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On relations between operators on $\mathbb{R}^N$, $\mathbb{T}^N$ and $\mathbb{Z}^N$

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Abstract. We study different discrete versions of maximal operators and g-functions arising from a convolution operator on $\mathbb{R}$. This allows us, in particular, to complete connections with the results of de Leeuw [L] and Kenig and Tomas [KT] in the setting of the groups $\mathbb{R}^N$, $\mathbb{T}^N$ and $\mathbb{Z}^N$.

§ 1. Introduction. The celebrated theorem of Carleson–Hunt ([C], [H]) asserts that the partial sums of the Fourier series of a periodic function g in $L^p(\mathbb{T})$, $1 < p \le \infty$, converge almost everywhere to g. One way for proving such a convergence result (see [H]) involves the establishment of strong or weak type (p,p) estimates for an associated maximal operator. In this case, this operator is

$$P^\sharp g(x) = \sup_{L=0,1,\ldots} \Big| \sum_{|k|\le L} \widehat{g}(k) e^{2\pi i k x} \Big|.$$

Instead of periodic functions and Fourier series one might consider the analogous result on $\mathbb{R}$ for Fourier integrals. More precisely, we introduce the maximal operator

$$C^\sharp g(x) = \sup_{t>0} \Big| \int_{-t}^{t} \widehat{f}(\xi) e^{2\pi i x \xi}\, d\xi \Big|.$$

Then well-known elementary arguments show that strong type (p,p), $1 < p < \infty$, for $P^\sharp$ is equivalent to that for $C^\sharp$ and, therefore, convergence almost everywhere of inverse Fourier integrals for functions in $L^p(\mathbb{R})$, $1 < p < \infty$, follows. We may also wonder whether there is an equivalent formulation, in the discrete case, for "Fourier integrals of sequences" in $l^p(\mathbb{Z})$. Toward this

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end, consider

$$D^\sharp a(n) = \sup_{0<t<1/2}\left|\int_{-t}^{t}\widehat{a}(\xi)e^{2\pi i n\xi}d\xi\right| = \sup_{0<t<1/2}\left|\sum_{m=0}^{\infty}\frac{\sin 2\pi m t}{\pi m}a(n-m)\right|.$$

In fact, the equivalence of strong type (p,p) for $P^\sharp$ and $D^\sharp$ has been established recently for all p in $(1,\infty)$ (see [CP], [AP] and [A]).

The equivalence of strong type (p,p) estimates for $P^\sharp$ and $C^\sharp$ is really a special case of a more general result: Let m be a bounded continuous function on $\mathbf{R}^N$. Let $\wedge$ denote the Fourier transform in $\mathbf{R}^N$, $\mathbf{T}^N$ or $\mathbf{Z}^N$, according to the setting. For $t>0$, define

$$(1)\qquad (C_t f)(x) = \int_{\mathbf{R}^N} m(t\xi)\widehat{f}(\xi)e^{2\pi i x\xi}\,d\xi\,,\qquad x\in\mathbf{R}^N,$$

for a function f defined in $\mathbf{R}^N$,

$$(2)\qquad (P_t g)(x) = \sum_{k\in\mathbf{Z}^N} m(tk)\widehat{g}(k)e^{2\pi i k x}\,,\qquad x\in\mathbf{T}^N,$$

for g a periodic function in $\mathbf{T}^N$, and

$$(3)\qquad (D_t a)(n) = \int_{[-1/2,1/2]^N} m(t\xi)P(\xi)e^{2\pi i n\xi}\,d\xi,\qquad n\in\mathbf{Z}^N,$$

for $a=(a(n))_n$ a sequence in $\mathbf{Z}^N$ and $P(\xi)=\sum_m a(m)e^{2\pi i m\xi}$.

In the Fourier multiplier theory setting, (1) represents the action of $m(t\,\cdot)$ as a multiplier on $\mathbf{R}^N$, (2) represents the action of the multiplier $(m(tn))_n$ on $\mathbf{Z}^N$, while (3) is the action of the periodic extension of the function $m(t\,\cdot)\chi_{[-1/2,1/2]^N}(\cdot)$ as a multiplier on $\mathbf{T}^N$. Observe that we are identifying $\mathbf{T}^N$ with the cube $Q=[-1/2,1/2]^N$ of $\mathbf{R}^N$. This identification is not arbitrary. As we shall see later (see Remark (iv)), if we choose to identify $\mathbf{T}^N$ with $[0,1)^N$ then Theorem 9 fails to hold. As before, let us also consider the maximal operators

$$C^*f(x)=\sup_{t>0}|C_tf(x)|\,,\quad P^*g(x)=\sup_{t>0}|P_tf(x)|\,,\quad D^*a(n)=\sup_{t>0}|D_ta(n)|\,.$$

Our main interest is to show how the L^p-boundedness of these three operators are related. Before doing so, let us consider some "classical" results of this nature.

A theorem of de Leeuw [L] says that for $1<p<\infty$,

$$\|C_1 f\|_{L^p(\mathbf{R}^N)}\le\|f\|_{L^p(\mathbf{R}^N)}$$

if and only if

$$\|P_t g\|_{L^p(\mathbf{T}^N)}\le\|g\|_{L^p(\mathbf{T}^N)}\,,$$

uniformly in $t>0$.

Kenig and Tomas (see [KT]) extended this result to the related maximal operators just described: for p in the same range,

$$\|C^*f\|_p\le\|f\|_p\quad\text{if and only if}\quad\|P^*g\|_p\le\|g\|_p\,.$$

Strictly speaking, all of this is true when m is "normalized", a condition that is less restrictive than continuity. In particular, if $m=\chi_{[-1,1]}$ then $C^*=C^\sharp$ and $P^*=P^\sharp$ (up to some trivial error term) and the Kenig–Tomas result applies to $C^\sharp$ and $P^\sharp$.

This describes a connection between Fourier multipliers (and maximal operators) on two groups: $\mathbf{T}^N$ and $\mathbf{R}^N$. In view of our example, it is then natural to try to relate more general Fourier multipliers on $\mathbf{Z}^N$ and $\mathbf{R}^N$. Let G be either $\mathbf{R}^N$, $\mathbf{T}^N$ or $\mathbf{Z}^N$ and let $\{M_u\}_u$ be a collection of bounded functions on G indexed by a set U. We say, following [KT], that $\{M_u\}_u$ is *p-maximal* on G if the operator $\sup_{u\in U}|(M_u\widehat{f})^\vee(x)|$ is of strong type (p,p). The [KT] result gives us an equivalence of p-maximality for $\mathbf{R}^N$ and $\mathbf{T}^N$. One of our results is the corresponding equivalence between $\mathbf{R}^N$ and $\mathbf{Z}^N$. As a consequence, we obtain an equivalence for p-maximality for $\mathbf{Z}^N$ and $\mathbf{T}^N$. That is, we have closed the following diagram:

(a) $\mathbf{R}^N$
$\{(m(t\,\cdot)\chi_{\mathbf{R}^N}\}_t$ p-maximal

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(b) $\mathbf{Z}^N$ (c) $\mathbf{T}^N$
$\{m(t\,\cdot)\chi_{[-1/2,1/2]^N}\}_t$ p-maximal $\Longleftrightarrow$ $\{m(t\,\cdot)\chi_{\mathbf{Z}^N}\}_t$ p-maximal

Letting $\check{m}=K$, we can write (at least formally) the operators (1) and (3) as convolutions:

$$(4)\quad (C_tf)(x)=(K_t*f)(x)\,,\qquad (D_ta)(n)=\sum_m a(m)(K_t*\operatorname{sinc})(n-m)\,,$$

where

$$\operatorname{sinc}(x)=\frac{\sin\pi x}{\pi x}\equiv\prod_{j=1}^{N}\frac{\sin\pi x_j}{\pi x_j}\,,\qquad x=(x_1,\dots,x_N)\,.$$

The function sinc arises naturally when we express D_t as a convolution operator; however, its role can be assumed by other functions φ with compactly supported Fourier transform. Therefore we will consider discrete operators more general than those in (4), namely,

$$(D_t^\varphi a)(n)=\sum_m a(m)(K_t*\varphi)(n-m),\qquad a=(a(m))_m.$$

When $\varphi = \operatorname{sinc}$ we shall drop φ in the notation. Also set

$$(D^{\varphi})^* a = \sup_{t>0} |D_t^{\varphi} a| .$$

We shall refer to results on $(D^{\varphi})^*$ as the "discrete case" since we deal with sequences, while references to the "continuous case" mean that we are working with the operators C^*. For m a bounded function on $\mathbf{R}^N$, let $M_p(m)$ denote its norm as a Fourier multiplier on $L^p(\mathbf{R}^N)$. The space of such multipliers is denoted by $M_p(\mathbf{R}^N)$.

One of our main results can be stated as follows:

THEOREM A. *Let $1 \le p \le \infty$ and $0 < q \le \infty$. Assume that $\widehat{\varphi} \in L^{\infty}(\mathbf{R}^N)$ satisfies:*

(a) *$\widehat{\varphi}$ is a multiplier on $L^p(\mathbf{R}^N)$,*
(b) $\operatorname{supp} \widehat{\varphi} \subset [-R, R]^N$, $R < 1$,
(c) *for some $\varepsilon > 0$ and $h \in C_0^{\infty}((-\varepsilon, \varepsilon)^N)$, $h \equiv 1$ on $[-\varepsilon/2, \varepsilon/2]^N$, $h/\widehat{\varphi} \in M_p(\mathbf{R}^N)$.*

Then, if the family of kernels comes from the dilations of a fixed kernel K, that is, $K_t(x) = t^{-N} K(t^{-1}x)$, we get

$$\| \| C_t(f) \|_q \|_p \le C \| f \|_p , \quad f \in L^p(\mathbf{R}^N) ,$$

if and only if

$$\| \| D_t^{\varphi}(a) \|_q \|_p \le C \| a \|_p , \quad a \in l^p(\mathbf{Z}^N) ,$$

where $\| \| g_t \|_q \|_p$ denotes the norm of the family of functions (resp. sequences) $(g_t)_t$ in the space $L^p(\mathbf{R}^N, L^q(\mathbf{R}^+, dt/t))$ (resp. $l^p(\mathbf{Z}^N, L^q(\mathbf{R}^+, dt/t))$).

Condition (c) is a local invertibility condition that will be more easily understood when we discuss the proof of the theorem. Throughout this paper we shall use the subscripts q and q' for the norm with respect to the variable t in the spaces $L^q(dt/t)$ and $L^{q'}(dt/t)$, and the subscripts p and s for the norm in the spaces $L^p(\mathbf{R}^N)$, $L^s(\mathbf{R}^N)$ or $l^p(\mathbf{Z}^N)$, $l^s(\mathbf{Z}^N)$.

The proof of this theorem is given in two sections. First we prove that the continuous case implies the discrete one, and then the converse. We do this since the two proofs are quite different; moreover, for each implication the hypotheses that we have to impose on φ are different. The fourth section is devoted to some applications and examples. Along similar lines, we will also get similar results for g-functions and nonisotropic dilations. Applications are given in Sections 5 and 6.

Finally, we would like to express our gratitude to Professors Javier Soria and Guido Weiss for their reading these notes more than carefully as well as for their numerous comments on an earlier draft. The second author also wants to thank Professor Fernando Soria for the useful conversations they had on the topic.

§ 2. Boundedness of discrete versions obtained from that of continuous ones. In this section, we do not need the dilation structure of $\mathbf{R}^N$. Some properties of functions of exponential type will be our main tools.

Denote by $K_t(x)$ a family of kernels not necessarily obtained by dilating a single kernel. We recall that, by definition, $K_t^{\varphi}(n) = (K_t * \varphi)(n)$ for a suitable distribution φ. We shall always assume joint measurability in (t, ξ) for the family of multipliers $M_t(\xi) = \widehat{K}_t(\xi)$. We shall write $\star$ for the convolution of two sequences.

THEOREM 1. *Let $1 \le p \le \infty$, $1 \le s \le \infty$ and $0 < q \le \infty$. Let $\widehat{\varphi} \in L^{\infty}(\mathbf{R}^N)$ be such that*

(a) *$\widehat{\varphi} \in M_s(\mathbf{R}^N)$ with norm $M_s(\widehat{\varphi})$,*
(b) *$\operatorname{supp} \widehat{\varphi} \subset [-R, R]^N$.*

Then

$$(5) \qquad \| \| K_t * f \|_q \|_p \le \| f \|_s , \quad f \in L^s(\mathbf{R}^N),$$

implies

$$(6) \qquad \| \| K_t^{\varphi} \star a \|_q \|_p \le A \| a \|_s , \quad a \in l^s(\mathbf{Z}^N),$$

where $A \le C M_s(\widehat{\varphi}) \max(1, R^{N(1/p+1-1/s)})$ and $C = C(p, q, s, N)$.

Remark. In the case $q = \infty$, the fact that the family $(K_t)_t$ is indexed by $\mathbf{R}^+$ is irrelevant; any set is convenient. This has already been observed by Bourgain (see [Bo] where he proves that (5) implies (6) in the case $p = s = 2$, $q = \infty$ and $\widehat{\varphi} = \chi_{[-1/2,1/2]}$).

In the case where $K_t(x) = t^{-N} K(t^{-1}x)$ we also have

PROPOSITION 2. *Under the same assumption on $\widehat{\varphi}$*

$$(7) \qquad \| K * f \|_p \le \| f \|_s \quad \text{implies} \quad \| K_1^{\varphi} \star a \|_p \le A \| a \|_s$$

with a similar estimate for A.

We only prove Theorem 1. Proposition 2 can be obtained by a similar argument. Introduce the set E_R of slowly increasing C^{∞} functions f with $\operatorname{supp} \widehat{f} \subset [-R, R]^N$. The elements of E_R are functions of exponential type R. We recall a well-known sampling theorem for such functions:

LEMMA 3. *Let $0 < p \le \infty$. Then there exists a constant $C = C(p, N)$ such that*

$$\sum_{n \in \mathbf{Z}^N} |g(n)|^p \le C^p \max(1, R^N) \int_{\mathbf{R}^N} |g(x)|^p \, dx ,$$

for any $g \in E_R$.

Proof. The argument for $N = 1$, found, for example, in [B, Ch. 6], can be adapted for establishing this lemma in N dimensions by using the subharmonicity of $|g(x+iy)|^p$ in each complex variable z_j, where $g(z) = g(z_1,\ldots,z_N) = g(x+iy)$ is the analytic extension of $g(x)$ to $\mathbf{C}^N$. However, for later purposes, it is convenient for us to present a sketch of the proof in the case $1 \le p \le \infty$.

Assume first that $R = 1$. We shall prove the inequality for $p = 1$; we then have it for all p by interpolation since the case $p = \infty$ is obvious. Let Ψ be such that $\widehat{\Psi}(\xi) = 1$ if $\xi \in [-1,1]^N$, $\operatorname{supp}\widehat{\Psi} \subset [-2,2]^N$, and $\widehat{\Psi} \in C^\infty$. Let $C = \sup_x \sum_{n\in\mathbf{Z}^N} |\Psi(x-n)| < \infty$. If $g \in E_1$, then $g = g * \Psi$. Thus,

$$\sum_{n\in\mathbf{Z}^N} |g(n)| = \sum_n \Big| \int_{\mathbf{R}^N} g(x)\Psi(n-x)\,dx\Big| \le \int_{\mathbf{R}^N} |g(x)| \sum_n |\Psi(n-x)|\,dx \le C \int_{\mathbf{R}^N} |g(x)|\,dx\,.$$

For the general case of $R > 0$, let α be the first integer greater than R. Let $g \in E_R$. Then $g \in E_\alpha$ and therefore $g(\cdot/\alpha) \in E_1$. Hence

$$\sum_{n\in\mathbf{Z}^N} |g(n/\alpha)|^p \le C \int_{\mathbf{R}^N} |g(x/\alpha)|^p dx = C\alpha^N \int_{\mathbf{R}^N} |g(x)|^p dx\,.$$

From this inequality the result is easily deduced. ∎

This lemma has the following useful generalization:

LEMMA 4. *Let $1 \le p \le \infty$, $0 < q \le \infty$. Then there is a constant $C = C(p,q,N)$ such that*

$$(8)\qquad \sum_{n\in\mathbf{Z}^N} \|g_t(n)\|_q^p \le C^p \max(1, R^N) \int_{\mathbf{R}^N} \|g_t(x)\|_q^p\,dx\,,$$

for any family $g_t(x)$, $t > 0$, of jointly measurable functions in E_R.

Proof. First consider the case $q \ge 1$. By usual duality arguments, it suffices to prove

$$(9)\qquad \int_{\mathbf{R}^+} \sum_{n\in\mathbf{Z}^N} |g_t(n)h_t(n)l(n)|\,\frac{dt}{t} \le C(\max(1,R^N))^{1/p} \|\|g_t\|_q\|_p\,,$$

for all finite sequences $(l(n))_n$ with $\|l\|_{p'} \le 1$, all finite sequences $(h_t(n))_n$ of functions on $\mathbf{R}^+$ with $\|h_t(n)\|_{q'} \le 1$ for all n and all families of functions g_t, $t > 0$, in E_R. Since the above sequences are finitely supported we may define

$$l(x) = \sum_{n\in\mathbf{Z}^N} l(n)\Psi(x-n) \quad\text{and}\quad h_t(x) = \sum_{n\in\mathbf{Z}^N} h_t(n)\Psi(x-n)$$

($t > 0$ fixed), where Ψ is, this time, a C^∞ function whose Fourier transform is supported in $[-2,2]^N$ and satisfies $\Psi(0) = 1$ and $\Psi(k) = 0$ for $k \in \mathbf{Z}^N \setminus \{0\}$. One can easily check that both $l(x)$ and $h_t(x)$ belong to E_2 and that $\|l(\cdot)\|_{p'} \le C$ and $\|h_t(x)\|_{q'} \le C$ for all $x \in \mathbf{R}^N$, where C depends only on p, q, N and Ψ. Now notice that for fixed t, $g_t(x)h_t(x)l(x) \in E_{R+4}$. Thus by Lemma 3 with $p = 1$, the left hand side of (9) does not exceed

$$C(R+4)^N \int_{\mathbf{R}^+}\int_{\mathbf{R}^N} |g_t(x)h_t(x)l(x)|\,dx\,\frac{dt}{t} \le C(R+4)^N \|l\|_{p'} \big(\sup_{x\in\mathbf{R}^N} \|h_t(x)\|_{q'}\big) \|\|g_t\|_q\|_p \le C(R+4)^N \|\|g_t\|_q\|_p\,.$$

For the case $q < 1$, the previous argument can be adapted in the following way. Let $r = p/q > 1$; then (8) is equivalent to

$$(10)\qquad \int_{\mathbf{R}^+} \sum_{n\in\mathbf{Z}^N} |g_t(n)|^q |l(n)|^q \frac{dt}{t} \le C^p \bigg(\int_{\mathbf{R}^N} \Big(\int_{\mathbf{R}^+} |g_t(x)|^q \frac{dt}{t}\Big)^{p/q} dx\bigg)^{q/p}$$

where $l = (l(n))_n$ is a finite sequence with $\|l\|_{qr'} \le 1$. As above, extend l to a function $l(x)$ in E_2 satisfying $\|l(\cdot)\|_{qr'} \le C$. To prove (10) proceed as before using Lemma 3 with exponent q and $R+2$ playing the role of R. ∎

The following result is central to our proof.

LEMMA 5. *Let φ be as in Theorem 1. Then $\varphi \in L^s$ and*

$$(11)\qquad \Big\| \sum_{m\in\mathbf{Z}^N} a(m)\varphi(\cdot - m)\Big\|_s \le C M_s(\widehat{\varphi}) \max(1, R^{N(1-1/s)}) \|a\|_s,$$

for all sequences $a = (a(m))_m$, where $C = C(s,N)$.

Proof. Assume $R = 1$. Let Ψ be as in the proof of Lemma 3. Since $\Psi \in L^s$ and $\varphi = \varphi * \Psi$, we obtain $\varphi \in L^s$ with norm less than or equal to $M_s(\widehat{\varphi})\|\Psi\|_s$. Also

$$\Big\| \sum_{m\in\mathbf{Z}^N} a(m)\varphi(\cdot - m)\Big\|_s = \Big\| \sum_{m\in\mathbf{Z}^N} a(m)\Psi(\cdot - m) * \varphi\Big\|_s \le M_s(\widehat{\varphi}) \Big\| \sum_{m\in\mathbf{Z}^N} a(m)\Psi(\cdot - m)\Big\|_s \le M_s(\widehat{\varphi}) C_s(\Psi)\|a\|_s\,,$$

as is easily seen from the properties of Ψ.

For the general case, we write $\varphi = \varphi * \Psi_{1/R}$ and we proceed as above. ∎

Proof of Theorem 1. Let $a = (a(n))_n$ be a sequence in l^s and define $f(x) = \sum_{n\in\mathbf{Z}^N} a(n)\varphi(x-n)$, which belongs to L^s by Lemma 5. Observe that $(K_t^\varphi \star a)(n) = (K_t * f)(n) = g_t(n)$, where $g_t = K_t * f \in E_R$, for all t.

Applying inequalities (8), (5) and (11) we obtain

$$\|\|\|K_t^\varphi \star a(n)\|_q\|_p \le C\max(1,R^{N/p})\|\|\|g_t(x)\|_q\|_p \le C\max(1,R^{N/p})\|f\|_s$$
$$\le C\max(1,R^{N/p})M_s(\widehat{\varphi})\max(1,R^{N(1-1/s)})\|a\|_s. \ \blacksquare$$

Remarks. (i) The dependence on R we exhibited suggests that R cannot be taken equal to ∞. For example, let $\widehat{\varphi}(\xi)=1$ everywhere and consider $K\in L^1$ such that $K\ge 0$, $K(n)=1$ for all $n\in \mathbf{Z}^N$. Then $f\to K*f$ is bounded on L^p, whereas, for $t=1$, $a\to K_t^\varphi\star a$ is not bounded on l^p since $K_t^\varphi(n)=K(n)=1$.

(ii) In view of embeddings between l^p spaces, converses of Theorem 1 and Proposition 2 are not true in general when $p\ne s$. See §3 for converses in the case $p=s$.

We also obtain weak type versions of Theorem 1 and Proposition 2.

THEOREM 6. *Under the hypotheses of Theorem* 1, $1<p<\infty$, *and* $1\le q\le\infty$, *the inequality*

(12) $$|\{x\in\mathbf{R}^N;\ \|K_t*f(x)\|_q>\lambda\}|\le(\|f\|_s/\lambda)^p,\quad \text{for all } \lambda>0,$$

implies

(13) $$\sharp\{n\in\mathbf{Z}^N;\ \|(K_t^\varphi\star a)(n)\|_q>\lambda\}\le A^p(\|a\|_s/\lambda)^p,\quad \text{for all } \lambda>0,$$

where $A\le C(p,q,s,N)M_s(\widehat{\varphi})\max(1,R^{N/p+N(1-1/s)})$.

PROPOSITION 7. *Under the hypotheses of Proposition* 2 *and* $1<p<\infty$, *the inequality*

(14) $$|\{x\in\mathbf{R}^N;\ |K*f(x)|>\lambda\}|\le(\|f\|_s/\lambda)^p,\quad \text{for all } \lambda>0,$$

implies

(15) $$\sharp\{n\in\mathbf{Z}^N;\ |(K_1^\varphi\star a)(n)|>\lambda\}\le A^p(\|a\|_s/\lambda)^p,\quad \text{for all } \lambda>0,$$

where A *is as in Theorem* 6.

Proofs of these two results are similar to the proof of Theorem 1. They will be obtained as a direct consequence of the following extension of Lemmas 3 and 4, which is interesting on its own.

LEMMA 8. *For* $1<p<\infty$, $1\le q\le\infty$ *there is a constant* $C=C(p,q,N)$ *such that*

(16) $$\sup_{\lambda>0}\sharp\{n\in\mathbf{Z}^N;\ |g(n)|>\lambda\}\lambda^p \le C^p\max(1,R^N)\sup_{\lambda>0}|\{x\in\mathbf{R}^N;\ |g(x)|>\lambda\}|\lambda^p,$$

and

(17) $$\sup_{\lambda>0}\sharp\{n\in\mathbf{Z}^N;\ \|g_t(n)\|_q>\lambda\}\lambda^p \le C^p\max(1,R^N)\sup_{\lambda>0}|\{x\in\mathbf{R}^N;\ \|g_t(x)\|_q>\lambda\}|\lambda^p,$$

for any $g\in E_R$ *and any family* $(g_t)_t$ *of jointly measurable functions in* E_R.

Proof. We restrict our attention to proving (17) when $R=1$. Let Ψ be as in the proof of Lemma 3 and introduce the sublinear operator

$$T(f_t)(k)=\|(f_t*\Psi)(k)\|_q.$$

In proving Lemma 4, we established that, for every function $f\in L^r(\mathbf{R}^N,L^q(\mathbf{R}^+,dt/t))$,

$$\|\|\|(f_t*\Psi)(n)\|_q\|_r\le C\bigg\|\bigg(\int_{\mathbf{R}^+}|\Psi*f_t(x)|^q\,\frac{dt}{t}\bigg)^{1/q}\bigg\|_r$$
$$\le C\|\Psi\|_{L^1}^{1+1/q'}\|\|\|f_t(x)\|_q\|_r;$$

that is, T maps $L^r(\mathbf{R}^N,L^q(\mathbf{R}^+,dt/t))$ into $l^r(\mathbf{Z}^N)$ for $1\le r\le\infty$, and therefore we have the result by interpolation together with the assumed properties of Ψ. For arbitrary R replace Ψ by $\Psi_{1/R}$, where Ψ is as above.

We thank L. Colzani for having pointed out to us that (16) also holds when $p=1$ and $N=1$. The argument of Colzani, which we present in an appendix, easily implies that, in dimension one, Proposition 7 holds when $p=1$ as well as Theorem 6 when $p=1$ and $q=\infty$. This is of particular interest when considering (maximal) operators of weak type (1, 1) (see §4).

§ 3. Converses.

In this section, the dilation structure of $\mathbf{R}^N$ plays a crucial role. We shall assume that $K_t(x)=t^{-N}K(t^{-1}x)$, $t>0$. However, there is no problem to extend the following results to the case of nonisotropic dilations. A basic hypothesis in §2 is that $\widehat{\varphi}\in M_p(\mathbf{R}^N)$. It is not surprising that, for the converse direction, we need a condition that is "close" to $1/\widehat{\varphi}\in M_p(\mathbf{R}^N)$.

THEOREM 9. *Let* $1\le p<\infty$ *and* $0<q\le\infty$. *Assume that* $\widehat{\varphi}$ *satisfies the following two conditions*:

(i) $\operatorname{supp}\widehat{\varphi}\subset[-R,R]^N$ *with* $R<1$ *and*

(ii) *for some* $\varepsilon>0$ *and* $h\in C^\infty((-\varepsilon,\varepsilon)^N)$, $h\equiv 1$ *on* $[-\varepsilon/2,\varepsilon/2]^N$, $h/\widehat{\varphi}\in M_p(\mathbf{R}^N)$.

Then the inequality

(18) $$\|\|\|K_t^\varphi\star a\|_q\|_p\le\|a\|_p,\quad a\in l^p(\mathbf{Z}^N),$$

implies

(19) $$\|\|\|K_t*f\|_q\|_p\le A\|f\|_p,\quad f\in L^p(\mathbf{R}^N),$$

where $A\le M_p(h/\widehat{\varphi})$.

Proof. To prove (19), we may assume, by density and rescaling, that $f \in \mathcal{S}(\mathbf{R}^N)$ satisfies $\operatorname{supp} \widehat{f} \subset [-\delta, \delta]^N$ where $\delta < \varepsilon/2$ and $\delta < 1 - R$, as long as our estimates do not depend on this assumption. Write $\widehat{f} = \widehat{f}h = (\widehat{f}h/\widehat{\varphi})\widehat{\varphi}$; for $x = n + u$, $n \in \mathbf{Z}^N$, $u \in [0, 1)^N$,

$$\widehat{f}(\xi)e^{2\pi i x\xi} = ((\widehat{f}h/\widehat{\varphi})(\xi)e^{2\pi i u\xi})\widehat{\varphi}(\xi)e^{2\pi i n\xi}$$
$$= \Big(\sum_{k\in\mathbf{Z}^N} \Big(\frac{\widehat{f}h}{\widehat{\varphi}}e^{2\pi i u\cdot}\Big)(\xi + k)\Big)\widehat{\varphi}(\xi)e^{2\pi i n\xi}.$$

The last equality holds because of our choice of δ and its relation with R (the terms with $k \neq 0$ vanish). The above series defines a 1-periodic function $P_u(\xi)$ whose Fourier coefficients are

$$a^u(m) = \int_{\mathbf{R}^N} (\widehat{f}h/\widehat{\varphi})(\xi)e^{2\pi i u\xi}e^{2\pi i m\xi}\, d\xi.$$

(Observe that the integrand is in $L^1(\mathbf{R}^N)$ since it is bounded with compact support.) Thus, we obtain the formula

$$(C_t f)(x) = (D_t^{\varphi} a^u)(n), \quad \text{for } x = n + u. \tag{20}$$

Therefore, by (20) and, then, (18),

$$\| \|C_t f\|_q \|_p^p = \int_{[0,1)^N} \sum_n \|(D_t^{\varphi} a^u)(n)\|_q^p\, du \le \int_{[0,1)^N} \|a^u\|_p^p\, du$$
$$= \int_{[0,1)^N} \sum_m \Big| \int_{\mathbf{R}^N} (\widehat{f}h/\widehat{\varphi})(\xi)e^{2\pi i u\xi}e^{2\pi i m\xi}\, d\xi\Big|^p du$$
$$= \int_{\mathbf{R}^N} \Big| \int_{\mathbf{R}^N} \widehat{f}(\xi)(h/\widehat{\varphi})(\xi)e^{2\pi i x\xi}\, d\xi\Big|^p dx \le M_p(h/\widehat{\varphi})^p\|f\|_p^p. \ \blacksquare$$

Remarks. (i) The dilation property for $\mathbf{R}^N$ is used when we assume that we can rescale f to be such that $\operatorname{supp} \widehat{f} \subset [-\delta, \delta]^N$. This is due to the fact that $f \to \|K_t * f\|_q$ commutes with dilations. More precisely, let f be in $L^p(\mathbf{R}^N)$ and $g(x) = R^N f(Rx)$ for $R > 0$; then

$$\|(K_t * g)(x)\|_q = R^N \|(K_t * f)(Rx)\|_q,$$

because of the invariance of the measure dt/t. It turns out that we may take nonisotropic dilations as well. These are given by a family of invertible matrices $A(t)$, $t > 0$, such that $A(ts) = A(t)A(s)$, $A(0) = 0$ and $\lim_{t\to\infty} \|A(t)x\| = \infty$ for all $x \in \mathbf{R}^N$. Let $Q = (\log t)^{-1} \log \det |A(t)|$ (Q is called the *homogeneous dimension*) and define $K_t(x) = t^{-Q}K(A(t)^{-1}x)$ and $C_t(f)(x) = K_t * f(x)$. Then, with exactly the same hypotheses on $\widehat{\varphi}$, Theorem 9 holds in this context.

(ii) In case of isotropic or nonisotropic dilations consider, for example, $\widehat{\varphi}(\xi) = 1$. Then $K_t^{\varphi}(n) = K_t(n)$ and the implication (18)⇒(19) is still true when $1 \le q \le \infty$. This is easily seen by approximating $C_t(f)(x)$ by Riemann sums. (Note that inequality (18) already takes care of the restriction of K_t to $\mathbf{Z}^N$.)

(iii) The first assumption in Theorem 9 is sharp in the following sense. Let $\widehat{\varphi}(\xi) = \chi_{[-1,1]}(\xi)$. Then for $K(x) = \text{p.v.}\frac{1}{\pi x}$, $C_t(f)(x) = H(f)(x)$ is the Hilbert transform of f while $D_t^{\varphi}(a) = 0$ for all sequences a. Thus, (18)⇒(19) is false for this φ.

(iv) The second assumption in Theorem 9, on the other hand, shows that the torus $\mathbf{T}^N$ cannot be identified with $[0,1)^N$ (as mentioned in the introduction). In this case, $\widehat{\varphi}$ is $\chi_{[0,1)^N}$ and the second assumption is not fulfilled.

(v) For a single convolution operator $f \to K * f$ and $K_t(x) = t^{-Q}K(A(t)^{-1}x)$, Theorem 9 takes the following form:

PROPOSITION 10. *Under the assumptions of Theorem* 9, $\|K_t^{\varphi} * a\|_p \le \|a\|_p$, *for all* $t > 0$, *implies* $\|K * f\|_p \le A\|f\|_p$.

(v) It is clear from relation (20) in the proof of Theorem 9 that weak type versions are easily obtained in the same way. The only new ingredient needed is the relation

$$|\{x \in \mathbf{R}^N ; |f(x)| > \lambda\}| = \int_{[0,1)^N} \sharp\{n \in \mathbf{Z}^N ; |f(n + u)| > \lambda\}\, du,$$

for any measurable function f. Let us state the result.

THEOREM 11. *Let* $1 \le p < \infty$, $0 < q \le \infty$. *Suppose* $\widehat{\varphi}$ *is as in Theorem* 9. *Then* $\|K_t^{\varphi} * a\|_{l^{p,\infty}(\mathbf{Z}^N)} \le \|a\|_p$, *for all* $t > 0$, *implies* $\|K * f\|_{L^{p,\infty}(\mathbf{R}^N)} \le A\|f\|_p$, *and* $\| \|K_t^{\varphi} * a\|_q \|_{l^{p,\infty}(\mathbf{Z}^N)} \le \|a\|_p$ *implies* $\| \|K_t * f\|_q \|_{L^{p,\infty}(\mathbf{R}^N)} \le A\|f\|_p$, *where* $A \le M_p(h/\widehat{\varphi})$ *and* $K_t(x) = t^{-Q}K(A(t)^{-1}x)$.

(vi) These implications can be considered to be of "transference type" (see [CW]). This is particularly clear when $\widehat{\varphi}$ is the characteristic function of $[-1/2, 1/2]^N$. In this case A in (19) becomes 1.

§4. Applications and examples. As seen in the introduction, conditions (a)–(c) of Theorem A are met by $\widehat{\varphi} = \chi_{[-1/2,1/2]^N}$. These conditions are also fulfilled by smooth φ; for example, $\varphi \in \mathcal{S}(\mathbf{R}^N)$, $\widehat{\varphi}(\xi) = 1$ on $[-1/2, 1/2]^N$, $\widehat{\varphi} \ge 0$ and $\operatorname{supp} \widehat{\varphi} \subset [-3/4, 3/4]^N$. For these φ's, the constants C and C' in Theorem A depend only on K and p, and C'/C is bounded away from 0 and ∞ by universal constants depending on $\widehat{\varphi}$ but not on p. Thus, smooth φ's can be used in many different situations for $1 \le p < \infty$.

We shall try to relate the discrete operator D_t^{φ} to a "natural" discrete version of the operator C_t. This will be achieved by selecting an appropriate φ. Most of the next examples are well known. Our point is to show that our method applies in many different situations.

EXAMPLE 1 (singular integrals). We first consider the Hilbert transform on $\mathbf{R}$ whose multiplier is $m(x) = -i\,\mathrm{sgn}(x)$. By choosing a smooth and even φ as above, we can compute explicitly $K_t^\varphi(m)$ and we obtain

$$K_t^\varphi(m) = \int_{\mathbf{R}} -i\,\mathrm{sgn}(t\xi)\widehat{\varphi}(\xi)e^{2\pi i m\xi}\,d\xi = 2\int_0^\infty \widehat{\varphi}(\xi)\sin(2\pi m\xi)\,d\xi = \frac{1}{\pi m} + O\left(\frac{1}{m^2}\right).$$

Thus, $D_t^\varphi = H_d + E$, where H_d denotes the discrete Hilbert transform and E is an operator bounded on all l^p, $1 \le p \le \infty$.

In N dimensions we obtain a general result of a similar nature:

PROPOSITION 12. *Suppose $K(x) = |x|^{-N}\Omega(x')$ for $x' = |x|^{-1}x \in \Sigma_{N-1}$, where $\Omega \in C^\varepsilon(\Sigma_{N-1})$, $\varepsilon > 0$, and $\int_{\Sigma_{N-1}} \Omega(x')\,dx' = 0$. Then, letting $K * f = \mathrm{p.v.}\ K * f$,*

$$\|K * f\|_p \le C\|f\|_p$$

if and only if

$$\Big\|\sum_{m\in\mathbf{Z}^N\setminus\{0\}} K(m)a(n-m)\Big\|_p \le C'\|a\|_p .$$

Sketch of the proof. Take $\varphi \in \mathcal{S}(\mathbf{R}^N)$ such that φ is even and $\int_{\mathbf{R}^N}\varphi = 1$. Then a direct consequence of standard computations in the theory of singular integrals (see [S], for example) is that

$$|K * \varphi_{1/s}(x') - K(x')| \le C/s^\varepsilon ,$$

uniformly for $x' \in \Sigma_{N-1}$. Now using the homogeneity of K we obtain

$$K_t^\varphi(m) = (K * \varphi)(m) = s^{-n}(K * \varphi_{1/s})(m')$$

for $s = |m|$, and, therefore, $K_t^\varphi(m) = K(m) + O(|m|^{-N-\varepsilon})$ for $m \ne 0$ and $K_t^\varphi(0) = 0$. This finishes the proof. ∎

EXAMPLE 2 (Hardy–Littlewood maximal operator). Let

$$a^*(n) = \sup_{N\in\mathbf{Z}^+}\Big|\frac{1}{2N+1}\sum_{m=-N}^{N} a(n-m)\Big|$$

be the (discrete) centered Hardy–Littlewood maximal operator applied to $(a(n))_n$. Our results apply to the kernel $K = \frac{1}{2}\chi_{[-1,1]}$. Elementary calculations show that, for $t = N + 1/2$, $N \in \mathbf{Z}^+$,

$$D_t^\varphi(a)(n) = \frac{1}{2N+1}\sum_{m=-N}^{N} Ta(n-m)$$

where T is a convolution operator bounded and invertible on all l^p, $1 \le p \le \infty$; its Fourier multiplier is the C^∞ nonvanishing 1-periodic function

$$M(\xi) = \sum_l \widehat{\varphi}(\xi + l)\frac{\sin(\pi\xi + \pi l)}{\pi\xi + \pi l}.$$

Then, for $1 < p < \infty$, strong type (p,p) equivalence between the two versions follows easily.

In their fundamental paper [HL], Hardy and Littlewood prove strong type (p,p) for the discrete version and, by an argument involving Riemann sums, they deduce strong type (p,p) for the continuous version. A converse can also be obtained by applying the continuous version to step functions. At this point, let us say that the "step function method" for maximal estimates works only for those operators having kernels such that

$$\sup_{t>0}|\{(K_t * \chi_{[-1/2,1/2]^N})(\cdot - u) - K_t(\cdot)\} \star a|$$

defines a bounded operator on $l^p(\mathbf{Z}^N)$, uniformly in $u \in [-1/2,1/2]^N$.

Let us be more explicit with the help of an example. Let

$$P_t(x) = \frac{1}{\pi}\frac{t}{t^2 + x^2}$$

be the Poisson kernel. A "natural" discrete version for the associated maximal operator is

$$(a(n))_n \to \left(\sup_{t>0}\Big|\sum_{m\ne 0}\frac{1}{\pi}\frac{t}{t^2+m^2}a(n-m)\Big|\right)_n .$$

Strong type (p,p) for this operator follows from that of the continuous version if we can prove

$$(*)\quad \sum_m \sup_{t>0}\Big|\sum_{n\ne 0}(P_t * \chi_{[-1/2,1/2]}(n+u) - P_t(n))a(m-n)\Big|^p \le C\sum_m |a(m)|^p$$

uniformly in $u \in [-1/2,1/2]$. Indeed, for $f(x) = \sum_n a(n)\chi_{[n-1/2,n+1/2]}(x)$, one can write

$$\int_{\mathbf{R}} \sup_{t>0}|P_t * f(x)|^p\,dx = \int_{-1/2}^{1/2}\sum_m \sup_{t>0}\Big|\sum_n a(m-n)P_t * \chi_{[-1/2,1/2]}(n+u)\Big|^p\,du$$

and this last quantity is bounded by $C\|f\|_p^p = C\|a\|_p^p$.

Inequality $(*)$ is not hard to establish since the properties of P_t lead to the estimate

$$|P_t * \chi_{[-1/2,1/2]}(n+u) - P_t(n)| \le Cn^{-2}$$

where C is independent of $t > 0$, $u \in [-1/2, 1/2]$ and $n \neq 0$. (For $n \le 10$ say, use crude estimates on each term and for $n \ge 10$ use Taylor's Formula.) This shows how the "step function method" applies. Note that our methods also apply very easily to this situation.

EXAMPLE 3 (convolution operators with compactly supported multipliers). If $\widehat{K}$ has compact support, then K_t^φ reduces to K_t for t large since $\widehat{\varphi}$ is 1 in a neighbourhood of 0. Thus, a consequence of our results is the following:

COROLLARY 13. *Suppose that $\widehat{K} = m$ has compact support. Then for* $1 < p < \infty$,

$$\|\sup_{t>0} |K_t * f|\|_p \le C\|f\|_p$$

is equivalent to

$$\Big\|\sup_{t\ge A} \sum_m K_t(m)a(n-m)\Big\|_p \le C\|a\|_p\,,$$

for some A depending on $\operatorname{supp}\widehat{K}$.

This gives us the following interesting applications. Let us start with $m(x) = \chi_{[-1/2,1/2]}(x)$. The continuous maximal operator is the operator $C^\sharp$ we mentioned in the introduction, and the discrete maximal operator is $D^\sharp$. That when $p = 2$ the "continuous" case implies the "discrete" case was first proved in [CP]. The converse was then established by K. Petersen (unpublished, personal communication). More recently, he and I. Assani have found a simple proof for this equivalence [AP]. They use, for this, elementary arguments and establish on the way a different equivalent formulation for $D^\sharp$. These arguments were extended in [A] to establish the equivalence when $p \in (1, \infty)$.

In N dimensions, if we take $m(x) = \chi_B(x)$ in $\mathbf{R}^N$, then the continuous operator is the maximal operator associated with the sphere multiplier. The discrete version of this operator is the following:

$$(a(m))_m \to \bigg(\sup_{t\ge1}\bigg|\frac{1}{t^{N/2}}\sum_{n\in\mathbf{Z}^N}\frac{J_{N/2}(|n|/t)}{|n|^{N/2}}a(m-n)\bigg|\bigg)_m\,.$$

As is already known, the continuous operator cannot be bounded if $p \neq 2$. Therefore, the previous discrete operator is not bounded for $p \neq 2$. For $p = 2$, this remains an open question.

A third example will be a discrete version for the maximal operator associated with the Bochner–Riesz multiplier of order α, $\sup_{t\ge1}|K_t^\alpha(x) * \cdot|$, with multiplier $\widehat{K}^\alpha(\xi) = (1-|\xi|^2)_+^\alpha$; this is the operator

$$(a(m))_m \to \bigg(\sup_{t\ge1}\bigg|\frac{1}{t^{N/2-\alpha}}\sum_{n\in\mathbf{Z}^N}\frac{J_{N/2+\alpha}(|n|/t)}{|n|^{N/2+\alpha}}a(m-n)\bigg|\bigg)_m\,.$$

§ 5. A characterization of the boundedness of Littlewood–Paley functions. Let φ satisfy the conditions of Theorem A. In the case $q = 2$, we can construct a positive definite matrix $B = (B(n,k))_{n,k\in\mathbf{Z}^N}$ depending on φ and such that the following equivalence holds:

$$\|g(f)\|_p = \bigg\|\bigg(\int_0^\infty |K_t * f|^2\,\frac{dt}{t}\bigg)^{1/2}\bigg\|_p \le C\|f\|_p$$

if and only if the operator

$$(a(m))_m \to \bigg(\Big(\sum_{n,k\in\mathbf{Z}^N} B(n,k)a(m-n)\overline{a}(m-k)\Big)^{1/2}\bigg)_m$$

is bounded on $l^p(\mathbf{Z}^N)$, with equivalent constants.

To see this we use the fact that the first condition is equivalent to the boundedness of the discrete operator

$$(a(m))_m \to \bigg(\bigg(\int_0^\infty \Big|\sum_n (K_t*\varphi)(n)a(m-n)\Big|^2\frac{dt}{t}\bigg)^{1/2}\bigg)_m\,.$$

Therefore, it is clear that the matrix B is given by

$$B(n,k) = \overline{B(k,n)} = \int_0^\infty (K_t*\varphi)(n)\overline{(K_t*\varphi)(k)}\,\frac{dt}{t}\,.$$

Moreover, in the case $\widehat{\varphi} = \chi_{[-1/2,1/2]^N}$ the coefficients $B(n,k)$ are the Fourier coefficients of the function defined in $\mathbf{R}^{2N}$ by

$$M(\xi,y) = \int_0^\infty m(t\xi)\overline{m(ty)}\,\frac{dt}{t}\,\chi_{[-1/2,1/2]^{2N}}(\xi,y).$$

A trivial consequence of the previous result is that a sufficient condition for the operator g to be bounded on $L^p(\mathbf{R}^N)$ is that the following bilinear map is bounded from $(l^p(\mathbf{Z}^N), l^p(\mathbf{Z}^N))$ into $l^{p/2}(\mathbf{Z}^N)$:

$$((a(m))_m, (b(m))_m) \to \Big(\sum_{n\le k} B(n,k)a(m-n)b(m-k)\Big)_m\,.$$

§ 6. Other dilations. So far all the examples presented involve families of multipliers arising from the usual dilation of a fixed kernel K. However, different dilations generate a wide variety of interesting operators that are different. One of these is the maximal parabolic operator (see for example [NRW1])

$$Mf(x) = \sup_{0\le r<\infty} \frac{1}{r}\int_{-r}^{r} f(x-\gamma(t))\,dt\,, \quad f \ge 0\,,$$

with $\gamma(t) = (t,t^2)$. In this case, it is very easy to see that $m(x) = \int_{-1}^{1} e^{-2\pi i x\gamma(s)}\,ds$, and then the discrete operator is given by

$$(a(p,q))_{p,q} \to \sup_{0<r<\infty}\left|\sum_{n,m}\left(\frac{1}{r}\int_{-r}^{r}\varphi(n-s)\varphi(m-s^2)\,ds\right)a(p-n,q-m)\right|.$$

Finally, one can check (applying Propositions 2 and 10) that the following result holds.

THEOREM 14. *Let $1<p<\infty$. If φ satisfies the same hypotheses as in Theorem A, then*

$$\left\|\text{p.v.}\int_{\mathbf{R}} D_t^{\varphi}(a)\,\frac{dt}{t}\right\|_p \le C\|a\|_p$$

if and only if

$$\left\|\text{p.v.}\int_{\mathbf{R}} C_t(f)\,\frac{dt}{t}\right\|_p \le C'\|f\|_p\,, \tag{21}$$

with equivalent constants.

A consequence of this result is the following. Let us consider the Parabolic Hilbert Transform in $\mathbf{R}^2$ (see [NRW2])

$$Hf(x) = \text{p.v.}\int_{\mathbf{R}} f(x-\gamma(t))\,\frac{dt}{t}\,, \quad \gamma(t) = (t,t^2)\,.$$

This operator is of the form (21) with $m(x) = e^{2\pi i(x_1+x_2)}$. Therefore, the already known boundedness of this operator on $L^p(\mathbf{R}^2)$ is equivalent to the boundedness on $l^p(\mathbf{Z}^2)$ of the convolution operator with the sequence

$$k(n,m) = \text{p.v.}\int_{\mathbf{R}} \varphi(n-t)\varphi(m-t^2)\,\frac{dt}{t}\,.$$

Similar results can be obtained for other dilations, say,

$$\gamma(t) = (|t|^{\alpha_1},\ldots,|t|^{\alpha_N}), \quad \alpha_j \ge 0\,.$$

Finally, we have to mention that in this case of the parabolic Hilbert transform, the behaviour of the sequence $k(n,n^2)$ is like $1/n^2$ when n goes to ∞. Thus, this convolution kernel is radically different from the discrete Hilbert transform along the parabola, defined by

$$H_p(a)(l,m) = \sum_{n\ne 0}\frac{1}{n}a(l-n,m-n^2)\,.$$

§ 7. Appendix. Proof of (16) when $p=1$. We have to show that

$$\sup_{\lambda>0}\sharp\{n\in\mathbf{Z}\,;\,|g(n)|>\lambda\} \le C\|g\|_{1,\infty} \tag{$*$}$$

for all $g \in E_1 \cap L^{1,\infty}$ (the case $R \ne 1$ can be dealt with similarly to the proof of Lemma 3). To do so, we shall use the function space Weak-H^1 for which we recall the basic facts.

Let Weak-H^1 be the space of tempered distributions for which the maximal function $f^*(x) = \sup_{t>0}|\varphi_t * f(x)|$ exists a.e. and belongs to Weak-$L^1 = L^{1,\infty}$. Here φ is assumed to be, say, a Schwartz function with $\int_{\mathbf{R}}\varphi(x)\,dx = 1$. The "norm" $\|f\|_{\text{Weak-}H^1}$ is defined by $\|f^*\|_{1,\infty}$. We also need two more norms on Weak-H^1, each equivalent to $\|f\|_{\text{Weak-}H^1}$.

First, let $f_t(x) = P_t * f(x)$ denote the harmonic extension of f to the upper half-plane and $f^\sharp(x) = \sup_{|y|\le t}|f_t(y)|$. Then $\|f\|_{\text{Weak-}H^1}$ is equivalent to $\|f^\sharp\|_{1,\infty}$. Second, the Hilbert transform maps boundedly Weak-H^1 into $L^{1,\infty}$. The second norm is then $\|f\|_{1,\infty} + \|Hf\|_{1,\infty} \sim \|f\|_{\text{Weak-}H^1}$. We refer to [FS] for details.

We then argue as follows: let $g \in E_1 \cap L^{1,\infty}$ and let $f(x) = e^{4\pi i x}g(x)$. Then $\widehat{f}(\xi) = \widehat{g}(\xi-2)$ (in the sense of distributions) so that $\operatorname{supp}\widehat{f} \subset [1,3]$. Therefore, $Hf = -if$. This implies that $f \in E_3 \cap \text{Weak-}H^1$. Moreover, it is enough to prove $(*)$ with f instead of g since $|f(x)| = |g(x)|$.

Let $e^{-2\pi|\xi|}$ be the Fourier transform of the Poisson kernel P_1 and let $m(\xi)$ be a C^∞ function with support in $[1/2,4]$ and with $m(\xi) = e^{2\pi\xi}$ for $\xi \in [1,3]$. Let next h be the distribution defined by $\widehat{h}(\xi) = m(\xi)\widehat{f}(\xi)$; since $\operatorname{supp}\widehat{h} \subset [1,3]$ and since m defines a multiplier on Weak-H^1 ($(\check{m}*f)^*$ is pointwise controlled by the maximal function f^*), $h \in E_3 \cap \text{Weak-}H^1$ and

$$\|h\|_{1,\infty} \le C\|h\|_{\text{Weak-}H^1} \le C\|f\|_{\text{Weak-}H^1}\,.$$

Now we observe that $f = P_1 * h = h_1$. Let $n \in \mathbf{Z}$ and $|x-n| \le 1/2$; it follows that $|f(n)| = |h_1(n)| \le h^*(x)$.

This and the characterization of Weak-H^1 by the maximal function $h^\sharp$

yield

$$\lambda\,\sharp\{n\,;\,|f(n)|>\lambda\}\le\lambda\,\sharp\{x\,;\,|h^*(x)|>\lambda\}\le C\|h\|_{\text{Weak-}H^1}$$
$$\le C\|f\|_{\text{Weak-}H^1}\le C\|f\|_{1,\infty}\,.\ ■$$

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On the rate of strong mixing in stationary Gaussian random fields

by

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Abstract. Rosenblatt showed that a stationary Gaussian random field is strongly mixing if it has a positive, continuous spectral density. In this article, spectral criteria are given for the rate of strong mixing in such a field.

A stationary random process $\{X_n\}$ is *strongly mixing* if, in a certain sense, its past and future are asymptotically independent. This idea was introduced by Rosenblatt [7] in connection with a central limit theorem. Kolmogorov and Rozanov [6] found a useful sufficient condition for a Gaussian process to be strongly mixing, namely, that it have a (strictly) positive, continuous spectral density. A necessary and sufficient condition was obtained by Helson and Sarason [3] (see also Sarason [9]). Ibragimov and Rozanov [4], and Khrushchev and Peller [5] are concerned with the rate at which strong mixing occurs as revealed by the smoothness of the spectral density.

The notion of strong mixing makes sense in the random field setting as well. Indeed, Rosenblatt [8] proved that a stationary Gaussian field $\{X_{mn}\}$ satisfies a strong mixing condition if it has a positive, continuous density; this is an exact analogue of the result in [6].

In this article, spectral criteria are derived for the rate at which strong mixing occurs in a stationary Gaussian field. First, the separation-of-variables technique of [2] is used to adapt the one-parameter methods in [5]. This yields mixing rates in which the roles of past and future are played by vertical halfplanes of the random field. This result is then extended to the case of halfplanes at rational slopes. Lastly, these ideas are used to investigate strong mixing in the full sense of [8].

1. The principal result. Let $\{X_{mn}\}$ be a stationary Gaussian random field on the integral lattice $\mathbf{Z}^2$, and μ its spectral measure on the torus $\mathbf{T}^2$.

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