yield

$$\lambda \sharp\{n\,;\, |f(n)| > \lambda\} \le \lambda \sharp\{x\,;\, |h^*(x)| > \lambda\} \le C\|h\|_{\text{Weak-}H^1} \le C\|f\|_{\text{Weak-}H^1} \le C\|f\|_{1,\infty}\,. \ \blacksquare$$

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On the rate of strong mixing in stationary Gaussian random fields

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Abstract. Rosenblatt showed that a stationary Gaussian random field is strongly mixing if it has a positive, continuous spectral density. In this article, spectral criteria are given for the rate of strong mixing in such a field.

A stationary random process $\{X_n\}$ is *strongly mixing* if, in a certain sense, its past and future are asymptotically independent. This idea was introduced by Rosenblatt [7] in connection with a central limit theorem. Kolmogorov and Rozanov [6] found a useful sufficient condition for a Gaussian process to be strongly mixing, namely, that it have a (strictly) positive, continuous spectral density. A necessary and sufficient condition was obtained by Helson and Sarason [3] (see also Sarason [9]). Ibragimov and Rozanov [4], and Khrushchev and Peller [5] are concerned with the rate at which strong mixing occurs as revealed by the smoothness of the spectral density.

The notion of strong mixing makes sense in the random field setting as well. Indeed, Rosenblatt [8] proved that a stationary Gaussian field $\{X_{mn}\}$ satisfies a strong mixing condition if it has a positive, continuous density; this is an exact analogue of the result in [6].

In this article, spectral criteria are derived for the rate at which strong mixing occurs in a stationary Gaussian field. First, the separation-of-variables technique of [2] is used to adapt the one-parameter methods in [5]. This yields mixing rates in which the roles of past and future are played by vertical halfplanes of the random field. This result is then extended to the case of halfplanes at rational slopes. Lastly, these ideas are used to investigate strong mixing in the full sense of [8].

1. The principal result. Let $\{X_{mn}\}$ be a stationary Gaussian random field on the integral lattice $\mathbf{Z}^2$, and μ its spectral measure on the torus $\mathbf{T}^2$.

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For any subset S of $\mathbf{Z}^2$, let $M(S)$ be the (closed) span in $L^2(\mu(e^{is}, e^{it}))$ of the set $\{e^{ims+int} : (m, n) \in S\}$. Define, for all nonempty subsets S and S' of $\mathbf{Z}$, the correlation coefficient

$$c(S, S') = \sup\{|\langle f, g\rangle| : f \in M(S),\ g \in M(S'),\ \|f\| \le 1,\ \|g\| \le 1\}.$$

(Let us write $c(S, S'; \mu)$ if the measure μ needs to be specified.) We say that $\{X_{mn}\}$ is *strongly mixing* if there exists a function $\varphi : [0, \infty) \to [0, 1]$ such that $\lim_{x\to\infty} \varphi(x) = 0$, and

$$c(S, S') \le \varphi(\mathrm{dist}(S, S'))$$

for all S and S'. (This definition is the spectral equivalent of that in [8].)

Take σ to be normalized Lebesgue measure on $\mathbf{T}$, and Λ_a the Hölder class of order a (see Section 4). The main result is

1.1. THEOREM. *Suppose that μ is absolutely continuous with respect to σ^2, and its density function w satisfies*

$$0 < k_1 \le w(e^{is}, e^{it}) \le k_2 < \infty$$

on $\mathbf{T}^2$ for some constants k_1 and k_2.

(i) *Strong mixing occurs with φ of the form $\varphi(x) = Ce^{-ax}$ if and only if w has an analytic continuation across $\mathbf{T}^2$.*

(ii) *Strong mixing occurs with φ of the form $\varphi(x) = Cx^{-a}$ if and only if $w \in \Lambda_a$.*

Remarks. That μ must be absolutely continuous for strong mixing follows from two applications of [2, 3.2]. But the boundedness assumptions on the density function w are not logically necessary. For instance, if strong mixing occurs with the measure $w\,d\sigma^2$, then it occurs with the measure $|P|^2 w d\sigma^2$, where P is any finite trigonometric sum; in the latter case, the density may vanish at some points on the torus. Rather, the boundedness assumptions reflect limitations of the present techniques.

The remaining sections are devoted to proving four lemmas, which together form a slightly stronger version of 1.1.

2. Vertical halfplanes. The first step is to borrow some of the one-parameter theory. This is achieved by considering $c(L, R_N)$, where L and R_N are "vertical halfplanes" in $\mathbf{Z}^2$:

$$L = \{(m, n) \in \mathbf{Z}^2 : m \le 0\}, \qquad R_N = \{(m, n) \in \mathbf{Z}^2 : m \ge N\}.$$

For then, the underlying shift occurs in the m-direction only, and the analysis will chiefly involve the variable coupled to m.

Accordingly, we will make use of the univariate Hölder classes, λ_a (these are more often denoted by Λ_a, but we reserve the latter symbol for the

bivariate Hölder classes in Section 4), defined as follows. Let m be a positive integer, $\varepsilon > 0$, and $f \in C(\mathbf{T})$. Put

$$\Delta_h^m f(e^{is}) = \sum_{j=0}^{m} \binom{m}{j} (-1)^j f(e^{i(s+jh)}),$$

$$\Omega^m(\varepsilon, f) = \sup_{|h|\le\varepsilon} \|\Delta_h^m f\|_{L^\infty}.$$

For $a > 0$, we say that $f \in \lambda_a$ if the quantity

$$\|f\|_{\lambda_a} = \|f\|_{L^\infty} + \sup_{\varepsilon>0} [\varepsilon^{-a} \cdot \Omega^m(\varepsilon, f)]$$

is finite, where m is the integer such that $a < m \le a + 1$. This is indeed a norm, under which λ_a is a complete Banach space. Properties and applications of these spaces are explored in [1] and [5]. In particular, let P_+ be the projection of $L^2(\mathbf{T}, \sigma)$ onto the Hardy space $H^2(\mathbf{T})$, and $P_- = I - P_+$ (σ denotes normalized Lebesgue measure on $\mathbf{T}$). Then

2.1. PROPOSITION. *Let $a > 0$, and $f \in L^\infty(\mathbf{T})$. If there exists a constant C such that*

$$\mathrm{dist}_{L^\infty(\mathbf{T})}(e^{i(N-1)s} f(e^{is}), H^\infty(\mathbf{T})) \le CN^{-a}$$

for all $N = 1, 2, \ldots$, then $P_- f \in \lambda_a$, and $\|P_- f\|_{\lambda_a} \le 3 \cdot 2^a C$.

Proof. This is the theorem on p. 68 of [5], with care taken to follow the constant. ∎

2.2. PROPOSITION. *Let $a > 0$, and $u \in H^\infty$. If $f \in P_- \lambda_a$, then $P_-(uf) \in P_- \lambda_a$, and $\|P_-(uf)\|_{\lambda_a} \le \|u\|_{H^\infty} \cdot \|f\|_{\lambda_a}$.*

Proof. See Example 1 on p. 109 of [5]. ∎

With that, we have the following criterion for strong mixing with respect to vertical halfplanes. The second marginal of μ is denoted by μ_2.

2.3. LEMMA. *Let $d\mu$ be of the form $wd(\sigma \times \mu_2)$, where $0 \le w(e^{is}, e^{it}) \le K$. If there exist positive constants C and a such that*

$$c(L, R_n) \le CN^{-a}, \qquad N = 1, 2, \ldots,$$

then there exists a constant M such that

$$\|w(\cdot, e^{it})\|_{\lambda_a} \le M, \qquad a.e.\,[\mu_2(e^{it})].$$

Proof. By [2, 3.1 and 3.2], the density w must satisfy the condition

$$\int_{\mathbf{T}} \log w(e^{is}, e^{it})\, d\sigma(e^{is}) > -\infty, \qquad \text{a.e.}\,[\mu_2(e^{it})].$$

Hence we can define the function $h(e^{is},e^{it})$ as in [2, equation (3-2)]. The hypothesis on $c(L,R_N)$ can, according to [2, 3.4], be expressed as

$$\operatorname{dist}_{L^\infty(\sigma(e^{is}))}\left(e^{i(N-1)s}\frac{\overline{h}(e^{is},e^{it})}{h(e^{is},e^{it})}, H^\infty(\sigma(e^{is}))\right) \le CN^{-a}, \qquad N=1,2,\ldots,$$

for μ_2-almost every fixed e^{it}. By 2.1, we have

$$\left\| P_-\left[\frac{\overline{h}(\cdot,e^{it})}{h(\cdot,e^{it})}\right]\right\|_{\lambda_a} \le 3\cdot 2^a C, \qquad \text{a.e.}\,[\mu_2(e^{it})].$$

Note that

$$P_-w(\cdot,e^{it}) = P_-|h(\cdot,e^{it})|^2 = P_-\{h(\cdot,e^{it})^2P_-[\overline{h}(\cdot,e^{it})/h(\cdot,e^{it})]\}.$$

Now 2.2 gives

$$\|P_-w(\cdot,e^{it})\|_{\lambda_a} \le K\|P_-[\overline{h}(\cdot,e^{it})/h(\cdot,e^{it})]\|_{\lambda_a}, \qquad \text{a.e.}\,[\mu_2(e^{it})].$$

Hence, for μ_2-almost every e^{it},

$$\begin{aligned}\|w(\cdot,e^{it})\|_{\lambda_a} &\le \|P_+w(\cdot,e^{it})\|_{\lambda_a} + \|P_-w(\cdot,e^{it})\|_{\lambda_a}\\ &\le K + 2\|P_-w(\cdot,e^{it})\|_{\lambda_a} \le K + 2\cdot 3\cdot 2^a KC. \ \blacksquare\end{aligned}$$

3. Halfplanes at rational slope. The last assertion can be extended to the case of halfplanes bounded by lines at rational slope. This is done by applying a change of variables which transforms the sloped case to an equivalent problem of vertical halfplanes.

Let r be a nonzero rational number, and fix the unique representation $r=p/q$, in lowest terms with q positive. Consider the index sets

$$U^r = \{(m,n)\in\mathbf{Z}^2 : mp-nq\le 0\},$$
$$V_N^r = \{(m,n)\in\mathbf{Z}^2 : mp-nq\ge N(p^2+q^2)\}, \qquad N=1,2,\ldots$$

Note that U^r and V_N^r are halfplanes of $\mathbf{Z}^2$, with boundaries at the rational slope r. Also, the distance between U^r and V_N^r is $N(p^2+q^2)^{1/2}$.

Consider the matrix

$$\Theta = \begin{bmatrix} p & -q\\ u & v\end{bmatrix}$$

where u and v are integers satisfying $\det\Theta = pv+qu = 1$. It determines a mapping of $\mathbf{Z}^2$ one-to-one and onto itself by

$$\Theta\begin{bmatrix} m\\ n\end{bmatrix} = \begin{bmatrix} mv+nq\\ -mu+np\end{bmatrix}.$$

In particular, the image of U^r and V_N^r under Θ are the vertical halfplanes L and $R_{N(p^2+q^2)}$, respectively.

Next we produce a measure μ_r on $\mathbf{T}^2$ so that Θ induces a Hilbert space isomorphism of $L^2(\mu)$ onto $L^2(\mu_r)$. This is done by setting $\widehat{\mu}_r(m,n) = \widehat{\mu}(mv+nq, np-mu)$ for all $(m,n)\in\mathbf{Z}^2$. These coefficients form a positive array, and indeed arise as the Fourier coefficients of a measure μ_r. In the case $d\mu = w\,d\sigma^2$, we have

$$\begin{aligned}\widehat{\mu}_r(m,n) &= \int_{\mathbf{T}^2} e^{i(mv+nq)s+i(np-mu)t}w(e^{is},e^{it})\,d\sigma^2(e^{is},e^{it})\\ &= \int_{\mathbf{T}^2} e^{ims+int}w(e^{i(ps+ut)},e^{i(vt-qs)})\,d\sigma^2(e^{is},e^{it})\end{aligned}$$

(using the fact that $\Theta\mathbf{T}^2=\mathbf{T}^2$). It follows from the action of Θ and the construction of μ_r that

3.1. PROPOSITION. $c(U^r,V_N^r;\mu) = c(L,R_{N(p^2+q^2)};\mu_r)$.

4. The proof of the principal result. The rational-slope criterion is extended in 4.1 to allow for arbitrary index sets S and S'; this demonstrates part of 1.1. The rest is established by using facts from approximation theory. As indicated earlier, we now bring in the bivariate Hölder classes, Λ_a. Let m be a positive integer, $\varepsilon>0$, and $f\in C(\mathbf{T}^2)$. Put

$$\Delta^m_{h,k}f(e^{is},e^{it}) = \sum_{j=0}^m \binom{m}{j}(-1)^j f(e^{i(s+jh)},e^{i(t+jk)}),$$
$$\Omega^m(\varepsilon,f) = \sup_{h^2+k^2\le\varepsilon^2}\|\Delta^m_{h,k}f\|_{L^\infty(\sigma^2)}.$$

For $a>0$, we say that $f\in\Lambda_a$ if the quantity

$$\|f\|_{\Lambda_a} = \|f\|_{L^\infty(\sigma^2)} + \sup_{\varepsilon>0}[\varepsilon^{-a}\cdot\Omega^m(\varepsilon,f)]$$

is finite, where m is the integer satisfying $a<m\le a+1$. And now

4.1. LEMMA. *Suppose that μ is absolutely continuous with respect to σ^2, and its density function w satisfies $0\le w(e^{is},e^{it})\le K$ on $\mathbf{T}^2$ for some constant K. If strong mixing occurs with φ of the form $\varphi(x)=Cx^{-a}$, where a and C are positive constants, then $w\in\Lambda_a$.*

Proof. First, assume that $0<a<1$. By hypothesis, strong mixing occurs with respect to vertical halfplanes; that is, $c(L,R_N)\le CN^{-a}$. Thus 2.3 provides a constant M_1 such that

$$\|w(e^{i(s+\varepsilon)},e^{it}) - w(e^{is},e^{it})\|_{L^\infty(\sigma^2(e^{is},e^{it}))} \le M_1|\varepsilon|^a.$$

A similar argument for horizontal halfplanes shows that

$$\|w(e^{is},e^{i(t+\delta)}) - w(e^{is},e^{it})\|_{L^\infty(\sigma^2(e^{is},e^{it}))} \le M_2|\delta|^a$$

for some M_2. Hence

$$\|w(e^{i(s+\varepsilon)}),e^{i(t+\delta)}) - w(e^{is},e^{it})\|_{L^\infty(\sigma^2(e^{is},e^{it}))} \le M(\varepsilon^2+\delta^2)^{a/2}$$

where $M = M_1 + M_2$. This shows that $w \in \Lambda_a$.

Next, assume that $a \geq 1$. Fix a nonzero rational r. Let the integers p, q, u, v, and the halfplanes U^r and V_N^r be defined as in Section 3. By hypothesis,

$$c(U^r, V_N^r; \mu) \leq C[N(p^2+q^2)^{1/2}]^{-a}.$$

Now 3.1 yields

$$c(L, R_{N(p^2+q^2)}; \mu_r) \leq C[N(p^2+q^2)^{1/2}]^{-a} = C(p^2+q^2)^{a/2}[N(p^2+q^2)]^{-a}.$$

Apply 2.3 to the density function for μ_r. This provides a constant M (independent of r) such that

$$\sup_{|h|\leq y} \Big\| \sum_{j=0}^{m} \binom{m}{j} (-1)^j w(e^{i[ps+ut+pjh]}, e^{i[vt-qs-qjh]}) \Big\|_{L^\infty(\sigma^2(e^{is},e^{it}))} \leq M(p^2+q^2)^{a/2} y^a,$$

where m is the integer such that $a < m \leq a+1$. By taking $\vartheta = -\operatorname{Arccot} r$, and $Y = (p^2+q^2)^{1/2} y$, this can be written

$$\sup_{|h|\leq Y} \Big\| \sum_{j=0}^{m} \binom{m}{j} (-1)^j w(e^{i(s+jh\cos\vartheta)}, e^{i(t+jh\sin\vartheta)}) \Big\|_{L^\infty(\sigma^2(e^{is},e^{it}))} \leq MY^a.$$

That is, w satisfies a smoothness condition in the rational-slope directions.

As for the other directions, fix $0 < b < 1$. Strong mixing occurs, *a fortiori*, with $\varphi(x) = Cx^{-b}$, so by the first part of this proof there exists a constant M' with

$$\sup_{\varepsilon^2+\delta^2\leq Y^2} \|w(e^{i(s+\varepsilon)}, e^{i(t+\delta)}) - w(e^{is}, e^{it})\|_{L^\infty(\sigma^2(e^{is},e^{it}))} \leq M'Y^b.$$

Hence for any real ε and δ,

$$\Big\| \sum_{j=0}^{m} \binom{m}{j} (-1)^j w(e^{i(s+j\varepsilon)}, e^{i(t+j\delta)}) \Big\|_{L^\infty(\sigma^2(e^{is},e^{it}))}$$
$$\leq \Big\| \sum_{j=0}^{m} \binom{m}{j} (-1)^j w(e^{i(s+j\varepsilon_0)}, e^{i(t+j\delta_0)}) \Big\|_{L^\infty(\sigma^2(e^{is},e^{it}))}$$
$$+ \sum_{j=0}^{m} \binom{m}{j} \|w(e^{i(s+j\varepsilon)}, e^{i(t+j\delta)}) - w(e^{i(s+j\varepsilon_0)}, e^{i(t+j\delta_0)})\|_{L^\infty(\sigma^2(e^{is},e^{it}))}$$
$$\leq M(\varepsilon_0^2+\delta_0^2)^{a/2} + 2^m m^{b/2} M'[(\varepsilon-\varepsilon_0)^2 + (\delta-\delta_0)^2]^{b/2},$$

whenever ε_0/δ_0 is rational. Let $\varepsilon_0 \to \varepsilon$ and $\delta_0 \to \delta$ along such values. This shows that $w \in \Lambda_a$. ■

The proof of a converse statement uses the following result about polynomial approximation in Λ_a. Let Q_δ be the collection of finite trigonometric sums of the form

$$\sum_{m^2+n^2<\delta^2} a_{mn} e^{ins+int}.$$

4.2. PROPOSITION. *Let $a > 0$, and $f \in L^\infty(\sigma^2)$. There exists a constant A such that*

$$\operatorname{dist}_{L^\infty(\sigma^2)}(f, Q_\delta) \leq A\delta^{-a}, \quad \delta > 0,$$

if and only if $f \in \Lambda_a$.

Proof. See [1, p. 188]. ■

A crude estimate, made possible by a boundedness assumption, now yields the rest of 1.1(ii).

4.3. LEMMA. *Suppose that μ is absolutely continuous with respect to σ^2, and its density function w satisfies $0 < K \leq w(e^{is}, e^{it})$ on $\mathbf{T}^2$ for some constant K. If $w \in \Lambda_a$ for some $a > 0$, then strong mixing occurs with φ of the form $\varphi(x) = Cx^{-a}$.*

Proof. Fix $\delta > 0$. Let S and S' be nonempty subsets of $\mathbf{Z}^2$ such that $\operatorname{dist}(S, S') \geq \delta$. For any finite trigonometric sums $f \in M(S)$, $g \in M(S')$ and $\psi \in Q_\delta$,

$$\Big|\int f\bar{g}w\, d\sigma^2\Big| = \Big|\int f\bar{g}(w-\psi)\, d\sigma^2 + \int f\bar{g}\psi\, d\sigma^2\Big|$$
$$= \Big|\int f\bar{g}(w-\psi)\, d\sigma^2 + 0\Big|$$
$$\leq \int |f\bar{g}|\, d\sigma^2 \cdot \|w-\psi\|_{L^\infty(\sigma^2)}$$
$$\leq \Big(\int |f|^2\, d\sigma^2\Big)^{1/2} \Big(\int |g|^2\, d\sigma^2\Big)^{1/2} \|w-\psi\|_{L^\infty(\sigma^2)}$$
$$\leq K^{-1} \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)} \|w-\psi\|_{L^\infty(\sigma^2)}.$$

This shows that if $\operatorname{dist}(S, S') \geq \delta$, then

$$c(S, S') \leq K^{-1} \operatorname{dist}_{L^\infty(\sigma^2)}(w, Q_\delta).$$

By 4.2, there exists a constant A such that

$$c(S, S') \leq K^{-1} \cdot A \cdot \delta^{-a}$$

whenever $\operatorname{dist}(S, S') \geq \delta$. This proves the claim. ■

Part (i) of 1.1 is elementary, as shown below.

4.4. LEMMA. *Suppose that μ is absolutely continuous with respect to σ^2, and its density function w satisfies $0 < K \leq w(e^{is}, e^{it})$ on $\mathbf{T}^2$ for some* *constant K. If w has an analytic continuation across $\mathbf{T}^2$, then strong mixing occurs with φ of the form $\varphi(x) = Ce^{-ax}$ for some positive constants a and C.*

Proof. By hypothesis, there exists r, $0 < r < 1$, such that w has a Laurent expansion

$$w(z_1, z_2) = \sum_{(m,n)\in\mathbf{Z}^2} a_{mn} z_1^m z_2^n,$$

which converges absolutely and uniformly in the annulus $A = \{(z_1, z_2) \in \mathbf{C}^2 : r \le |z_j| \le 1/r,\ j = 1, 2\}$. Then $|a_{mn}| \le M r^{|m|+|n|}, (m, n) \in \mathbf{Z}^2$, for some M.

As in the proof of 4.3,

$$c(S, S') \le K^{-1} \operatorname{dist}_{L^\infty(\sigma^2)}(w, Q_\delta)$$

whenever $\operatorname{dist}(S, S') \ge \delta$. But then

$$\begin{aligned}\operatorname{dist}_{L^\infty(\sigma^2)}(w, Q_\delta) &\le \Big\| \sum_{m^2+n^2>\delta^2} a_{mn} e^{ims+int} \Big\|_{L^\infty(\sigma^2)} \le \sum_{m^2+n^2>\delta^2} |a_{mn}| \\ &\le \sum_{m^2+n^2>\delta^2} M r^{|m|+|n|} \le 4M(1-r)^{-2} e^{-\delta\sqrt{2}\log(1/r)}. \ \blacksquare\end{aligned}$$

4.5. Lemma. *Suppose that μ is absolutely continuous with respect to σ^2, and its density function is w. If strong mixing occurs with φ of the form $\varphi(x) = Ce^{-ax}$, where a and C are positive constants, then w has an analytic continuation across $\mathbf{T}^2$.*

Proof. By hypothesis,

$$\begin{aligned}|\widehat{w}_{m,n}| &= \Big| \int e^{ims+int} w(e^{is}, e^{it})\, d\sigma(e^{is}, e^{it}) \Big| \\ &\le \Big(\int |e^{ims+int}|^2 w(e^{is}, w^{it})\, d\sigma^2(e^{is}, e^{it}) \Big)^{1/2} \\ &\quad \times \Big(\int |1|^2 w(e^{is}, e^{it})\, d\sigma^2(e^{is}, e^{it}) \Big)^{1/2} \cdot Ce^{-a\sqrt{m^2+n^2}} \\ &= C \int w\, d\sigma^2 \cdot e^{-a\sqrt{m^2+n^2}}, \quad (m, n) \in \mathbf{Z}^2.\end{aligned}$$

It follows that $w(z_1, z_2) = \sum_{(m,n)\in\mathbf{Z}^2} \widehat{w}_{m,n} z_1^m z_2^n$ is a Laurent series converging in the annulus $A = \{(z_1, z_2) \in \mathbf{C}^2 : r \le |z_j| \le 1/r,\ j = 1, 2\}$ where $e^{-a\sqrt{2}} < r < 1$. $\blacksquare$

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