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The Muckenhoupt class $A_1(\mathbf{R})$

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Abstract. It is shown that the Muckenhoupt structure constants for f and f^* on the real line are the same.

Introduction. In a previous paper, [4], one of the authors has shown that if a function f lies in $A_p(Q)$, $Q \subset \mathbf{R}^n$, with constant c, then the nonincreasing rearrangement of f, f^*, lies in $A_p((0,|Q|))$ with another constant, c_1, depending on n, p and c. In Theorem 1 we prove that in the special case $n = 1$, the constant c_1 can be taken as c, which of course is optimal. To do this we will need a covering lemma. We also show, by means of an example, that this result is not true in dimensions higher than one. As a consequence of the theorem we obtain Lemma 2 with Corollary 1, a refinement of a lemma by Muckenhoupt [2]. This is also proved in a more direct way. Theorem 2 describes another property of the weights in the class $A_1(\mathbf{R})$.

Notations. We let $⨍_I f(x)\,dx$ stand for the mean value of f over I. For an interval $\mathcal{J}$ we will use the notation $A_1(\mathcal{J}; c)$ for the class of locally integrable functions f such that for every subinterval I of $\mathcal{J}$ we have

$$(1) \qquad ⨍_I f(x)\,dx \le c \operatorname*{ess\,inf}_I f(x)\,.$$

For any subinterval I of $(0,1)$ let f_I denote the restriction of f to I and f_I^* the corresponding nonincreasing (left-continuous) rearrangement.

Theorems and proofs

THEOREM 1. *Suppose that f is a function in $A_1((0,1); c)$ and I a subin-*

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terval of $(0,1)$. *Then for each* $\mathcal{I} \subset I$

$$\fint_{\mathcal{I}} f_I^*(u)\,du \le c \operatorname*{ess\,inf}_{\mathcal{I}} f_I^*(x). \tag{2}$$

Conversely, if (2) *holds for all local rearrangements* f_I^* *of* f, *then* $f \in A_1((0,1);c)$.

The implication of (1) is thus a set of inequalities for the functions f_I, $I \subset (0,1)$, with the common constant c. These inequalities completely characterize $A_1((0,1);c)$.

We stress the fact that, contrary to most of the recent literature (see e.g. [1], [3]), we require the nonincreasing rearrangement f^* to be continuous on the left. By this condition $f^*(t)$ is uniquely determined for each $t>0$ and (2) implies

$$\fint_0^t f^*(u)\,du \le c \operatorname*{ess\,inf}_{(0,t)} f^*(x) = cf^*(t). \tag{2'}$$

We will first give a short proof under the extra assumption that f, and therefore also f^*, are continuous.

Fix an $I \subset (0,1)$. Let E_λ be the open set $\{x \in I;\ f(x) > \lambda\}$. If $\lambda \ge \fint_I f$ it can be written as the union of disjoint open intervals, $E_\lambda = \bigcup \omega_\nu$, where $f(x) = \lambda$ at the endpoints of the intervals except possibly at the endpoints of I. Therefore

$$\int_{E_\lambda} f_I(x)\,dx = \sum_\nu \int_{\omega_\nu} f_I(x)\,dx \le \sum_\nu c\lambda|\omega_\nu| = c\lambda|E_\lambda|.$$

Dividing this inequality by $|E_\lambda|$ and using the fact that f^* is continuous we find that

$$\fint_0^{|E_\lambda|} f_I^*(t)\,dt = \frac{1}{|E_\lambda|}\int_{E_\lambda} f_I(x)\,dx \le c\lambda \le c \operatorname*{ess\,inf}_{t\in(0,|E_\lambda|)} f_I^*(t).$$

Thus we have proved statement (2′) for $f^* = f_I^*$ in case $t = |E_\lambda|$ for some λ.

Now take an arbitrary $t \in I$. Put $f_I^*(t) = \lambda_1$, $t_1 = \min\{t;\ f_I^*(t) = \lambda_1\}$. Then $|E_{\lambda_1}| = t_1$ and

$$\begin{aligned}\fint_0^t f_I^*(t)\,dt &= \frac{t_1}{t}\fint_0^{|E_{\lambda_1}|} f_I^*(t)\,dt + \frac{t-t_1}{t}\lambda_1 \le \frac{t_1}{t}c\lambda_1 + \frac{t-t_1}{t}\lambda_1 \\ &\le cf_I^*(t) = c\min_{(0,t)} f_I^*(u).\end{aligned}$$

This means that (2′) holds with $f^* = f_I^*$. Since f_I^* is nonincreasing, this implies (2). In fact, for an arbitrary interval $(a,b) \subset I$ we have

$$\fint_a^b f_I^*(t)\,dt \le \fint_0^b f_I^*(t)\,dt \le cf_I^*(b) = c \operatorname*{ess\,inf}_{(a,b)} f_I^*(t),$$

which proves (2).

The implication (2)⇒(1) is immediate. We choose $\mathcal{I} = I$ in (2). Then the stars may be removed and we conclude that $f \in A_1((0,1);c)$.

When f is not continuous, we will use the following covering lemma as a substitute for continuity:

LEMMA 1. *Let* E *be a measurable bounded subset of* $\mathbb{R}$ *and* $\varepsilon > 0$. *Then there exist a sequence* $\{\omega_\nu\}_{\nu=1}^\infty$ *of intervals with disjoint interiors and a subset* E_1 *of* E *with the properties that* $|E_1| = |E|$ *and*

$$\text{(3)}\qquad \begin{aligned}&\text{(i)}\quad E_1 \subset \bigcup_{\nu=1}^\infty \omega_\nu,\\ &\text{(ii)}\quad (1-\varepsilon)|\omega_\nu| \le |\omega_\nu \cap E| < |\omega_\nu|, \quad \nu = 1,2,\ldots\end{aligned}$$

Proof. First we choose as ω_1 a closed interval of maximal length satisfying (ii). Suppose then that the intervals $\omega_1,\ldots,\omega_n$ are chosen. Take as ω_{n+1} a closed interval of maximal length with interior disjoint from $\bigcup_{\nu=1}^n \omega_\nu$ and satisfying (ii). Put

$$E_1 = \Big(\bigcup_{\nu=1}^\infty \omega_\nu\Big) \cap E.$$

We have to prove that $|E_1| = |E|$. If this were not true, there would exist a density point x of the set $E \setminus E_1 = E_2$. Put $\omega(x,\delta) = (x-\delta, x+\delta)$ for $\delta > 0$ and suppose that there exists a δ such that $|\omega(x,\delta) \cap E_2| = |\omega(x,\delta)|$. Then ω_ν cannot be a subset of $\omega(x,\delta)$ for any ν. Therefore, if $\omega(x,\delta)$ intersects $\bigcup_{\nu=1}^\infty \omega_\nu$, it has to intersect at most two intervals $\omega_{\nu_1} \ni (x-\delta)$ and $\omega_{\nu_2} \ni (x+\delta)$. Then $\omega_{\nu_1} \cup \omega(x,\delta) \cup \omega_{\nu_2}$ is a candidate for an interval that should have been chosen in the process. This contradiction shows that

$$|\omega(x,\delta) \cap E_2| < |\omega(x,\delta)| \quad \text{for every } \delta > 0.$$

Since x is a density point of E_2, there exists a δ_0 such that

$$|\omega(x,\delta) \cap E_2| \ge (1-\varepsilon)|\omega(x,\delta)| \quad \text{for } \delta < \delta_0.$$

All of these intervals $\omega(x,\delta)$ have to intersect intervals of $\{\omega_\nu\}_{\nu=1}^\infty$ (otherwise they could be adjoined). However, each one of them can intersect at most two bigger intervals from $\{\omega_\nu\}_{\nu=1}^\infty$, say $\omega_{\nu_1} \ni (x-\delta)$ and $\omega_{\nu_2} \ni (x+\delta)$ with $|\omega_{\nu_k}| \ge |\omega(x,\delta)|$, $k = 1,2$. Notice that x lies in neither of these closed intervals and therefore it is possible to enlarge at least one of them by adjoining the smallest $\omega(x,\delta)$ that has a common endpoint with ω_{ν_1} or

ω_{ν_2}. This contradicts the construction process. Thus we have proved that $|E_2| = 0$, i.e. $|E_1| = |E|$.

Proof of Theorem 1. f_I^* is the nonincreasing rearrangement of f_I and is uniquely determined when we require that f_I^* is continuous on the left.

We take an arbitrary $t \in (0, |I|)$ and let E_t be a subset of I with measure t such that $f_I(x) \le f_I^*(t)$ for $x \notin E_t$. Then we use the covering lemma above to cover almost every point of E_t, a set we denote by $E_{t,1}$, by a union of intervals with disjoint interiors:

$$E_{t,1} \subset \bigcup_{\nu=1}^{\infty} \omega_\nu ,$$

such that for every positive integer ν

$$(4) \qquad (1-\varepsilon)|\omega_\nu| \le |\omega_\nu \cap E_t| < |\omega_\nu| .$$

Since the second inequality is strict, ω_ν contains a set of positive measure in the complement of E_t and we have

$$\operatorname*{ess\,inf}_{\omega_\nu} f_I(x) \le f_I^*(t)$$

and therefore, using (1) and (4), we obtain

$$\int_0^t f_I^*(u)\,du = \int_{E_t} f_I(x)\,dx \le \sum_{\nu=1}^{\infty} \int_{\omega_\nu} f_I(x)\,dx$$
$$\le c \sum_{\nu=1}^{\infty} |\omega_\nu| f_I^*(t) \le \frac{c}{1-\varepsilon} t f_I^*(t) .$$

Thus

$$⨍_0^t f_I^*(u)\,du \le \frac{c}{1-\varepsilon} f_I^*(t) .$$

Since $\varepsilon > 0$ was arbitrary, we may let ε tend to 0 to obtain (2′) with $f^* = f_I^*$. The proof is now completed in the same way as we did when f was continuous.

The conclusion (2), implying that we have the same constant c for f^* and f, is in general not possible to achieve in higher dimensions as we will see in the following counterexample in two dimensions.

Counterexample. *There exists a function f which belongs to $A_1(Q)$ with constant c, but f^* does not belong to $A_1((0, |Q|); c)$.*

Proof. Let $1/2 < l \le 1 - s$ and define $f : [0,1] \times [0,1] \to \mathbf{R}$ by

$$f(x) = f(x_1, x_2) = \begin{cases} 1 & \text{if } l \le x_1 \le 1, \\ c^2 & \text{in } (0,s) \times (0,s) \cup (0,s) \times (1-s, 1), \\ c & \text{elsewhere.} \end{cases}$$

Then we have, for every square Q,

$$⨍_Q f(x)\,dx \Big/ \operatorname*{ess\,inf}_Q f(x) \le c(1 + s^2(c-1)/l^2) .$$

On the other hand,

$$\sup_a ⨍_0^a f^*(t)\,dt \Big/ \operatorname*{ess\,inf}_{(0,a)} f^*(t) = ⨍_0^l f^*(t) dt = c(1 + 2s^2(c-1)/l) = c_1 .$$

Since $l > 1/2$, it follows that $c_1 > c$. In fact, with this construction, we can have c_1 arbitrarily close to $2c$ (for c large enough and l close to 1).

It is obvious that the same type of construction will work also in $\mathbf{R}^n$, $n > 2$.

From the theorem above we will now draw a conclusion about higher integrability of f. We will use the following lemma:

Lemma 2. *Let $c \ge 1$ be a constant and let g be a nonnegative, nonincreasing function on $(0,1]$ satisfying*

$$(5) \qquad ⨍_0^s g(t)\,dt \le c g(s) \quad \textit{for } s \in (0,1] .$$

Then g lies in $L^p(0,1)$ for $p < c/(c-1)$ and

$$(6) \qquad ⨍_0^s g^p(t)\,dt \le \frac{1}{c^{p-1}(c+p-cp)} \Big(⨍_0^s g(t)\,dt \Big)^p \quad \textit{for } s \in (0,1] .$$

Remark. The constant on the right in (6), as well as the upper bound of p, cannot be improved. In fact, the function $g(t) = (1/c)t^{1/c-1}$ is an extremal function, which gives equality in (6) and lies in L^p if and only if $p < c/(c-1)$. For $c = 1$, g has to be constant in $(0,1)$ and equality in (6) prevails trivially.

Proof. Without loss of generality we may assume that g is continuous on the left. We start by proving that g satisfies the inequality

$$(7) \qquad \int_0^s g^p(t)\,dt \le \frac{g^{p-1}(s)}{c+p-cp} \int_0^s g(t)\,dt .$$

Put $G(t)=\int_0^t g(x)dx$ and let ε be a small positive number. We integrate by parts and use the facts that $G(t)\le ctg(t)$ and $g'(t)\le 0$ to get

$$(8)\qquad \int_\varepsilon^s g^p(t)\,dt = \int_\varepsilon^s g^{p-1}(t)\,dG(t)$$

$$= [g^{p-1}(t)G(t)]_\varepsilon^s - (p-1)\int_\varepsilon^s G(t)g^{p-2}(t)\,dg(t)$$

$$\le g^{p-1}(s)G(s) - g^{p-1}(\varepsilon)G(\varepsilon) - c(p-1)\int_\varepsilon^s tg^{p-1}(t)\,dg(t).$$

More integration by parts yields

$$(9)\qquad \int_\varepsilon^s tg^{p-1}(t)\,dg(t) = \frac{1}{p}\Big(sg^p(s) - \varepsilon g^p(\varepsilon) - \int_\varepsilon^s g^p(t)\,dt\Big).$$

Now, combining (8) and (9) we obtain

$$\Big(1-\frac{c(p-1)}{p}\Big)\int_\varepsilon^s g^p(t)\,dt$$

$$\le g^{p-1}(s)G(s) - g^{p-1}(\varepsilon)G(\varepsilon) - \frac{c(p-1)}{p}(sg^p(s) - \varepsilon g^p(\varepsilon)).$$

Since g is nonincreasing, we have $G(\varepsilon)\ge \varepsilon g(\varepsilon)$. We also use the fact that $p<c/(c-1)$ implies $c(p-1)/p<1$. Therefore

$$\frac{c(p-1)}{p}\varepsilon g^p(\varepsilon) < \frac{c(p-1)}{p}G(\varepsilon)g^{p-1}(\varepsilon) < G(\varepsilon)g^{p-1}(\varepsilon),$$

which gives

$$\Big(1-\frac{c(p-1)}{p}\Big)\int_\varepsilon^s g^p(t)\,dt \le g^{p-1}(s)G(s) - \frac{c(p-1)}{p}sg^p(s).$$

By (5), $csg(s)\ge G(s)$ and thus

$$(c+p-cp)\int_\varepsilon^s g^p(t)\,dt \le g^{p-1}(s)G(s).$$

By letting ε tend to zero we obtain (7).

Now we will use (7) to prove (6). Since g is nonincreasing, we have $g(s)\le \fint_0^s g(t)\,dt$. This together with (7) implies (6) but without the factor c^{1-p}. However, with some extra effort, we can achieve the optimal constant. For simplicity we may choose $s=1$ and also assume $G(1)=\int_0^1 g(t)\,dt=1$. We want to show that $g(1)$ in (7) can be replaced by c^{-1}. From (5) it is evident that $g(1)\ge c^{-1}$. If $g(1)=c^{-1}$, then the inequalities (6) and (7) are the same. Assume therefore that

$$g(1)=\lim_{t\to 1^-} g(t) = a > c^{-1}.$$

For δ satisfying $0<\delta<(ac-1)/ac$ we construct auxiliary functions g_δ, with the following properties: (i) $g_\delta(t)\le g(t)$, (ii) $g_\delta(t)$ converges to $g(t)$ in $(0,1)$ as $\delta\to 0$, (iii) $g_\delta(1)=c^{-1}$ and (iv) g_δ satisfies the requirements of Lemma 2. To be more specific, we define

$$g_\delta(t)=\begin{cases} g(t), & 0<t\le 1-\delta,\\ \dfrac{1}{c}+\dfrac{a-c^{-1}}{\delta}(1-t), & 1-\delta<t\le 1.\end{cases}$$

Then g_δ is nonincreasing and trivially satisfies (i)–(iii). To verify (iv) it remains to show that g_δ satisfies (5). This property is inherited from g for $0<s\le 1-\delta$. It is easy to check that $g_\delta(s)\ge (cs)^{-1}$ in $(1-\delta,1]$ and for s in this interval we have

$$(10)\qquad \int_0^s g_\delta(t)\,dt \le \int_0^s g(t)\,dt \le 1 \le csg_\delta(s).$$

It is therefore justified to use inequality (7) with $g=g_\delta$. We get

$$\int_0^1 g_\delta^p(t)\,dt \le \frac{1}{c^{p-1}(c+p-cp)}\Big(\int_0^1 g_\delta(t)\,dt\Big)^p.$$

Now we let δ tend to 0 and by dominated convergence we obtain (6).

We can now give an improved version of Muckenhoupt's lemma [2, p. 213], where we do not have to require f to be decreasing and also obtain the best constant.

COROLLARY 1. *Suppose that f is a function in $A_1(\mathbf{R})$ with constant $c\ge 1$. Then for every $p<c/(c-1)$ and every interval I*

$$(11)\qquad \fint_I f^p(x)dx \le \frac{1}{c^{p-1}(c+p-cp)}\Big(\fint_I f(x)\,dx\Big)^p.$$

Proof. Without loss of generality we may assume that $|I|=1$ and $\fint_I f(x)\,dx=1$. Then, by Theorem 1,

$$\fint_0^t f^*(u)\,du \le cf^*(t),$$

where f^* is the nonincreasing rearrangement of the restriction of f to I. We apply Lemma 2 to obtain (11) with $f=f^*$. The stars may be deleted and the corollary is proved.

The next theorem describes a very nice and sharp property of A_1-weights, which is also easy to remember.

THEOREM 2. *If $f \in A_1((0,1);c)$, then, for every $p < c/(c-1)$,*

$$f^p \in A_1((0,1);c_p) \quad \text{with } c_p = \frac{c}{c+p-cp}.$$

Proof. Substitute the defining inequality for $A_1((0,1);c)$ into (11) and use the fact that $\operatorname{ess\,inf}_I f^p(x) = (\operatorname{ess\,inf}_I f(x))^p$.

Another way to proceed from the assumption $f \in A_1$ would be to try to make the function f continuous. A convolution method will not work. However, there is another type of mean value that has the advantage that it preserves the constant c even in several dimensions. We use it in the following lemma.

LEMMA 3. *Suppose f is in $A_1(Q_0)$ with constant c, i.e.*

$$\fint_Q f(x)\,dx \le c \operatorname*{ess\,inf}_Q f(x) \quad \text{for every } Q \subset Q_0. \tag{12}$$

Put

$$f_t(x) = \fint_{Q_0} f(x(1-t)+yt)\,dy, \quad t \in (0,1).$$

Then $f_t(x)$ is continuous in Q_0 and $f_t(x)$ satisfies (12) with the same constant c.

Proof. A change of the order of integration gives

$$\begin{aligned}\fint_Q f_t(x)\,dx &= \fint_{Q_0} dy \fint_Q f(x(1-t)+yt)\,dx \\ &= \fint_{Q_0} dy \fint_{Q_{t,y}} f(z)\,dz \le c \fint_{Q_0} \operatorname*{ess\,inf}_{x\in Q_{t,y}} f(x)\,dy,\end{aligned} \tag{13}$$

where $Q_{t,y} \subset Q_0$ is the cube $\{z = x(1-t)+yt; x \in Q\}$. By the integral representation of f_t it follows that f_t is continuous. We find that for some x_0

$$\inf_{x\in Q} f_t(x) = f_t(x_0) = \fint_{Q_0} f(x_0(1-t)+yt)\,dy \ge \fint_{Q_0} \operatorname*{ess\,inf}_{x\in Q_{t,y}} f(x)\,dy, \tag{14}$$

since $x_0(1-t)+yt \in Q_{t,y}$. A combination of (13) and (14) now gives the desired result.

Second proof of Corollary 1. We may assume that $I = Q_0 = (0,1)$. We conclude from Lemma 3 that f_t is continuous. By our first short proof of the theorem it follows that f_t^* satisfies the hypothesis of Lemma 2.

Proceeding as in the previous proof of Corollary 1 we find that

$$\int_0^1 f_t^p(u)\,du \le \frac{1}{c^{p-1}(c+p-cp)}\Big(\int_0^1 f_t(u)du\Big)^p. \tag{15}$$

We have

$$f_t(x) = \frac{1}{t^n}\int_{Q_t(x)} f(z)\,dz, \quad \text{where} \quad Q_t(x) = \{z = x(1-t)+yt\,;\, y \in Q\},$$

which is a cube containing x and with measure t^n. Therefore, we have $f_t(x) \to f(x)$ a.e. as $t \to 0$ and Fatou's lemma gives

$$\limsup_{t\to 0}\int_0^1 f_t^p(u)\,du \ge \int_0^1 \limsup_{t\to 0} f_t^p(u)\,du = \int_0^1 f^p(u)\,du.$$

On the other hand, by the dominated convergence theorem

$$\begin{aligned}\int_0^1 f_t(u)\,du &= \int_0^1 dx \int_0^1 f(x(1-t)+yt)\,dy = \int_0^1 dy \int_0^1 f(x(1-t)+yt)\,dx \\ &= \int_0^1 \frac{dy}{1-t}\int_{ty}^{ty+1-t} f(z)\,dz \to \int_0^1 f(z)\,dz \quad \text{as } t \to 0.\end{aligned}$$

Taking $\limsup_{t\to 0}$ in (15) therefore completes this proof of Corollary 1.

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