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Closed operators affiliated with a Banach algebra of operators

by

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Abstract. Let $\mathcal{B}$ be a Banach algebra of bounded linear operators on a Banach space X. If S is a closed operator in X such that $(\lambda - S)^{-1} \in \mathcal{B}$ for some number λ, then S is affiliated with $\mathcal{B}$. The object of this paper is to study the spectral theory and Fredholm theory relative to $\mathcal{B}$ of an operator which is affiliated with $\mathcal{B}$. Also, applications are given to semigroups of operators which are contained in $\mathcal{B}$.

1. Introduction. Let X be a Banach space, and denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. If S is a closed operator in X such that $(\lambda - S)^{-1} \in \mathcal{B}(X)$ for some $\lambda \in \mathbf{C}$, then useful information concerning the spectral and Fredholm properties of S can be derived from the corresponding properties of $(\lambda - S)^{-1}$. For example, from the holomorphic operational calculus of $(\lambda - S)^{-1}$ one can define the holomorphic operational calculus of S; see [8, pp. 599–604].

Now let $\mathcal{B} \subseteq \mathcal{B}(X)$ be some Banach algebra of operators. Define a closed operator S in X to be *affiliated with* $\mathcal{B}$ if $\exists\lambda \in \mathbf{C}$ such that $(\lambda - S)^{-1} \in \mathcal{B}$. Again, in this more general situation, interesting and useful information concerning S can be derived from properties of $(\lambda - S)^{-1}$ relative to $\mathcal{B}$. In this paper we are mainly interested in closed operators affiliated with one of the following algebras:

I. *Jörgen's algebras*. Let X and Y be Banach spaces which along with a bounded nondegenerate bilinear form, $\langle\cdot,\cdot\rangle$, comprise a dual system; see [11, p. 43]. Let $\mathcal{A} = \mathcal{A}(X, Y)$ be the set of all $T \in \mathcal{B}(X)$ such that $\exists T^\dagger \in \mathcal{B}(Y)$ with

$$\langle Tx, y\rangle = \langle x, T^\dagger y\rangle \quad (x \in X,\ y \in Y).$$

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Then $\mathcal{A}$ is a Banach algebra of operators with norm $\|T\| = \max(\|T\|, \|T^\dagger\|)$ where the norms on the right are the usual operator norms [11, pp. 45–46]. The Banach algebra $\mathcal{A}$ is used in various ways in the study of linear integral operators ([11] and [12]). The spectral and Fredholm theory of an operator $T \in \mathcal{A}$ is well understood ([2] and [11]) and leads to useful information concerning the relationship between T and $T^\dagger$.

II. *The algebra* $\mathcal{B}_{p,s}$. Let (Ω, μ) be a σ-finite measure space. Fix $1 \leq p < s \leq \infty$. Let $T \in \mathcal{B}_{p,s}$ if T is a bounded linear operator on $L^p \cap L^s$ and has continuous extensions T_p on L^p and T_s on L^s. Here L^p is the usual Lebesgue space $L^p(\Omega, \mu)$ when $1 \leq p < \infty$. For the case $s = \infty$, L^s will be understood to be the closure of $L^p \cap L^s$ in the norm $\|\cdot\|_\infty$, and T_s, the bounded extension of T to L^s. The space $\mathcal{B}_{p,s}$ is a Banach algebra with norm $\|T\| = \max(\|T_p\|, \|T_s\|)$. The spectral and Fredholm properties of $T \in \mathcal{B}_{p,s}$ relative to $\mathcal{B}_{p,s}$ have been studied in [4], and to some extent, in [5]. If $T \in \mathcal{B}_{p,s}$, then by the Riesz Convexity Theorem, for all $r \in [p, s]$, T has a unique extension $T_r \in \mathcal{B}(L^r)$. One motivation here is that from properties of T relative to $\mathcal{B}_{p,s}$, one can draw interesting conclusions concerning T_r for all $r \in [p, s]$; see [4].

Since the spectral and Fredholm theory relative to the Banach algebras $\mathcal{A}(X, Y)$ and $\mathcal{B}_{p,s}$ is well understood, we use this information to study closed operators which are affiliated with one of these algebras. One could also apply the general theory developed in Section 2 and results in [1] to study closed operators affiliated with the Banach algebra of all regular operators on a Banach lattice, but we do not pursue this direction here.

In the last section of this paper, we study semigroups of operators which have infinitesimal generators affiliated with a Banach algebra of operators, $\mathcal{B}$. This general theory has interesting applications when $\mathcal{B} = \mathcal{A}(X, Y)$ and when $\mathcal{B} = \mathcal{B}_{p,s}$.

When R is an operator in X, then we use the notation:

$\mathcal{N}(R) \equiv$ the null space of R;
$\mathcal{R}(R) \equiv$ the range of R;
$\sigma(R) \equiv$ the spectrum of R;
$\operatorname{ind}(R) \equiv$ the index of R (when this makes sense);
$\|R\| \equiv$ the operator norm of R (when R is bounded);
$\operatorname{nul}(R) \equiv \dim(\mathcal{N}(R))$;
$\operatorname{def}(R) \equiv \dim(X/\mathcal{R}(R))$.

2. Spectral and Fredholm theory of an affiliated operator. Throughout this section $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach algebra of operators with $\mathcal{B} \subseteq \mathcal{B}(X)$, $I \in \mathcal{B}$, and $\|T\|_{\mathcal{B}} \geq \|T\|$ (the operator norm of T) for all $T \in \mathcal{B}$.

Assume S is a closed linear operator with domain $D(S)$ in X. Let

$$\varrho_{\mathcal{B}}(S) = \{\lambda \in \mathbb{C} : (\lambda - S)^{-1} \in \mathcal{B}\}.$$

The notation $(\lambda - S)^{-1}$ will always be understood to mean

$$(\lambda - S)(\lambda - S)^{-1}x = x \quad \text{for all } x \in X\text{; and}$$
$$(\lambda - S)^{-1}(\lambda - S)x = x \quad \text{for all } x \in D(S).$$

DEFINITION 1. A closed operator S is *affiliated with* $\mathcal{B}$ if $\varrho_{\mathcal{B}}(S)$ is nonempty.

Assume S and $\mathcal{B}$ are as above. When S is affiliated with $\mathcal{B}$, we usually assume for simplicity that $S^{-1} \in \mathcal{B}$. Formulas or results in the general situation can always be easily derived from those that hold in this special case.

Define the *$\mathcal{B}$-spectrum* of S by $\sigma_{\mathcal{B}}(S) = \mathbb{C} \setminus \varrho_{\mathcal{B}}(S)$.

THEOREM 2. *Assume S is a closed operator in X with $T = S^{-1} \in \mathcal{B}$.*

(1) *For $\lambda \neq 0$, $\lambda \in \varrho_{\mathcal{B}}(S) \Leftrightarrow \lambda^{-1} \in \varrho_{\mathcal{B}}(T)$, and in this case,*

$$(\lambda - S)^{-1} = -\lambda^{-1}T(\lambda^{-1} - T)^{-1}.$$

(2) $\sigma_{\mathcal{B}}(S) = \{\lambda^{-1} : \lambda \in \sigma_{\mathcal{B}}(T) \setminus \{0\}\}$.
(3) *$\varrho_{\mathcal{B}}(S)$ is open and $\sigma_{\mathcal{B}}(S)$ is closed.*

Proof. Parts (2) and (3) easily follow from (1). We prove (1). For $\lambda \neq 0$ we have $(\lambda - S)T = -\lambda(\lambda^{-1} - T)$ and $T(\lambda - S) = -\lambda(\lambda^{-1} - T)$ on $D(S)$. Therefore if $\lambda^{-1} - T$ is invertible in $\mathcal{B}$, then $(\lambda - S)^{-1} = -\lambda^{-1}T(\lambda^{-1} - T)^{-1}$. Conversely, assume $(\lambda - S)^{-1} = R \in \mathcal{B}$. Note that $T + R = T[(\lambda - S) + S]R = \lambda TR$. Then

$$(\lambda^{-1} - T)(\lambda - \lambda^2 R) = I - \lambda T - \lambda R + \lambda^2 TR = I - \lambda T - \lambda R + \lambda[T + R] = I.$$

Also, $TR = RT$, so $(\lambda^{-1} - T)^{-1} = \lambda - \lambda^2 R \in \mathcal{B}$.

It is worthwhile to note that when S is a closed densely defined operator on X which is affiliated with $\mathcal{B}$, then a $\mathcal{B}$-operational calculus can be defined for S. The definition and properties of this operational calculus are stated exactly as in the case where $\mathcal{B} = \mathcal{B}(X)$ as in [8, pp. 599–602]. In this case, when f is analytic on some open set containing $\sigma_{\mathcal{B}}(S)$ and at ∞, then $f(S) \in \mathcal{B}$.

Next we consider the main topic of this section, Fredholm theory relative to $\mathcal{B}$. In order to have a useful Fredholm theory in $\mathcal{B}$, we must assume that $\mathcal{B}$ contains sufficiently many operators of finite rank. If $x \in X$ and $\alpha \in X'$, where X' is the dual space of X, then let $\alpha \otimes x$ denote the operator on X defined by

$$(\alpha \otimes x)(y) = \alpha(y)x \quad (y \in X).$$

We will assume that $\mathcal{B}$ has the following property:

(#) There exists a total subspace Y of X' such that $\alpha \otimes x \in \mathcal{B}$ for all $x \in X$ and all $\alpha \in Y$.

We omit the proof of the following proposition.

Proposition 3. *Assume $\mathcal{B} \subseteq \mathcal{B}(X)$ as before and that $\mathcal{B}$ satisfies (#).*

(1) *$\mathcal{B}$ is a primitive Banach algebra.*

(2) *For $\alpha \in Y$, $\alpha \otimes X = \{\alpha \otimes x : x \in X\}$ is a minimal left ideal of $\mathcal{B}$.*

The *socle* of a primitive algebra $\mathcal{B}$, denoted by $\mathrm{soc}(\mathcal{B})$, is the smallest left ideal of $\mathcal{B}$ which contains all minimal left ideals of $\mathcal{B}$ [7, Def. 8, p. 156]. In fact, $\mathcal{B}$ being primitive implies that $\mathrm{soc}(\mathcal{B})$ is an ideal of $\mathcal{B}$. This ideal is easily identified for the algebras under consideration. Basic properties of $\mathrm{soc}(\mathcal{B})$ are discussed in [6, BA.3] or [7, pp. 154–160].

We will make use of the general Fredholm theory in the primitive Banach algebra $\mathcal{B}$ as defined and studied in [6]; see especially pp. 29–35. There Fredholm theory is developed relative to $\mathrm{soc}(\mathcal{B})$. The next proposition identifies $\mathrm{soc}(\mathcal{B})$ in our present situation.

Proposition 4. *Assume $\mathcal{B}$ satisfies (#). Let $\mathcal{F}_{\mathcal{B}}$ be the set of all finite rank operators in $\mathcal{B}$. Then $\mathcal{F}_{\mathcal{B}} = \mathrm{soc}(\mathcal{B})$.*

Proof. Since the algebra $\mathcal{B}$ is primitive, $\mathrm{soc}(\mathcal{B})$ is a minimal ideal of $\mathcal{B}$. As $\mathcal{F}_{\mathcal{B}}$ is an ideal of $\mathcal{B}$, it follows that $\mathrm{soc}(\mathcal{B}) \subseteq \mathcal{F}_{\mathcal{B}}$.

Conversely, as noted in Proposition 3(2), for $\alpha \in Y$, $\alpha \otimes X$ is a minimal left ideal of $\mathcal{B}$, so $\alpha \otimes X \subseteq \mathrm{soc}(\mathcal{B})$. Therefore

$$(*) \qquad \alpha \otimes x \in \mathrm{soc}(\mathcal{B}) \quad (x \in X,\ \alpha \in Y).$$

Assume $F \in \mathcal{F}_{\mathcal{B}}$ and that $\{x_1, \ldots, x_n\}$ is a basis for the range of F. The argument in [12, Lemma 4.13, p. 41] shows that $\exists\{\alpha_1, \ldots, \alpha_n\} \subseteq Y$ with $\alpha_k(x_j) = \delta_{kj}$, $1 \le k, j \le n$. Let $E = \sum_{k=1}^{n} \alpha_k \otimes x_k \in \mathcal{B}$. Then E is a projection with $EF = F$. By (*), $E \in \mathrm{soc}(\mathcal{B})$, so $F = EF \in \mathrm{soc}(\mathcal{B})$.

Assume $\mathcal{B}$ satisfies (#). Fix $\alpha \in Y$, $\alpha \neq 0$, and let L be the minimal left ideal of $\mathcal{B}$, $L = \alpha \otimes X$. For $T \in \mathcal{B}$, define $\widehat{T} : L \to L$ by $\widehat{T}(\alpha \otimes x) = \alpha \otimes Tx$. Let $V : L \to X$ be the bicontinuous isomorphism of L onto X defined by $V(\alpha \otimes x) = x$, $x \in X$. Then $V^{-1}TV = \widehat{T}$, so T and $\widehat{T}$ are similar. The $\mathcal{B}$-Fredholm properties of $T \in \mathcal{B}$ relative to $\mathcal{F}_{\mathcal{B}} = \mathrm{soc}(\mathcal{B})$ are defined in terms of $\widehat{T}$ in [6, pp. 30–31]. In particular, if T is $\mathcal{B}$-Fredholm, then $\widehat{T}$ is a Fredholm operator on L, and the *$\mathcal{B}$-index* of T is defined to be $\mathrm{ind}(\widehat{T})$. Thus in our case, since T is similar to $\widehat{T}$, the $\mathcal{B}$-index of T will be just $\mathrm{ind}(\widehat{T}) = \mathrm{ind}(T)$ (as an operator on X). We use this fact in what follows.

For the remainder of this section we assume that $\mathcal{B}$ is a unital Banach subalgebra of $\mathcal{B}(X)$ which satisfies (#).

Let $\mathcal{K}_{\mathcal{B}}$ be any closed nonzero inessential ideal of $\mathcal{B}$ [6, F.3.12, p. 42]; $\mathcal{K}_{\mathcal{B}}$ may be taken to be the closure of $\mathcal{F}_{\mathcal{B}}$ in $\mathcal{B}$, for example. In the primitive Banach algebra $\mathcal{B}$, Fredholm theory in $\mathcal{B}$ relative to $\mathcal{F}_{\mathcal{B}}$ is equivalent to Fredholm theory in $\mathcal{B}$ relative to $\mathcal{K}_{\mathcal{B}}$. The key fact here is that by [6, BA 2.4], for $T \in \mathcal{B}$, $\exists V, W \in \mathcal{B}$ and $\exists F, G \in \mathcal{F}_{\mathcal{B}}$ such that $TV = I - F$ and $WT = I - G$ if and only if the same holds for T with $F, G \in \mathcal{K}_{\mathcal{B}}$. We summarize our notation involving Fredholm theory relative to $\mathcal{B}$:

$$\Phi(\mathcal{B}) = \{T \in \mathcal{B} : T \text{ is invertible in } \mathcal{B} \text{ relative to } \mathcal{F}_{\mathcal{B}} \text{ (or } \mathcal{K}_{\mathcal{B}})\};$$

$$\Phi^0(\mathcal{B}) = \{T \in \Phi(\mathcal{B}) : \mathrm{ind}(T) = 0\};$$

$$\omega_{\mathcal{B}}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi(\mathcal{B})\};$$

$$W_{\mathcal{B}}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi^0(\mathcal{B})\}.$$

Let S be a closed operator in X. Next we extend the $\mathcal{B}$-Fredholm theory to S. As part of the usual definition of being Fredholm for a closed operator S in X, it is required that $D(S)$ be dense in X. We do not make this requirement. In our terminology, S is a Fredholm in X if S is Fredholm as an operator from $\overline{D(S)}$ into X in the usual sense; see [15, p. 162]. Let $\Phi(Z, W)$ be the collection of all Fredholm operators densely defined in Z with values in W. Thus in this notation, we consider S Fredholm in X if $S \in \Phi(\overline{D(S)}, X)$. We use the notation $\Phi^0(Z, W)$ for the set of all $S \in \Phi(Z, W)$ such that $\mathrm{ind}(S) = 0$.

Definition 5. Let S be an operator in X. Then S is *$\mathcal{B}$-Fredholm* if $\exists V, W \in \mathcal{B}$ and $F, G \in \mathcal{K}_{\mathcal{B}}$ with

$$SV = I - F \quad \text{on } X, \quad \text{and} \quad WS = I - G \quad \text{on } D(S).$$

If S is $\mathcal{B}$-Fredholm, then we write $S \in \Phi(\mathcal{B})$; if $S \in \Phi(\mathcal{B})$ and $S \in \Phi^0(\overline{D(S)}, X)$, then we write $S \in \Phi^0(\mathcal{B})$.

Theorem 6. *Assume $T = S^{-1} \in \mathcal{B}$. For $\lambda \neq 0$*

$$\lambda - S \in \Phi(\mathcal{B}) \iff \lambda^{-1} - T \in \Phi(\mathcal{B}).$$

In this case $\mathcal{R}(\lambda - S) = \mathcal{R}(\lambda^{-1} - T)$, $\mathcal{N}(\lambda - S) = T\mathcal{N}(\lambda^{-1} - T)$, and $\mathrm{ind}(\lambda - S) = \mathrm{ind}(\lambda^{-1} - T)$.

Proof. For $\lambda \neq 0$,

$$(*) \qquad (\lambda - S)T = -\lambda(\lambda^{-1} - T).$$

From (*) it follows immediately that $\mathcal{R}(\lambda - S) = \mathcal{R}(\lambda^{-1} - T)$ and $\mathcal{N}(\lambda - S) = T\mathcal{N}(\lambda^{-1} - T)$. Furthermore, $\lambda - S \in \Phi(\overline{D(S)}, X)$ is clearly equivalent to $\lambda^{-1} - T$ being Fredholm on X by applying the usual criteria [15, p. 162] and (*). When these two operators are Fredholm, then that $\mathrm{ind}(\lambda - S) = \mathrm{ind}(\lambda^{-1} - T)$ follows from these observations.

Now assume $\lambda^{-1} - T \in \Phi(\mathcal{B})$, so $\exists V, W \in \mathcal{B}$ and $F, G \in \mathcal{K}_{\mathcal{B}}$ with

$$(\lambda^{-1} - T)V = I - F \quad \text{and} \quad W(\lambda^{-1} - T) = I - G.$$

Using (*), it is immediate that

$$(\lambda - S)(-\lambda^{-1}TV) = I - F \quad \text{and} \quad (-\lambda^{-1}WT)(\lambda - S) = I - G \quad \text{on } D(S).$$

Thus, $\lambda - S \in \Phi(\mathcal{B})$.

Conversely, assume $\exists V, W \in \mathcal{B}$ and $F, G \in \mathcal{K}_{\mathcal{B}}$ such that

$$(\lambda - S)V = I - F \quad \text{and} \quad W(\lambda - S) = I - G \quad \text{on } D(S).$$

Note that $\lambda TV = T[(\lambda - S) + S]V = T(I - F) + V$. Therefore

$$\begin{aligned}(\lambda^{-1} - T)(\lambda - \lambda^2 V) &= I - \lambda T - \lambda V + \lambda^2 TV \\ &= I - \lambda T - \lambda V + \lambda[T + V - TF] = I - \lambda TF.\end{aligned}$$

Similarly, $\lambda WT = W + T - GT$, and $(\lambda - \lambda^2 W)(\lambda^{-1} - T) = I - \lambda GT$. This proves $\lambda^{-1} - T \in \Phi(\mathcal{B})$.

LEMMA 7. *Assume $T \in \mathcal{B}$, $\lambda \neq 0$, and $\lambda - T \in \Phi^0(\mathcal{B})$. Then $\exists G \in \mathcal{F}_{\mathcal{B}}$ such that $\lambda - T + \mu GT$ is invertible in $\mathcal{B}$ for all $\mu \neq 0$.*

Proof. We assume for convenience that $\lambda = 1$. By Propositions 3 and 4 we can apply the Fredholm theory in a primitive Banach algebra as developed in [6, F1 and F2]. First by [6, F.1.10] there exist idempotents $P, Q \in \mathcal{F}_{\mathcal{B}}$ such that

$$(I - T)\mathcal{B} = (I - P)\mathcal{B} \quad \text{and} \quad \mathcal{B}(I - T) = \mathcal{B}(I - Q).$$

Set $E = QT \in \mathcal{F}_{\mathcal{B}}$. Note that since $TQ = Q$, we have $E^2 = QTQT = QT = E$. Also, $QE = E$ and $EQ = Q$. Now $(I - Q)E = 0$, so $\mathcal{B}(I - Q) \subseteq \mathcal{B}(I - E)$. Again, $(I - E)Q = 0$, so $\mathcal{B}(I - E) \subseteq \mathcal{B}(I - Q)$.

By [6, F.2.11] we can choose $F \in P\mathcal{B}E$ such that $I - T + \mu F$ is invertible in $\mathcal{B}$ for all $\mu \neq 0$. Since $F = FE = FQT$, setting $G = FQ$, we have

$$I - T + \mu GT \text{ is invertible in } \mathcal{B} \text{ for all } \mu \neq 0.$$

THEOREM 8. *Assume $T = S^{-1} \in \mathcal{B}$. For $\lambda \neq 0$, the following are equivalent*:

(1) $\lambda - S \in \Phi^0(\mathcal{B})$.
(2) $\lambda^{-1} - T \in \Phi^0(\mathcal{B})$.
(3) $\exists G \in \mathcal{F}_{\mathcal{B}}$ *such that* $(\lambda - S + \mu G)^{-1} \in \mathcal{B}$ *for all* $\mu \neq 0$.
(4) $\exists K \in \mathcal{K}_{\mathcal{B}}$ *such that* $(\lambda - S + K)^{-1} \in \mathcal{B}$.

Proof. By Theorem 6, (1) and (2) are equivalent. Clearly (3)⇒(4). Also, if (4) holds then $\lambda - S + K \in \Phi^0(\mathcal{B})$, so $\lambda - S \in \Phi^0(\mathcal{B})$. We complete the proof by showing that (2)⇒(3).

Assuming (2) holds, by Lemma 7, $\exists G \in \mathcal{F}_{\mathcal{B}}$ such that $\lambda^{-1} - T + \mu GT$ is invertible in $\mathcal{B}$ for all $\mu \neq 0$. Now assuming $\mu \neq 0$,

$$(\lambda - S + \mu G)T = -\lambda(\lambda^{-1} - T - \lambda^{-1}\mu GT)$$

is invertible in $\mathcal{B}$. If R is the inverse of this operator in $\mathcal{B}$ for some μ, then $(\lambda - S + \mu G)TR = I$. Also, $TR(\lambda - S + \mu G)(TRx) = TRx$ for all $x \in X$. Since $\mathcal{R}(TR) = D(S)$, $TR(\lambda - S + \mu G) = I$ on $D(S)$. This proves (3).

Next we define and verify some properties of two essential spectra for S relative to $\mathcal{B}$.

DEFINITION 9.

$$\omega_{\mathcal{B}}(S) = \{\lambda \in \mathbb{C} : \lambda - S \notin \Phi(\mathcal{B})\}, \qquad W_{\mathcal{B}}(S) = \{\lambda \in \mathbb{C} : \lambda - S \notin \Phi^0(\mathcal{B})\}.$$

The $\omega_{\mathcal{B}}(S)$ is the *Fredholm essential spectrum* of S relative to $\mathcal{B}$, and $W_{\mathcal{B}}(S)$ is the *Weyl essential spectrum* of S relative to $\mathcal{B}$. We use the notation $\omega(S)$ and $W(S)$ when $\mathcal{B} = \mathcal{B}(X)$.

When S is affiliated with $\mathcal{B}$, then clearly $\omega_{\mathcal{B}}(S) \subseteq W_{\mathcal{B}}(S) \subseteq \sigma_{\mathcal{B}}(S)$.

THEOREM 10. *Assume S is affiliated with $\mathcal{B}$ with $T = (S - \lambda_0)^{-1} \in \mathcal{B}$.*

(1) $\sigma_{\mathcal{B}}(S) = \{(\lambda - \lambda_0)^{-1} : \lambda \in \sigma_{\mathcal{B}}(T),\ \lambda \neq \lambda_0\}$.
(2) $\omega_{\mathcal{B}}(S) = \{(\lambda - \lambda_0)^{-1} : \lambda \in \omega_{\mathcal{B}}(T),\ \lambda \neq \lambda_0\}$.
(3) $W_{\mathcal{B}}(S) = \{(\lambda - \lambda_0)^{-1} : \lambda \in W_{\mathcal{B}}(T),\ \lambda \neq \lambda_0\}$.
(4) $W_{\mathcal{B}}(S) = \bigcap\{\sigma_{\mathcal{B}}(S + K) : K \in \mathcal{K}_{\mathcal{B}}\}$.

Proof. Part (1) follows from Theorem 2. Parts (2) and (3) are an immediate result of Theorem 6.

To prove (4) first note that for any $K \in \mathcal{K}_{\mathcal{B}}$,

$$\lambda - S \in \Phi^0(\mathcal{B}) \Leftrightarrow \lambda - S - K \in \Phi^0(\mathcal{B}).$$

This is a direct result of the definition, Definition 5, and a standard property of the index [15, Theorem 2.1, p. 167] (or use Theorem 8(4)). It follows that for any $K \in \mathcal{K}_{\mathcal{B}}$, $W_{\mathcal{B}}(S) \subseteq \bigcap\{\sigma_{\mathcal{B}}(S + K) : K \in \mathcal{K}_{\mathcal{B}}\}$. On the other hand, if $\lambda \notin W_{\mathcal{B}}(S)$, then by Theorem 8(4), $\exists K \in \mathcal{K}_{\mathcal{B}}$ with $\lambda - (S + K)$ invertible in $\mathcal{B}$. Therefore $\lambda \notin \sigma_{\mathcal{B}}(S + K)$. This proves the opposite inclusion.

3. Operators affiliated with a Jörgens algebra.

Let $\langle X, Y\rangle$ be a dual system. Dual systems are used in the theory of linear integral operators ([11] and [12]). Many linear differential operators have an inverse which is an integral operator. This is one motivation for studying closed operators affiliated with the algebra $\mathcal{A}(X, Y)$. In this section we develop information concerning such operators using the general theory presented in Section 2.

If Z is a subspace of X and W is a subspace of Y, then let

$$Z^\perp = \{y \in Y : \langle z, y\rangle = 0 \text{ for all } z \in Z\},$$
$$W^\perp = \{x \in X : \langle x, w\rangle = 0 \text{ for all } w \in W\}.$$

The subspace Z is *Y-total* if $Z^\perp = \{0\}$ and W is *X-total* if $W^\perp = \{0\}$.

Now we define the adjoint of an operator S.

DEFINITION 11. Let S be an operator in X with $D(S)$ Y-total. Define $y \in Y$ to be in $D(S^\dagger)$ if $\exists w \in Y$ such that $\langle Sx, y\rangle = \langle x, w\rangle$ for all $x \in D(S)$. For $y \in D(S^\dagger)$ and w as above, set $S^\dagger y = w$.

Note that $S^\dagger$ is well defined since $D(S)$ is assumed to be Y-total. Also, by definition

$$\langle Sx, y\rangle = \langle x, S^\dagger y\rangle \quad (x \in D(S),\ y \in D(S^\dagger)).$$

PROPOSITION 12. *Assume $(S, D(S))$ is a closed operator in X with $D(S)$ Y-total. Assume S is affiliated with $\mathcal{A}(X,Y)$. Then $D(S^\dagger)$ is X-total and $S^\dagger$ is a closed operator in Y.*

Proof. We may assume $T \in \mathcal{A}(X,Y)$ with $S^{-1} = T$. For any $y \in Y$, $x \in D(S)$, $\langle Sx, T^\dagger y\rangle = \langle TSx, y\rangle = \langle x, y\rangle$ so $\mathcal{R}(T^\dagger) \subseteq D(S^\dagger)$. Also, $\mathcal{R}(T^\dagger)$ is X-total since if $\langle x, T^\dagger y\rangle = 0$ for all $y \in Y$, then $\langle Tx, y\rangle = 0$ for all $y \in Y$, so $Tx = 0$. But then $0 = STx = x$. That $S^\dagger$ is closed is easy to verify.

We mention one situation where an operator may not have an adjoint in the usual sense, but does have an adjoint with respect to a natural dual system.

EXAMPLE 13. Let X be a nonreflexive Banach space. Assume S is an operator in X' with $D(S)$ X-total in X'. Consider the dual system $\langle X', X\rangle$ with the obvious bilinear form $\langle \alpha, x\rangle = \alpha(x)$ $(\alpha \in X', x \in X)$. Define $D(S^\dagger)$ and $S^\dagger$ as in Definition 11. In this situation $S^\dagger$ is called the *preconjugate* of S. This is a useful concept in the theory of ordinary differential operators; see [9, p. 126].

Now we note that $\mathcal{A}(X,Y)$ does satisfy condition (#) postulated in Section 2. First, Y can be considered as a subspace of X' via the identification $y(x) = \langle x, y\rangle$ $(y \in Y, x \in X)$. For $x \in X$ and $y \in Y$, define as before $(y \otimes x)(z) = \langle z, y\rangle x$ $(z \in X)$. It is easy to check that $(y \otimes x)^\dagger$ is the operator

$$(y \otimes x)^\dagger(w) = \langle x, w\rangle y \quad (w \in Y).$$

Therefore the collection $\{y \otimes x : x \in X,\ y \in Y\} \subseteq \mathcal{A}$, so $\mathcal{A}$ satisfies (#).

The socle of $\mathcal{A}$ is easily identified as $\mathrm{soc}(\mathcal{A}) = \mathcal{F}_{\mathcal{A}} = \mathrm{span}\{y \otimes x : x \in X,\ y \in Y\}$. We take as $\mathcal{K}_{\mathcal{A}}$ the ideal

$$\mathcal{K}_{\mathcal{A}} = \{T \in \mathcal{A} : \text{both } T \text{ and } T^\dagger \text{ are compact}\}.$$

The following result is the main theorem concerning operators affiliated with a Jörgens algebra.

THEOREM 14. *Assume S is a closed operator with $D(S)$ Y-total in X. Also assume $S^{-1} = T \in \mathcal{A} = \mathcal{A}(X,Y)$.*

(1) *For any λ, if $(\lambda - S)^{-1} \in \mathcal{A}$, then $(\lambda - S^\dagger)^{-1} = ((\lambda - S)^{-1})^\dagger$.*

(2) *For $\lambda \neq 0$,*

$$\lambda \in \varrho_{\mathcal{A}}(S) \Leftrightarrow (\lambda - S)^{-1} \in \mathcal{B}(X) \text{ and } (\lambda - S^\dagger)^{-1} \in \mathcal{B}(Y).$$

(3) *For $\lambda \neq 0$,*

$$\lambda - S \in \Phi(\mathcal{A}) \Leftrightarrow \begin{cases} \lambda - S \in \Phi(\overline{D(S)}, X), \text{ and} \\ \lambda - S^\dagger \in \Phi(\overline{D(S^\dagger)}, Y), \text{ and} \\ \mathrm{ind}(\lambda - S) = -\mathrm{ind}(\lambda - S^\dagger). \end{cases}$$

(4) *For $\lambda \neq 0$,*

$$\lambda - S \in \Phi^0(\mathcal{A}) \Leftrightarrow \begin{cases} \lambda - S \in \Phi^0(\overline{D(S)}, X), \text{ and} \\ \lambda - S^\dagger \in \Phi^0(\overline{D(S^\dagger)}, Y). \end{cases}$$

(5) *For $\lambda \neq 0$, if $\lambda - S \in \Phi(\mathcal{A})$, then*

(a) $\mathcal{N}(\lambda - S) = \mathcal{R}(\lambda - S^\dagger)^\perp$;
(b) $\mathcal{N}(\lambda - S)^\perp = \mathcal{R}(\lambda - S^\dagger)$;
(c) $\mathcal{N}(\lambda - S^\dagger) = \mathcal{R}(\lambda - S)^\perp$;
(d) $\mathcal{N}(\lambda - S^\dagger)^\perp = \mathcal{R}(\lambda - S)$;
(e) $\mathrm{nul}(\lambda - S) = \mathrm{def}(\lambda - S^\dagger)$;
(f) $\mathrm{nul}(\lambda - S^\dagger) = \mathrm{def}(\lambda - S)$.

Proof. To prove (1) it suffices to show that $(S^\dagger)^{-1} = T^\dagger$. For $x \in D(S)$ and $y \in Y$, $\langle Sx, T^\dagger y\rangle = \langle TSx, y\rangle = \langle x, y\rangle$. This shows $T^\dagger y \in D(S^\dagger)$ and $S^\dagger(T^\dagger y) = y$. To show $T^\dagger = (S^\dagger)^{-1}$, it remains to verify that $S^\dagger$ is one-to-one on $D(S^\dagger)$. Suppose $y \in D(S^\dagger)$ with $S^\dagger y = 0$. Then for all $x \in D(S)$, $0 = \langle x, S^\dagger y\rangle = \langle Sx, y\rangle \Rightarrow y = 0$. This proves (1).

To prove (2), first assume $\lambda \neq 0$ and $(\lambda - S)^{-1} \in \mathcal{A}$. Then certainly $(\lambda - S)^{-1} \in \mathcal{B}(X)$, and applying (1), $(\lambda - S^\dagger)^{-1} = ((\lambda - S)^{-1})^\dagger \in \mathcal{B}(Y)$. Conversely, assume $(\lambda - S)^{-1} \in \mathcal{B}(X)$ and $(\lambda - S^\dagger)^{-1} \in \mathcal{B}(Y)$. If $x \in X$, $y \in Y$, then $\exists u \in D(S), v \in D(S^\dagger)$ such that $x = (\lambda - S)u$ and $y = (\lambda - S^\dagger)v$. Then

$$\langle(\lambda - S)^{-1}x, y\rangle = \langle u, (\lambda - S^\dagger)v\rangle = \langle(\lambda - S)u, v\rangle = \langle x, (\lambda - S^\dagger)^{-1}y\rangle.$$

Thus $(\lambda - S)^{-1} \in \mathcal{A}$ and $((\lambda - S)^{-1})^\dagger = (\lambda - S^\dagger)^{-1}$.

By Theorem 6 we have for $\lambda \neq 0$, setting $\mu = \lambda^{-1}$, $\lambda - S \in \Phi(\mathcal{A}) \Leftrightarrow \mu - T \in \Phi(\mathcal{A})$. By [2, Theorem 2.5],

$$(6)\quad \mu - T \in \Phi(\mathcal{A}) \Leftrightarrow \begin{cases} \mu - T \in \Phi(X), \text{ and} \\ \mu - T^\dagger \in \Phi(Y), \text{ and} \\ \operatorname{ind}(\mu - T) = -\operatorname{ind}(\mu - T^\dagger). \end{cases}$$

Also we have

(7) $(\mu - T)S = -\lambda^{-1}(\lambda - S)$ on $D(S)$, and $(\mu - T^\dagger)S^\dagger = -\lambda^{-1}(\lambda - S^\dagger)$ on $D(S^\dagger)$.

Assume (6) holds. Now $S \in \Phi^o(\overline{D(S)}, X)$, so by (7), $\lambda - S \in \Phi(\overline{D(S)}, X)$ and $\operatorname{ind}(\lambda - S) = \operatorname{ind}(\mu - T)$ [15, Theorem 1.3, p. 163]. Similarly, $\lambda - S^\dagger \in \Phi(\overline{D(S)^\dagger}, Y)$ and $\operatorname{ind}(\lambda - S^\dagger) = \operatorname{ind}(\mu - T^\dagger)$. Thus, using (6), $\operatorname{ind}(\lambda - S) = -\operatorname{ind}(\lambda - S^\dagger)$. This proves (3).

Part (4) is an immediate consequence of (3).

Now we prove (5). Since $\lambda \neq 0$ and $\lambda - S \in \Phi(\mathcal{A})$, we have, as shown in the proof of (3), that for $\mu = \lambda^{-1}$, $\mu - T \in \Phi(X)$, $\mu - T^\dagger \in \Phi(Y)$, and $\operatorname{ind}(\mu - T) = -\operatorname{ind}(\mu - T^\dagger)$. It follows from [11, Theorem 5.16, p. 111] that:

(i) $\mathcal{N}(\mu - T) = \mathcal{R}(\mu - T^\dagger)^\perp$;
(ii) $\mathcal{N}(\mu - T^\dagger) = \mathcal{R}(\mu - T)^\perp$;
(iii) $\operatorname{nul}(\mu - T) = \operatorname{def}(\mu - T^\dagger)$;
(iv) $\operatorname{nul}(\mu - T^\dagger) = \operatorname{def}(\mu - T)$.

Let $\{x_1, \ldots, x_n\}$ be a basis for $\mathcal{N}(\mu - T)$. By [12, Lemma 4.3, p. 41] $\exists\{y_1, \ldots, y_n\} \subseteq Y$ such that $\langle x_k, y_j\rangle = \delta_{kj}$, $1 \leq k, j \leq n$. If $\lambda_k \in \mathbb{C}$ and $\lambda_1 y_1 + \ldots + \lambda_n y_n \in \mathcal{N}(\mu - T)^\perp$, then $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$. Using (i), this implies that $\operatorname{codim}(\mathcal{R}(\mu - T^\dagger)^{\perp\perp}) = \operatorname{codim}(\mathcal{N}(\mu - T)^\perp) \geq n$. By (iii), $\operatorname{codim}(\mathcal{R}(\mu - T^\dagger)) = n$. Since $\mathcal{R}(\mu - T^\dagger) \subseteq \mathcal{R}(\mu - T^\dagger)^{\perp\perp}$, it follows that $\mathcal{R}(\mu - T^\dagger) = \mathcal{R}(\mu - T^\dagger)^{\perp\perp} = \mathcal{N}(\mu - T)^\perp$. This proves

(v) $\mathcal{N}(\mu - T)^\perp = \mathcal{R}(\mu - T^\dagger)$.

A similar argument shows

(vi) $\mathcal{N}(\mu - T^\dagger)^\perp = \mathcal{R}(\mu - T)$.

Now as shown in Theorem 6, $\mathcal{R}(\lambda - S) = \mathcal{R}(\mu - T)$, $\mathcal{N}(\lambda - S) = T\mathcal{N}(\mu - T)$, $\mathcal{R}(\lambda - S^\dagger) = \mathcal{R}(\mu - T^\dagger)$, and $\mathcal{N}(\lambda - S^\dagger) = T^\dagger\mathcal{N}(\mu - T^\dagger)$. Therefore (i)–(vi) imply that (a)–(f) hold.

Let S be an operator affiliated with $\mathcal{A}(X, Y)$, and assume $D(S)$ is Y-total in X. In this situation the operator $S^\dagger$ can be defined, although S may have no adjoint on X' in the usual sense. One advantage gained from the existence of $S^\dagger$ is that equations determined by S have corresponding related adjoint equations. For example, we note the following application of Theorem 14.

Corollary 15. *Let S be as in Theorem 14, and assume that $\lambda - S$ is $\mathcal{A}$-Fredholm. Consider the equations:*

$$\text{(h)}\quad (\lambda - S)x = 0; \qquad \text{(g)}\quad (\lambda - S)x = z;$$
$$\text{(h}^\dagger\text{)}\quad (\lambda - S^\dagger)y = 0; \qquad \text{(g}^\dagger\text{)}\quad (\lambda - S^\dagger)y = w.$$

The solution spaces of (h) and (h†) are the finite-dimensional spaces $\mathcal{N}(\lambda - S)$ and $\mathcal{N}(\lambda - S^\dagger)$ respectively. For $z \in X$, equation (g) has a solution exactly when $z \in \mathcal{N}(\lambda - S^\dagger)^\perp$. For $w \in Y$, equation (g†) has a solution exactly when $w \in \mathcal{N}(\lambda - S)^\perp$. Furthermore, when $\lambda - S \in \Phi^0(\mathcal{A})$, then the dimensions of the solution spaces of (h) and (h†) are the same and the codimensions of $\mathcal{R}(\lambda - S)$ and $\mathcal{R}(\lambda - S^\dagger)$ are the same.

The Fredholm of index zero case is of special interest. We summarize the results from the general theory, Theorem 8, for a Jörgens algebra.

Theorem 16. *Assume S is a closed operator with $D(S)$ Y-total and $S^{-1} = T \in \mathcal{A} = \mathcal{A}(X, Y)$.*

(1) *For $\lambda \neq 0$, the following are equivalent.*
  (i) $\lambda - S \in \Phi^0(\mathcal{A})$.
  (ii) $\exists G \in \mathcal{F}_{\mathcal{A}}$ *such that* $(\lambda - S + \mu G)^{-1} \in \mathcal{B}(X)$ *and* $(\lambda - S^\dagger - \mu G^\dagger)^{-1} \in \mathcal{B}(Y)$ *for all* $\mu \neq 0$.
  (iii) $\exists K \in \mathcal{K}_{\mathcal{A}}$ *such that* $(\lambda - S + K)^{-1} \in \mathcal{B}(X)$ *and* $(\lambda - S^\dagger + K^\dagger)^{-1} \in \mathcal{B}(Y)$.

(2) $W_{\mathcal{A}}(S) = \bigcap\{\sigma_{\mathcal{A}}(S + K) : K \in \mathcal{K}_{\mathcal{A}}\}$.

When S has an inverse $S^{-1} = T \in \mathcal{A}$, then any special spectral properties that T might have relative to $\mathcal{A}$ have consequences for S. The next proposition illustrates this.

Proposition 17. *Assume S is a closed operator in X, $D(S)$ is Y-total, and S is affiliated with $\mathcal{A}(X, Y)$. If $\sigma(S)$ and $\sigma(S^\dagger)$ are both countable, then $\sigma(S) = \sigma(S^\dagger)$.*

Proof. Assume $S^{-1} = T \in \mathcal{A}$. By Theorem 14(2), $\sigma_{\mathcal{A}}(S) = \sigma(S) \cup \sigma(S^\dagger)$, which is countable. By Theorem 2(2), $\sigma_{\mathcal{A}}(S) = \{\lambda^{-1} : \lambda \in \sigma_{\mathcal{A}}(T) \setminus \{0\}\}$, so $\sigma_{\mathcal{A}}(T)$ is countable. Consider the two maps $\varphi : \mathcal{A} \to \mathcal{B}(X)$ and $\psi : \mathcal{A} \to \mathcal{B}(Y)$ defined by $\varphi(R) = R$ and $\psi(R) = R^\dagger$ for $R \in \mathcal{A}$. Applying [4, Theorem 4.5], any isolated point of $\sigma_{\mathcal{A}}(T)$ is in $\sigma(\varphi(T))$ and $\sigma(\psi(T))$. Since $\sigma_{\mathcal{A}}(T)$ is countable, the isolated points in $\sigma_{\mathcal{A}}(T)$ form a dense subset, so $\sigma_{\mathcal{A}}(T) \subseteq \sigma(\varphi(T))$ and $\sigma_{\mathcal{A}}(T) \subseteq \sigma(\psi(T))$. The opposite inclusions always hold, so $\sigma_{\mathcal{A}}(T) = \sigma(T) = \sigma(T^\dagger)$. From this it follows that $\sigma(S) = \sigma(S^\dagger)$.

Now we prove a proposition which we use in what follows. This result is of independent interest; it adds to the information given in [3, Theorem 4].

PROPOSITION 18. *Assume Y is a Banach space and X is a closed subspace of Y. Let $W \in \mathcal{B}(Y)$, and suppose $W(Y) \subseteq X$. Denote by W_0 the restriction of W to X, so $W_0 \in \mathcal{B}(X)$.*

(1) $\mathcal{R}(I - W_0) = \mathcal{R}(I - W) \cap X$.

(2) $\mathcal{R}(I - W_0)$ *is closed in* $X \Leftrightarrow \mathcal{R}(I - W)$ *is closed in* Y.

Proof. Clearly, $\mathcal{R}(I-W_0) \subseteq \mathcal{R}(I-W) \cap X$. If $(I-W)g = f$ and $f \in X$, then as $W(g) \in X$, we have $g \in X$. Therefore $\mathcal{R}(I-W) \cap X \subseteq \mathcal{R}(I-W_0)$.

To prove (2), first note that if $\mathcal{R}(I - W)$ is closed, then (1) implies $\mathcal{R}(I - W_0)$ is closed. Conversely, assume $\mathcal{R}(I - W_0)$ is closed. Let $N = \mathcal{N}(I - W_0)$, and note that $N = \mathcal{N}(I - W)$. Define V_0 and V by

$$V : (Y/N) \to Y, \quad V(y + N) = y + W(y),$$
$$V_0 : (X/N) \to X, \quad V_0(x + N) = x + W_0(x).$$

If $\mathcal{R}(I - W) = \mathcal{R}(V)$ is not closed, then for every n, $\exists y_n$ such that $\|V(y_n + N)\| < n^{-1}\|y_n + N\|$. Therefore $\exists\{z_n\} \subseteq Y$, $\|z_n + N\| = 1$, with $\|z_n + W(z_n)\| = \|V(z_n + N)\| \to 0$. Setting $v_n = -W(z_n)$, we have $\{v_n\} \subseteq X$ and $\|z_n - v_n\| \to 0$. Now $\|(z_n + N) - (v_n + N)\| \to 0$, so we may assume $\|v_n + N\| = 1$, $n \geq 1$. Also,

$$\|V_0(v_n+N)\| = \|v_n-W_0(v_n)\| \leq \|(v_n-z_n)+W(v_n-z_n)\|+\|z_n+W(z_n)\| \to 0.$$

This contradicts the fact that $\exists m > 0$ such that $\|V_0(v + N)\| \geq m\|v + N\|$ for all $v \in X$.

When S is closed and densely defined in X and has a densely defined adjoint S' in X', then S and S' share many properties. For example: $\sigma(S) = \sigma(S')$, $\omega(S) = \omega(S')$, and $\mathcal{R}(S)$ is closed if and only if $\mathcal{R}(S')$ is closed. In the general situation relative to a dual system, the spectral theory of S and $S^\dagger$ can be almost completely unrelated. In particular, any of the relationships listed above may fail when $S^\dagger$ is substituted in place of S'.

Next we consider an interesting special case where the spectral and Fredholm properties of S and $S^\dagger$ are related in the same strong ways as those of S and S'. Let Ω be a locally compact Hausdorff space which is σ-compact and let μ be a positive regular Borel measure on Ω. Denote by $BC = BC(\Omega)$ the Banach space of all bounded continuous complex-valued functions on Ω. Set $L^1 = L^1(\Omega, \mu)$ and $L^\infty = L^\infty(\Omega, \mu)$. Assume that μ satisfies the condition: If U is a nonempty open set, then $\mu(U) > 0$. This condition implies that if $f \in BC$ and $f = 0$ μ-a.e. on Ω, then $f \equiv 0$ on Ω. Thus the embedding of BC into L^∞ is one-to-one. Now $\langle BC, L^1\rangle$ is a dual system with the natural form

$$\langle f, g\rangle = \int_\Omega fg\, d\mu \quad (f \in BC,\ g \in L^1).$$

We consider operators affiliated with $\mathcal{A}(BC, L^1)$ which have a special, but common, property. Assume S is a closed operator in BC with $D(S)$ L^1-total in BC and also that:

(a) $D(S^\dagger)$ is dense in L^1; and

(b) $S^{-1} = T \in \mathcal{A}(BC, L^1)$ and $(T^\dagger)'(L^\infty) \subseteq BC$.

THEOREM 19. *Let S and T be as above, so in particular* (a) *and* (b) *hold.*

(1) $\sigma(S) = \sigma(S^\dagger) = \sigma_{\mathcal{A}}(S)$.

(2) $W(S) = W(S^\dagger) = W_{\mathcal{A}}(S)$.

(3) $\omega(S) = \omega(S^\dagger) = \omega_{\mathcal{A}}(S)$.

(4) $\mathcal{R}(\lambda - S)$ *is closed in* $BC \Leftrightarrow \mathcal{R}(\lambda - S^\dagger)$ *is closed in* L^1.

(5) $\lambda - S \in \Phi(\overline{D(S)}, X) \Leftrightarrow \lambda - S^\dagger \in \Phi(\overline{D(S^\dagger)}, Y)$, *and in this case,*

(a) $\mathcal{N}(\lambda - S)^\perp = \mathcal{R}(\lambda - S^\dagger)$; *and*

(b) $\mathcal{N}(\lambda - S^\dagger)^\perp = \mathcal{R}(\lambda - S)$.

Proof. First we note two key facts:

(A) The spectral and Fredholm properties of $T^\dagger$ on L^1 and $(T^\dagger)'$ on L^∞ are related in ways well known in standard operator theory; a reference is [11, Corollary 3, p. 91], for example.

(B) The spectral and Fredholm properties of T and $(T^\dagger)'$ are identical. This follows from assumption (b) and [3, Theorem 4].

Parts (1), (2), (3), and (5) of the theorem follow by combining (A), (B) with the statement and proof of Theorem 14.

To see (4), first apply Proposition 18 with $Y = L^\infty$, $X = BC$, $W = \lambda(T^\dagger)'$, and $W_0 = \lambda T$, $\lambda \neq 0$. Thus, for $\mu = \lambda^{-1}$, $\mathcal{R}(\mu - (T^\dagger)')$ is closed if and only if $\mathcal{R}(\mu - T)$ is closed. Now apply the theorem that $\mathcal{R}(\mu - T^\dagger)$ is closed if and only if $\mathcal{R}(\mu - (T^\dagger)')$ is closed [9, Theorem IV.1.2, p. 95]. This yields the conclusion:

$$\mathcal{R}(\mu - T) \text{ is closed} \Leftrightarrow \mathcal{R}(\mu - T^\dagger) \text{ is closed.}$$

Finally, as noted before, $\mathcal{R}(\lambda - S) = \mathcal{R}(\mu - T)$ and $\mathcal{R}(\lambda - S^\dagger) = \mathcal{R}(\mu - T^\dagger)$, so this proves (4).

Let S be a closed differential operator in $BC = BC(\Omega)$ having an inverse T which is an integral operator on BC. Very often T will have a natural extension $\overline{T}$ on $L^\infty(\Omega)$ (given by the same kernel that determines T) with the property $\overline{T}(L^\infty) \subseteq BC$. In this situation it is usual that properties (a) and (b) are satisfied, so that Theorem 19 holds.

4. Operators affiliated with $\mathcal{B}_{p,s}$. Let $\mathcal{B}_{p,s}$ be the algebra of operators introduced in Example II of the Introduction. The theory of spectral and Fredholm properties of an operator in $\mathcal{B}_{p,s}$ relative to the algebra $\mathcal{B}_{p,s}$ is developed in [4], so this information can be used to study properties of an operator S in $L^p \cap L^s$ with $S^{-1} \in \mathcal{B}_{p,s}$. This study is the object of this section. We need some basic facts concerning minimal closed extensions.

PROPOSITION 20. *Assume Y is a Banach space and X is a dense subspace of Y which is a Banach space continuously embedded in Y. Assume S is a closed operator in X with $S^{-1} = T \in \mathcal{B}(X)$ and that T has a bounded extension $\overline{T} \in \mathcal{B}(Y)$.*

(1) *S has a minimal closed extension $\overline{S}$ in Y if and only if $\mathcal{N}(\overline{T}) = \{0\}$.*
(2) *When $\mathcal{N}(\overline{T}) = \{0\}$, then $(\overline{S})^{-1} = \overline{T}$.*

Proof. First assume $\mathcal{N}(\overline{T}) = \{0\}$. Suppose $\{x_n\} \subseteq D(S)$, $\|x_n\|_Y \to 0$ and $\|S(x_n) - y\|_Y \to 0$. Then $\|x_n - \overline{T}(y)\|_Y = \|\overline{T}(Sx_n - y)\|_Y \to 0$, so $\overline{T}(y) = 0$. Therefore $y = 0$, so S has a minimal closure in Y.

Conversely, assume $y \in \mathcal{N}(\overline{T})$, $y \neq 0$. Choose $\{y_n\} \subseteq X$ such that $y_n \to y$ in Y. Then $\exists\{x_n\} \subseteq D(S)$ with $Sx_n = y_n$, and $\|x_n\|_Y = \|\overline{T}S(x_n)\|_Y = \|\overline{T}(S(x_n) - y)\|_Y \to 0$ and $\|Sx_n - y\|_Y \to 0$. This proves that S is not closable in Y.

Now assume $\mathcal{N}(\overline{T}) = \{0\}$ so $\overline{S}$ exists in Y. Let $y \in Y$ be arbitrary. Choose $\{x_n\} \subseteq X$ with $x_n \to y$ in Y, so $\|Tx_n - \overline{T}y\|_Y \to 0$ and $\|ST(x_n) - y\|_Y = \|x_n - y\|_Y \to 0$, so $\overline{T}y \in D(\overline{S})$ and $\overline{S}\,\overline{T}y = y$. Finally, assume $y \in D(\overline{S})$. Then $\exists\{x_n\} \subseteq D(S)$ with $x_n \to y$ in Y and $S(x_n) \to \overline{S}(y)$ in Y. Therefore $x_n = TS(x_n) = \overline{T}(Sx_n) \to \overline{T}\,\overline{S}y$ in Y, so $\overline{T}\,\overline{S}y = y$.

COROLLARY 21. *Assume S is a closed operator in $L^p \cap L^s$ and $(\lambda - S)^{-1} = R \in \mathcal{B}_{p,s}$. If for some $r \in [p, s]$, S has a minimal closure S_r in L^r, then $(\lambda - S_r)^{-1} = R_r$. Also, S_r exists exactly when $\mathcal{N}(R_r) = \{0\}$.*

Throughout this section S is an operator in $L^p \cap L^s$. When for some $r \in [p, s]$, S has a minimal closed extension in L^r, then this extension is denoted by S_r. Assuming $S^{-1} = T \in \mathcal{B}_{p,s}$, the next corollary notes the basic relationship between the operators S_r and T_r.

COROLLARY 22. *Let S be closed in $L^p \cap L^s$ and $S^{-1} = T \in \mathcal{B}_{p,s}$. Assume S_r exists for some $r \in [p, s]$.*

(1) *For $\lambda \neq 0$, $(\lambda - S_r)T_r = -\lambda(\lambda^{-1} - T_r)$.*
(2) *$\mathcal{R}(\lambda - S_r) = \mathcal{R}(\lambda^{-1} - T_r)$ and $T_r\mathcal{N}(\lambda^{-1} - T_r) = \mathcal{N}(\lambda - S_r)$. In particular, $\dim(\mathcal{N}(\lambda - S_r)) = \dim(\mathcal{N}(\lambda^{-1} - T_r))$.*

Proof. The key fact here is that $S_r^{-1} = T_r$ by Corollary 21. Thus, for $\lambda \neq 0$, $(\lambda - S_r)T_r = \lambda T_r - I = -\lambda(\lambda^{-1} - T_r)$. Part (2) follows immediately from (1) and $S_r^{-1} = T_r$.

Next we look at conditions which imply that S is affiliated with $\mathcal{B}_{p,s}$.

THEOREM 23. *Let S be a closed operator in $L^p \cap L^s$ such that S has minimal closures S_p on L^p and S_s on L^s. S is affiliated with $\mathcal{B}_{p,s}$ if and only if for some $\lambda \in \mathbb{C}$, $\lambda - S$, $\lambda - S_p$, and $\lambda - S_s$ have bounded inverses on $L^p \cap L^s$, L^p, and L^s, respectively. In this case, setting $R = (\lambda - S)^{-1}$, then $R_p = (\lambda - S_p)^{-1}$ and $R_s = (\lambda - S_s)^{-1}$.*

Proof. First assume S is affiliated with $\mathcal{B}_{p,s}$. By definition $\exists\lambda$ such that $(\lambda - S)^{-1} \in \mathcal{B}_{p,s}$. Set $R = (\lambda - S)^{-1}$. It follows from Corollary 21 that $R_p = (\lambda - S_p)^{-1}$ and $R_s = (\lambda - S_s)^{-1}$.

Conversely, assume $R = (\lambda - S)^{-1}$, $V = (\lambda - S_p)^{-1}$, and $W = (\lambda - S_s)^{-1}$ all exist for some λ. For $f \in L^p \cap L^s$, $R(f) \in D(S)$ and $f = (\lambda - S)Rf = (\lambda - S_p)Rf = (\lambda - S_s)Rf$. Thus, $Rf = Vf = Wf$. This implies $R \in \mathcal{B}_{p,s}$.

The algebra $\mathcal{B}_{p,s}$ satisfies conditon (#) in Section 2. To see this let q be the conjugate index of p and let t be the conjugate index of s. If $X = L^p \cap L^s$ and $Y = L^q \cap L^t$, then the operators

$$\{g \otimes f : f \in X,\ g \in Y\} \subseteq \mathcal{B}_{p,s},$$

where $g \otimes f$ acts on $L^p \cap L^s$ in the usual way:

$$(f \otimes g)(h) = \Big(\int_\Omega hf\,d\mu\Big)g \quad (h \in L^p \cap L^s).$$

We use the notation

$$\mathcal{F}_{p,s} = \{F \in \mathcal{B}_{p,s} : \mathcal{R}(F) \text{ is finite-dimensional in } L^p \cap L^s\}.$$

Also, let $\mathcal{K}_{p,s}$ be the closed inessential ideal consisting of all $T \in \mathcal{B}_{p,s}$ such that T_r is compact on L^r for all $r \in [p, s]$. These are the ideals in $\mathcal{B}_{p,s}$ relative to which Fredholm theory in $\mathcal{B}_{p,s}$ is developed.

The next theorem describes the basic $\mathcal{B}_{p,s}$-spectral theory and $\mathcal{B}_{p,s}$-Fredholm theory for a closed operator in $L^p \cap L^s$. Concerning the Fredholm part of the theory, it is assumed that $p \neq 1$ or $s \neq \infty$. Fredholm theory can be developed relative to $\mathcal{B}_{1,\infty}$, but we omit this case here to avoid technical complications.

When the measure space is finite, then all of the conditions stated in Theorem 24 simplify since $L^p \cap L^s = L^s$. In this case parts (2) and (3) of the theorem also hold for $p = 1$ and $s = \infty$.

THEOREM 24. *Let S be a closed operator in $L^p \cap L^s$ such that S_p and S_s exist. Assume $S^{-1} = T \in \mathcal{B} = \mathcal{B}_{p,s}$. For $\lambda \neq 0$:*

$$(1)\ \lambda \in \varrho_{\mathcal{B}}(S) \Leftrightarrow \begin{cases} (\lambda - S_p)^{-1} \in \mathcal{B}(L^p), \text{ and} \\ (\lambda - S_s)^{-1} \in \mathcal{B}(L^s), \text{ and} \\ (\lambda - S)^{-1} \in \mathcal{B}(L^p \cap L^s). \end{cases}$$

(2) *Assume either $p \neq 1$ or $s \neq \infty$. Then*

$$\lambda - S \in \Phi(\mathcal{B}) \Leftrightarrow \begin{cases} \lambda - S_p \in \Phi(\overline{D(S_p)}, L^p), \text{ and} \\ \lambda - S_s \in \Phi(\overline{D(S_s)}, L^s), \text{ and} \\ \lambda - S \in \Phi(\overline{D(S)}, L^p \cap L^s), \text{ and} \\ \operatorname{ind}(\lambda - S_p) = \operatorname{ind}(\lambda - S_s) = \operatorname{ind}(\lambda - S). \end{cases}$$

(3) *Assume either $p \neq 1$ or $s \neq \infty$. Then*

$$\lambda - S \in \Phi^0(\mathcal{B}) \Leftrightarrow \begin{cases} \lambda - S_p \in \Phi^0(\overline{D(S_p)}, L^p), \text{ and} \\ \lambda - S_s \in \Phi^0(\overline{D(S_s)}, L^s), \text{ and} \\ \lambda - S \in \Phi^0(\overline{D(S)}, L^p \cap L^s). \end{cases}$$

Proof. By Theorem 2

(i) $$\lambda \in \varrho_{\mathcal{B}}(S) \Leftrightarrow \lambda^{-1} \in \varrho_{\mathcal{B}}(T).$$

Also, by the characterization of spectrum in $\mathcal{B}_{p,s}$, [4, Theorem 5.1], we have

(ii) $$\mu \in \varrho_{\mathcal{B}}(T) \Leftrightarrow \begin{cases} (\mu - T_p)^{-1} \in \mathcal{B}(L^p), \text{ and} \\ (\mu - T_s)^{-1} \in \mathcal{B}(L^s), \text{ and} \\ (\mu - T)^{-1} \in \mathcal{B}(L^p \cap L^s). \end{cases}$$

Then (1) follows from (i), (ii), and Corollary 21.

The proof of (2) proceeds along similar lines. First by Theorem 6

(iii) $$\lambda - S \in \Phi(\mathcal{B}) \Leftrightarrow \lambda^{-1} - T \in \Phi(\mathcal{B}).$$

From [4, Theorem 5.6] we have

(iv) $$\mu - T \in \Phi(\mathcal{B}) \Leftrightarrow \begin{cases} \mu - T_p \in \Phi(L^p), \text{ and} \\ \mu - T_s \in \Phi(L^s), \text{ and} \\ \mu - T \in \Phi(L^p \cap L^s), \text{ and} \\ \operatorname{ind}(\mu - T_p) = \operatorname{ind}(\mu - T_s) = \operatorname{ind}(\mu - T). \end{cases}$$

Set $S = S_o$, $T = T_o$, and $L^o = L^p \cap L^s$. We have the additional fact from Corollary 22 that

(v) $$\mathcal{R}(\lambda - S_r) = \mathcal{R}(\lambda^{-1} - T_r), \quad \dim(\mathcal{N}(\lambda - S_r)) = \dim(\mathcal{N}(\lambda^{-1} - T_r)) \quad (r = o, p, s).$$

It follows from the defining properties of a Fredholm operator [15, p. 162] that $\lambda - S_r \in \Phi(\overline{D(S_r)}, L^r)$ if and only if $\lambda^{-1} - T_r \in \Phi(L^r)$, and in this case, $\operatorname{ind}(\lambda - S_r) = \operatorname{ind}(\lambda^{-1} - T_r)$, $r = o, p, s$. Therefore (2) follows from this argument and (iv).

Part (3) is a consequence of (2).

There is an elementary condition on a closed operator S in $L^p \cap L^s$ which assures that S_r exists for all $r \in [p, s]$. In order to prove that this condition suffices, we need some basic properties of an operator $T \in \mathcal{B}_{p,s}$. We state these in the following note.

NOTE. *Assume $T \in \mathcal{B}_{p,s}$ and fix $r \in [p, s]$.*

(1) *If $f \in L^p \cap L^r$, then $T_p(f) = T_r(f)$. If $f \in L^r \cap L^s$, then $T_r(f) = T_s(f)$.*

(2) *If $f \in L^r$, then $f = f_1 + f_2$ where $f_1 \in L^p \cap L^r$ and $f_2 \in L^r \cap L^s$. Also, $T_r(f) = T_p(f_1) + T_s(f_2)$.*

Proof. Assume $f \in L^p \cap L^r$. We may assume $f \geq 0$. Choose a sequence of simple functions $\{s_n\}$ with $0 \leq s_n \leq f$ and $s_n \uparrow f$ pointwise on Ω. Since $|s_n - f|^r \leq 2^r |f|^r$ and $|s_n - f|^p \leq 2^p |f|^p$, it follows from the Dominated Convergence Theorem that $\|s_n - f\|_r \to 0$ and $\|s_n - f\|_p \to 0$. Now $\{s_n\} \subseteq L^p \cap L^r$, so $T_p(s_n) = T_r(s_n)$ for $n \geq 1$. Therefore $T_p(f) = T_r(f)$. The proof of the second statement in (1) is the same.

To prove (2) assume $f \in L^r$ and set $E = \{x \in \Omega : |f(x)| \leq 1\}$. Letting χ_E and χ_{E^c} denote the characteristic functions of E and the complement of E, we have $f_2 = f\chi_E \in L^s \cap L^r$, $f_1 = f\chi_{E^c} \in L^p \cap L^r$, and $f = f_1 + f_2$. Also, $T_r(f) = T_p(f_1) + T_s(f_2)$ by (1).

THEOREM 25. *Assume S is a closed operator in $L^p \cap L^s$, S_p and S_s exist, and S is affiliated with $\mathcal{B}_{p,s}$. In addition assume $D(S_p) \cap D(S_s) = D(S)$. Then for all $r \in [p, s]$, S_r exists.*

Proof. Fix r, $p < r < s$. We may assume $S^{-1} = T \in \mathcal{B}_{p,s}$. By Corollary 21 it suffices to show that $\mathcal{N}(T_r) = \{0\}$. Suppose $f \in \mathcal{N}(T_r)$. As noted above, $f = f_1 + f_2$ where $f_1 \in L^r \cap L^p$ and $f_2 \in L^r \cap L^s$. Therefore $0 = T_r(f) = T_r(f_1) + T_r(f_2) = T_p(f_1) + T_s(f_2)$. By Corollary 21, $S_p^{-1} = T_p$ and $S_s^{-1} = T_s$. Therefore $T_p(-f_1) = T_s(f_2) \in D(S_p) \cap D(S_s) = D(S)$ (by assumption). Therefore $\exists g \in L^p \cap L^s$ such that $T(g) = T_p(-f_1) = T_s(f_2)$. Then $g = S(Tg) = S_p T_p(-f_1) = -f_1 = S_s T_s(f_2) = f_2$. Thus, $f = f_1 + f_2 = 0$.

When the measure μ is finite, then $L^s \cap L^p = L^s$. Thus $S = S_s$ and S_p is an extension of S_s. Therefore $D(S_s) \cap D(S_p) = D(S)$ automatically in this case. Similarly, when (Ω, μ) is special discrete (see [4]) it will always follow that $D(S_p) \cap D(S_s) = D(S)$.

One of the main objects of studying the spectral and Fredholm theory of a closed operator S relative to $\mathcal{B}_{p,s}$ is that by using this tool, properties that S, S_p, and S_s have in common can often be "interpolated" to S_r for all $r \in [p, s]$. The next two results are applications of this type. Again, the situation for the finite measure case is much simpler.

THEOREM 26. *Assume S is a closed operator in $L^p \cap L^s$ and S_p and S_s exist. Assume $D(S_p) \cap D(S_s) = D(S)$, and $S^{-1} = T \in \mathcal{B}_{p,s}$. Set $S_o = S$ and $L^o = L^p \cap L^s$. For $\lambda \neq 0$:*

(1) *Consider the property*

$$\text{(A)} \qquad (\lambda - S_r)^{-1} \in \mathcal{B}(L^r).$$

If (A) *holds for* $r = p, s, o$, *then* (A) *holds for all* $r \in [p, s]$.

In (2) *and* (3) *assume* $p \neq 1$ *or* $s \neq \infty$.

(2) *Consider the properties*

$$\text{(B)} \qquad \lambda - S_r \in \Phi(\overline{D(S_r)}, L^r) \quad \text{and} \quad \operatorname{ind}(\lambda - S_r) = \operatorname{ind}(\lambda - S).$$

If (B) *holds for* $r = p, s, o$, *then* (B) *holds for all* $r \in [p, s]$.

(3) *Consider the property*

$$\text{(C)} \qquad \lambda - S_r \in \Phi^0(\overline{D(S_r)}, L^r).$$

If (C) *holds for* $r = p, s, o$, *then* (C) *holds for all* $r \in [p, s]$.

Proof. First note that by Theorem 25, S_r exists for all $r \in [p, s]$.

Given that $(\lambda - S_r)^{-1} \in \mathcal{B}(L^r)$ for $r = o, p, s$, it follows from Theorem 24 that $\lambda \in \varrho_{\mathcal{B}}(S)$. Set $R = (\lambda - S)^{-1} \in \mathcal{B}_{p,s}$. Then by Corollary 21, for all $r \in [p, s]$, $(\lambda - S_r)^{-1} \in R_r \in \mathcal{B}(L^r)$. This proves (1).

Now assume $\lambda - S_r \in \Phi(\overline{D(S_r)}, L^r)$ and $\operatorname{ind}(\lambda - S_r) = \operatorname{ind}(\lambda - S)$ for $r = o, p, s$. Then Theorem 24 shows that $\lambda - S \in \Phi(\mathcal{B})$, and also, by Theorem 6, $\lambda^{-1} - T \in \Phi(\mathcal{B})$. Therefore [4, Theorem 3.3] implies that $\lambda^{-1} - T_r \in \Phi(L^r)$ and $\operatorname{ind}(\lambda^{-1} - T_r) = \operatorname{ind}(\lambda^{-1} - T)$ for all $r \in [p, s]$. Furthermore, by Corollary 22, we have $\mathcal{R}(\lambda - S_r) = \mathcal{R}(\lambda^{-1} - T_r)$ and $\dim(\mathcal{N}(\lambda - S_r)) = \dim(\mathcal{N}(\lambda^{-1} - T_r))$ for all $r \in [p, s]$. Therefore $\lambda - S_r \in \Phi(\overline{D(S_r)}, L^r)$ and $\operatorname{ind}(\lambda - S_r) = \operatorname{ind}(\lambda - S)$ for all $r \in [p, s]$.

Part (3) is a direct consequence of (2).

THEOREM 27. *Assume all the hypotheses of Theorem* 26. *Assume* $\lambda \neq 0$ *and* $p \neq 1$ *or* $s \neq \infty$.

(1) *If* $\lambda - S_r \in \Phi^0(\overline{D(S_r)}, L^r)$ *for* $r = o, p, s$, *then* $\exists K \in \mathcal{K}_{p,s}$ *such that* $(\lambda - S_r + K_r)^{-1} \in \mathcal{B}(L^r)$ *for all* $r \in [p, s]$ *and* $(\lambda - S + K)^{-1} \in \mathcal{B}(L^p \cap L^s)$.

(2) $W_{\mathcal{B}}(S) = \bigcap\{\sigma_{\mathcal{B}}(S + K) : K \in \mathcal{K}_{p,s}\}$.

Proof. The assumptions in (1) imply that $\lambda - S \in \Phi^0(\mathcal{B})$ by Theorem 24(3). By Theorem 8, $\exists K \in \mathcal{K}_{p,s}$ such that $(\lambda - S + K)^{-1} \in \mathcal{B}$. Therefore the conclusions in (1) follow from Corollary 21.

Part (2) is immediate from Theorem 10(4).

The final result in this section is another application of spectral theory relative to $\mathcal{B}_{p,s}$. Again, assuming $S^{-1} \in \mathcal{B}_{p,s}$, certain spectral properties of S^{-1} force corresponding properties of S. The result below is not the most general possible, but it has the virtue of being both elementary and interesting.

PROPOSITION 28. *Assume* S *is closed in* $L^p \cap L^s$, S_p *and* S_s *exist, and* $S^{-1} = T \in \mathcal{B}_{p,s}$.

(1) *Assume the underlying measure space has finite measure. For all* $r \in [p, s]$

$$\sigma(S_r) \subseteq \sigma(S_p) \cup \sigma(S_s).$$

(2) *If* $\sigma(S_p)$ *and* $\sigma(S_s)$ *are countable, then* $\sigma(S_r) = \sigma(S_p)$ *for all* $r \in [p, s]$ *such that* S_r *exists.*

Proof. The statements in (1) and (2) hold for T_r by [4, Theorem 5.1 and Corollary 5.2]. Since $\sigma(S_r) = \{\lambda^{-1} : \lambda \neq 0,\ \lambda \in \sigma(T_r)\}$ whenever S_r exists (Corollary 22), then (1) and (2) hold for S_r.

5. Semigroups affiliated with an algebra of operators. Let $\mathcal{B} \subseteq \mathcal{B}(X)$ be a Banach algebra of operators, and let $\mathcal{C}$ be a subset of $\mathcal{B}$. A semigroup of operators on X, $\{E(t)\}_{t \geq 0}$, is *affiliated with* $\mathcal{C}$ if $\{E(t)\} \subseteq \mathcal{C}$. In the first part of this section we develop a theory of semigroups affiliated with certain subsets $\mathcal{C}$ and the corresponding infinitesimal generators. We make no attempt at full generality; we deal only with the most basic aspects of the theory. Most of this theory is a straightforward modification of what happens in the situation where $\mathcal{C} = \mathcal{B}(X)$. In fact, we directly use and modify the arguments in Chapter X of M. Schechter's book [15]. Later in this section applications are given in the cases when $\mathcal{B}$ is a Jörgens algebra and when $\mathcal{B} = \mathcal{B}_{p,s}$.

The strong convergence of certain sequences of operators on X is an important ingredient in the development of classical semigroup theory. Thus it is no surprise that $\mathcal{C}$ must satisfy a closure property with respect to strong convergence. We deal with this property first.

DEFINITION 29. A subset $\mathcal{C} \subseteq \mathcal{B}$ has the *strong convergence property*, denoted by SCP, if $\{T_n\} \subseteq \mathcal{C}$, $\|T_n\|_{\mathcal{B}} \leq M$ for $n \geq 1$, and $T_n x \to Tx$ for all $x \in X$, implies that $T \in \mathcal{C}$ and $\|T\|_{\mathcal{B}} \leq M$.

Both $\mathcal{B}_{p,s}$ and $\mathcal{A}(X, Y)$ satisfy the SCP on the relevant Banach space. It is straightforward to verify this for $\mathcal{B}_{p,s}$.

PROPOSITION 30. $\mathcal{B}_{p,s}$ *satisfies the SCP on* $L^p \cap L^s$.

Proof. The complete norm on $L^p \cap L^s$ is

$$\|f\|_{p,s} = \max(\|f\|_p, \|f\|_s) \quad (f \in L^p \cap L^s).$$

Assume $\{T_n\} \subseteq \mathcal{B} = \mathcal{B}_{p,s}$, $T \in \mathcal{B}(L^p \cap L^s)$, $\|T_n\|_{\mathcal{B}} \leq M$ for $n \geq 1$, and $\|T_n(f) - T(f)\|_{p,s} \to 0$ for all $f \in L^p \cap L^s$. Then for each $f \in L^p \cap L^s$,

$$M\|f\|_p \geq \|T_n f\|_p \to \|Tf\|_p \quad \text{and} \quad M\|f\|_s \geq \|T_n f\|_s \to \|Tf\|_s.$$

Therefore $T \in \mathcal{B}$ and $\|T\|_{\mathcal{B}} \leq M$.

The situation with respect to the SCP property is more complicated in case of the Jörgens algebra. Before going further, we need to be precise concerning the properties the set $\mathcal{C}$ must satisfy. One necessary property is the SCP discussed above. We also need that $\mathcal{C}$ is closed under certain algebraic operations.

DEFINITION 31. Let $\mathcal{B}$ be a Banach algebra with unit 1. A subset $\mathcal{C} \subseteq \mathcal{B}$ is a *power closed cone* (p.c.c.) if:

(i) $1 \in \mathcal{C}$;
(ii) $T, R \in \mathcal{C}$ and $a \geq 0$, $b \geq 0 \Rightarrow aT + bR \in \mathcal{C}$; and
(iii) $T \in \mathcal{C}$, $n \geq 1 \Rightarrow T^n \in \mathcal{C}$.

Note that if $\mathcal{C}$ is a p.c.c. in $\mathcal{B}$ and $\mathcal{C}$ is closed in $\mathcal{B}$, then when $T \in \mathcal{C}$, $\{a_k\}_{k\geq 0} \subseteq [0,\infty)$, and $\sum_{k=0}^{\infty} a_k T^k$ converges in $\mathcal{B}$, then the sum of this series is in $\mathcal{C}$. In particular, $T \in \mathcal{C} \Rightarrow e^T = \exp(T) \in \mathcal{C}$.

Let $\mathcal{A} = \mathcal{A}(X,Y)$ be the Jörgens algebra relative to the dual system $\langle X, Y\rangle$. Let $X \oplus Y$ be the direct sum of X and Y with norm $\|x \oplus y\| = \max(\|x\|, \|y\|)$ where $x \in X$, $y \in Y$. Define a mapping $T \to \widetilde{T}$ of $\mathcal{A}$ into $\mathcal{B}(X \oplus Y)$ by $\widetilde{T} = T \oplus T^\dagger$ where $(T \oplus T^\dagger)(x \oplus y) = Tx \oplus T^\dagger y$. It is easy to check that $T \to \widetilde{T}$ is a linear isometry with the property that $(T^n)^\sim = (\widetilde{T})^n$ for $n \geq 1$.

PROPOSITION 32. *Assume the setting above. Then $\widetilde{\mathcal{A}} \subseteq \mathcal{B}(X \oplus Y)$ is a closed p.c.c. and satisfies the SCP on $X \oplus Y$.*

Proof. Assume $\{T_n\} \subseteq \mathcal{A}$ with $\widetilde{T}_n \to S$ in the strong operator topology in $\mathcal{B}(X \oplus Y)$. It follows that for all $x \in X$ and all $y \in Y$, $\{T_n(x)\}$ is Cauchy and $\{T_n^\dagger(y)\}$ is Cauchy. Define $T \in \mathcal{B}(X)$ and $R \in \mathcal{B}(Y)$ by

$$T(x) = \lim_{n\to\infty} T_n(x), \qquad R(y) = \lim_{n\to\infty} T_n^\dagger(y).$$

Therefore for $x \in X$, $y \in Y$,

$$\langle Tx, y\rangle = \lim \langle T_n x, y\rangle = \lim \langle x, T_n^\dagger y\rangle = \langle x, Ry\rangle.$$

Thus, $R = T^\dagger$, and it follows that $T \oplus T^\dagger = S$. Finally, by the construction it follows that if $\|\widetilde{T}_n\| = \max(\|T_n\|, \|T_n^\dagger\|) \leq M$ for $n \geq 1$, then $\|\widetilde{T}\| \leq M$.

That $\widetilde{\mathcal{A}}$ is a p.c.c. is easily verified.

The next two results form the basis for a theory of semigroups of operators affiliated with a closed p.c.c.

THEOREM 33. *Let $\mathcal{B} \subseteq \mathcal{B}(X)$ be a Banach algebra of operators. Assume $\mathcal{C} \subseteq \mathcal{B}$ is a closed p.c.c. in $\mathcal{B}$ with the SCP relative to X. Let S be an operator in X, and assume $\exists a \geq 0$ such that for $\lambda > 0$:*

(1) $(\lambda - S)^{-1} \in \mathcal{C}$; *and*
(2) $\|(\lambda - S)^{-1}\|_{\mathcal{B}} \leq (a+\lambda)^{-1}$.

Let $\{E(t)\}$ be the semigroup of operators in $\mathcal{B}(X)$ associated with S. Then $\{E(t)\} \subseteq \mathcal{C}$ and $\|E(t)\|_{\mathcal{B}} \leq e^{-at}$, $t \geq 0$.

Proof. We modify and use the proof of [15, Theorem 3.1, p. 229]. For $\lambda > 0$ set

$$T_\lambda = \lambda S(\lambda - S)^{-1} = -\lambda + \lambda^2(\lambda - S)^{-1}.$$

For $t \geq 0$, by (1)

$$\exp(tT_\lambda) = \exp(-t\lambda)\exp(t\lambda^2(\lambda - S)^{-1}) \in \mathcal{C}.$$

Also, using (2),

$$\begin{aligned}\|\exp(tT_\lambda)\|_{\mathcal{B}} &\leq \exp(-t\lambda)\exp(t\lambda^2\|(\lambda - S)^{-1}\|_{\mathcal{B}})\\ &\leq \exp(-t\lambda)\exp(t\lambda^2(a+\lambda)^{-1}) = \exp(-t\lambda a(a+\lambda)^{-1}).\end{aligned}$$

By the proof of [15, Theorem 3.1, p. 229], $\exp(tT_\lambda)x \to E(t)x$ for all $x \in X$ and all $t \geq 0$. By assumption $\mathcal{C}$ has the SCP, so $E(t) \in \mathcal{C}$ and $\|E(t)\|_{\mathcal{B}} \leq e^{-at}$ for $t \geq 0$.

Next we prove a converse by modifying the proof in [15, Theorem 4.2, p. 238].

THEOREM 34. *Let $\mathcal{B} \subseteq \mathcal{B}(X)$ be a Banach algebra of operators. Assume that $\mathcal{C} \subseteq \mathcal{B}$ is a closed p.c.c. in $\mathcal{B}$ with the SCP relative to X. Let $\{E(t)\}$ be a family in $\mathcal{C}$ having the properties:*

(a) $E(s)E(t) = E(s+t)$, $s \geq 0$, $t \geq 0$;
(b) $E(0) = I$;
(c) $\exists a \geq 0$ *such that* $\|E(t)\|_{\mathcal{B}} \leq e^{-at}$ *for* $t \geq 0$;
(d) $E(t)x$ *is continuous on* $[0,\infty)$ *for each* $x \in X$.

Then $E(t)$ has an infinitesimal generator W affiliated with $\mathcal{C}$, $(\lambda - W)^{-1} \in \mathcal{C}$ for all $\lambda > 0$, and $\|(\lambda - W)^{-1}\|_{\mathcal{B}} \leq (a+\lambda)^{-1}$ for all $\lambda > 0$.

Proof. We essentially follow the proof of [15, Theorem 4.2, p. 238], modifying the arguments when necessary. For $s > 0$, let $W(s) = s^{-1}(E(s) - I) \in \mathcal{B}$. As shown in [15, p. 237] the infinitesimal generator of $\{E(t)\}$ has domain $D(W) = \{x \in X : \lim_{s\to 0^+} W(s)x \text{ exists}\}$ and $Wx = \lim_{s\to 0^+} W(s)x$ for $x \in D(W)$. For $\lambda > 0$, $s > 0$,

$$\lambda - W(s) = s^{-1}(\lambda s - E(s) + I) = [s^{-1}(\lambda s + 1)][I - (E(s)(\lambda s+1)^{-1})].$$

Using (c), we have $\|E(s)(\lambda s+1)^{-1}\|_{\mathcal{B}} < 1$. Therefore $[I - E(s)(\lambda s+1)^{-1}]$ is invertible in $\mathcal{B}$ and is given by the series

$$[I - E(s)(\lambda s+1)^{-1}]^{-1} = \sum_{k=0}^{\infty}(\lambda s+1)^{-k}E(s)^k \in \mathcal{C},$$

and

$$||I - E(s)(\lambda s+1)^{-1}||_{\mathcal{B}} \le \sum_{k=0}^{\infty}(\lambda s+1)^{-k}(e^{-as})^k = \frac{1}{1-e^{-as}(\lambda s+1)^{-1}} = \left(\frac{\lambda s+1}{\lambda s+1-e^{-as}}\right).$$

Therefore for $s > 0$, $\lambda > 0$, $\lambda - W(s)$ is invertible in $\mathcal{B}$, $(\lambda - W(s))^{-1} \in \mathcal{C}$, and

$$||(\lambda - W(s))^{-1}||_{\mathcal{B}} \le \left(\frac{s}{\lambda s+1}\right)\left(\frac{\lambda s+1}{\lambda s+1-e^{-as}}\right) = \frac{s}{\lambda s+1-e^{-as}} \le \lambda^{-1}.$$

Now we follow the argument of Schechter [15, pp. 239–240] (with $W(s)$ in place of $A(s)$ and W in place of A). Near the bottom of p. 239 there is the conclusion that for all $y \in D(W)$, $||(\lambda - W)(\lambda - W(s))^{-1}y - y|| \to 0$ as $s \to 0^+$. Also, as shown in the proof, $(\lambda - W)^{-1} \in \mathcal{B}(X)$. Therefore for all $y \in D(W)$, $||(\lambda - W(s))^{-1}y - (\lambda - W)^{-1}y|| \to 0$ as $s \to 0^+$. It follows from [15, Lemma 3.2, p. 230] that $\lim_{s\to 0^+}(\lambda - W(s))^{-1}x = (\lambda - W)^{-1}x$ for all $x \in X$. Therefore since $\mathcal{C}$ has the SCP, $(\lambda - W)^{-1} \in \mathcal{C}$, and also,

$$||(\lambda - W)^{-1}||_{\mathcal{B}} \le \lim_{s\to 0^+}\left(\frac{s}{\lambda s+1-e^{-as}}\right) = (\lambda + a)^{-1}.$$

Now we apply Theorem 33 to the case of a Jörgens algebra, $\mathcal{A}(X,Y)$. The result gives a condition under which an operator S is an infinitesimal generator of a semigroup of operators $\{E(t)\}$, and at the same time, $S^\dagger$ is the infinitesimal generator of $\{E(t)^\dagger\}$.

THEOREM 35. *Let $\mathcal{C} \subseteq \mathcal{A}(X,Y)$ be such that $\widetilde{\mathcal{C}} = \{T \oplus T^\dagger : T \in \mathcal{C}\}$ is a closed p.c.c. in $\mathcal{B}(X \oplus Y)$ with the SCP on $X \oplus Y$. Assume S is a closed densely defined operator in X with $S^\dagger$ densely defined in Y. Suppose $\exists a \ge 0$ such that $(\lambda - S)^{-1} \in \mathcal{C}$ and $||(\lambda - S)^{-1}||_{\mathcal{A}} \le (a+\lambda)^{-1}$ for $\lambda > 0$. Then $\exists\{E(t)\} \subseteq \mathcal{C}$ with $\{E(t)\}$ a strongly continuous semigroup on X with infinitesimal generator S, and $\{E(t)^\dagger\}$ is a strongly continuous semigroup on Y with infinitesimal generator $S^\dagger$.*

Proof. Set $D(\widetilde{S}) = D(S) \oplus D(S^\dagger)$ in $X \oplus Y$, and define

$$\widetilde{S}(x \oplus y) = Sx \oplus S^\dagger y \quad (x \oplus y \in D(\widetilde{S})).$$

By Theorem 14(1) for all $\lambda > 0$, $((\lambda - S)^{-1})^\dagger = (\lambda - S^\dagger)^{-1}$. Therefore for $\lambda > 0$, $((\lambda - S)^{-1})^\sim = (\lambda - S)^{-1} \oplus (\lambda - S^\dagger)^{-1}$, and by hypothesis, $||((\lambda - S)^{-1})^\sim|| \le (a+\lambda)^{-1}$. By Theorem 33, $\exists\{F(t)\}$ a strongly continuous semigroup on $X \oplus Y$ with infinitesimal generator $\widetilde{S} = S \oplus S^\dagger$ such that $\{F(t)\} \subseteq \widetilde{\mathcal{C}}$. By definition of $\widetilde{\mathcal{C}}$, $F(t)$ has the form $F(t) = E(t) \oplus E(t)^\dagger$ $(t \ge 0)$ where $\{E(t)\} \subseteq \mathcal{C}$. The conclusions of the theorem now follow easily.

By Proposition 32, Theorem 35 applies in the case where $\mathcal{C} = \mathcal{A}(X,Y)$. The next example describes another situation where Theorem 35 applies.

EXAMPLE 36. Let X and Y be Banach lattices which are also a dual system $\langle X,Y\rangle$. Let $\mathcal{C} = \{T \in \mathcal{A}(X,Y) : T$ and $T^\dagger$ are positive $\}$. For information on Banach lattices and positive operators see [14], especially pp. 233–246. It is easy to verify that $\widetilde{\mathcal{C}}$ is a closed p.c.c. in $\mathcal{B}(X \oplus Y)$. Assume S is a closed densely defined operator with the properties:

(1) $S^\dagger$ is densely defined in Y;
(2) $(\lambda - S)^{-1} \in \mathcal{C}$, $\lambda > 0$; and
(3) $\exists a \ge 0$ such that $||(\lambda - S)^{-1}||_{\mathcal{A}} \le (a+\lambda)^{-1}$, $\lambda > 0$.

By Theorem 35, the strongly continuous semigroup $\{E(t)\}$ on X associated with S is in $\mathcal{C}$ and $\{E(t)^\dagger\}$ has infinitesimal generator $S^\dagger$.

Next we consider an important topic, the adjoint semigroup. The theory of the adjoint semigroup is part of the standard theory of semigroups of operators. The treatment we give here is close to that in [10, pp. 422–426]. Another approach to the adjoint semigroup can be found in [14, pp. 16–18].

Before proving the main theorem on adjoint semigroups, we need a preliminary result. Assume that S is a closed densely defined operator in X and $\exists\{\lambda_n\} \subseteq \mathbb{C}$ with $|\lambda_n| \to \infty$ such that $(\lambda_n - S)^{-1} \in \mathcal{B}(X)$ for $n \ge 1$ and $||(\lambda_n - S)^{-1}|| \to 0$ as $n \to \infty$. Let S' be the usual adjoint operator in X'. Consider the case where $D(S')$ is not dense in X'. Set $Y = \overline{D(S')}$. Now $\langle X,Y\rangle$ is a dual system with the obvious bilinear form: $\langle x,y\rangle = y(x)$, $x \in X$, $y \in Y$. The fact that this form is nondegenerate follows from the property that $D(S')$ is X-total [15, Theorem 4.1, p. 177]. Form $S^\dagger$ for the dual system $\langle X,Y\rangle$.

PROPOSITION 37. (1) *$S^\dagger$ is densely defined on Y.*

(2) *If $(\lambda - S)^{-1} \in \mathcal{B}(X)$, then $(\lambda - S^\dagger)^{-1} \in \mathcal{B}(Y)$ and $||(\lambda - S^\dagger)^{-1}|| \le ||(\lambda - S')^{-1}||$.*

Proof. First assume $(\lambda - S)^{-1} \in \mathcal{B}(X)$. For $\alpha \in X'$, $S'(\lambda - S')^{-1}\alpha = \lambda(\lambda - S')^{-1}\alpha - \alpha$. Since $(\lambda - S')^{-1}\alpha \in D(S')$ for all α, it follows that

$$(3)\quad S'(\lambda - S')^{-1}y \in \overline{D(S')} \quad (y \in \overline{D(S')}).$$

If $x \in D(S)$, $y \in Y = \overline{D(S')}$, then $\langle Sx, (\lambda - S')^{-1}y\rangle = \langle x, S'(\lambda - S')^{-1}y\rangle = \langle x,z\rangle$ for some $z \in Y$ by (3). Therefore if $(\lambda - S)^{-1} \in \mathcal{B}(X)$, then $w = (\lambda - S')^{-1}y \in D(S^\dagger)$ for all $y \in Y$ and $S^\dagger w = S'w$. Therefore $(\lambda - S^\dagger)^{-1}$ exists and $(\lambda - S^\dagger)^{-1}y = (\lambda - S')^{-1}y$ for $y \in Y$. Furthermore, $||(\lambda - S^\dagger)^{-1}|| \le ||(\lambda - S')^{-1}||$, and this proves (2).

To prove (1), let $y \in D(S')$. Then

$$\begin{aligned}||\lambda_n(\lambda_n - S')^{-1}y - y|| &= ||(\lambda_n - S')^{-1}(\lambda_n - S')y + (\lambda_n - S')^{-1}S'y - y|| \\ &= ||(\lambda_n - S')^{-1}S'y|| \to 0 \quad \text{as } n \to \infty .\end{aligned}$$

Since $(\lambda_n - S')^{-1}y \in D(S^\dagger)$ as shown above, this proves $y \in \overline{D(S^\dagger)}$. Therefore $D(S') \subseteq \overline{D(S^\dagger)}$, and so $Y = \overline{D(S^\dagger)}$.

The following theorem is the basic result concerning the adjoint semigroup. It is an easy consequence of Proposition 37 and Theorem 35.

THEOREM 38. *Let S be closed and densely defined in X. Assume $\exists a \geq 0$ such that*

$$(\lambda - S)^{-1} \in \mathcal{B}(X) \quad \text{and} \quad ||(\lambda - S)^{-1}|| \leq (a + \lambda)^{-1}, \quad \lambda > 0 .$$

Set $Y = \overline{D(S')}$ and $\mathcal{A} = \mathcal{A}(X, Y)$ (as above). Then $(\lambda - S)^{-1} \in \mathcal{A}$ and $||(\lambda - S)^{-1}||_{\mathcal{A}} \leq (a + \lambda)^{-1}$ for $\lambda > 0$. The adjoint $S^\dagger$ is densely defined in Y and $\sigma(S) = \sigma(S^\dagger)$. If $\{E(t)\}$ is the semigroup in $\mathcal{B}(X)$ associated with S, then $\{E(t)\} \subseteq \mathcal{A}$ and $\{E(t)^\dagger\}$ is a strongly continuous semigroup on Y with infinitesimal generator $S^\dagger$.

Now we give our final application of Theorem 33. In this case the setting is the algebra $\mathcal{B}_{p,s}$. The aim is to provide a condition which implies that a semigroup $\{E(t)\} \subseteq \mathcal{B}_{p,s}$ has the property that $\{E(t)_r\}$ is a strongly continuous semigroup on L^r for all $r \in [p, s]$. Some related results are given in [13, §4] and [14, pp. 335–336].

Recall the notation

$$||g||_{p,s} = \max(||g||_p, ||g||_s) \quad (g \in L^p \cap L^s) .$$

THEOREM 39. *Let $\mathcal{C}$ be a closed p.c.c. in $\mathcal{B}_{p,s}$. Assume S is a closed densely defined operator on $L^p \cap L^s$ such that $\exists a \geq 0$ such that for $\lambda > 0$,*

(1) $(\lambda - S)^{-1} \in \mathcal{C}$; *and*

(2) $||(\lambda - S)^{-1}||_{\mathcal{B}} \leq (a + \lambda)^{-1}$.

Let $\{E(t)\}$ be the strongly continuous semigroup on $L^p \cap L^s$ with infinitesimal generator S. Then

(a) $\{E(t)\} \subseteq \mathcal{C}$.

(b) $||E(t)||_{\mathcal{B}} \leq e^{-at}$, $t \geq 0$.

(c) *For $r \in [p, s]$, $\{E(t)_r\}$ is a strongly continuous semigroup on L^r.*

(d) *For $r \in [p, s]$, S has a minimal closure S_r on L^r and S_r is the infinitesimal generator of $\{E(t)_r\}$.*

Proof. Parts (a) and (b) follow immediately from Theorem 33.

Now consider the semigroup $\{E(t)_r\}$ on L^r. We know that for all $f \in L^p \cap L^s$, $\lim_{t\to 0+} ||E(t)f - f||_{p,s} = 0$. Since $||g||_r \leq ||g||_{p,s}$ for $g \in L^p \cap L^s$, it follows that

$$\lim_{t\to 0+} ||E(t)_r f - f||_r = 0$$

for all $f \in L^p \cap L^s$. Let $g \in L^r$ and $\varepsilon > 0$. Choose $f \in L^p \cap L^s$ such that $||g - f||_r < \varepsilon$. Then

$$\begin{aligned}||E(t)_r g - g||_r &\leq ||E(t)_r f - f||_r + ||E(t)_r(g - f)||_r + ||f - g||_r \\ &\leq ||E(t)_r f - f||_r + 2\varepsilon .\end{aligned}$$

Therefore

$$\lim_{t\to 0+} ||E(t)_r g - g||_r = 0 .$$

Now let W_r be the infinitesimal generator of $E(t)_r$ on L^r. Let $f \in D(S)$, so

$$\lim_{t\to 0+} \left|\left| \frac{E(t)f - f}{t} - S(f) \right|\right|_{p,s} = 0$$

by definition [15, p. 236]. Therefore

$$\lim_{t\to 0+} \left|\left| \frac{E(t)f - f}{t} - S(f) \right|\right|_{r} = 0 ,$$

and this implies $f \in D(W_r)$ and $W_r(f) = S(f)$. This proves W_r is an extension of S on L^r. It follows that S_r exists. Now by (1) and Corollary 21, $(\lambda - S_r)^{-1}$ exists for all $\lambda > 0$. Therefore for $\lambda > 0$ sufficiently large both $(\lambda - S_r)^{-1}$ and $(\lambda - W_r)^{-1}$ exist. Thus, $D(S_r) = D(W_r)$.

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Weighted norm inequalities on spaces of homogeneous type

by

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Abstract. We give a characterization of the weights (u,w) for which the Hardy–Littlewood maximal operator is bounded from the Orlicz space $L_\Phi(u)$ to $L_\Phi(w)$. We give a characterization of the weight functions w (respectively u) for which there exists a nontrivial u (respectively $w>0$ almost everywhere) such that the Hardy–Littlewood maximal operator is bounded from the Orlicz space $L_\Phi(u)$ to $L_\Phi(w)$.

1. Preliminaries and main results. The main objective of this paper is to study weight pairs (u,w) for which the Hardy–Littlewood maximal operator is bounded from the Orlicz space $L_\Phi(u)$ to $L_\Phi(w)$ in the context of spaces of homogeneous type. Some work in this direction was done in [1]–[3], [4]–[9], [11]–[15]. With this aim, we introduce some notations.

Let X be a set. A nonnegative symmetric function $d(x,y)$ defined on $X\times X$ will be called a *quasi-distance* if there exists an absolute constant D such that

$$d(x,y)\le D(d(x,z)+d(z,y))$$

for every $x,y,z\in X$, and $d(x,y)=0$ if and only if $x=y$. Let μ be a positive measure defined on a σ-algebra of subsets of X which contains balls $B(x,r)=\{y\,;d(x,y)<r\}$. Now we say that (X,d,μ) is a *space of homogeneous type* if X is a set endowed with a quasi-distance d and a positive measure μ such that:

(i) The family $\{B(x,r)\,;x\in X,\ r>0\}$ is a basis of the topology of X;

(ii) There exists a natural number N such that for any $x\in X$ and $r>0$ the ball $B(x,r)$ contains at most N points x_i with $d(x_i,x_j)\ge \frac12 r$;

(iii) μ is a doubling Borel measure, i.e., there exists a constant D such that $0<\mu(B(x,2r))\le D\mu(B(x,r))$ for all $x\in X$ and $r>0$.

Hereafter, we shall suppose that the continuous functions with compact support are dense in $L^p(X,d\mu)$ for $1\le p<\infty$.

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