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DEPARTMENT OF MATHEMATICS AND STATISTICS
WRIGHT STATE UNIVERSITY
DAYTON, OHIO 45435
U.S.A.

Received May 4, 1990
Revised version May 7, 1991 (2684)

STUDIA MATHEMATICA 101 (2) (1992)

A noncommutative version of a Theorem of Marczewski for submeasures

by

PAOLO DE LUCIA (Napoli) and PEDRO MORALES (Sherbrooke, Qué.)

Abstract. It is shown that every monocompact submeasure on an orthomodular poset is order continuous. From this generalization of the classical Marczewski Theorem, several results of commutative Measure Theory are derived and unified.

1. Introduction. According to the well known theorems of Aleksandrov [2], von Neumann [22] and Marczewski [19], a mild regularity condition is sufficient for the σ-additivity of a real-valued set function defined on a family of sets. One of the purposes of this paper is to unify these apparently unrelated results via an extension of the Marczewski Theorem to submeasures on an orthomodular poset. Incidentally, we indicate that one of the particular interests of the noncommutative Measure Theory is its relevance to the Hilbert space formulation of Quantum Mechanics (see [16], [23], [28] and [31]).

The paper is organized as follows: In Section 2 we give some elementary notions of orthoposets and uniform semigroups, and we define some pertinent classes of functions from an orthoposet into a uniform semigroup. Section 3 introduces the notion of an approximating paving for the aforesaid kind of functions, and this notion is illustrated with appropriate examples. In the next section we extend properly the notion of compact measure of Marczewski to the noncommutative setting, and we establish the first of the main results of this paper. We also deduce, as by-products, several results bearing the names of Aleksandrov [2], K. P. S. Bhaskara Rao and M. Bhaskara Rao [5], Glicksberg [13], Huneycutt [14], Kluvánek [17], Marczewski [19], Millington [20], von Neumann [22] and Topsøe [30]. In the last

1991 *Mathematics Subject Classification*: 28B10, 28C15, 46C10, 81C20.

The research of the first-named author was partially supported by the Consiglio Nazionale delle Ricerche (Italy).

The research of the second-named author was partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

Section 5 we introduce the notion of regularity, and we establish the second of the main results of this paper. This theorem improves a noncommutative result of Béaver and Cook [3] and yields, as corollary, the σ-additivity of an additive regular $[0,+\infty]$-valued set function defined on an s-class of sets.

2. Preliminaries. Let D be a nonempty subset of a partially ordered set. If the supremum (resp. infimum) of D exists, it will be denoted by $\bigvee D$ (resp. $\bigwedge D$). In particular, we shall write: $\bigvee\{a,b\} = a \vee b$, $\bigwedge\{a,b\} = a \wedge b$, $\bigvee\{a_i : i \in I\} = \bigvee_{i\in I} a_i$, $\bigvee\{a_i : i \in \{0,1,\ldots,n\}\} = \bigvee_{i=0}^n a_i$, where $n \in \omega = \{0,1,2\ldots\}$, etc.

A *bounded poset* is a quadruplet $(L,\le,0,1)$ where $(L,\le)$ is a partially ordered set, 0 is the least element of L, 1 is the greatest element of L and $0 \ne 1$.

Let $L = (L,\le,0,1)$ be a bounded poset. An *orthocomplementation* on L is a function $'$ from L into L satisfying the following conditions:

(i) $'$ is idempotent.
(ii) $'$ is decreasing.
(iii) For all $a \in L$, $a \wedge a'$ exists and it is equal to 0.

From these axioms it follows that $0' = 1$, $1' = 0$ and, for all $a \in L$, $a \vee a'$ exists and it is equal to 1.

We call a bounded poset with an orthocomplementation an *orthoposet*. An *ortholattice* is an orthoposet which is also a lattice. An *orthomodular lattice* is an ortholattice $L = (L,\le,',0,1)$ satisfying the orthomodular law: If $a,b \in L$ and $a \le b$, then $b = a\vee(a'\wedge b)$. A *Boolean algebra* is an ortholattice satisfying the distributive law: If $a,b,c \in L$, then $a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)$.

An orthoposet $L = (L,\le,',0,1)$ is called *an orthomodular poset* if the following conditions hold:

(i) If $a,b \in L$ and $a \le b'$, then $a \vee b$ exists.
(ii) L satisfies the orthomodular law.

It is obvious that every orthomodular lattice is an orthomodular poset and every Boolean algebra is an orthomodular lattice.

Let $L = (L,\le,',0,1)$ be an orthoposet. Consider the following binary relation $\perp$ on L: $a \perp b$ if $a \le b'$. It is clear that $\perp$ is symmetric and $a \perp a$ implies $a = 0$. If $a \in L$ and B is a nonempty subset of L, we write $a \perp B$ if $a \perp b$ for every $b \in B$. If $b \perp B \setminus \{b\}$ for every $b \in B$, we say that B is an *orthogonal* set.

Let L be an orthomodular poset. Consider the following binary relation C on L: aCb if there exists an orthogonal subset $\{u,v,w\}$ of L such that $a = u \vee v$ and $b = u \vee w$. It is easy to see that, if aCb, then aCb', $a'Cb$, $a'Cb'$ and the elements $a\vee b$ and $a\wedge b$ exist. If D is a nonempty subset of L,

we write $C(D) = \{a \in L : aCb \text{ for every } b \in D\}$. Clearly $0,1 \in C(D)$, and $C(L) = L$ if L is a Boolean algebra.

Three important examples of orthomodular posets are the following:

1) Let H be a Hilbert space over $\mathbf{R}$ or $\mathbf{C}$ with inner product $\langle\cdot,\cdot\rangle$. Let $L(H)$ be the set of all closed vector subspaces of H. Consider the function $\perp$ from $L(H)$ into $L(H)$ defined by the formula: $M \to M^\perp = \{x \in H : \langle x,y\rangle = 0 \text{ for all } y \in M\}$. Then $(L(H),\subseteq,\perp,\{0\},H)$ is a complete orthomodular lattice, where $\bigwedge_{i\in I} M_i = \bigcap_{i\in I} M_i$ and $\bigvee_{i\in I} M_i =$ closed span of $\bigcup_{i\in I} M_i$ for every family $(M_i)_{i\in I}$ in $L(H)$ (see [15, Proposition 1, p. 65]). Since $C(L(H)) = \{\{0\},H\}$, $L(H)$ is not a Boolean algebra if $\dim(H) \ge 2$.

2) Let Ω be a nonempty set and let 2^Ω denote its power set. If ^{c} denotes the usual set complementation, then $(2^\Omega,\subseteq,{}^c,\emptyset,\Omega)$ is a complete Boolean algebra and it will be denoted by 2^Ω.

3) A subset $\mathcal{C}$ of 2^Ω is called an *s-class* in Ω if the following conditions are satisfied:

(i) $\emptyset \in \mathcal{C}$.
(ii) If $A \in \mathcal{C}$, then $A^c \in \mathcal{C}$.
(iii) Every finite disjoint union of elements of $\mathcal{C}$ belongs to $\mathcal{C}$.

If $\mathcal{C}$ is an s-class in Ω, then $(\mathcal{C},\subseteq,{}^c,\emptyset,\Omega)$ is an orthomodular poset which is not necessarily a Boolean algebra (see [26]).

Let E be a nonempty subset of 2^Ω. Then the symbol $A(E)$ will denote the Boolean subalgebra of 2^Ω generated by E.

If L_1 and L_2 are two orthoposets, then the product orthoposet $L_1 \times L_2$ is defined in the obvious way.

If L is an orthoposet, any subset of L containing 0 is called a *paving* in L.

For more details concerning orthomodular posets or orthomodular lattices we refer to [4], [15] and [24].

A *uniform semigroup* is a quadruplet $(S,+,0,\mathcal{U})$ where $(S,+)$ is a commutative semigroup, 0 is the neutral element for $+$ and $\mathcal{U}$ is a uniformity on S such that the function $(x,y) \to x+y$ from $S\times S$ into S is uniformly continuous.

Let $S = (S,+,0,\mathcal{U})$ be a uniform semigroup and let P be the set of all continuous pseudo-metrics p on S such that $p(x+z,y+z) \le p(x,y)$ for all $x,y,z \in S$ (semi-invariant property). It is well known that the set $\{\{(x,y) \in S\times S : p(x,y) < \varepsilon\} : p \in P \text{ and } \varepsilon > 0\}$ is a subbase for the uniformity $\mathcal{U}$. If, further, S is a group, then the elements of P can be chosen invariant.

Two important examples of uniform semigroups are the following:

1) Any commutative topological group is a uniform semigroup.

2) Let $\mathcal{U}_\infty$ be the uniformity on $[0,+\infty]$ generated by the semi-invariant pseudo-metric

$$p_\infty(x,y) = \left|\frac{x}{1+x} - \frac{y}{1+y}\right|$$

on $[0,+\infty]$ with the convention that $+\infty/(1+(+\infty)) = 1$. Then $([0,+\infty], +, 0, \mathcal{U}_\infty)$ is a Hausdorff uniform semigroup which will be denoted by S_∞.

Let L be an orthoposet, let S be a Hausdorff uniform semigroup and let $\lambda : L \to S$ be a function such that $\lambda(0) = 0$. We say that

(i) λ is *additive* if, for every finite orthogonal sequence $(a_i)_{0\le i\le n}$, $n \in \omega$, in L such that $\bigvee_{i=0}^n a_i$ exists, we have $\lambda(\bigvee_{i=0}^n a_i) = \sum_{i=0}^n \lambda(a_i)$.

(ii) λ is *σ-additive* if, for every orthogonal sequence $(a_i)_{i\in\omega}$ in L such that $\bigvee_{i\in\omega} a_i$ exists, we have $\lambda(\bigvee_{i\in\omega} a_i) = \lim_{n\to\infty} \sum_{i=0}^n \lambda(a_i)$.

(iii) λ is *s-bounded* if, for every orthogonal sequence $(a_i)_{i\in\omega}$ in L, we have $\lim_{i\to\infty} \lambda(a_i) = 0$.

(iv) λ is *order continuous* if, for every decreasing sequence $(a_i)_{i\in\omega}$ in L such that $\bigwedge_{i\in\omega} a_i = 0$, we have $\lim_{i\to\infty} \lambda(a_i) = 0$.

Let now $S = S_\infty$. We say that

(v) λ is *subadditive* if, for every finite sequence $(a_i)_{0\le i\le n}$, $n \in \omega$, in L such that $\bigvee_{i=0}^n a_i$ exists, we have $\lambda(\bigvee_{i=0}^n a_i) \le \sum_{i=0}^n \lambda(a_i)$.

(vi) λ is *σ-subadditive* if, for every sequence $(a_i)_{i\in\omega}$ in L such that $\bigvee_{i\in\omega} a_i$ exists, we have $\lambda(\bigvee_{i\in\omega} a_i) \le \sum_{i=0}^\infty \lambda(a_i)$.

(vii) λ is a *submeasure* if λ is increasing and subadditive.

Remark. Let L be a Boolean subalgebra of 2^Ω and let $S = \mathbf{R}$ or $\mathbf{C}$. Using Lemmas III.1.5 and III.1.6 of [8], it is easy to show that every additive bounded set function $\lambda : L \to S$ is s-bounded.

Let now L be a Boolean algebra, let S be a Hausdorff uniform semigroup, let $p \in P$ and let $\lambda : L \to S$ be an additive function. The function $\lambda_p : L \to S_\infty$ defined by $\lambda_p(a) = \sup\{p(\lambda(b), 0) : b \in L \text{ and } b \le a\}$ is called the *p-semivariation* of λ. It is easy to verify that λ_p is a submeasure dominating $p(\lambda(\cdot), 0)$. Moreover, λ_p is s-bounded if λ is s-bounded.

3. Approximating pavings. Let $L_0 = (L_0, \le, ', 0, 1)$ be an orthoposet, let L be a suborthomodular poset of L_0, let $S = (S, +, 0, \mathcal{U})$ be a Hausdorff uniform semigroup and let $\lambda : L \to S$ be a function such that $\lambda(0) = 0$. A paving F in $L \times L_0$ is called an *approximating paving for* λ if, for every $a \in L$ and every $U \in \mathcal{U}$, there exists $(b, c) \in F$ such that $b \le c \le a$ and $\lambda(d) \in U[0]$ whenever $d \in L$ and $d \le a \wedge b'$.

The following example is due to Marczewski [19, p. 116]: Let L be the Boolean subalgebra of $2^{[0,1]}$ of all finite unions of intervals of the form $[\alpha, \beta[$, where $0 \le \alpha < \beta \le 1$, let $S = \mathbf{R}$ and let $\lambda([\alpha,\beta[) = \beta - \alpha$. If a set E of L has the form $E = \bigcup_{i=0}^n I_i$, where the I_i are disjoint intervals of the type described, we put $\lambda(E) = \sum_{i=0}^n \lambda(I_i)$. Then λ is well defined, additive and the paving $F = \{(E, K) : E \in L \text{ and } K \text{ is a finite union of closed subintervals of } [0,1]\}$ in $L \times 2^{[0,1]}$ is an approximating paving for λ.

If S is a Hausdorff uniform semigroup, it is shown in [21] that every S-valued Baire measure on a locally compact Hausdorff space X has $\{(\bigcup_{i=0}^\infty K_i, \bigcup_{i=0}^\infty K_i)$: each K_i is a compact G_δ subset of $X\}$ as an approximating paving. Further, if S is a Hausdorff commutative topological group, it is shown in [29] that every S-valued Borel measure on a Polish space X has $\{(K, K) : K \text{ is a compact subset of } X\}$ as an approximating paving.

The following two propositions yield more examples:

PROPOSITION 3.1 (see [3]). *Let H be an infinite-dimensional separable Hilbert space over $\mathbf{C}$ with inner product $\langle\cdot,\cdot\rangle$ and let $\lambda : L(H) \to [0, +\infty[$ be a σ-additive function. Then λ has $L(H) \times \{M \in L(H) : \dim(M) < +\infty\}$ as an approximating paving.*

Proof. By the Gleason Theorem [12] (see also [16]) there exists a unique positive bounded linear operator $T : H \to H$ of trace class such that $\operatorname{tr}(T) = 1$ and $\lambda(\cdot) = \lambda(H) \cdot \operatorname{tr} TP(\cdot)$ where $P(\cdot)$ is the projection of H onto $(\cdot)$. For fixed $\varepsilon > 0$, $M \in L(H)$, $\dim(M) = +\infty$, let $(e_i)_{i\in\omega}$ be an orthonormal basis in M. Then

$$\operatorname{tr} TP(M) = \sum \langle Te_i, e_i\rangle = \lim_{n\to\infty} \sum_{i=0}^n \langle Te_i, e_i\rangle$$

and the closed span N of $\{e_0, e_1, \ldots, e_n\}$, with a suitably large n, satisfies $(N, N) \in F$, $\operatorname{tr} T(P(M) \wedge P(N)^\perp) < \varepsilon$.

A subset $\mathcal{F}$ of 2^Ω is called a *δ-paving* in 2^Ω if the following conditions are satisfied:

(i) $\emptyset, \Omega \in \mathcal{F}$.

(ii) If $(F_i)_{i\in\omega}$ is a sequence in $\mathcal{F}$, then $\bigcap_{i=0}^\infty F_i \in \mathcal{F}$.

(iii) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cup F_2 \in \mathcal{F}$.

Clearly (i) and (ii) imply that, if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$.

Let $\mathcal{F}$ be a δ-paving in 2^Ω. A function $f : \Omega \to \mathbf{R}$ is called *$\mathcal{F}$-continuous* if, for every $\alpha \in \mathbf{R}$, the sets $f^{-1}(]-\infty, \alpha])$ and $f^{-1}([\alpha, +\infty[)$ belong to $\mathcal{F}$. Trivially every real-valued constant function on Ω is $\mathcal{F}$-continuous. Let $f_1, f_2 : \Omega \to \mathbf{R}$ be two $\mathcal{F}$-continuous functions and let $\alpha, \beta \in \mathbf{R}$. It is easy to see that βf_1 is $\mathcal{F}$-continuous, and from the identities $|f_1|^{-1}(]-\infty, \alpha]) = f_1^{-1}(]-\infty, \alpha]) \cap f_1^{-1}([-\alpha, +\infty[)$ and $|f_1|^{-1}([\alpha, +\infty[) = f_1^{-1}([\alpha, +\infty[) \cup f_1^{-1}(]-\infty, -\alpha])$ it follows that $|f_1|$ is $\mathcal{F}$-continuous. If $\mathbf{Q}$

denotes the set of rational numbers, the identity

$$\{x \in \Omega : (f_1 + f_2)(x) < \alpha\} = \bigcup_{\gamma\in\mathbf{Q}} (\{x \in \Omega : f_1(x) < \gamma\} \cap \{x \in \Omega : f_2(x) < \alpha - \gamma\}),$$

the similar identity for $>$ and the De Morgan law imply that $f_1 + f_2$ is $\mathcal{F}$-continuous. Hence $\max(f_1, f_2) = \frac{1}{2}(f_1 + f_2 + |f_1 - f_2|)$ and $\min(f_1, f_2) = \frac{1}{2}(f_1 + f_2 - |f_1 - f_2|)$ are also $\mathcal{F}$-continuous.

A subset Z of Ω is called an *$\mathcal{F}$-zero-set* if there exists an $\mathcal{F}$-continuous function $f : \Omega \to \mathbf{R}$ such that $0 \le f \le 1$ and $Z = f^{-1}(0)$. The set of all $\mathcal{F}$-zero-sets will be denoted by $Z(\mathcal{F})$. Clearly $\emptyset, \Omega \in Z(\mathcal{F})$.

LEMMA 3.2. *Let $\mathcal{F}$ be a δ-paving in 2^Ω. Then*

(a) $Z(\mathcal{F}) \subseteq \mathcal{F}$.

(b) $Z_1, Z_2 \in Z(\mathcal{F}) \Rightarrow Z_1 \cup Z_2,\ Z_1 \cap Z_2 \in Z(\mathcal{F})$.

(c) *Every element of $Z(\mathcal{F})$ can be written as a countable intersection of sets belonging to $Z(\mathcal{F})^c = \{G \subseteq \Omega : G^c \in Z(\mathcal{F})\}$.*

Proof. (a) Let $Z \in Z(\mathcal{F})$. Then $Z = f^{-1}(0)$ where $f : \Omega \to \mathbf{R}$ is an $\mathcal{F}$-continuous function and $0 \le f \le 1$. Since $f^{-1}(0) = f^{-1}(]-\infty, 0]) \cap f^{-1}([0, +\infty[)$ it follows that $Z \in \mathcal{F}$.

(b) For $i = 1, 2$ write $Z_i = f_i^{-1}(0)$ where $f_i : \Omega \to \mathbf{R}$ is $\mathcal{F}$-continuous and $0 \le f_i \le 1$. Since $Z_1 \cup Z_2 = \{x \in \Omega : \min(f_1, f_2)(x) = 0\}$ and $Z_1 \cap Z_2 = \{x \in \Omega : \min(f_1 + f_2, 1)(x) = 0\}$ it follows that $Z_1 \cup Z_2,\ Z_1 \cap Z_2 \in Z(\mathcal{F})$.

(c) Let $Z \in Z(\mathcal{F})$, $Z = f^{-1}(0)$ with f as above. For every $n \in \omega$ let $G_n = \{x \in \Omega : f(x) < 1/(n+1)\}$. Then $Z = \bigcap_{n=0}^\infty G_n$. Since

$$G_n^c = \left\{x \in \Omega : f(x) \ge \frac{1}{n+1}\right\} = \left\{x \in \Omega : \frac{1}{n+1} - \min\left(f, \frac{1}{n+1}\right)(x) = 0\right\},$$

it follows that $G_n \in Z(\mathcal{F})^c$.

LEMMA 3.3. *Let $\mathcal{F}$ be a δ-paving in 2^Ω and let $L = A(Z(\mathcal{F}))$. Then every element of L can be written as a countable union of sets of $Z(\mathcal{F})$.*

Proof. Let $Z(\mathcal{F})_\sigma = \{\bigcup_{i=0}^\infty Z_i : Z_i \in Z(\mathcal{F})\}$ and $Z(\mathcal{F})^c_\delta = \{\bigcap_{i=0}^\infty G_i : G_i \in Z(\mathcal{F})^c\}$. From Lemma 3.2(b) it follows that $Z(\mathcal{F})_\sigma$ and $Z(\mathcal{F})^c_\delta$ are closed under the formation of finite unions. Since a set belongs to $Z(\mathcal{F})_\sigma$ if and only if its complement belongs to $Z(\mathcal{F})^c_\delta$, it follows that $L_1 = \{B \in 2^\Omega : B \in Z(\mathcal{F})_\sigma \text{ and } B \in Z(\mathcal{F})^c_\delta\}$ is a Boolean subalgebra of 2^Ω. By Lemma 3.2(c), L_1 contains $Z(\mathcal{F})$. So $L \subseteq L_1$.

Taking into account the Remark of Section 2, the following proposition improves Lemma 1 of [2, p. 605]:

PROPOSITION 3.4. *Let $\mathcal{F}$ be a δ-paving in 2^Ω, let $L = A(Z(\mathcal{F}))$, let S be a Hausdorff uniform semigroup and let $\lambda : L \to S$ be a σ-additive s-bounded set function. Then λ has $L \times Z(\mathcal{F})$ as an approximating paving.*

Proof. Let $p \in P$ and consider the p-semivariation λ_p of λ. Then λ_p is an s-bounded submeasure on L. We shall show that λ_p is σ-subadditive. Let $(B_i)_{i\in\omega}$ be a sequence in L such that $B = \bigcup_{i=0}^\infty B_i$ belongs to L. We may assume that the B_i are pairwise disjoint. Let C be an arbitrary set in L. Then $B \cap C = \bigcup_{i=0}^\infty (B_i \cap C)$. Since λ is σ-additive and p is continuous and semi-invariant, we have

$$p(\lambda(B \cap C), 0) = p\Big(\lim_{n\to\infty} \sum_{i=1}^n \lambda(B_i \cap C), 0\Big) = \lim_{n\to\infty} p\Big(\sum_{i=0}^n \lambda(B_i \cap C), 0\Big) \le \lim_{n\to\infty} \sum_{i=0}^n p(\lambda(B_i \cap C), 0) \le \sum_{i=0}^\infty \lambda_p(B_i),$$

and therefore $\lambda_p(B) \le \sum_{i=0}^\infty \lambda_p(B_i)$.

From Theorem 5.3 of [7, p. 280] it follows that λ_p is order continuous.

To prove that $L \times Z(\mathcal{F})$ is an approximating paving for λ, let B be a set in L and let $U \in \mathcal{U}$. We may suppose that $U = \bigcap_{j=0}^m \{(u, v) \in S \times S : p_j(u, v) < \varepsilon\}$ where $p_j \in P$ and $\varepsilon > 0$.

By Lemma 3.3 there exists a sequence $(Z_i)_{i\in\omega}$ in $Z(\mathcal{F})$ such that $B = \bigcup_{i=0}^\infty Z_i$. Put $B_n = \bigcup_{i=0}^n Z_i$ for every $n \in \omega$. By Lemma 3.2 every B_n belongs to $Z(\mathcal{F})$. Since $(B \setminus B_n)_{n\in\omega}$ is a decreasing sequence in L whose intersection is empty, it follows that $\lim_{n\to\infty} \lambda_{p_j}(B \setminus B_n) = 0$. Thus, for every $j = 0, 1, \ldots, m$ there exists $n_j \in \omega$ such that $n \in \omega$ and $n \ge n_j$ imply $p_\infty(\lambda_{p_j}(B \setminus B_n), 0) < \varepsilon/(1 + \varepsilon)$, and therefore $\lambda_{p_j}(B \setminus B_n) < \varepsilon$ whenever $n \ge n_j$. Let $k = \max_{0\le j\le m} n_j$. Then $\lambda_{p_j}(B \setminus B_k) < \varepsilon$ for all $j = 0, 1, \ldots, m$.

Clearly $(B_k, B_k) \in L \times Z(\mathcal{F})$. Let D be an element of L such that $D \subseteq B \setminus B_k$. Then $p_j(\lambda(D), 0) \le \lambda_{p_j}(D) \le \lambda_{p_j}(B \setminus B_k) < \varepsilon$ for all $j = 0, 1, \ldots, m$. Hence $\lambda(D) \in U[0]$.

4. Monocompact submeasures.

Let $L_0 = (L_0, \le, ', 0, 1)$ be an orthoposet. A paving K in L_0 is called *monocompact* if, for every decreasing sequence $(a_i)_{i\in\omega}$ in K such that $\bigwedge_{i\in\omega} a_i = 0$, there exists $n \in \omega$ such that $a_n = 0$.

The following list gives some interesting examples of monocompact pavings:

1. Let H be a Hilbert space over $\mathbf{R}$ or $\mathbf{C}$ and let $K = \{M \in L(H) : \dim(M) < +\infty\}$. Then K is a monocompact paving in $L(H)$. In fact, let $(M_i)_{i\in\omega}$ be a decreasing sequence in K such that $\bigwedge_{i\in\omega} M_i = \{0\}$. For every $i \in \omega$ write $k_i = \dim(M_i)$. Then $(k_i)_{i\in\omega}$ is a decreasing sequence in ω such

that $\lim_{i\to\infty} k_i = 0$. So there exists $n \in \omega$ such that $k_n = 0$.

2. Following Marczewski [19] a *compact class* in Ω is a subset K of 2^Ω satisfying the following condition: If $(A_i)_{i\in\omega}$ is a sequence in K such that $\bigcap_{i=0}^n A_i \neq \emptyset$ for every $n \in \omega$, then $\bigcap_{i=0}^\infty A_i \neq \emptyset$. It is clear that if K is a compact class in Ω, then $K \cup \{\emptyset\}$ is a monocompact paving in 2^Ω.

3. A topological space is said to be *pseudo-compact* if every continuous real-valued function on it is bounded. For example, if $\beta(\cdot)$ denotes the Čech–Stone compactification, then $\beta(\mathbf{R})\setminus(\beta(\omega)\setminus\omega)$ is a Hausdorff pseudo-compact space which is not countably compact (see [9, Example 3.10.29] and [11]).

Let Ω be a pseudo-compact topological space and let $\mathcal{F}$ be the set of all closed subsets of 2^Ω. Then $Z(\mathcal{F})$ is a monocompact paving in 2^Ω. In fact, $\mathcal{F}$ is a δ-paving in 2^Ω and, since Ω is pseudo-compact, it is easy to see that $Z(\mathcal{F})$ is the set of all zero-sets in Ω. Then the implication (i)$\Rightarrow$(viii) of [27, Theorem 2.3] assures that $Z(\mathcal{F})$ is a compact class in Ω.

4. Let $\mathcal{F}$ be a δ-paving in 2^Ω. Following Aleksandrov [1, p. 314] the pair $(\Omega, \mathcal{F})$ is called a *space*. A space $(\Omega, \mathcal{F})$ is said to be $\mathcal{F}$-*compact* if, for every sequence $(F_i)_{i\in\omega}$ in $\mathcal{F}$ such that $\Omega = \bigcup_{i=0}^\infty F_i^c$, there exists $n \in \omega$ such that $\Omega = \bigcup_{i=0}^n F_i^c$. It is clear that a δ-paving $\mathcal{F}$ in 2^Ω is monocompact if and only if the space $(\Omega, \mathcal{F})$ is $\mathcal{F}$-compact.

5. Let $\mathcal{C}$ be a nonempty subset of $\Omega^\omega \times \Omega$. Following von Neumann [22] we say that the pair $(\Omega, \mathcal{C})$ is a *space* if $\mathcal{C}$ satisfies the following conditions:

(i) If $((x_i)_{i\in\omega}, x) \in \mathcal{C}$ and $((x_i)_{i\in\omega}, y) \in \mathcal{C}$, then $x = y$.

(ii) If $((x_i)_{i\in\omega}, x) \in \mathcal{C}$ and $(x_{k_i})_{i\in\omega}$ is a subsequence of $(x_i)_{i\in\omega}$, then $((x_{k_i})_{i\in\omega}, x) \in \mathcal{C}$.

For example, let X be a topological space, let Y be a Hausdorff topological space, and let Ω be the set of all continuous functions from X into Y. Consider the following subset $\mathcal{C}$ of $\Omega^\omega \times \Omega$: $((f_i)_{i\in\omega}, f) \in \mathcal{C} \Leftrightarrow$ the sequence $(f_i(x_i))_{i\in\omega}$ in Y converges to $f(x)$ whenever $(x_i)_{i\in\omega}$ is a sequence in X converging to x. Clearly $\mathcal{C} \neq \emptyset$ and the pair $(\Omega, \mathcal{C})$ satisfies (i). Using the argument of [18, p. 198] we can show that $(\Omega, \mathcal{C})$ satisfies (ii). We note that this "continuous convergence" is not topological in general (see [10]).

Let $(\Omega, \mathcal{C})$ be a space and let F be a nonempty subset of Ω. We say that F is $\mathcal{C}$-*closed* if whenever $(x_i)_{i\in\omega}$ is a sequence in F for which there exists $x \in \Omega$ such that $((x_i)_{i\in\omega}, x) \in \mathcal{C}$, we have $x \in F$. We assume that $\emptyset$ is $\mathcal{C}$-closed. A subset C of Ω is called $\mathcal{C}$-*compact* if

(a) C is $\mathcal{C}$-closed.

(b) For every sequence $(x_i)_{i\in\omega}$ in C there exists a subsequence $(x_{k_i})_{i\in\omega}$ of $(x_i)_{i\in\omega}$ and an element $x \in \Omega$ such that $((x_{k_i})_{i\in\omega}, x) \in \mathcal{C}$.

It is obvious that $\emptyset$ is $\mathcal{C}$-compact.

Let $(\Omega, \mathcal{C})$ be a space and let $K = \{C \subseteq \Omega : C \text{ is } \mathcal{C}\text{-compact}\}$. Then K is

a monocompact paving in 2^Ω. In fact, using Theorems 10.1.16 and 10.1.18 of [22] it is easy to show that K is a compact class in Ω.

Now, let L_0 be an orthoposet, let L be a suborthomodular poset of L_0, let S be a Hausdorff uniform semigroup and let $\lambda : L \to S$ be a function such that $\lambda(0) = 0$. We say that λ is *monocompact* if there exists a monocompact paving K in L_0 such that $C(L) \times K$ is an approximating paving for λ.

For the proof of the first main result of this paper, we need the following lemma:

LEMMA 4.1. *Let $L = (L, \leq, ', 0, 1)$ be an orthomodular poset and let $(b_i)_{i\in\omega}$ be a decreasing sequence in $C(L)$. Then, for every $m \in \omega$, $\bigvee_{i=0}^m (b_{i-1} \wedge b_i')$ exists (where $b_{-1} = 1$) and it is equal to b_m'.*

Proof. Since $(C(L), \leq, ', 0, 1)$ is a Boolean algebra (see [4, Exercise VIII.2]), $\bigvee_{i=0}^m (b_{i-1} \wedge b_i')$ exists for every $m \in \omega$. We shall show the formula

$$(*) \qquad \bigvee_{i=0}^m (b_{i-1} \wedge b_i') = b_m'$$

by induction on m. For $m = 0$, the formula $(*)$ is trivial. Suppose that $(*)$ holds for m. Then

$$\bigvee_{i=0}^{m+1} (b_{i-1} \wedge b_1') = \Big(\bigvee_{i=0}^{m} (b_{i-1} \wedge b_1')\Big) \vee (b_m \wedge b_{m+1}') = b_m' \vee (b_m \wedge b_{m+1}')$$
$$= (b_m' \vee b_m) \wedge (b_m' \vee b_{m+1}') = 1 \wedge b_{m+1}' = b_{m+1}',$$

and $(*)$ holds for $m + 1$.

THEOREM 4.2. *Let $L_0 = (L_0, \leq, ', 0, 1)$ be an orthoposet and let L be a suborthomodular poset of L_0. Then every monocompact submeasure on L is order continuous.*

Proof. Let $\lambda : L \to S_\infty$ be a submeasure and let K be a monocompact paving in L_0 such that $C(L) \times K$ is an approximating paving for λ. Let $(a_i)_{i\in\omega}$ be a decreasing sequence in L such that $\bigwedge_{i\in\omega} a_i = 0$ and let $U \in \mathcal{U}_\infty$. We may suppose that $U = \{(u, v) \in [0, +\infty[\times [0, +\infty[: p_\infty(u, v) < \varepsilon\}$ where $\varepsilon > 0$. For every $i \in \omega$, put $U_i = \{(u, v) \in [0, +\infty[\times [0, +\infty[: p_\infty(u, v) < \varepsilon/2^{i+1}\}$. Then $U_i \in \mathcal{U}_\infty$ and $\sum_{i=0}^n U_i[0] \subseteq U[0]$ for all $n \in \omega$.

Using the monocompactness of λ, we can construct inductively a sequence $(b_i)_{i\in\omega}$ in $C(L)$ and a sequence $(c_i)_{i\in\omega}$ in K such that $b_i \leq c_i \leq a_i \wedge b_{i-1}$ and $\lambda(a_i \wedge b_{i-1} \wedge b_i') \in U_i[0]$ for every $i \in \omega$, where $b_{-1} = 1$. Then $(c_i)_{i\in\omega}$ is a decreasing sequence in K such that $c_i \leq a_i$ for all $i \in \omega$. So $\bigwedge_{i\in\omega} c_i = 0$. Since K is a monocompact paving in L_0, there exists $n \in \omega$ such that $c_n = 0$, and therefore $b_n = 0$.

Since $(a_i \wedge b_{i-1} \wedge b_i')_{i\in\omega}$ is an orthogonal sequence in L, $\bigvee_{i=0}^m (a_i \wedge b_{i-1} \wedge b_i')$ exists for all $m \in \omega$. Let $m \in \omega$ be such that $m \geq n$. We shall show that $a_m \le \bigvee_{i=0}^m (a_i \wedge b_{i-1} \wedge b_i')$. Let $i \in \{0,1,\dots,m\}$. Since $a_i C b_j$ for all $j \in \{-1,0,1,\dots,m\}$, we deduce, by [4, Remark VIII.2.15], that $a_i C(b_{i-1}\wedge b_i')$ and, because $a_m C a_i$, it follows that $a_m C(a_i\wedge b_{i-1}\wedge b_i')$. Then [4, Theorem VIII.2.14] implies that

$$a_m \wedge \Big(\bigvee_{i=0}^m (a_i \wedge b_{i-1}\wedge b_i')\Big) = \bigvee_{i=0}^m (a_m \wedge b_{i-1}\wedge b_i') = a_m \wedge \Big(\bigvee_{i=0}^m (b_{i-1}\wedge b_i')\Big).$$

Since $(b_i)_{i\in\omega}$ is a decreasing sequence in $C(L)$, Lemma 4.1 implies that $\bigvee_{i=0}^m (b_{i-1}\wedge b_i') = b_m'$. But $b_m' \ge b_n' = 1$. So $a_m \wedge (\bigvee_{i=0}^m (a_i\wedge b_{i-1}\wedge b_i')) = a_m$, and therefore $a_m \le \bigvee_{i=0}^m (a_i \wedge b_{i-1}\wedge b_i')$.

Since λ is a submeasure, we have $\lambda(a_m) \le \lambda(\bigvee_{i=0}^m (a_i \wedge b_{i-1} \wedge b_i')) \le \sum_{i=0}^m \lambda(a_i\wedge b_{i-1}\wedge b_i') \in \sum_{i=0}^m U_i[0] \subseteq U[0]$ for $m \ge n$. So $\lim_{m\to\infty}\lambda(a_m) = 0$.

COROLLARY 4.3. *Let L be a Boolean subalgebra of 2^Ω, let S be a Hausdorff uniform semigroup and let $\lambda : L \to S$. If λ is additive and monocompact, then λ is σ-additive.*

Proof. Let $p \in P$. Then the p-semivariation λ_p is a submeasure on L.

We shall show that λ_p is monocompact. Let K be a monocompact paving in 2^Ω such that $L\times K$ is an approximating paving for λ. Let $A \in L$ and let $U \in \mathcal{U}_\infty$. We may suppose that $U = \{(u,v) \in [0,\infty[\times[0,+\infty[: p_\infty(u,v) < \varepsilon/(1+\varepsilon)\}$ where $\varepsilon > 0$. Then $V = \{(x,y)\in S\times S : p(x,y) < \varepsilon/2\}$ belongs to $\mathcal{U}$. So there exists $(B,C) \in L\times K$ such that $B\subseteq C\subseteq A$ and $\lambda(D) \in V[0]$ whenever $D\in L$ and $D \subseteq A\setminus B$. Hence $D \in L$ and $D\subseteq A\setminus B$ imply $p(\lambda(D),0) < \varepsilon/2$. Then $\lambda_p(A\setminus B) \le \varepsilon/2 < \varepsilon$, and therefore $\lambda_p(D) < \varepsilon$. Consequently, $\lambda_p(D) \in U[0]$ whenever $D \in L$ and $D \subseteq A\setminus B$.

By Theorem 4.2, λ_p is order continuous. Since $p\in P$ is arbitrary and λ_p dominates $p(\lambda(\cdot),0)$, it follows that λ is order continuous, and therefore σ-additive.

Remarks 4.4. (a) The classical Marczewski Theorem [19, 4(i)] and a topological group-valued result of Millington [20, Lemma 4.1] are immediate consequences of Corollary 4.3.

(b) The first statement following Lemma 1 of Topsøe [30] is obviously contained in Corollary 4.3.

(c) Let Ω be a topological space. For a real- or complex-valued set function λ defined on a Boolean subalgebra L of 2^Ω, the regularity of λ in the sense of [8, Definition III.5.11] implies that λ has $L\times\{F\subseteq\Omega : F$ is closed in $\Omega\}$ as an approximating paving. Then Theorem III.5.12 of [8] is contained in Corollary 4.3.

(d) In our terminology, Theorem 2.3.4 of [5] can be stated as follows: Let L be a Boolean subalgebra of 2^Ω, let K be a compact subclass of L and let $\lambda : L \to [0,+\infty[$ be an additive set function such that $\lambda(A) = \sup\{\lambda(C) : C\in K$ and $C\subseteq A\}$ for all $A\in L$. Then λ is σ-additive. It is obvious that this theorem is contained in Corollary 4.3.

(e) Following Aleksandrov [2, pp. 567–568] a *charge* on a space $(\Omega,\mathcal{F})$ is an additive bounded real-valued set function λ on $L = A(\mathcal{F})$ satisfying the following regularity condition: For every $A\in L$ and every $\varepsilon>0$, there exists $F\in\mathcal{F}$ such that $F\subseteq A$ and $|\lambda(A)-\lambda(F)|<\varepsilon$. The classical Aleksandrov Theorem [2, Theorem 1, p. 590] can be stated as follows: Let $(\Omega,\mathcal{F})$ be a $\mathcal{F}$-compact space. Then every charge on $(\Omega,\mathcal{F})$ is σ-additive. It is clear that this theorem is an immediate consequence of Corollary 4.3 and Example 4 above taking into account Theorem 2 of [2, p. 571].

(f) The following interesting result appears implicitly in Glicksberg's paper [13, pp. 256–258]: Let Ω be a pseudo-compact topological space, let $Z(\Omega)$ be the set of all zero-sets in Ω, let $L = A(Z(\Omega))$ and let $\lambda : L\to\mathbf{R}$ be an additive bounded set function such that $\lambda(A) = \sup\{\lambda(Z) : Z\in Z(\Omega)$ and $Z\subseteq A\}$ for all $A\in L$. Then λ is σ-additive. First we note that $Z(\Omega)$ is a δ-paving in 2^Ω (see [11, pp. 14–16]). Then, by Examples 3 and 4 above, $(\Omega, Z(\Omega))$ is a $Z(\Omega)$-compact space. To deduce the Glicksberg Theorem from the Aleksandrov Theorem, it suffices to observe that the regularity of λ implies that λ is a charge on $(\Omega, Z(\Omega))$.

COROLLARY 4.5. *Let $\mathcal{R}$ be a ring in Ω, let K be a compact class in Ω, let S be a Hausdorff uniform semigroup and let $\lambda : \mathcal{R}\to S$ be an additive set function satisfying the following regularity condition: For every $A\in\mathcal{R}$ and every $U\in\mathcal{U}$, there exist $B\in\mathcal{R}$ and $C\in K$ such that $B\subseteq C\subseteq A$ and $\lambda(D)\in U[0]$ whenever $D\in\mathcal{R}$ and $D\subseteq A\setminus B$. Then λ is σ-additive.*

Proof. Let $(E_i)_{i\in\omega}$ be a disjoint sequence in $\mathcal{R}$ such that $\Omega_0 = \bigcup_{i=0}^\infty E_i$ belongs to $\mathcal{R}$. Let $L = \{A\cap\Omega_0 : A\in\mathcal{R}\}$ and $H = \{C\in K : C\subseteq\Omega_0\}$. Then L is a Boolean subalgebra of 2^{Ω_0} contained in $\mathcal{R}$ and H is a compact class in Ω_0. Let $\mu = \lambda|L$. Then it is easy to verify that $\mu : L\to S$ is an additive monocompact set function. By Corollary 4.3, μ is σ-additive. So $\lambda(\Omega_0) = \mu(\Omega_0) = \lim_{n\to\infty}\sum_{i=0}^n \mu(E_i) = \lim_{n\to\infty}\sum_{i=0}^n\lambda(E_i)$. Thus λ is σ-additive.

Remarks 4.6. (a) A paving $\mathcal{Q}$ in 2^Ω is said to be a *quasi-ring* in Ω if the difference and intersection of two elements of Q can be written as finite disjoint unions of sets of $\mathcal{Q}$. Clearly every semiring in Ω (called half-ring by von Neumann [22]) is a quasi-ring in Ω.

The following recent result of Kluvánek appears in [17, Proposition 1.12]: Let $\mathcal{Q}$ be a quasi-ring in Ω, let $\mathcal{C}$ be a compact class in Ω and let $\mu : \mathcal{Q}\to[0,+\infty[$ be an additive set function satisfying the following regularity condition: For every $A\in\mathcal{Q}$ and every $\varepsilon>0$, there exist $B\in\mathcal{Q}$ and $C\in\mathcal{C}$ such that $B\subseteq C\subseteq A$ and $\mu(A)-\mu(B)<\varepsilon$. Then μ is σ-additive.

Let $\mathcal{R}$ be the ring in Ω generated by $\mathcal{Q}$. Then it is easy to verify that $\mathcal{R}$

is the set of all finite unions of elements of $\mathcal{Q}$. Let λ be the usual additive extension of μ to $\mathcal{R}$. We can consider λ as a function from $\mathcal{R}$ into S_∞. Let $K = \{\bigcup_{i=0}^n C_i : C_i \in \mathcal{C},\ n \in \omega\}$. Then K is a compact class in Ω (see [19, 2(iii)]). Moreover, it is easy to show that λ satisfies the regularity condition of Corollary 4.5. Thus the Kluvánek Theorem is an immediate consequence of Corollary 4.5.

(b) The following result of von Neumann appears in [22, Theorem 10.1.20]: Let $(\Omega,\mathcal{C})$ be a space, let $\mathcal{Q}$ be a semiring in Ω and let $\mu : \mathcal{Q} \to [0,+\infty[$ be an additive set function satisfying the following regularity condition: For every $A \in \mathcal{Q}$ and every $\varepsilon > 0$, there exist $B, D \in \mathcal{Q}$, a $\mathcal{C}$-compact subset C of Ω and a $\mathcal{C}$-closed subset F of Ω such that $B \subseteq C \subseteq A \subseteq F^c \subseteq D$, $\mu(B) > \mu(A) - \varepsilon$ and $\mu(D) < \mu(A) + \varepsilon$. Then μ is σ-additive.

Let $\mathcal{R}$ be the ring in Ω generated by $\mathcal{Q}$, and let λ be the usual additive extension of μ to $\mathcal{R}$. We can consider λ as a function from $\mathcal{R}$ to S_∞. By Example 5 above, the set of all $\mathcal{C}$-compact subsets of Ω is a compact class in Ω. Thus the von Neumann Theorem is an immediate consequence of Corollary 4.5.

(c) Now let Ω be a Hausdorff topological space, let $\mathcal{Q}$ be a semiring in Ω, let $\mathcal{R}$ be the ring in Ω generated by $\mathcal{Q}$. Let S be a Hausdorff commutative topological group and let $\mu : \mathcal{Q} \to S$ be a set function such that $\mu(\emptyset) = 0$. For every $p \in P$, we define the set functions $(\mu_R)_p, (\mu_D)_p : 2^\Omega \to S_\infty$ by the formulas: $(\mu_R)_p(B) = \sup\{p(\mu(A), 0) : A \in \mathcal{Q}$ and $A \subseteq B\}$ and $(\mu_D)_p(B) = \sup\{p(\sum_{i=0}^n \mu(A_i), 0) : A_i \in \mathcal{Q},\ A_i \cap A_j = \emptyset$ if $i \neq j$, $\bigcup_{i=0}^n A_i \subseteq B, n \in \omega\}$.

It is easy to verify that $(\mu_R)_p(\emptyset) = 0$ and $(\mu_R)_p(\cdot)$ dominates $p(\mu(\cdot), 0)$ on $\mathcal{Q}$. Suppose that μ is additive, and let λ be the usual additive extension of μ to $\mathcal{R}$. Then it can be shown that $(\mu_D)_p|\mathcal{R}$ is increasing and subadditive and $(\lambda_R)_p|\mathcal{R} = (\mu_D)_p|\mathcal{R}$.

We say that μ is *μ_D-regular* if, for every $A \in \mathcal{Q}$, every $p \in P$ and every $U \in \mathcal{U}_\infty$ there exist $B, D \in \mathcal{Q}$, a closed countably compact subset C of Ω and an open subset G of Ω such that $B \subseteq C \subseteq A \subseteq G \subseteq D$ and $(\mu_D)_p(D \setminus B) \in U[0]$.

The following interesting result of Huneycutt appears in [14, Theorem 2.1]: Let Ω be a Hausdorff topological space, let $\mathcal{Q}$ be a semiring in Ω and let S be a Hausdorff commutative topological group. If $\mu : \mathcal{Q} \to S$ is additive and μ_D-regular, then μ is σ-additive.

Let λ be the usual additive extension of μ to $\mathcal{R}$. Since the set $\mathcal{C} = \{C \subseteq \Omega : C$ is a closed countably compact subset of $\Omega\}$ is a compact class in Ω (see [9, Theorem 3.10.2, p. 258]), it follows that $K = \{\bigcup_{i=0}^n C_i : C_i \in \mathcal{C}$ and $n \in \omega\}$ is also a compact class in Ω. To see that the Huneycutt Theorem is contained in Corollary 4.5 it remains to show the regularity of λ.

Let $\mathcal{U}$ be the two-sided uniformity on S. Let $A \in \mathcal{R}$ and let $U \in \mathcal{U}$. We may suppose that

$$U = \bigcap_{i=0}^n \{(x,y) \in S \times S : p_i(x,y) < \varepsilon\}$$

where $p_i \in P$ and $\varepsilon > 0$. For every $k \in \omega$, let

$$U_k = \left\{(u,v) \in S_\infty \times S_\infty : p_\infty(u,v) < \frac{\varepsilon}{2^{k+1} + \varepsilon}\right\}.$$

Then $U_k \in \mathcal{U}_\infty$ for all $k \in \omega$. Write $A = \bigcup_{j=0}^m A_j$ where $A_j \in \mathcal{Q}$ and $A_k \cap A_j = \emptyset$ if $k \neq j$. Fix $i \in \{0,1,\ldots,n\}$. Since μ is μ_D-regular, for every $j = 0,1,\ldots,m$ there exist $B_j \in \mathcal{Q}$ and $C_j \in \mathcal{C}$ such that $B_j \subseteq C_j \subseteq A_j$ and $(\mu_D)_{p_i}(A_j \setminus B_j) \in U_j[0]$. Let $B = \bigcup_{j=0}^m B_j$ and $C = \bigcup_{j=0}^m C_j$. Then $B \in \mathcal{R}$, $C \in K$ and $B \subseteq C \subseteq A$. Let $E \in \mathcal{R}$ be such that $E \subseteq A \setminus B$. Then

$$\begin{aligned} p_i(\lambda(E), 0) &\le (\lambda_R)_{p_i}(E) = (\mu_D)_{p_i}(E) \le (\mu_D)_{p_i}(A \setminus B) \\ &= (\mu_D)_{p_i}\Big(\bigcup_{j=0}^m (A_j \setminus B_j)\Big) \le \sum_{j=0}^m (\mu_D)_{p_i}(A_j \setminus B_j) \\ &< \sum_{j=0}^m \frac{\varepsilon}{2^{j+1}} < \varepsilon. \end{aligned}$$

So $p_i(\lambda(E), 0) < \varepsilon$ for all $i \in \{0,1,\ldots,n\}$, and therefore $\lambda(E) \in U[0]$.

5. Regular submeasures. Let $L = (L, \le, ', 0, 1)$ be an orthomodular poset. A paving K in L is called *regular* if the following conditions are satisfied:

(i) Every countable subset of K has an infimum in L.

(ii) For every $b \in K$ and every sequence $(b_i)_{i\in\omega}$ such that $\bigwedge_{i\in\omega} b_i \perp b$, there exists $n \in \omega$ such that $\bigwedge_{i=0}^n b_i \perp b$.

For example, if F is a compact class in Ω, then $\{\bigcap_{i=0}^\infty F_i : F_i \in F\}$ is a regular paving in 2^Ω.

Let S be a Hausdorff uniform semigroup and let $\lambda : L \to S$ be a function such that $\lambda(0) = 0$. We say that λ is *regular* if there exists a regular paving K in L such that $\Delta(K) = \{(b,b) : b \in K\}$ is an approximating paving for λ. Sometimes we say that λ is *K-regular.*

We are in a position to state the second of the main results of this paper:

THEOREM 5.1. *Let L be an orthomodular poset. Then every regular submeasure on L is σ-subadditive.*

Proof. Let $\lambda : L \to S_\infty$ be a submeasure and let K be a regular paving in L such that $\Delta(K) = \{(b,b) : b \in K\}$ is an approximating paving for λ.

Let $(a_i)_{i\in\omega}$ be a sequence in L such that $\bigvee_{i\in\omega} a_i$ exists and let $\varepsilon > 0$. Write $a = \bigvee_{i\in\omega} a_i$,

$$U = \left\{(u,v) \in [0,+\infty[\times[0,+\infty[: p_\infty(u,v) < \frac{\varepsilon}{1+\varepsilon}\right\},$$
$$U_i = \left\{(u,v) \in [0,+\infty[\times[0,+\infty[: p_\infty(u,v) < \frac{\varepsilon}{2^{i+1}+\varepsilon}\right\} \quad \text{for all } i \in \omega .$$

By the regularity of λ there exists $b \in K$ such that $b \le a$ and $\lambda(a \wedge b') \in U[0]$. Also, for every $i \in \omega$, there exists $b_i \in K$ such that $b_i \le a_i'$ and $\lambda(a_i' \wedge b_i') \in U_i[0]$.

Thus $\bigwedge_{i\in\omega} b_i \le \bigwedge_{i\in\omega} a_i' = a' \le b'$, and therefore $\bigwedge_{i\in\omega} b_i \perp b$. Since K is regular, there exists $n \in \omega$ such that $\bigwedge_{i=0}^n b_i \perp b$, and therefore $\bigwedge_{i=0}^n b_i \le b'$. Since $a = b \vee (a \wedge b')$, $b_i' = a_i \vee (a_i' \wedge b_i')$, $\lambda(a\wedge b') < \varepsilon$ and $\lambda(a_i' \wedge b_i') < \varepsilon/2^{i+1}$ for all $i \in \omega$, we have

$$\lambda(a) \le \lambda(b) + \varepsilon \le \lambda\Big(\bigvee_{i=0}^n b_i'\Big) + \varepsilon \le \sum_{i=0}^n \lambda(b_i') + \varepsilon$$
$$\le \sum_{i=0}^\infty \lambda(b_i') + \varepsilon \le \sum_{i=0}^\infty \lambda(a_i) + 2\varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, we get $\lambda(a) \le \sum_{i=0}^\infty \lambda(a_i)$.

The following corollary improves [3, Theorem, p. 134]:

COROLLARY 5.2. *Let L be an orthomodular poset and let K be a regular paving in L. If $\lambda : L \to S_\infty$ is an additive K-regular function such that $\lambda(a \vee b) \le \lambda(a) + \lambda(b)$ for all $a, b \in K' = \{c : c' \in K\}$, then λ is σ-additive.*

Proof. Since λ is increasing, it suffices to show that λ is subadditive and to apply Theorem 5.1.

We note first that, by induction, it is easy to show that $\lambda(\bigvee_{i=0}^n c_i) \le \sum_{i=0}^n \lambda(c_i)$ for every finite sequence $(c_i)_{0\le i\le n}$, $n \in \omega$, in K'.

Let $(a_i)_{0\le i\le n}$, $n \in \omega$, be a finite sequence in L such that $\bigvee_{i=0}^n a_i$ exists and let $\varepsilon > 0$. Write $a = \bigvee_{i=0}^n a_i$, and define U and U_i for $i = 0, 1, \ldots, n$ as in the previous proof. By the regularity of λ there exists $b \in K$ such that $b \le a$ and $\lambda(a \wedge b') \in U[0]$. Also, for every $i = 0, 1, \ldots, n$, there exists $b_i \in K$ such that $b_i \le a_i'$ and $\lambda(a_i' \wedge b_i') \in U_i[0]$. Thus

$$\lambda(a) \le \lambda(b) + \varepsilon \le \lambda\Big(\bigvee_{i=0}^n b_i'\Big) + \varepsilon \le \sum_{i=0}^n \lambda(b_i') + \varepsilon$$
$$\le \sum_{i=0}^n \left\{\lambda(a_i) + \frac{\varepsilon}{2^{i+1}}\right\} + \varepsilon \le \sum_{i=0}^n \lambda(a_i) + 2\varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, we get $\lambda(a) \le \sum_{i=0}^n \lambda(a_i)$.

LEMMA 5.3. *Let L be an orthomodular poset and let $\lambda : L \to S_\infty$ be an additive function. If $a, b \in L$ and aCb, then $\lambda(a \vee b) \le \lambda(a) + \lambda(b)$.*

Proof. By [4, Theorems VIII.2.8, VIII.2.10 and VIII.2.11], $a' \wedge b$ exists and $a \vee b = a \vee (a' \wedge b)$. So $\lambda(a \vee b) = \lambda(a) + \lambda(a' \wedge b) \le \lambda(a) + \lambda(b)$.

COROLLARY 5.4. *Let L be an orthomodular poset and let K be a regular paving in L such that $K \subseteq C(K)$. Then every additive K-regular function $\lambda : L \to S_\infty$ is σ-additive.*

Proof. This a trivial consequence of Corollary 5.2 and Lemma 5.3 noting that aCb implies $a'Cb'$.

COROLLARY 5.5. *If L is an s-class in Ω and $\lambda : L \to S_\infty$ is an additive regular function, then λ is σ-additive.*

Proof. Let K be a regular paving in L such that λ is K-regular. Since $C(K) = \{A \subseteq \Omega : A \cap B \in K \text{ for all } B \in K\}$, the conclusion follows from Corollary 5.4.

Acknowledgements. This paper is a portion of a joint research undertaken during the stay of the first author at the Département de mathématiques et d'informatique de l'Université de Sherbrooke and the stay of the second author at the Department of Mathematics of the University of Reading. Both authors gratefully acknowledge the kind hospitality of the host institutions. Special thanks are due to Professors J. Dubois and J. D. M. Wright.

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DIPARTIMENTO DI MATEMATICA E
APPLICAZIONI "R. CACCIOPPOLI"
UNIVERSITÀ DEGLI STUDI DI NAPOLI
VIA MEZZOCANNONE, 8
I-80134 NAPOLI, ITALY

DÉPARTEMENT DE MATHÉMATIQUES
ET D'INFORMATIQUE
UNIVERSITÉ DE SHERBROOKE
SHERBROOKE, QUÉBEC, CANADA J1K 2R1

Received February 12, 1990
Revised version April 12, 1991
(2653)

STUDIA MATHEMATICA 101 (2) (1992)

A strong mixing condition for second-order stationary random fields

by

RAYMOND CHENG (Louisville, KY)

Abstract. Let $\{X_{mn}\}$ be a second-order stationary random field on $\mathbb{Z}^2$. Let $\mathcal{M}(L)$ be the linear span of $\{X_{mn} : m \le 0,\ n \in \mathbb{Z}\}$, and $\mathcal{M}(R_N)$ the linear span of $\{X_{mn} : m \ge N,\ n \in \mathbb{Z}\}$. Spectral criteria are given for the condition $\lim_{N\to\infty} c_N = 0$, where c_N is the cosine of the angle between $\mathcal{M}(L)$ and $\mathcal{M}(R_N)$.

1. Introduction. Suppose that $\{X_n\}_{n=-\infty}^{\infty}$ is a stationary process on the probability space $(\Omega, \mathcal{B}, \nu)$. A classical (linear) prediction problem is to estimate X_n, $n \ge 1$, based on the past of the process; that is, to find X in the linear span $\mathcal{P}$ of $\{\ldots, X_{-2}, X_{-1}, X_0\}$ for which the mean error $(\int |X - X_n|^2\, d\nu)^{1/2}$ is a minimum (see [4], [5], [17]). A variation on this idea is to replace X_n by the span $\mathcal{F}_n$ of $\{X_n, X_{n+1}, X_{n+2}, \ldots\}$, and to investigate the linear dependence between the subspaces $\mathcal{P}$ and $\mathcal{F}_n$. This class of problems is addressed in, for instance, [6], [8]–[11], [16], [18], [20]. These concerns, in turn, admit a multitude of generalizations.

In this article, we consider prediction problems in which the process is replaced by a random field, $\{X_{mn}\}_{\mathbb{Z}^2}$. For any subset S of $\mathbb{Z}^2$, we define $\mathcal{M}(S)$ to be the linear span of $\{X_{mn} : (m,n) \in S\}$; such spaces play roles analogous to $\mathcal{P}$ and $\mathcal{F}_n$. Now the issue is to understand the dependence between $\mathcal{M}(S_1)$ and $\mathcal{M}(S_2)$. In particular, we seek descriptions of those fields for which the dependence tends to zero as the distance between the generating sets S_1 and S_2 increases to infinity in some way—a sort of "strong mixing" condition. As in the case of processes on $\mathbb{Z}$, we pass to the spectral domain and apply techniques from function theory. This yields spectral criteria for strong mixing to occur.

1991 *Mathematics Subject Classification*: 60G60, 60G25.
Key words and phrases: stationary random field, prediction theory, strong mixing.
This research was supported by NSF Grant DMS-87 01212 at the University of Virginia, by the NSF Graduate Fellowship Program, and by the Commonwealth Fellowship Program in the state of Virginia.