and only if for some $\overline{x} \in X$, $\|h(\cdot,\overline{x})\|_{(\Phi,w)} < \infty$, where we write $h(x,\overline{x}) = (1 + \mu(B(\overline{x}, d(x,\overline{x}))))^{-1}$.*

Now let us say a little about the existence of a weight function σ for which (1) holds. If u is a weight function such that

$$\int \Phi(Mf) u \, d\mu \le C \int \Phi(|f|) u \, d\mu$$

for all $f \in L_\Phi(u)$, then $\sigma = 1$ satisfies (1). An N-function Φ is said to satisfy the Δ' *condition* if there exists a constant C such that

$$C^{-1}\Phi(st) \le \Phi(s)\Phi(t) \le C\Phi(st)$$

for $s, t > 0$. We will show that if Φ satisfies the Δ' condition and $\varrho(1/u)$ is a doubling measure, then (1) holds when $\sigma = \varrho(1/u)$. Recall that M_σ is a bounded operator on $L_\Phi(\sigma)$ when σ is a doubling measure and that

$$C^{-1} t\varrho(t) \le \Phi(\varrho(t)) \le Ct\varrho(t)$$

for $t > 0$. Hence $\Phi(\sigma)u = \Phi(\varrho(1/u))u \approx \varrho(1/u) = \sigma$,

$$\begin{aligned} \int \Phi(\sigma M_\sigma(f)) u \, d\mu &\le C \int \Phi(\sigma)\Phi(M_\sigma(f)) u \, d\mu \\ &\le C \int \Phi(M_\sigma(f))\sigma \, d\mu \le C \int \Phi(|f|)\sigma \, d\mu \\ &\le C \int \Phi(\sigma|f|) u \, d\mu , \end{aligned}$$

and (1) holds when $\sigma = \varrho(1/u)$. For example, $\Phi(t) = t^p$ for some $p > 1$ and $\sigma = u^{-1/p-1}$. For a general N-function Φ, we still do not know the exact condition on a weight function u such that a weight function σ exists for which (1) holds.

Theorem 1 is an outgrowth of the one in [11], [13], where $\Phi(t) = t^p$ for some $p > 1$ was considered on the Euclidean space $\mathbf{R}^n$ or on a space of homogeneous type. It is still new and meaningful for a general Orlicz space even when $X = \mathbf{R}^n$, though the condition (1) is not very computable. J. L. Rubio de Francia ([12]) and L. Carleson and P. Jones ([3]) considered the existence of a weight function w such that M is bounded from $L_\Phi(u)$ to $L_\Phi(w)$ when $\Phi(t) = t^p$ for some $p > 1$ and $X = \mathbf{R}^n$. We consider the similar question on a space of homogeneous type in Theorem 2. Theorem 3 is a generalization of the results in [7], [14], [15], where $\Phi(t) = t^p$ for some $p > 1$ was considered. Here we want to point out that the normal condition in [15] is not essential because for a space of homogeneous type there exists a normal space with the same topology.

In this paper, the same letter C will be used to denote constants which may be different at different occurrences.

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2. Proof of Theorems

Proof of Theorem 1. First, we assume (2) holds. Fix a nonnegative function $f \in L_\Phi(u)$. For each integer k, let K_k be an arbitrary compact subset of $\{2^k < Mf \le 2^{k+1}\}$. By the compactness of K_k we can find a finite collection of balls $\{B_j^k\}$ such that

$$K_k \subseteq \bigcup_j B_j^k \quad \text{and} \quad \mu(B_j^k)^{-1} \int_{B_j^k} |f(x)|\, d\mu(x) > 2^k .$$

Put $E_1^k = B_1^k \cap K_k$, $E_j^k = (B_j^k \setminus \bigcup_{s<j} B_s^k) \cap K_k$ for $j > 1$. The E_j^k's are obviously pairwise disjoint and $\bigcup_j E_j^k = K_k$. Since $K_k \subseteq \{2^k < Mf \le 2^{k+1}\}$, we see that for arbitrary $n \in \mathbf{N}$,

$$\int_{\bigcup_{k=-n}^n K_k} \Phi(Mf(x))w(x)\, d\mu(x) \le C \sum_{j,k} \Phi(2^k) w(E_j^k)$$

$$\le C \sum_{j,k} w(E_j^k) \Phi\left(\frac{\sigma(B_j^k)}{\mu(B_j^k)} \frac{1}{\sigma(B_j^k)} \int_{B_j^k} (f\sigma^{-1})\sigma\, d\mu \right).$$

Let $\Gamma(t) = \{(j,k)\,;\ -n \le k \le n,\ \sigma(B_j^k)^{-1} \int_{B_j^k} (f\sigma^{-1})\sigma\, d\mu > t\}$ for $t > 0$ and

$$G(t) = \bigcup_{(j,k)\in\Gamma(t)} B_j^k .$$

Obviously $G(t) \subset \{M_\sigma(f/\sigma) > t\}$. By using a covering lemma in ([4], p. 69), we can find a subfamily $\{B_i^t\}$ of $\{B_j^k\}_{(j,k)\in\Gamma(t)}$ and a constant D such that the B_i^t's are pairwise disjoint and for every B_j^k there exists B_i^t such that $B_j^k \subset \overline{B}_i^t$, where $\overline{B}_i^t$ is the ball with the same center as B_i^t and with radius D times that of B_i^t. Recall that the E_j^k's are pairwise disjoint and $G(t) \subset \{M_\sigma(f/\sigma) > t\}$. Hence we have

$$\int_{\bigcup_{k=-n}^n K_k} \Phi(Mf(x))w(x)\, d\mu(x)$$

$$\le C \int_0^\infty t^{-1}\, dt \sum_{(j,k)\in\Gamma(t)} \Phi\left(\frac{\sigma(B_j^k)}{\mu(B_j^k)} t\right) w(E_j^k)$$

$$\le C \int_0^\infty t^{-1}\, dt \sum_i \sum_{B_j^k \subset \overline{B}_i^t} \Phi\left(\frac{\sigma(B_j^k)}{\mu(B_j^k)} t\right) w(E_j^k)$$

$$\le C \int_0^\infty t^{-1}\, dt \sum_i \int_{\overline{B}_i^t} \Phi(M(\sigma t \chi_{\overline{B}_i^t}))w\, d\mu$$

$$\le C \int_0^\infty t^{-1}\, dt \sum_i \int_{\overline{B}_i^t} \Phi(\sigma t) u\, d\mu$$

$$\le C \int_0^\infty t^{-1}\, dt \sum_i \int_{B_i^t} \Phi(\sigma t) u\, d\mu$$

$$\le C \int_0^\infty t^{-1}\, dt \int_{G(t)} \Phi(\sigma t) u\, d\mu$$

$$\le C \int_0^\infty t^{-1}\, dt \int_{\{M_\sigma(f/\sigma)>t\}} \Phi(t\sigma) u\, d\mu$$

$$\le C \int \Phi(\sigma M_\sigma(f/\sigma))u\, d\mu \le C \int \Phi(|f|)u\, d\mu ,$$

where the fourth inequality follows from (2), the fifth inequality follows from the facts that $u\, d\mu$ is a doubling measure and

$$\int \Phi(\sigma t M_\sigma(\chi_B))w\, d\mu \le C \int_B \Phi(t\sigma) u\, d\mu ,$$

and the last inequality follows from (1). Hence we have proved that

$$\int \Phi(Mf)w\, d\mu \le C \int \Phi(|f|)u\, d\mu$$

and

$$\|Mf\|_{(\Phi,w)} \le C\|f\|_{(\Phi,u)} .$$

Therefore (2) implies M is bounded from $L_\Phi(u)$ to $L_\Phi(w)$.

To prove the converse, we note that

$$\|M(t\chi_B\sigma)\|_{(\Phi,w)} \le C\|t\chi_B\sigma\|_{(\Phi,u)}$$

and

$$\int_B \Phi(M(t\chi_B\sigma))w\,d\mu \le C\int_B \Phi(t\sigma)u\,d\mu\,.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. First we assume that there exists a measurable function w which is positive, finite almost everywhere and such that M is bounded from $L_\Phi(u)$ to $L_\Phi(w)$. Then

$$\int_{B_1} \Phi(tM(\chi_B\sigma))w\,d\mu \le C\int_B \Phi(t\sigma)u\,d\mu\,.$$

Define $E_j = \{x\in B_1\,;\, w(x)/u(x) > 2^{-j}\}$, $j\in\mathbb{N}$. Then $\bigcup_{j\ge 1} E_j = B_1$ and

$$\int_{E_j} \Phi(tM(\chi_B\sigma))u\,d\mu \le 2^j \int_{E_j} \Phi(tM(\chi_B\sigma))w\,d\mu$$

$$\le C\int_{B_1} \Phi(tM(\chi_B\sigma))w\,d\mu \le C\int_B \Phi(t\sigma)u\,d\mu\,.$$

Hence (3) holds.

Now we prove that (3) implies M is bounded from $L_\Phi(u)$ to $L_\Phi(w)$ for some nontrivial w. First we observe that there exists a family of balls $\{B(x_i,r_i)\}$ which is a covering of X such that x is contained in $B_i = B(x_i,r_i)$ for at most M different i for every $x\in X$, where M is a fixed integer independent of x. For every B_i, by (3), we can find a family of subsets $\{E_{ij}\}$ which is a covering of B_i such that

$$d_{ij} = \sup_{t>0}\sup_{B\supset B_i}\Big(\int_B \Phi(t\sigma)u\,d\mu\Big)^{-1}\int_{E_{ij}} \Phi(tM(\chi_B\sigma))u\,d\mu < \infty\,.$$

Define

$$u_{B_i} = \inf_{t>0}\frac{\Phi(t\sigma)u}{\Phi(tM(\chi_{\widetilde{B}_i}\sigma))}h_i\,,$$

where $\widetilde{B}_i = B(x_i, 5D^2 r_i)$, $h_i = \sum_{j>1} c_j\chi_{E_{ij}\setminus\bigcup_{s<j}E_{is}} + c_1\chi_{E_{i1}}$ and $c_j = 2^{-j}(d_{ij}+1)^{-1}$ for $j\ge 1$. Now let us prove

$$\int \Phi(Mf)u_{B_i}\,d\mu \le C\int \Phi(|f|)u\,d\mu\,.$$

Using the same notation as in the proof of Theorem 1, we write

$$\Gamma(t) = \Gamma_1(t)\cup\Gamma_2(t)\,,$$

where

$$\Gamma_1(t) = \{(j,k)\in\Gamma(t)\,;\, B_j^k\cap\widetilde{B}_i^c \ne\emptyset,\, B_j^k\cap B_i\ne\emptyset\}\,.$$

Then we have

$$\int_{\bigcup_{k=-n}^n K_k} \Phi(Mf)u_{B_i}\,d\mu$$

$$\le C\int_0^\infty \frac{dt}{t}\sum_{(j,k)\in\Gamma_2(t)} \Phi\Big(\frac{\sigma(B_j^k)}{\mu(B_j^k)}t\Big)u_{B_i}(E_j^k)$$

$$+ C\int_0^\infty \frac{dt}{t}\sum_{(j,k)\in\Gamma_1(t)} \Phi\Big(\frac{\sigma(B_j^k)}{\mu(B_j^k)}t\Big)u_{B_i}(E_j^k)$$

$$= I_1 + I_2\,.$$

Since the support of u_{B_i} is B_i, we get

$$I_1 \le C\int_0^\infty \frac{dt}{t}\sum_{(j,k)\in\Gamma_2(t),\,B_j^k\subset\widetilde{B}_i} \Phi\Big(\frac{\sigma(B_j^k)}{\mu(B_j^k)}t\Big)u_{B_i}(E_j^k)$$

$$\le \int_0^\infty \frac{dt}{t}\sum_{(j,k)\in\Gamma_2(t)}\int_{E_j^k} \Phi(t\sigma)u\,d\mu$$

$$\le C\int_0^\infty \frac{dt}{t}\int_{\{M_\sigma(f/\sigma)>t\}} \Phi(t\sigma)u\,d\mu \le C\int \Phi(|f|)u\,d\mu\,.$$

Therefore it suffices to prove

$$I_2 \le C\int \Phi(|f|)u\,d\mu\,.$$

Let $r = \max_{(j,k)\in\Gamma_2(t)} r(B_j^k) = r(B(x_{j_1}^{k_1}, r))$, where $r(B_j^k)$ denotes the radius of B_j^k. Since $(j,k)\in\Gamma_1(t)$, we have $r\ge r(B_i)$. Therefore there exists a constant D such that

$$B(x_{j_1}^{k_1}, Dr) \supset \bigcup_{(j,k)\in\Gamma_2(t)} B_j^k\cup B_i$$

and we get

$$\sum_{(j,k)\in\Gamma_1(t)} \Phi\Big(\frac{\sigma(B_j^k)}{\mu(B_j^k)}t\Big)u_{B_i}(E_j^k) \le \int_{B_i} \Phi(M(\chi_{B(x_{j_1}^{k_1},Dr)}\sigma t))uh_i\,d\mu$$

$$\le \sum_j c_j\int_{E_{ij}} \Phi(M(\chi_{B(x_{j_1}^{k_1},Dr)}\sigma t))u\,d\mu \le \sum_j c_j d_{ij}\int_{B(x_{j_1}^{k_1},Dr)} \Phi(\sigma t)u\,d\mu$$

$$\le C\int_{B(x_{j_1}^{k_1},r)} \Phi(\sigma t)u\,d\mu \le C\int_{\{M_\sigma(f/\sigma)>t\}} \Phi(\sigma t)u\,d\mu\,,$$

where the fourth inequality follows from the fact that σ is a doubling measure and

$$\|\sigma M_\sigma(t\chi_{B(x_{j_1}^{k_1},r)})\|_{(\Phi,u)} \le C\|\sigma t\chi_{B(x_{j_1}^{k_1},r)}\|_{(\Phi,u)}.$$

Hence we have

$$I_2 \le C\int_0^\infty \frac{dt}{t}\int_{\{M_\sigma(f/\sigma)>t\}} \Phi(t\sigma)u\,d\mu \le C\int \Phi(|f|)u\,d\mu.$$

Until now we have proved the following:

$$\int \Phi(Mf)u_{B_i}\,d\mu \le e_i\int \Phi(|f|)u\,d\mu$$

for some positive constant e_i. Write

$$w = \sum e_i^{-1}2^{-i}u_{B_i}.$$

Then w is finite and positive almost everywhere and

$$\int \Phi(Mf)w\,d\mu \le 2\int \Phi(|f|)u\,d\mu.$$

Hence Theorem 2 holds.

Proof of Theorem 3. First, we show that if M is bounded from $L_\Phi(u)$ to $L_\Phi(w)$ then $\|h(\cdot,\overline{x})\|_{(\Phi,w)} < \infty$ for some $\overline{x}\in X$. Because u is a measurable function and is finite almost everywhere, there exists a set E such that $0 < u(E) < \infty$, $\mu(E) > 0$ and $E\subset B(\overline{x},R)$ for some $\overline{x}\in X$ and $0 < R < \infty$. Let $f = \chi_E$ be the characteristic function of E. Observe that $Mf(x) \ge Ch(x,\overline{x})$ for all $x\in X$ and some small constant C. Then we get

$$\|h(\cdot,\overline{x})\|_{(\Phi,w)} \le C\|Mf\|_{(\Phi,w)} \le C\|f\|_{(\Phi,u)} < \infty.$$

Now let us prove that if $\|h(\cdot,\overline{x})\|_{(\Phi,w)} < \infty$ for some $\overline{x}\in X$ then M is bounded from $L_\Phi(u)$ to $L_\Phi(w)$ for some nontrivial u. Observe that

$$\begin{aligned} Mf(x) &\le \sup\Big\{\mu(B)^{-1}\int_B |f|\,d\mu\,;\ \mu(B)\le C_0h(x,\overline{x})^{-1}\Big\}\\ &\quad+\sup\Big\{\mu(B)^{-1}\int_B |f|\,d\mu\,;\ \mu(B) > C_0h(x,\overline{x})^{-1}\Big\}\\ &= M_1f(x) + M_2f(x),\end{aligned}$$

where C_0 is a sufficiently small constant to be chosen later. Therefore the matter reduces to proving the following:

$$\|M_1f\|_{(\Phi,w)} \le C\|f\|_{(\Phi,u_1)}, \tag{4}$$

$$\|M_2f\|_{(\Phi,w)} \le C\|f\|_{(\Phi,u_2)}, \tag{5}$$

for some nonnegative measurable functions u_1 and u_2 which are finite almost everywhere.

We prove (5) first. Observe that $M_2f(x) \le Ch(x,\overline{x})\int_X |f|\,d\mu$. Hence we have

$$\|M_2f\|_{(\Phi,w)} \le C\int_X |f|\,d\mu\,\|h(\cdot,\overline{x})\|_{(\Phi,w)} \le C\int_X |f|\,d\mu.$$

Let $u_2 = \sum_{i\ge1} c_i\chi_{B(\overline{x},r_i)\setminus B(\overline{x},r_{i-1})} + c_0\chi_{B(\overline{x},r_0)}$, where r_i is chosen so that $\mu(B(\overline{x},r_i)) \le 2^i$ and $\bigcup_{i\ge0}B(\overline{x},r_i) = X$, and the c_i are large constants to be chosen later. Then

$$\int_X \Psi(u_2^{-1})u_2\,d\mu \le \sum_{i=0}^\infty \mu(B(\overline{x},r_i))\Psi(c_i^{-1})c_i \le \sum_{i\ge0}2^ic_i\Psi(c_i^{-1}).$$

Because $\Psi(s)/s\to0$ as $s\to0$, we can choose c_i such that $c_i\Psi(c_i^{-1}) \le 2^{-2i}$ for $i\ge0$. Then $\|u_2^{-1}\|_{(\Psi,u_2)} < \infty$. Obviously u_2 is finite and positive almost everywhere. Therefore by the Hölder inequality

$$\|M_2f\|_{(\Phi,w)} \le C\int_X |f|\,d\mu \le C\|f\|_{(\Phi,u_2)}\|u_2^{-1}\|_{(\Psi,u_2)} \le C\|f\|_{(\Phi,u_2)}.$$

Hence (5) holds.

Now we prove (4). Define

$$M_3f(x) = \sup_{B\ni x}\Big\{\mu(B)^{-1}\int_B |f|\,d\mu\,;\ \mu(B)\le C_0^{-1}h(x,\overline{x})^{-1}\Big\},$$

where C_0 is chosen as in the definition of M_1 and M_2. Then we can prove the following in the same way as in [15], [7] by choosing C_0 sufficiently small:

$$w(\{M_1f > t\}) \le Ct^{-1}\int |f|M_3w\,d\mu$$

for all $t > 0$ and

$$\|M_1f\|_\infty \le C\|f\|_\infty.$$

Hence by the interpolation theorem (see [5, Theorem 2.17]) we get

$$\|M_1f\|_{(\Phi,w)} \le C\|f\|_{(\Phi,M_3w)}.$$

Therefore the matter reduces to proving that $M_3w < \infty$ almost everywhere. When $\mu(X) < \infty$, we see that $h(x,\overline{x})$ is bounded below away from 0 and bounded above, and that $w(X) < \infty$. Therefore on account of the weak type (1,1) boundedness of the Hardy–Littlewood maximal operator M, we deduce that $M_3w < \infty$ almost everywhere. When $\mu(X) = \infty$, for every $x\in B(\overline{x},n)$ ($n\in\mathbb{N}$) and for any ball B containing x with $\mu(B)\le C_0^{-1}h(x,\overline{x}) \le Cn$ we have $B\subset B(\overline{x},C_1)$ for some constant C_1 independent of B. Therefore

$$M_3w(x) \le M(w\chi_{B(\overline{x},C_1)})(x)$$

for all $x\in B(\overline{x},n)$ and so $M_3w(x) < \infty$ almost everywhere. Hence (4) holds and Theorem 3 holds.

3. Some remarks. We say the pair (u,w) on a space of homogeneous type satisfies the *A_Φ-condition* if there exists a positive constant C such that for every ball $B=B(x,r)$ and every positive s,

$$\Big(\mu(\overline{B})^{-1}\int\limits_{\overline{B}} u\,d\mu\Big)s\varphi\Big(\mu(B)^{-1}\int\limits_{B}\varrho(1/sw)\,d\mu\Big)\le C\,,$$

where $\overline{B}=B(x,5D^2r)$.

We can prove the following in the same way as in [5], [9].

THEOREM 4. *Let (X,d,μ) be a space of homogeneous type. Then*

$$w(\{Mf>t\})\le C\Phi(t)^{-1}\int\limits_X \Phi(|f|)u\,d\mu \tag{6}$$

for all $t>0$ if and only if the weight pair (u,w) satisfies the A_Φ-condition.

THEOREM 5. *Given a weight u on X, there exists a weight $w>0$ almost everywhere such that* (6) *holds if and only if for every ball B the following conditions are satisfied*:

(i) *For almost every $x\in X$*

$$\sup_{s>0} s\varphi(M(\varrho(s^{-1}w^{-1}\chi_B)))(x)<\infty\,.$$

(ii) $$\sup_{s>0}\sup_{B_1\supset B} s\varphi\Big(\mu(B_1)^{-1}\int\limits_{B_1}\varrho((s\mu(B_1)w)^{-1})\,d\mu\Big)<\infty\,,$$

where the second supremum is taken over all balls B_1 containing B.

Also we have

THEOREM 6. *Let w be a nonnegative measurable function. Then there exists $u\ge 0$ finite almost everywhere such that* (6) *holds if and only if for some $\overline{x}\in X$*

$$w(\{h(x,\overline{x})>t\})\le C\inf_{s>0}\Phi(s)/\Phi(st)$$

for all $t>0$ and a constant C independent of t.

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