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Decomposable multipliers and applications to harmonic analysis

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Abstract. For a multiplier on a semisimple commutative Banach algebra, the decomposability in the sense of Foiaş will be related to certain continuity properties and growth conditions of its Gelfand transform on the spectrum of the multiplier algebra. If the multiplier algebra is regular, then all multipliers will be seen to be decomposable. In general, an important tool will be the hull-kernel topology on the spectrum of the typically nonregular multiplier algebra. Our investigation involves various closed subalgebras of the multiplier algebra and includes perturbation results of Wiener–Pitt type for the invertibility of multipliers. Under suitable topological assumptions on the spectrum of the given Banach algebra, we shall characterize decomposable multipliers, Riesz multipliers, and multipliers with natural or countable spectrum. Most of these results are new even in the case of convolution operators given by measures on a locally compact abelian group. We shall obtain various classes of measures for which the corresponding convolution operators are decomposable both on the measure algebra and on the group algebra. Moreover, the spectral properties of a convolution operator will be related to the behavior of the Fourier–Stieltjes transform of the underlying measure on the dual group and on the spectrum of the measure algebra. Finally, it will be shown that the decomposability of convolution operators behaves nicely with respect to absolute continuity and singularity of measures.

Introduction. Some rather significant examples of the development of spectral theory outside the realm of Hilbert space are found among convolution operators on group algebras. Even here it is surprising how spotty and sketchy progress has been. One early result is due to Akemann [4], who characterized the compact and also the weakly compact left convolution operators on the group algebra $L_1(G)$ for a compact group G as those operators for which the corresponding measure is absolutely continuous with respect to Haar measure on G. For general locally compact abelian groups, Colojoară and Foiaş [10] considered the multiplication operator given by a fixed element of a regular semisimple commutative Banach algebra, of which

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the group algebra $L_1(G)$ for a locally compact abelian group G is an example. They showed that any such multiplication operator is decomposable in the sense of Foiaş [10], [25]. Their approach is similar to the one of the present paper in that they approached the specific from the general, using the theory of regular Banach algebras. They also posed a problem which is a guiding principle for the present work: to describe all the measures in the measure algebra $M(G)$ which define decomposable convolution operators. In 1982, independently, Albrecht [5] and Eschmeier [11] gave some answers. They showed that, for any nondiscrete locally compact abelian group, there will always exist a nondecomposable convolution operator. They also proved that any measure on G whose continuous part is absolutely continuous with respect to Haar measure on G will induce a decomposable convolution operator on $L_1(G)$. A somewhat stronger result was obtained by the present authors in [18], but a measure-theoretic characterization of all those measures for which the corresponding convolution operator is decomposable on $L_1(G)$ or on $M(G)$ is still missing.

It turns out that this problem is related to the classical inversion problem for measures μ on G whose Fourier–Stieltjes transform $\widehat{\mu}$ on the dual group Γ is bounded away from zero. This inversion problem dates back to work of Beurling [7] and Wiener–Pitt [26] from 1938. In the same vein, let us also mention the investigation of Zafran [27] on measures μ on G which have a natural spectrum in the sense that $\sigma(\mu) = \widehat{\mu}(\Gamma)^-$. Our results on decomposable convolution operators in the last section of this paper will shed some new light on these problems.

Recently, an intrinsic characterization of the decomposability of multiplication operators on an arbitrary semisimple commutative Banach algebra A has been given in [19]: decomposability of the operator L_a of multiplication by an element $a \in A$ is equivalent to continuity of the Gelfand transform $\widehat{a}$ in the hull-kernel topology of the spectrum $\Delta(A)$ (see also [20]). This description of decomposability has some immediate consequences for permanence: sums, products, and uniform limits of such decomposable elements will be decomposable. This is the starting point of the present paper. The hull-kernel description of decomposability allows a unified approach to the questions that we have alluded to, based in the general theory of semisimple commutative Banach algebras. Thus the objects of study will often be subalgebras and ideals of the algebra A and of its multiplier algebra $M(A)$, rather than individual elements. Particularly definitive results will be obtained for two classes of multipliers that satisfy certain growth conditions: the class $M_0(A)$ of all multipliers on A whose Gelfand transforms on $\Delta(A)$ vanish at infinity, and the class $M_{00}(A)$ of all multipliers on A whose Gelfand transforms on $\Delta(M(A))$ vanish outside $\Delta(A)$. In the classical case of the group algebra $A = L_1(G)$ for a locally compact abelian group G, these classes correspond to certain important ideals $M_0(G)$ and $M_{00}(G)$ of the measure algebra $M(G)$.

The section headlines indicate the program of this paper. In the preliminary first section, we collect some basic definitions and facts on decomposable operators and multipliers. The next section contains our main results on decomposable multipliers on a semisimple commutative Banach algebra A. We discuss the relationship between the decomposability of a multiplier on A, the hull-kernel continuity of its Gelfand transform on the spectra $\Delta(A)$ and $\Delta(M(A))$, and a natural spectral property for multipliers in the spirit of Zafran [27]. Typical results include the following: under the strong assumption that the multiplier algebra $M(A)$ be regular, all multipliers are decomposable on A, whereas, under the weaker assumption that A be regular, a multiplier T in $M_0(A)$ is decomposable on A if and only if T belongs to $M_{00}(A)$. Moreover, if A is regular, then the greatest regular Banach subalgebra of $M_0(A)$ is seen to be precisely $M_{00}(A)$. Finally, from the fact that all decomposable multipliers have natural spectrum, we derive a general Wiener–Pitt type criterion [26] for the invertibility of multipliers on a regular Banach algebra with a bounded approximate identity.

In the third section, we impose topological assumptions on the spectrum of the algebra A. If $\Delta(A)$ is scattered in the Gelfand topology, then the decomposability of a multiplier $T \in M_0(A)$ is equivalent to the hull-kernel continuity of its Gelfand transform on $\Delta(M(A))$, to the natural spectrum property of T, and also to the countability of the spectrum of T. And if the topological condition on $\Delta(A)$ is strengthened to discreteness, then we can add to this list of equivalences the condition that the multiplier $T \in M_0(A)$ be a Riesz operator on A. These results improve recent work of Aiena [3] and apply directly to the case of group algebras on compact abelian groups.

In the last section, we investigate convolution operators given by regular Borel measures on an arbitrary locally compact abelian group G. By means of the greatest regular subalgebra of the measure algebra $M(G)$, we identify some classes of measures for which the corresponding convolution operators are decomposable both on $L_1(G)$ and on $M(G)$. We also describe the spectral maximal spaces of these operators in simple terms and show that decomposability is preserved under absolute continuity. For measures on G whose Fourier–Stieltjes transforms vanish at infinity in the dual group, we obtain various characterizations of decomposability, thus extending work of Albrecht [5]. We also locate certain singular measures which induce nondecomposable convolution operators and characterize a class of Riesz product measures for which convolution is decomposable on the circle group. Moreover, we improve some classical results due to Beurling [7], Wiener–Pitt [26], and Zafran [27] and correct recent work of Aiena [2] on the characterization of Riesz multipliers on $L_1(G)$.

1. Preliminaries on decomposable operators and multipliers. Given a complex Banach space X, let $\mathcal{L}(X)$ denote the Banach algebra of all continuous linear operators on X. An operator $T \in \mathcal{L}(X)$ is called *decomposable* if, for every open covering $\{U, V\}$ of the complex plane $\mathbb{C}$, there exists a pair of T-invariant closed linear subspaces Y and Z of X such that $Y + Z = X$, $\sigma(T|Y) \subseteq U$, and $\sigma(T|Z) \subseteq V$, where σ denotes the spectrum. We refer to Theorem IV.4.28 of [25] for various characterizations and to the monographs [10] and [25] for a thorough discussion of decomposable operators.

For a decomposable operator $T \in \mathcal{L}(X)$ and a closed subset F of $\mathbb{C}$, let $X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\}$ denote the corresponding *spectral maximal space*, where $\sigma_T(x) \subseteq \mathbb{C}$ is the local spectrum of T at the point $x \in X$, i.e. the complement of the set of all those $\lambda \in \mathbb{C}$ for which there exist an open neighborhood U of λ in $\mathbb{C}$ and an analytic function $f : U \to X$ such that $(T - \mu)f(\mu) = x$ holds for all $\mu \in U$. The spaces $X_T(F)$ are hyperinvariant closed linear subspaces of X (cf. [10]).

Next, given a commutative complex Banach algebra A with or without identity, let $\Delta(A)$ stand for the spectrum of A, i.e. the set of all nontrivial multiplicative linear functionals on A. For each $a \in A$, let $\widehat{a} : \Delta(A) \to \mathbb{C}$ denote the corresponding Gelfand transform given by $\widehat{a}(\varphi) := \varphi(a)$ for all $\varphi \in \Delta(A)$. On $\Delta(A)$ we shall have to consider both the Gelfand and the hull-kernel topology. The latter is determined by the Kuratowski closure operation $\mathrm{cl}(E) := \mathrm{hul}(\ker(E)) := \{\psi \in \Delta(A) : \psi(u) = 0$ for all $u \in A$ with $\varphi(u) = 0$ for each $\varphi \in E\}$ for all $E \subseteq \Delta(A)$. The hull-kernel topology is always coarser than the Gelfand topology on $\Delta(A)$, and they coincide if and only if the algebra A is regular. Consequently, for some $a \in A$ the Gelfand transform $\widehat{a}$ will not be hull-kernel continuous on $\Delta(A)$ whenever the algebra A is nonregular. For further information concerning the hull-kernel topology, we refer to [9] and [22]. We shall frequently use the following result from [20]. Some related results can be found in [19].

1.1. Proposition. *Let A be a semisimple commutative complex Banach algebra, and consider an algebraic homomorphism $\Phi : A \to \mathcal{L}(X)$. Then for every $a \in A$ for which the Gelfand transform $\widehat{a}$ is continuous with respect to the hull-kernel topology of $\Delta(A)$, the corresponding operator $T := \Phi(a) \in \mathcal{L}(X)$ is decomposable.*

Finally, we shall use some standard results from the general theory of multipliers as presented, for instance, in [17]. Throughout this paper, let A be a semisimple commutative Banach algebra over $\mathbb{C}$, and let $M(A)$ denote the corresponding *multiplier algebra* consisting of all mappings $T : A \to A$ with the property $T(u)v = uT(v)$ for all $u, v \in A$. Every $T \in M(A)$ satisfies $T(uv) = T(u)v$ for all $u, v \in A$; moreover, $M(A)$ is a semisimple closed commutative subalgebra of $\mathcal{L}(A)$ containing the identity operator I (cf. [17]).

For each $a \in A$, let $L_a : A \to A$ denote the corresponding multiplication operator on A given by $L_a(u) := au$ for all $u \in A$. By semisimplicity, we can identify A with the ideal $\{L_a : a \in A\}$ of $M(A)$. Note, however, that the norm of A may be strictly greater than the operator norm inherited from $M(A)$ and that A need not be closed in $M(A)$.

By the results of Section 1.4 in [17], the spectrum $\Delta(M(A))$ of the multiplier algebra can be represented as the disjoint union of $\Delta(A)$ and $H(A)$, where $\Delta(A)$ is canonically embedded in $\Delta(M(A))$ and $H(A)$ denotes the hull of A in $\Delta(M(A))$. When $\Delta(A)$ is regarded as a subset of $\Delta(M(A))$, the hull-kernel topology of $\Delta(A)$ coincides with the relative hull-kernel topology induced by $\Delta(M(A))$; the same result holds with respect to the Gelfand topology. Obviously, $\Delta(A)$ is hull-kernel and hence Gelfand open in $\Delta(M(A))$. Moreover, it is easily seen that $\Delta(A)$ is hull-kernel dense in $\Delta(M(A))$, but this is certainly not always true for the Gelfand topology. Next let

$$M_0(A) := \{T \in M(A) : \widehat{T}|\Delta(A) \text{ vanishes at infinity in the Gelfand topology of } \Delta(A)\},$$

$$M_{00}(A) := \{T \in M(A) : \widehat{T} \equiv 0 \text{ on } H(A)\} = \ker(\mathrm{hul}(A)).$$

Clearly, $M_0(A)$ and $M_{00}(A)$ are closed ideals in $M(A)$ with $A \subseteq M_{00}(A) \subseteq M_0(A)$. In general, A is strictly contained in $M_{00}(A)$, as a result of Hewitt–Zuckerman [16] shows: in any nondiscrete locally compact abelian group G, they construct a singular measure μ for which the convolution square $\mu * \mu$ is absolutely continuous. Since $(\mu * \mu)^\wedge = \mu^\wedge \cdot \mu^\wedge$, this implies $\mu \in M_{00}(L_1(G)) \setminus L_1(G)$. Also the inclusion $M_{00}(A) \subseteq M_0(A)$ may be strict, as we shall see after the next proposition. Further information will be obtained in 2.5.

We close this section by recalling some elementary spectral properties of multipliers. As usual, the point and the residual spectrum of a continuous linear operator T on a given Banach space will be denoted by $\sigma_p(T)$ and $\sigma_r(T)$, respectively; and $\sigma(T, B)$ stands for the spectrum of an element T of a Banach algebra B.

1.2. Proposition. *For each $T \in M(A)$ we have:*

(a) *$\sigma(T) = \sigma(T, M(A)) = \widehat{T}(\Delta(M(A)))$; and $\sigma(T) = \sigma(T, M_0(A))$ whenever $T \in M_0(A)$.*

(b) *$\sigma_p(T) \subseteq \widehat{T}(\Delta(A)) \subseteq \sigma_p(T) \cup \sigma_r(T)$.*

(c) *If $\widehat{T}$ is hull-kernel continuous on $\Delta(M(A))$, then $\sigma(T) = \widehat{T}(\Delta(A))^-$.*

(d) *If A has no unit and $T \in M_0(A)$, then $\sigma(T) \supseteq \widehat{T}(\Delta(A)) \cup \{0\} = \widehat{T}(\Delta(A))^-$.*

(e) *If A has no unit and $T \in M_{00}(A)$, then $\sigma(T) = \widehat{T}(\Delta(A)) \cup \{0\} = \widehat{T}(\Delta(A))^-$.*

Proof. Statement (a) is clear from Corollary 1.1.1 of [17], Theorem 2 of [3], and elementary Gelfand theory. (b) has been shown in Theorem 3 of [3]. (c) follows immediately from (a) and the hull-kernel denseness of $\Delta(A)$ in $\Delta(M(A))$. Finally, because of $\Delta(M(A)) = \Delta(A) \cup H(A)$, (d) and (e) are easy consequences of (a).

A multiplier $T \in M(A)$ is said to have *natural spectrum* if $\sigma(T) = \widehat{T}(\Delta(A))^-$. In the case of the group algebra $A = L_1(G)$ for a locally compact abelian group G, the systematic investigation of multipliers with natural spectrum dates back to Zafran [27]. It is noted in [27] that there are always multipliers $T \in M_0(L_1(G))$ with nonnatural spectrum, except for the trivial case that G is discrete. In view of 1.2, this implies that $M_{00}(A) \neq M_0(A)$ whenever $A = L_1(G)$ and G is a nondiscrete locally compact abelian group. Of course, this fact is also closely related to the inversion problem for measures in the measure algebra $M(G)$, which has a long tradition in classical Fourier analysis; see for instance [7], [13], [26] and, in particular, 8.2.6 of [12]. In the following, we shall obtain some further information about multipliers with natural spectrum.

2. Decomposable multipliers on semisimple commutative Banach algebras. We first prove that all decomposable multipliers on a semisimple commutative Banach algebra A have natural spectrum. The converse is not true in general: for group algebras over certain groups, Albrecht [5] has given an example of a nondecomposable multiplier which does have a natural spectrum. On the other hand, we shall show in 3.1 that, under a topological assumption on $\Delta(A)$, decomposability characterizes the multipliers in $M_0(A)$ with natural spectrum.

2.1. PROPOSITION. *Assume that $T \in M(A)$ is decomposable. Then $\widehat{T}|\Delta(A)$ is hull-kernel continuous on $\Delta(A)$ and $\sigma(T) = \widehat{T}(\Delta(A))^-$.*

Proof. Suppose that the restriction $\widehat{T}|\Delta(A)$ is not hull-kernel continuous on $\Delta(A)$. Then there exists a closed subset F of $\mathbb{C}$ such that $E := \{\varphi \in \Delta(A) : \widehat{T}(\varphi) \in F\}$ is not hull-kernel closed in $\Delta(A)$. Let $\psi \in \mathrm{cl}(E) \setminus E$, where $\mathrm{cl}(E) = \mathrm{hul}(\ker(E))$ denotes the hull-kernel closure of E with respect to $\Delta(A)$, and consider $\lambda := \widehat{T}(\psi) \notin F$. By decomposability, there exist T-invariant closed linear subspaces Y and Z of A such that $\sigma(T|Y) \subseteq \mathbb{C}\setminus\{\lambda\}$, $\sigma(T|Z) \subseteq \mathbb{C}\setminus F$, and $Y+Z = A$. From $\sigma(T|Y) \subseteq \mathbb{C}\setminus\{\lambda\}$ we conclude that for each $u \in Y$ there exists some $v \in Y$ such that $u = (T-\lambda I)v$, which implies $\psi(u) = \psi(Tv) - \lambda\psi(v) = \widehat{T}(\psi)\psi(v) - \lambda\psi(v) = 0$. Thus $\psi \equiv 0$ on Y. On the other hand, given an arbitrary $\varphi \in E$, we note that $\mu := \widehat{T}(\varphi) \in F$ so that $\mu \notin \sigma(T|Z)$. Consequently, for each $u \in Z$ there exists some $v \in Z$ such that $u = (T - \mu I)v$, which implies that $\varphi(u) = \varphi(Tv) - \mu\varphi(v) = \widehat{T}(\varphi)\varphi(v) - \mu\varphi(v) = 0$ and hence $\varphi \equiv 0$ on Z. From $\psi \in \mathrm{cl}(E)$ we conclude that also $\psi \equiv 0$ on Z. Since $Y + Z = A$, it follows that the functional $\psi \in \Delta(A)$ vanishes identically on A. This is a contradiction and shows that $\widehat{T}|\Delta(A)$ is hull-kernel continuous on $\Delta(A)$. To show that T has natural spectrum, let $C^*(\Delta(A))$ denote the Banach algebra of all Gelfand continuous bounded complex-valued functions on $\Delta(A)$, endowed with the supremum norm. Moreover, let $R : A \to C^*(\Delta(A))$ denote the Gelfand transform given by $R(u) := \widehat{u}$ for all $u \in A$, and let $S : C^*(\Delta(A)) \to C^*(\Delta(A))$ denote the operator given by multiplication by $\widehat{T}$. Then obviously $RT = SR$. Since T is decomposable and since R is injective by the semisimplicity of A, we conclude from Lemma 1 of [11] that $\sigma(T) \subseteq \sigma(S)$. But it is trivial that $\sigma(S) \subseteq \widehat{T}(\Delta(A))^- \subseteq \sigma(T)$. Hence T has natural spectrum.

In general, the hull-kernel continuity of the restriction $\widehat{T}|\Delta(A)$ does not imply that T is decomposable. Indeed, if A is regular, then every $T \in M(A)$ has the property that $\widehat{T}|\Delta(A)$ is hull-kernel continuous, since the Gelfand and the hull-kernel topology coincide on $\Delta(A)$. But, as mentioned above, even in the case of the group algebra of an arbitrary nondiscrete locally compact abelian group, we know from [27] that there are multipliers with nonnatural spectrum, and by 2.1 such a multiplier cannot be decomposable. On the other hand, it follows immediately from 1.1 that a multiplier $T \in M(A)$ is decomposable under the stronger assumption that $\widehat{T}$ is hull-kernel continuous on $\Delta(M(A))$.

2.2. PROPOSITION. *For each $T \in M(A)$, the following assertions are equivalent:*

(a) *$T \in M_0(A)$ and $\widehat{T}$ is hull-kernel continuous on $\Delta(M(A))$.*

(b) *$T \in M_{00}(A)$ and $\widehat{T}|\Delta(A)$ is hull-kernel continuous on $\Delta(A)$.*

Proof. (a)⇒(b). To show that $\widehat{T}$ vanishes on $H(A)$, let $\varepsilon > 0$ be arbitrarily given. Then (a) implies that $E := \{\varphi \in \Delta(A) : |\widehat{T}(\varphi)| \geq \varepsilon\}$ is a Gelfand compact hull in $\Delta(A)$. Hence, by Theorem 3.6.7 of [22], the kernel of E in A is a modular ideal in A. Consequently, by Lemma 3.1.15 of [22], there exists some $u \in A$ such that $\widehat{u} \equiv 1$ on E. Since the element $T(u) \in A$ satisfies $|T(u)^\wedge| = |\widehat{T}\widehat{u}| \geq \varepsilon$ on the hull E, we may apply Theorem 3.6.15 of [22] to obtain some $v \in A$ such that $\widehat{T}\widehat{u}\widehat{v} \equiv 1$ on E. We conclude that the multiplier $S := I - T(uv) \in M(A)$ satisfies $\widehat{S} \equiv 0$ on E and $\widehat{S} \equiv 1$ on $H(A)$. Therefore $\mathrm{cl}(E) \cap H(A) = \emptyset$, where $\mathrm{cl}(E) \subseteq \Delta(M(A))$ denotes the hull-kernel closure of E in $\Delta(M(A))$. Now suppose that $|\widehat{T}(\psi)| > \varepsilon$ for some $\psi \in H(A)$. Then, by the hull-kernel continuity of $\widehat{T}$ on $\Delta(M(A))$, there exists some hull-kernel open neighborhood U of ψ in $\Delta(M(A))$ such that $|\widehat{T}(\varphi)| > \varepsilon$

for all $\varphi \in U$. Since $\Delta(A)$ is hull-kernel dense in $\Delta(M(A))$, it follows that $U \cap V \cap \Delta(A)$ is nonempty for each hull-kernel open neighborhood V of ψ in $\Delta(M(A))$. Since $U \cap \Delta(A) \subseteq E$, we conclude that $V \cap E \neq \emptyset$ for every hull-kernel open neighborhood V of ψ in $\Delta(M(A))$ and hence $\psi \in \mathrm{cl}(E)$. Since $\mathrm{cl}(E) \cap H(A) = \emptyset$, this is a contradiction and shows that $|\widehat{T}(\psi)| \leq \varepsilon$ for all $\psi \in H(A)$ and all $\varepsilon > 0$, hence that $\widehat{T} \equiv 0$ on $H(A)$.

(b)$\Rightarrow$(a). To prove the hull-kernel continuity of $\widehat{T}$ on $\Delta(M(A))$, let F be an arbitrary closed nonempty subset of $\mathbb{C}$, and let $E := \{\varphi \in \Delta(M(A)) : \widehat{T}(\varphi) \in F\}$ be its preimage under $\widehat{T}$ in $\Delta(M(A))$. To show that E is hull-kernel closed in $\Delta(M(A))$, we consider two cases. If $0 \in F$, it follows from $\widehat{T} \equiv 0$ on $H(A)$ that $\Delta(M(A)) \setminus E = \{\varphi \in \Delta(A) : \widehat{T}(\varphi) \in \mathbb{C} \setminus F\} \subseteq \Delta(A)$. Since $\Delta(A)$ is hull-kernel open in $\Delta(M(A))$, the hull-kernel continuity of $\widehat{T}|\Delta(A)$ implies that $\Delta(M(A)) \setminus E$ is hull-kernel open in $\Delta(M(A))$. Hence, if $0 \in F$, the set E is a hull in $\Delta(M(A))$. If $0 \notin F$, we have $\varepsilon := \inf\{|\lambda| : \lambda \in F\} > 0$. Since $\widehat{T}|\Delta(A)$ is hull-kernel continuous and vanishes at infinity, the set $D := \{\varphi \in \Delta(A) : |\widehat{T}(\varphi)| \geq \varepsilon\}$ is a Gelfand compact hull in $\Delta(A)$. Hence, using exactly the same arguments as in the proof of the implication (a) $\Rightarrow$ (b), we obtain first $u \in A$ such that $\widehat{u} \equiv 1$ on D and then $v \in A$ such that $\widehat{T}\widehat{u}\widehat{v} \equiv 1$ on D. In particular, $\widehat{T}\widehat{u}\widehat{v} \equiv 1$ on the subset E of D. Again, it follows that the multiplier $S := I - T(uv)$ satisfies both $\widehat{S} \equiv 0$ on E and $\widehat{S} \equiv 1$ on $H(A)$. This implies that $\mathrm{cl}(E) \cap H(A) = \emptyset$ and hence $\mathrm{cl}(E) \subseteq \Delta(A)$, where $\mathrm{cl}(E)$ is the hull-kernel closure of E in $\Delta(M(A))$. By assumption, E is a hull in $\Delta(A)$, so the inclusion $\mathrm{cl}(E) \subseteq \Delta(A)$ implies that $E = \mathrm{cl}(E)$. This completes the proof of the hull-kernel continuity of $\widehat{T}$ on $\Delta(M(A))$.

The following theorem subsumes the main result of [19], which characterizes the decomposability of a multiplication operator in terms of the hull-kernel continuity of its Gelfand transform on $\Delta(A)$. It also shows that, for a multiplier $T \in M_0(A)$, the decomposability of T on A is equivalent to the decomposability of the corresponding multiplication operator L_T on $M(A)$. This observation provides a partial solution to one of the problems posed in [18, p. 50]. However, even in the case of the group algebra $A = L_1(G)$ for a locally compact abelian group G, it is an interesting open problem whether the decomposability of a multiplier always carries over from $L_1(G)$ to $M(G)$.

2.3. THEOREM. *For an arbitrary $T \in M(A)$, consider the following statements*:

(a) *$\widehat{T}$ is hull-kernel continuous on $\Delta(M(A))$.*

(b) *$L_T : M(A) \to M(A)$ is decomposable.*

(c) *$T : A \to A$ is decomposable.*

(d) *$\widehat{T}|\Delta(A)$ is hull-kernel continuous on $\Delta(A)$.*

Then, for every $T \in M(A)$, we have (a)$\Leftrightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d). *Moreover, if $T \in M_{00}(A)$, then also* (d)$\Rightarrow$(a) *so that all the conditions* (a)–(d) *are equivalent in this case. Furthermore, for each $T \in M_0(A)$, we have* (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c). *Finally, a multiplier T in $M_0(A)$ is decomposable on A if and only if T belongs to $M_{00}(A)$ and satisfies* (d).

Proof. For arbitrary $T \in M(A)$, the equivalence of the conditions (a) and (b) is clear from Theorem 1.2 of [19]. Actually, it is also an immediate consequence of 1.1 and 2.1, applied to the left regular representation Φ of $M(A)$ and the operator L_T. Similarly, the implications (a)$\Rightarrow$(c)$\Rightarrow$(d) follow from 1.1 and 2.1, where 1.1 is applied to the inclusion mapping Φ from $M(A)$ into $\mathcal{L}(A)$. For $T \in M_{00}(A)$, the implication (d)$\Rightarrow$(a) is clear from 2.2. Moreover, the same result will prove the very last assertion, once for every $T \in M_0(A)$ the conditions (a)–(c) are seen to be equivalent. Hence it remains to show that every decomposable multiplier $T \in M_0(A)$ satisfies condition (b). By Theorem 3.2 of [18], this assertion can be reformulated as follows: for every pair of spectral maximal spaces Y and Z of the operator $T : A \to A$ with $\sigma(T|Y) \cap \sigma(T|Z) = \emptyset$, we have to find an operator $R \in M(A)$ such that $R \equiv 0$ on Y and $R \equiv I$ on Z. Now, given such a pair of spectral maximal spaces Y and Z, we observe that the sets

$$E := \{\varphi \in \Delta(A) : \widehat{T}(\varphi) \in \sigma(T|Y)\},$$
$$F := \{\varphi \in \Delta(A) : \widehat{T}(\varphi) \in \sigma(T|Z)\}$$

are hulls in $\Delta(A)$, since $\widehat{T}$ is hull-kernel continuous on $\Delta(A)$ by 2.1. Moreover, since $\sigma(T|Y)$ and $\sigma(T|Z)$ are disjoint, we may as well assume that 0 does not belong to $\sigma(T|Z)$. Then, with $\varepsilon := \inf\{|\lambda| : \lambda \in \sigma(T|Z)\}$, we obtain $|\varphi(T)| \geq \varepsilon > 0$ for all $\varphi \in F$. Because of $T \in M_0(A)$, we conclude that F is Gelfand compact in $\Delta(A)$. Since E and F are disjoint, by Corollary 3.6.10 of [22] there exists some $e \in A$ such that $\widehat{e} \equiv 0$ on E and $\widehat{e} \equiv 1$ on F. We claim that $R := L_e \in M(A)$ has the desired properties. To show that $R \equiv 0$ on Y, let $u \in Y$ be arbitrarily given, and consider some $\varphi \in \Delta(A) \setminus E$. Since $\lambda := \varphi(T) \notin \sigma(T|Y)$, there exists a $v \in Y$ such that $u = (T - \lambda I)v$, which implies that $\varphi(u) = (\varphi(T) - \lambda)\varphi(v) = 0$. Therefore $\widehat{u} \equiv 0$ on $\Delta(A) \setminus E$ and hence $\widehat{e}\widehat{u} \equiv 0$ on $\Delta(A)$. By semisimplicity, this implies $Ru = 0$ and therefore $R \equiv 0$ on Y. A similar argument will show that $R \equiv I$ on Z, which completes the proof.

The preceding theorem includes the following results. If $M(A)$ is regular, then every multiplier $T \in M(A)$ is decomposable on A. Moreover, if A is regular, then a multiplier $T \in M_0(A)$ is decomposable on A if and only if T belongs to $M_{00}(A)$.

2.4. THEOREM. *Let J be an $M(A)$-invariant closed linear subspace of A, and consider a multiplier $T \in M(A)$ for which $\widehat{T}$ is hull-kernel continuous on $\Delta(M(A))$. Then the restriction $S := T|J \in \mathcal{L}(J)$ is decomposable on J and satisfies $\sigma(S) = \widehat{S}(\Delta(J))^-$. Moreover, the spectral maximal spaces of S are given for all closed $F \subseteq \mathbf{C}$ by*

$$J_S(F) = \bigcap_{\lambda\in\mathbf{C}\setminus F} (T-\lambda I)(J) = \{u \in J : \operatorname{supp}\widehat{u} \subseteq \widehat{T}^{-1}(F)\},$$

where $\Delta(J)$ is canonically embedded in $\Delta(M(A))$ and $\operatorname{supp}\widehat{u}$ denotes the closure of the set $\{\varphi \in \Delta(J) : \varphi(u) \neq 0\}$ with respect to the Gelfand topology of $\Delta(J)$.

Proof. Consider the homomorphism $\Phi : M(A) \to \mathcal{L}(J)$ given by $\Phi(R) := R|J$ for all $R \in M(A)$. Then $S = \Phi(T)$ is decomposable on J by 1.1 and has natural spectrum by 2.1. Since J is an ideal in $M(A)$, Theorem 2.6.6 of [22] shows that $\Delta(J)$ can be canonically embedded in $\Delta(M(A))$ and that $\Delta(M(A)) = \Delta(J) \cup H(J)$, where $H(J)$ denotes the hull of J in $\Delta(M(A))$. For each closed $F \subseteq \mathbf{C}$, it is easily seen that

$$J_S(F) \subseteq \bigcap_{\lambda\in\mathbf{C}\setminus F} (T-\lambda I)(J) \subseteq Z_S(F) := \{u \in J : \operatorname{supp}\widehat{u} \subseteq \widehat{T}^{-1}(F)\}.$$

To prove the reverse inclusions, fix an arbitrary open neighborhood U of F and choose an open subset V of $\mathbf{C}$ such that $U \cup V = \mathbf{C}$ and $F \cap \overline{V} = \emptyset$. Since $\widehat{T}$ is hull-kernel continuous on $\Delta(M(A))$ the proof of Theorem 1 in [20] shows that there exists an $R \in M(A)$ such that $\sigma(S|\overline{R(J)}) \subseteq U$ and $\sigma(S|\overline{(I-R)(J)}) \subseteq V$. By spectral maximality, this implies $R(J) \subseteq J_S(\overline{U})$ and $(I-R)(J) \subseteq J_S(\overline{V})$. Moreover, for $u \in Z_S(F)$ we obtain

$$(I-R)(u) \in Z_S(F) \cap J_S(\overline{V}) \subseteq Z_S(F) \cap Z_S(\overline{V}) = \{0\},$$

since F and $\overline{V}$ are disjoint and J is semisimple. From $(I-R)(u) = 0$ we conclude that $u = R(u) \in J_S(\overline{U})$. Thus $Z_S(F) \subseteq J_S(\overline{U})$ and hence $Z_S(F) \subseteq J_S(F)$, as desired.

In connection with the preceding result, one may wonder which closed ideals J in A are invariant under $M(A)$. In addition to the trivial case $J = A$, every ideal in A which is the intersection of maximal modular ideals is certainly $M(A)$-invariant, as can be easily seen from Theorem 1.2.4 of [17]. This criterion is appropriate for the class of N-algebras [22], but not group algebras because of the problem of spectral synthesis. On the other hand, every closed ideal in a Banach algebra with a (not necessarily bounded) approximate identity is obviously invariant under all multipliers. Thus 2.4 does apply to arbitrary closed ideals of the group algebra $A = L_1(G)$ for a locally compact abelian group G.

We now turn to certain subalgebras which induce decomposable multipliers. By a *Banach subalgebra* of A we mean a subalgebra of A which is endowed with some Banach algebra topology. Since we assume A to be semisimple, such a topology is necessarily finer than the relative topology induced by the norm of A. Nonclosed Banach subalgebras exist in abundance. For instance, A is always a Banach subalgebra of $M(A)$, but not necessarily closed in $M(A)$. Moreover, if $\Phi : B \to A$ is an algebra homomorphism from a Banach algebra B into A, then the range $\Phi(B)$ is easily seen to be a Banach subalgebra of A, which need not be closed in A.

As in [19] and [20], one can show that, among all regular Banach subalgebras of A, there is a greatest one, denoted by $\operatorname{Reg}(A)$. Moreover, this *greatest regular subalgebra* is closed in the spectral radius norm on A. If A is self-adjoint and has a minimal approximate identity, then, by Theorem 1.8.3 of [17], the *derived algebra* A_0 introduced by Helgason is a regular Banach subalgebra of A and hence contained in $\operatorname{Reg}(A)$. Standard examples in the group algebra setting show that the derived algebra A_0 may be much smaller than $\operatorname{Reg}(A)$.

Finally, let $\operatorname{Dec}(A) := \{a \in A : L_a : A \to A \text{ is decomposable}\}$. By 2.3, $\operatorname{Dec}(A)$ consists of all $a \in A$ for which $\widehat{a}$ is hull-kernel continuous on $\Delta(A)$. In particular, it follows that $\operatorname{Dec}(A)$ is a closed subalgebra of A and that $\operatorname{Dec}(A)$ is even closed in the spectral radius norm on A; see also [5] and [6] where $\operatorname{Dec}(A)$ has been studied by different techniques. 1.1 implies that $\operatorname{Reg}(A)$ is contained in $\operatorname{Dec}(A)$. Also, by 1.1 and 2.3, it is easily seen that $\operatorname{Reg}(A) = \operatorname{Dec}(A)$ whenever $\operatorname{Dec}(A)$ is an ideal in A, but we do not know if this identity holds in general. Of course, $\operatorname{Reg}(A) = \operatorname{Dec}(A) = A$ whenever A is regular, but even in this case it is interesting to investigate $\operatorname{Reg}(B)$ and $\operatorname{Dec}(B)$ for certain subalgebras B of the typically nonregular multiplier algebra $M(A)$. Here, again, the motivating example is the group algebra $A = L_1(G)$.

2.5. THEOREM. *Let $DM(A) := \{T \in M(A) : T : A \to A$ is decomposable$\}$ denote the set of decomposable multipliers on A. Then we have:*

(a) *$DM(A)$ is a closed subalgebra of $M(A)$ if A has a bounded approximate identity.*

(b) $\operatorname{Reg}(M(A)) \subseteq \operatorname{Dec}(M(A)) \subseteq DM(A)$.

(c) $\operatorname{Reg}(M_0(A)) \subseteq \operatorname{Dec}(M_0(A)) = DM(A) \cap M_0(A) \subseteq M_{00}(A)$.

(d) *$DM(A) \cap M_0(A)$ is a closed subalgebra of $M_{00}(A)$.*

(e) $\operatorname{Reg}(M_0(A)) = \operatorname{Dec}(M_0(A)) = DM(A) \cap M_0(A) = M_{00}(A)$ *when A is regular.*

(f) *A is regular and $\Delta(M_0(A)) = \Delta(A)$ if and only if $M_0(A)$ is regular if and only if A is regular and $M_0(A) = M_{00}(A)$.*

Proof. (a) follows from Theorem 2.6 of [5], while (b) and (c) are easy consequences of 1.1 and 2.3. Moreover, (d) is clear from the very last characterization in 2.3. Now assume that A is regular. Since $M_{00}(A)$ and A have the same spectrum, it follows that $M_{00}(A)$ is regular and hence contained in $\mathrm{Reg}(M_0(A))$. Thus, all the inclusions of (c) are identities when A is regular; this proves statement (e). Finally, (f) is an immediate consequence of (c) and Theorem 3 of [20]. Indeed, if A regular and $\Delta(M_0(A)) = \Delta(A)$, then $M_0(A)$ is obviously regular. If $M_0(A)$ is regular, then (c) shows that $M_0(A) = DM(A) \cap M_0(A) = M_{00}(A)$. In particular, it follows that, for each $a \in A$, the multiplication operator $L_a \in M_0(A)$ is decomposable on A. By Theorem 3 of [20], this yields the regularity of A. Finally, $M_0(A) = M_{00}(A)$ implies $\Delta(M_0(A)) = \Delta(M_{00}(A)) = \Delta(A)$. The assertion follows.

Statement (f) improves results of Birtel [8], who has shown by different methods that regularity of $M_0(A)$ forces A to be regular and $\Delta(M_0(A)) = \Delta(A)$. The preceding theorem indicates that arbitrary decomposable multipliers in $M(A)$ are much harder to handle than those in $M_0(A)$. Indeed, the proof of statement (a) due to Albrecht [5] requires the spectral theory of several commuting operators and makes essential use of Cohen's factorization theorem [9], whereas the corresponding result for $M_0(A)$ given in assertion (d) is obtained by more elementary techniques and without any additional assumption on A. Thus it would be interesting to know when equalities occur in statement (b) of Theorem 2.5.

As an application, we finish this section with a general criterion for the invertibility of multipliers. As we shall see in 4.3, this generalizes classical results of Beurling [7] and of Wiener–Pitt [26] on the Fourier–Stieltjes transform of measures. Note that condition (2) of Theorem 2.6 is fulfilled with the choice $R = T$ and $S = V = 0$, when T is a decomposable multiplier on A satisfying (1). Thus, for small perturbations of decomposable multipliers, condition (1) is sufficient to guarantee invertibility in $M(A)$.

2.6. THEOREM. *Let A be a regular semisimple commutative Banach algebra with a bounded approximate identity, and consider a multiplier $T \in M(A)$ for which*

$$\inf\{|\widehat{T}(\varphi)| : \varphi \in \Delta(A)\} > 0 . \tag{1}$$

Assume that $T = R + S$ for a pair of multipliers $R, S \in M(A)$, where R is decomposable on A and the spectral radius $r(S)$ of the operator S satisfies the condition

$$r(S) < \inf\{|(\widehat{R} + \widehat{V})(\varphi)| : \varphi \in \Delta(A)\} \quad \textit{for some } V \in M_{00}(A). \tag{2}$$

Then T is invertible in $M(A)$.

Proof. From (a) and (e) of 2.5 we know that $DM(A)$ is a subalgebra of $M(A)$ which contains $M_{00}(A)$. Hence $R + V$ is decomposable on A, which

implies

$$\sigma(R+V) = (\widehat{R} + \widehat{V})(\Delta(A))^{-} , \tag{3}$$

by 2.1. From condition (2) we conclude that $0 \notin \sigma(R+V)$ so that $R+V$ is invertible in $M(A)$ by assertion (a) of 1.2. Furthermore, we have

$$\begin{aligned} r(S(R+V)^{-1}) &\le r(S)r((R+V)^{-1}) = r(S)\sup\{|\lambda| : \lambda \in \sigma((R+V)^{-1})\} \\ &= r(S)\sup\{|\lambda^{-1}| : \lambda \in \sigma(R+V)\} \\ &= r(S)/\inf\{|(\widehat{R} + \widehat{V})(\varphi)| : \varphi \in \Delta(A)\} < 1 , \end{aligned}$$

where we made use of the conditions (2) and (3) and of Proposition 5.6 of [9]. Since $r(S(R+V)^{-1}) < 1$, it follows from the standard geometric series argument that the operator $I + S(R+V)^{-1}$ is invertible in $M(A)$. Next observe that

$$\begin{aligned} T = R + V + S - V &= (I + S(R+V)^{-1} - V(R+V)^{-1})(R+V) \\ &= W(I + S(R+V)^{-1})(R+V) , \end{aligned}$$

where the operator $W : A \to A$ is given by

$$W := I - V(R+V)^{-1}(I + S(R+V)^{-1})^{-1} = I - V(T+V)^{-1} = T(T+V)^{-1} .$$

Since by assumption $V \in M_{00}(A)$ and since $M_{00}(A)$ is an ideal in $M(A)$, we conclude from 2.5 that $W \in I + M_{00}(A) \subseteq DM(A)$. Again by 2.1, the decomposability of W implies that $\sigma(W) = \widehat{W}(\Delta(A))^{-}$. But from condition (1) it is clear that

$$\begin{aligned} &\inf\{|\widehat{W}(\varphi)| : \varphi \in \Delta(A)\} \\ &\quad \ge \inf\{|\widehat{T}(\varphi)| : \varphi \in \Delta(A)\}/\sup\{|(\widehat{T} + \widehat{V})(\varphi)| : \varphi \in \Delta(A)\} > 0 , \end{aligned}$$

so that W is invertible in $M(A)$. Thus T is the product of three invertible operators in $M(A)$. This completes the proof.

3. The case of scattered and discrete spectra $\Delta(A)$.

Again, let A be a semisimple commutative Banach algebra over $\mathbf{C}$. In this section, we shall obtain some further characterizations of decomposable multipliers in $M_0(A)$ by imposing additional assumptions on the spectrum $\Delta(A)$.

Recall that a locally compact Hausdorff space Ω is *scattered* (or dispersed) if every nonempty compact subset of Ω contains an isolated point. For the basic facts on scattered spaces, we refer to [21] and [23]. It is easily seen that every discrete space is scattered and that every scattered space is totally disconnected. Moreover, Ω is scattered if and only if each continuous complex-valued function on Ω that vanishes at infinity has countable range. This characterization will be one principal tool; its proof follows easily from the Main Theorem of [21] and Theorem 2 of [23], applied to the one-point compactification of Ω.

The other tool is the observation that any semisimple commutative Banach algebra with scattered spectrum $\Delta(A)$ is regular. This is not hard to check by means of the Shilov idempotent theorem [9]. Alternatively, one can use Theorem 2.3 of [18] to conclude that every multiplication operator on A is decomposable whenever $\Delta(A)$ is totally disconnected. Then Theorem 3 of [20] yields the regularity of A.

3.1. THEOREM. *Assume that the spectrum $\Delta(A)$ is scattered in the Gelfand topology. Then, for each $T \in M_0(A)$, the following statements are equivalent*:

(a) $\widehat{T}$ *is hull-kernel continuous on* $\Delta(M(A))$.
(b) $T : A \to A$ *is decomposable.*
(c) $T : A \to A$ *has a natural spectrum* $\sigma(T) = \widehat{T}(\Delta(A))^-$.
(d) $\sigma(T)$ *is countable.*
(e) $\widehat{T} \equiv 0$ *on* $H(A)$.

Moreover, $\mathrm{Reg}(M_0(A))=\mathrm{Dec}(M_0(A))=\{T \in M_0(A) : T$ *satisfies* (a)–(e)$\}$ $=M_{00}(A)$.

Proof. Note that $\widehat{T}(\Delta(A))^- \subseteq \widehat{T}(\Delta(A)) \cup \{0\}$, since $T \in M_0(A)$, and that $\widehat{T}|\Delta(A)$ has countable range, since $\Delta(A)$ is scattered. Hence we obtain (c)⇒(d), and the implications (a)⇒(b)⇒(c) follow from 2.3 and 2.1. Conversely, (d)⇒(b) is clear, since every operator with totally disconnected spectrum is decomposable; see, for instance, Theorem 3.1.19 and Example 3.1.20 of [10]. Moreover, the implication (b)⇒(e) holds again by 2.3. It remains to see that (e) implies (a) and that $\mathrm{Reg}(M_0(A))$ has the stated description. All this follows immediately from 2.3 and 2.5 since A is regular.

Under the stronger assumption that the spectrum $\Delta(A)$ is discrete, the characterization of 3.1 can be extended to include Riesz operators. Recall that an operator $T \in \mathcal{L}(X)$ on a complex Banach space X is said to be a *Riesz operator* if for each $\lambda \in \mathbb{C}\setminus\{0\}$ the dimension of the kernel $\ker(T-\lambda I)$ and the codimension of the range $(T-\lambda I)(X)$ in X are both finite. There are various equivalent definitions of Riesz operators [14], but for our purposes this one is the most expedient. Since Riesz operators have countable spectrum [14], it follows that all Riesz operators are decomposable (cf. [10]). For certain multipliers we can prove the converse.

3.2. THEOREM. *Assume that the spectrum $\Delta(A)$ is discrete in the Gelfand topology. Then a multiplier $T \in M_0(A)$ satisfies the equivalent conditions* (a)–(e) *from Theorem* 3.1 *if and only if $T : A \to A$ is a Riesz operator.*

Proof. It remains to show that every decomposable multiplier $T \in M_0(A)$ is a Riesz operator when $\Delta(A)$ is discrete. First note that, given an arbitrary $\varepsilon > 0$, the set $E_\varepsilon := \{\varphi \in \Delta(A) : |\varphi(T)| \geq \varepsilon\}$ is compact in $\Delta(A)$

because of $T \in M_0(A)$. By the discreteness of $\Delta(A)$, this implies that E_ε is finite. Since we know from Theorem 3.1 that the multiplier $T \in M_0(A)$ has natural spectrum, we conclude that $\sigma(T)$ is either finite or countable with 0 as the only accumulation point. Hence, given an arbitrary $\lambda \in \sigma(T)$ with $\lambda \neq 0$, we can find closed sets $F, G \subseteq \mathbb{C}$ with $F \cap \sigma(T) = \{\lambda\}$ and $\lambda \notin G$ such that $\mathbb{C}$ is covered by the interiors of F and G. Since T is decomposable on A, we obtain $A = A_T(F) + A_T(G)$. Also, $A_T(F) = A_T(F \cap \sigma(T)) = A_T(\{\lambda\})$ and consequently $A_T(F) \cap A_T(G) = A_T(\{\lambda\} \cap G) = A_T(\emptyset) = \{0\}$. Hence $A = A_T(\{\lambda\}) \oplus A_T(G)$ holds as a direct sum decomposition. Moreover, since

$$\ker(T-\lambda I) \subseteq A_T(\{\lambda\}) \quad \text{and} \quad A_T(G) \subseteq (T-\lambda I)(A_T(G)) \subseteq (T-\lambda I)(A),$$

all that remains to show is that $A_T(\{\lambda\})$ is finite-dimensional. Now, since $\widehat{T}$ vanishes at infinity and $\Delta(A)$ is discrete, it is clear that $E := \{\varphi \in \Delta(A) : \varphi(T) = \lambda\}$ is finite. It follows that the space Y_E of all complex-valued functions on $\Delta(A)$, vanishing on $\Delta(A) \setminus E$, is of finite dimension. Since the Gelfand transform is injective and maps $A_T(\{\lambda\})$ into Y_E, we conclude that $A_T(\{\lambda\})$ is finite-dimensional. Thus the dimension of $\ker(T-\lambda I)$ and the codimension of $(T-\lambda I)(A)$ in A are both finite for each $\lambda \in \sigma(T)$ with $\lambda \neq 0$, so that T is a Riesz operator.

3.3. Remarks. (a) With certain assumptions on the family $I(A)$ of isometric multipliers from A onto A, some of the characterizations given in 3.2 have been obtained by Aiena in his recent paper [3]. If $I(A)$ satisfies the following condition:

(4) $I(A)$ separates the points of $\Delta(A)$ and is compact in the strong operator topology,

then Theorem 10 of [3] shows that a multiplier $T \in M_0(A)$ has natural spectrum if and only if T is a Riesz operator on A. Since, by Theorem 1.6.4 of [17], condition (4) forces the spectrum $\Delta(A)$ to be discrete, this result is contained in 3.2. The approach in [3] depends heavily on the more restrictive assumption (4).

(b) Aiena [3] also investigates the algebraic properties of the family $\mathcal{C}$ of all multipliers $T \in M_0(A)$ with natural spectrum. Again under assumption (4), Theorem 11 of [3] shows that $\mathcal{C}$ is a closed ideal in $M_0(A)$ with discrete spectrum $\Delta(\mathcal{C})$. This generalizes a main result of Zafran [27] in the classical setting $A = L_1(G)$ for a compact abelian group G. Now, 3.1 sheds some new light on these results, since, even under the weaker assumption that $\Delta(A)$ is scattered, we obtain the identity $\mathcal{C} = M_{00}(A)$ so that $\mathcal{C}$ is a closed ideal in $M_0(A)$ with spectrum $\Delta(\mathcal{C}) = \Delta(A)$.

(c) Given a complex Banach space X, let $\mathcal{A}(X)$ denote the class of all operators $T \in \mathcal{L}(X)$ for which the restriction $T|Y$ to any T-invariant closed linear subspace Y of infinite dimension is not bijective [1]. It is clear from Theorem 52.8 of [14] that $\mathcal{A}(X)$ contains the class $\mathcal{R}(X)$ of all Riesz operators on X, and it seems interesting to investigate how different these two classes can be. In the case $X = L_1(\mathbf{T})$ for the circle group $\mathbf{T}$, Aiena [1] has found an example of a convolution operator on X which belongs to $\mathcal{A}(X)$, but not to $\mathcal{R}(X)$. By means of the preceding theorems, we can obtain a more precise version of this result: indeed, if A is any semisimple commutative Banach algebra with discrete spectrum, then $\mathcal{R}(A) \cap M_0(A) = M_{00}(A)$ by Theorem 3.2 whereas $\mathcal{A}(A) \cap M_0(A) = M_0(A)$ by Theorem 7 of [3]. Since discreteness of $\Delta(A)$ implies that A is regular, we conclude from 2.5 that $\mathcal{R}(A) \neq \mathcal{A}(A)$ whenever $M_0(A)$ is not regular. This covers the case $A = L_1(G)$ for an arbitrary compact abelian group G.

(d) In connection with 3.2, it is natural to ask for a characterization of the class of compact multipliers on A. If A has a bounded approximate identity, it is not hard to show that every compact multiplier on A has to be a multiplication operator on A. Banach algebras on which all multiplication operators are compact were studied by Kaplansky under the name of completely continuous algebras; see also the discussion of the larger class of compact Banach algebras in [9]. Hence, if A is completely continuous and has a bounded approximate identity, then a multiplier $T \in M(A)$ is compact if and only if T is a multiplication operator on A.

(e) Concerning the topological assumptions in 3.1 and 3.2, it is clear that $\Delta(A)$ may be scattered without being discrete. But for the group algebra $A = L_1(G)$ for a locally compact abelian group G, the spectrum $\Delta(A)$ is scattered if and only if it is discrete, since $\Delta(A)$ can be identified with the dual group of G and since it is easily seen from the principal structure theorem for locally compact abelian groups [24] that such a group is scattered if and only if it is discrete. Consequently, both 3.1 and 3.2 apply to a group algebra $A = L_1(G)$ exactly when G is compact and abelian.

4. Decomposable convolution operators.

Throughout this section, let G denote an arbitrary locally compact abelian group with dual group Γ. The corresponding group algebra $A = L_1(G)$ is a regular semisimple commutative Banach algebra with a bounded approximate identity and spectrum $\Delta(A) \cong \Gamma$ (cf. [24]). Moreover, via convolution, its multiplier algebra $M(A)$ can be canonically identified with the measure algebra $M(G)$ consisting of all regular complex Borel measures on G (cf. [17]). With this identification, $M_{00}(A)$ becomes the subalgebra $M_{00}(G)$ of all measures $\mu \in M(G)$ whose Fourier–Stieltjes transforms $\mu^\wedge$ on $\Delta(M(G))$ vanish outside Γ, and $M_0(A)$ becomes the subalgebra $M_0(G)$ of all measures in $M(G)$ whose Fourier–Stieltjes transforms on the dual group Γ vanish at infinity. Identifying $L_1(G)$ with the ideal $M_a(G)$ of all measures on G which are absolutely continuous with respect to Haar measure on G, we have $L_1(G) \cong M_a(G) \subseteq M_{00}(G) \subseteq M_0(G) \subseteq M(G)$. Finally, for each μ in $M(G)$, let $T_\mu : M(G) \to M(G)$ denote the corresponding convolution operator given by $T_\mu(\nu) := \mu * \nu$ for all $\nu \in M(G)$. From 1.2, 2.3 , and 2.4 we obtain immediately:

4.1. THEOREM. *For each $\mu \in M(G)$, the operator T_μ is decomposable on $M(G)$ if and only if $\mu^\wedge$ is hull-kernel continuous on $\Delta(M(G))$. Moreover, in this case, T_μ is also decomposable on $X := L_1(G)$ and we have $\sigma(T_\mu) = \sigma(T_\mu|L_1(G)) = \mu^\wedge(\Gamma)^-$. The spectral maximal spaces of the restriction $T := T_\mu|X$ are given for all closed $F \subseteq \mathbf{C}$ by*

$$X_T(F) = \bigcap_{\lambda \in \mathbf{C}\setminus F} (T - \lambda I)(X) = \{u \in L_1(G) : \operatorname{supp} \widehat{u} \subseteq \widehat{\mu}^{-1}(F)\},$$

where the Fourier transforms are taken with respect to the dual group Γ.

Although we do not know of a measure-theoretic characterization of the decomposable multipliers on $M(G)$ or $L_1(G)$, we can use the inclusion $\operatorname{Reg}(M(G)) \subseteq \operatorname{Dec}(M(G))$ from 2.5 to identify classes of measures to which 4.1 applies. Indeed, since $L_1(G)$ is regular, 2.5 shows that $M_{00}(G)$ is a regular closed subalgebra of $M(G)$ and hence contained in $\operatorname{Dec}(M(G))$. Similarly, if H is a closed subgroup of G and if $L_1(H)$ is canonically identified with the space of all measures on G which are concentrated on H and absolutely continuous with respect to Haar measure on H, then $L_1(H)$ is a regular closed subalgebra of $M(G)$ and therefore contained in $\operatorname{Dec}(M(G))$. Since all measures in this subalgebra are singular with respect to Haar measure on G whenever the subgroup H is nontrivial, it follows that $\operatorname{Reg}(M(G))$ and hence $\operatorname{Dec}(M(G))$ may contain singular measures. Finally, since the subalgebra $M_d(G)$ of all discrete measures on G may be identified with $L_1(G_d)$ where G_d stands for G with the discrete topology, we have $M_d(G) \subseteq \operatorname{Reg}(M(G)) \subseteq \operatorname{Dec}(M(G))$. Now, 2.5 and 4.1 yield the following result.

4.2. COROLLARY. *Let $\mu \in M_{00}(G) + M_d(G) + L_1(H)$ where H is a closed subgroup of G. Then T_μ is decomposable both on $M(G)$ and on $L_1(G)$ and satisfies $\sigma(T_\mu) = \sigma(T_\mu|L_1(G)) = \mu^\wedge(\Gamma)^-$.*

This result applies, in particular, to all measures on G whose continuous part is absolutely continuous and therefore generalizes classical results of Beurling [7] for the real line $G = \mathbf{R}$ and of Hartman [13] for the circle group $G = \mathbf{T}$. The fact that all measures in $M_a(G) + M_d(G)$ have natural spectrum can be reformulated as follows: if a measure μ in $M_a(G) + M_d(G)$ satisfies $|\mu^\wedge| \geq \delta > 0$ on Γ for some $\delta > 0$, then μ is invertible in $M(G)$. We next show that this criterion is stable under small perturbations by singular measures.

4.3. COROLLARY. *Let $\mu \in M(G)$, and let $\mu = \mu_a + \mu_d + \mu_s$ be the decomposition of μ into its absolutely continuous, discrete, and singular part. Assume that $|\mu^\wedge| \geq \delta > 0$ on Γ for some $\delta > 0$ and that the spectral radius of the singular part satisfies either*

$$(5)\quad r(\mu_s) < \inf\{|\widehat{\mu}_d(\gamma)| : \gamma \in \Gamma\} \quad or \quad r(\mu_s) < \inf\{|(\widehat{\mu}_a + \widehat{\mu}_d)(\gamma)| : \gamma \in \Gamma\}.$$

Then μ is invertible in $M(G)$.

Proof. Let T, R, S, V denote the operators of convolution on $L_1(G)$ given by μ, $\mu_a + \mu_d$, μ_s, and $-\mu_a$, respectively. Then $V \in M_{00}(G)$, and R is decomposable on $L_1(G)$ by 4.2. Hence, under the first assumption on $r(\mu_s)$ in condition (5), the assertion is clear from 2.6. In the second case, 2.6 has to be applied with the choice $V := 0$.

Note that the second condition in (5) is fulfilled whenever $\mu_s = 0$, which shows that 4.3 improves results of Beurling [7] and of Hartman [13]. On the other hand, the first condition in (5) is fulfilled whenever the total variation norm of the singular part satisfies $\|\mu_s\| < \inf\{|\mu_d^\wedge(\gamma)| : \gamma \in \Gamma\}$, which shows that 4.3 contains a classical result due to Wiener–Pitt [26] for the case $G = \mathbf{R}$. It follows from the counterexamples given in Theorem 3 of [26] that the condition on μ_s cannot be weakened in general. We next use the fact that the decomposable multipliers form a closed subalgebra of the multiplier algebra to prove that decomposability is preserved under absolute continuity.

4.4. THEOREM. *Let $\mu \in M(G)$ be such that T_μ is decomposable on $L_1(G)$, and assume that $\nu \in M(G)$ is absolutely continuous with respect to μ. Then T_ν is decomposable on $L_1(G)$. The same conclusion holds for decomposability on $M(G)$.*

Proof. We first consider the case of decomposable convolution operators on $M(G)$. Hence assume that T_μ is decomposable on $M(G)$ and consider an arbitrary character γ in the dual group Γ. Since it is easily seen that $(\gamma\mu) * (\gamma\lambda) = \gamma(\mu * \lambda)$ holds for all $\lambda \in M(G)$, it follows that the continuous linear operator $U_\gamma : M(G) \to M(G)$ given by $U_\gamma(\lambda) := \gamma\lambda$ for all $\lambda \in M(G)$ satisfies $T_{\gamma\mu}U_\gamma = U_\gamma T_\mu$ on $M(G)$. Since U_γ is invertible on $M(G)$, we conclude that the operators T_μ and $T_{\gamma\mu}$ are similar on $M(G)$, which implies that the decomposability on $M(G)$ carries over from T_μ to $T_{\gamma\mu}$. Since $\mathrm{Dec}(M(G))$ is a subalgebra of $M(G)$, we obtain $f\mu \in \mathrm{Dec}(M(G))$ for every trigonometric polynomial f on G, i.e. for every linear combination of continuous characters on G. Since, by Lemma 31.4 of [15], the trigonometric polynomials on G are dense in $L_1(|\mu|)$ and since $\mathrm{Dec}(M(G))$ is known to be closed in $M(G)$, we conclude that $f\mu \in \mathrm{Dec}(M(G))$ for all $f \in L_1(|\mu|)$, where $|\mu|$ denotes the total variation of μ. Hence, by the Radon–Nikodym theorem, T_ν is decomposable on $M(G)$ whenever ν is absolutely continuous with respect to μ. For decomposability on $L_1(G)$, the argument is basically the same, except that we use the fact that the decomposable multipliers on $L_1(G)$ form a closed subalgebra of $M(G)$; this follows from assertion (a) of 2.5.

We now turn to the more tractable subalgebra $M_0(G)$ of $M(G)$. For this the characterization of decomposable convolution operators in 4.1 can be improved. The following result generalizes work of Albrecht [5] and Zafran [27]. Its proof is an immediate consequence of 1.2, 2.3, 2.5, and 3.2.

4.5. THEOREM. *For each $\mu \in M_0(G)$, the following statements are equivalent:*

(a) *$\widehat{\mu}$ is hull-kernel continuous on $\Delta(M(G))$.*
(b) *$T_\mu : M(G) \to M(G)$ is decomposable.*
(c) *$T_\mu : L_1(G) \to L_1(G)$ is decomposable.*
(d) *$\mu \in M_{00}(G)$.*

Moreover, we have $\mathrm{Reg}(M_0(G)) = \mathrm{Dec}(M_0(G)) = M_{00}(G)$. If, in addition, G is compact, then, for each $\mu \in M_0(G)$, the statements (a)–(d) *are also equivalent to each of the following assertions:*

(e) *$T_\mu : L_1(G) \to L_1(G)$ has a natural spectrum $\sigma(\mu) = \sigma(T_\mu|L_1(G)) = \widehat{\mu}(\Gamma)^-$.*
(f) *$T_\mu : L_1(G) \to L_1(G)$ has countable spectrum.*
(g) *$T_\mu : L_1(G) \to L_1(G)$ is a Riesz operator.*

In the following, we shall modify some techniques developed by Zafran [27] to locate certain singular measures which induce nondecomposable convolution operators.

4.6. LEMMA. *If a measure $\nu \in M(G)$ is absolutely continuous with respect to some $\mu \in M_{00}(G)$, then also $\nu \in M_{00}(G)$.*

Proof. Given an arbitrary $\gamma \in \Gamma$, we consider, as in the proof of 4.4, the automorphism U_γ on $M(G)$ given by $U_\gamma(\lambda) := \gamma\lambda$ for all $\lambda \in M(G)$. U_γ induces a homeomorphism U_γ^* on $\Delta(M(G))$. Since $U_\gamma^*(\chi) = \gamma^{-1}\chi$ for all $\chi \in \Gamma$, we have $U_\gamma^*(\Gamma) = \Gamma$ so that $U_\gamma^*(\Delta(M(G)) \setminus \Gamma) = \Delta(M(G)) \setminus \Gamma$. Therefore $\mu \in M_{00}(G)$ implies that $\gamma\mu = U_\gamma(\mu) \in M_{00}(G)$ and hence $f\mu \in M_{00}(G)$ for all trigonometric polynomials f on G. Since, by Lemma 31.4 of [15], these polynomials are dense in $L_1(|\mu|)$, the Radon–Nikodym theorem implies that $\nu \in M_{00}(G)$.

4.7. LEMMA. *Let $\mu \in M(G)$ be a measure on G such that the convolution powers μ^n are all singular with respect to Haar measure on G, for all $n \in \mathbf{N}$. Then μ is singular with respect to each ν in $M_{00}(G)$.*

Proof. Suppose, to the contrary, that there exists some $\nu \in M_{00}(G)$ which is not singular with respect to μ. By 4.6 we may assume that ν is nonnegative. From the Lebesgue decomposition theorem, we obtain a pair of nonnegative measures $\alpha, \beta \in M(G)$ such that $\nu = \alpha + \beta$, $\alpha \ll \mu$, and $\beta \perp \mu$. Because of our assumption on ν, it is clear that α is nonzero. Moreover, since $0 \le \alpha \le \nu$ and since $\nu \in M_{00}(G)$, we conclude again from 4.6 that $\alpha \in M_{00}(G)$. Finally, the basic assumption on μ and $\alpha \ll \mu$ imply that

$$\alpha^n \ll \mu^n \quad \text{and hence} \quad \alpha^n \perp \lambda \text{ for all } n \in \mathbb{N}, \tag{6}$$

where λ denotes the Haar measure on G. Since $M_{00}(G)$ and $L_1(G)$ have the same spectrum, the quotient algebra $M_{00}(G)/L_1(G)$ is radical. Therefore, the equivalence class $[\alpha] := \alpha + L_1(G)$ in this algebra is quasi-nilpotent. Since (6) implies that

$$\|\alpha^n + f\| = \|\alpha^n\| + \|f\| \ge \|\alpha^n\| \quad \text{for all } f \in L_1(G) \text{ and } n \in \mathbb{N},$$

we conclude that $\|[\alpha]^n\| = \|\alpha^n\|$ for all $n \in \mathbb{N}$, which shows that α is quasi-nilpotent in $M(G)$. Since $M(G)$ is semisimple, we obtain the desired contradiction $\alpha = 0$.

4.8. COROLLARY. *Let $\mu \in M_0(G)$ be a nonzero measure such that μ^n is singular with respect to Haar measure on G for all $n \in \mathbb{N}$. Then T_μ is not decomposable on $L_1(G)$.*

Actually, it follows from 4.5 and 4.7 that the measure μ in 4.8 is singular with respect to each $\nu \in M_0(G)$ for which the corresponding convolution operator is decomposable on $L_1(G)$. Measures in $M_0(G)$ for which all convolution powers are singular with respect to Haar measure on G arise naturally in the context of Riesz product measures; see [27] and Chapter 7 of [12]. The following characterization of decomposable convolution operators given by certain Riesz product measures on the unit circle $\mathbf{T}$ follows immediately from 4.5 in combination with Theorem 3.9 of [27].

4.9. EXAMPLE. Consider a sequence of nonzero real numbers $a_k \in [-1,1]$ such that $a_k \to 0$ as $k \to \infty$. Moreover, let $q > 3$ be a fixed real number, and consider a sequence of integers $n_k \in \mathbb{N}$ such that $n_{k+1} \ge q n_k$ for all $k \in \mathbb{N}$. Finally, let μ denote the measure on $\mathbf{T}$ represented by the Riesz product $\prod_{k=1}^{\infty}(1 + a_k \cos(n_k x))$, $0 \le x \le 2\pi$. Then $\mu \in M_0(\mathbf{T})$. Moreover, μ satisfies the equivalent conditions (a)–(g) of Theorem 4.5 for the circle group $G = \mathbf{T}$ if and only if $\sum_{k=1}^{\infty} |a_k|^n < \infty$ for some $n \in \mathbb{N}$.

4.10. Remark. Assume that G is a compact abelian group. Then 4.5 shows that the class of measures $\mu \in M_0(G)$ for which T_μ is a Riesz operator on $L_1(G)$ coincides with the ideal $M_{00}(G)$. Consequently, if a measure $\mu \in M(G)$ satisfies the condition

$$\mu^n \in L_1(G) \quad \text{for some } n \in \mathbb{N}, \tag{7}$$

then T_μ is a Riesz operator on $L_1(G)$. As noted by Aiena in [2], this result can also be derived from the fact that each measure in $L_1(G)$ induces a compact convolution operator on $L_1(G)$ (see for instance [4]). A partial converse follows immediately from 4.8. Indeed, if for a measure $\mu \in M_0(G)$ the operator T_μ is a Riesz operator on $L_1(G)$, then there exists an $n \in \mathbb{N}$ such that the corresponding convolution power of μ is not singular with respect to Haar measure on G. The main result of [2] claims that even more is true, namely that condition (7) characterizes Riesz multipliers on $L_1(G)$. However, the following argument will show that this result is wrong whenever the underlying group G is nondiscrete. Suppose that Aiena's result were correct. Then the preceding characterization of Riesz multipliers in terms of $M_{00}(G)$ implies that every $\mu \in M_{00}(G)$ satisfies condition (7). Hence the quotient algebra $M_{00}(G)/L_1(G)$ contains only nilpotent elements, which by a result of Grabiner forces this quotient algebra to be nilpotent (see Theorem 46.3 of [9]). We conclude that there exists some $n \in \mathbb{N}$ such that $\mu^n \in L_1(G)$ for all $\mu \in M_{00}(G)$. But this is impossible unless G is discrete, since it follows, for instance, from Corollary 7.2.4 of [12] that, for every nondiscrete locally compact abelian group G and every $n \in \mathbb{N}$, there exists some probability measure $\mu \in M(G)$ such that $\mu^n \notin L_1(G)$, but $\mu^{n+1} \in L_1(G)$ and hence $\mu \in M_{00}(G)$. This observation disproves the result of [2].

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