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A weak molecule condition for certain Triebel–Lizorkin spaces

by

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Abstract. A weak molecule condition is given for the Triebel–Lizorkin spaces $\dot{F}_p^{\alpha,q}$, with $0 < \alpha < 1$ and $1 < p, q < \infty$. As an easy corollary, one may deduce, by atomic-molecular methods, a Triebel–Lizorkin space "T1" Theorem of Han and Sawyer, and Han, Jawerth, Taibleson and Weiss, for Calderón–Zygmund kernels $K(x, y)$ which are not assumed to satisfy any regularity condition in the y variable.

0. Introduction and statement of results. In recent years, there has been considerable interest in two problems concerning function spaces, or more generally, spaces of distributions. The first is the decomposition of such spaces into building blocks (usually referred to as "atoms" and "molecules"). The second is the boundedness on such spaces of various types of operators, for example Calderón–Zygmund, or (C–Z), operators, which are generalizations of the classical principal value convolution operators studied by Calderón and Zygmund (e.g. [CZ]). The two problems are in some sense related: to show that a linear operator T is bounded on some distribution space, it is often enough to consider the action of T on the relatively simple "atoms", and show that T maps atoms uniformly into "molecules".

The class of Triebel–Lizorkin spaces is one class of distribution spaces which has been studied by means of atomic and molecular decompositions. Let φ be a Schwartz function such that its Fourier transform $\widehat{\varphi}$ is supported on the annulus $\{1/2 \le |\xi| \le 2\}$, with $|\widehat{\varphi}(\xi)| \ge c > 0$ if $3/5 \le |\xi| \le 5/3$. One can define the homogeneous Triebel–Lizorkin space $\dot{F}_p^{\alpha,q}$, $0 < q \le \infty$, $0 < p < \infty$, $\alpha \in \mathbf{R}$, as the space of tempered distributions modulo polynomials (because all the moments of φ are zero) with the norm (or quasi-norm, if p or $q < 1$)

$$(0.1)\qquad \|f\|_{\dot{F}_p^{\alpha,q}} \equiv \Big\|\Big(\sum_{k=-\infty}^{\infty} (2^{k\alpha}|\varphi_k * f|)^q\Big)^{1/q}\Big\|_p ,$$

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where $\varphi_k(x) \equiv 2^{kn}\varphi(2^k x)$. We obtain an equivalent norm if we choose a different φ with the same properties. Frazier and Jawerth [FJ] have shown that the Triebel–Lizorkin spaces have a decomposition into smooth atoms. For $0 < \alpha < 1$, and $1 < p, q < \infty$, a smooth atom a_Q associated with a dyadic cube Q is a C_0^∞ function with the following properties:

$$\text{supp}\, a_Q \subseteq 3Q\,, \tag{0.2}$$

$$\int a_Q(x)\,dx = 0\,, \tag{0.3}$$

$$\|a_Q\|_\infty \le |Q|^{-1/2}\,, \tag{0.4}$$

$$\|\nabla a_Q\|_\infty \le |Q|^{-1/2-1/n}\,, \tag{0.5}$$

where $3Q$ is the cube concentric with Q having side length 3 times as large. Every $f \in \dot F_p^{\alpha,q}$, $0 < \alpha < 1$, $1 < p, q < \infty$, can be written as

$$f = \sum s_Q a_Q\,, \tag{0.6}$$

where the sum runs over all dyadic cubes, and where the coefficients s_Q satisfy

$$\Big\|\Big(\sum_Q (|Q|^{-\alpha/n}|s_Q|\widetilde\chi_Q)^q\Big)^{1/q}\Big\|_{L^p} \le C\|f\|_{\dot F_p^{\alpha,q}}\,. \tag{0.7}$$

Here $\widetilde\chi_Q \equiv |Q|^{-1/2}\chi_Q$. Conversely, if f has the decomposition (0.6), then (0.1) is bounded by a constant times the left side of (0.7).

Frazier and Jawerth [FJ] also found a molecular decomposition of the Triebel–Lizorkin spaces. The idea is that if

$$f = \sum_{Q \text{ dyadic}} s_Q m_Q\,,$$

where m_Q is a molecule associated to the cube Q, then $\|f\|_{\dot F_p^{\alpha,q}}$ is bounded by a constant times the left side of (0.7). An immediate corollary is that if T is, say, a Calderón–Zygmund operator with $Ta_Q = m_Q$, then T is bounded on $\dot F_p^{\alpha,q}$.

We now define (this definition will be justified later) a *weak* (δ,ε) *molecule* m_Q associated to a dyadic cube Q as a function satisfying

$$|m_Q(x)| \le |Q|^{-1/2}(1 + l(Q)^{-1}|x - x_Q|)^{-n}\,, \tag{0.8}$$

where n = dimension, $l(Q)$ = side length of Q, and x_Q is the "lower left corner" of Q, and

$$|m_Q(x) - m_Q(y)| \le |Q|^{-1/2}(l(Q)^{-1}|x-y|)^\delta \times\{(1 + l(Q)^{-1}|x - x_Q|)^{-n-\varepsilon} + (1 + l(Q)^{-1}|y - x_Q|)^{-n-\varepsilon}\} \tag{0.9}$$

if $|x - y| \le l(Q)$, or if $|x - x_Q| \ge 10\sqrt{n}\,|x - y|$.

The smoothness condition (0.9) is the same as in the Frazier–Jawerth theory (although it seems to have first been explicitly formulated in this way in [FHJW]), except that previously (0.9) was assumed to hold in general, not just for $|x - y|$ sufficiently small. What is new here is the size condition (0.8). In the Frazier–Jawerth theory, (0.8) was replaced by the same condition but with the order of decay $-n-\gamma$, for some $\gamma > 0$, whereas the present molecules only decay as $|x|^{-n}$ at infinity. This may seem at first glance like a superficial difference, but of course $-n$ is the "critical index" for decay at infinity to achieve integrability in $\mathbf{R}^n$. Furthermore, as an application, the weak molecule condition introduced here will enable us to prove easily, via the atomic-molecular approach, a result of Han and Sawyer, and also Han, Jawerth, Taibleson, and Weiss, that Calderón–Zygmund operators are bounded on $\dot F_p^{\alpha,q}$, $0 < \alpha < 1$, $1 < p, q < \infty$, under significantly weakened regularity assumptions on the kernel. Specifically, suppose that T is a continuous linear mapping from $\mathcal D$ to $\mathcal D'$ associated to a kernel $K(x,y)$ in the sense that

$$\langle Tf, g\rangle \equiv \int\int g(x)K(x,y)f(y)\,dy\,dx\,,$$

where $g, f \in C_0^\infty$ with disjoint supports. Suppose also that

$$|K(x,y)| \le C|x-y|^{-n}\,, \tag{0.10}$$

$$|K(x,y) - K(x',y)| \le C|x - x'|^\varepsilon|x-y|^{-n-\varepsilon} \quad \text{whenever } |x-y| \ge 2|x-x'| \text{ with } 0 < \varepsilon \le 1\,. \tag{0.11}$$

Note that we assume no smoothness in the second (y) variable of $K(x,y)$. We also assume that T satisfies that *Weak Boundedness Property* (WBP):

$$\langle g, Tf\rangle \le CR^n(\|f\|_\infty + R\|\nabla f\|_\infty)(\|g\|_\infty + R\|\nabla g\|_\infty) \tag{0.12}$$

for all $f, g \in C_0^\infty$ with support in a ball of radius R. Y. S. Han and E. T. Sawyer [HS] and also Han, Jawerth, Taibleson, and Weiss [HJTW] have proved the following important result:

THEOREM (0.13). *Suppose $K(x,y)$ satisfies* (0.10) *and* (0.11), *and suppose that the corresponding operator T satisfies $T1 = 0$ and WBP* (0.12). *Then, for $0 < \alpha < \varepsilon$, and $1 < p, q < \infty$, we have*

$$\|Tf\|_{\dot F_p^{\alpha,q}} \le C\|f\|_{\dot F_p^{\alpha,q}}\,. \tag{0.14}$$

Remark. By interpolation, Theorem (0.13) implies an analogous (and earlier) result of Lemarié [L] for Besov spaces.

The proof in [HS] is a direct one based on a Littlewood–Paley argument found in work of David, Journé and Semmes [DJS] and Coifman, David, Meyer and Semmes [CDMS] concerning L^2 boundedness of C–Z operators. This Littlewood–Paley approach had also been used by the present author and Han [HH] to extend the "$T1$" arguments of [DJS] and [CDMS] to Besov and Triebel–Lizorkin spaces under weak regularity conditions on the kernel defined in terms of an L^r or L^1 modulus of continuity. Of course, there is an implicit connection between the Littlewood–Paley approach and the atomic approach (see e.g. [FJ]), and this connection provided some of the motivation for the present work.

Under the stronger assumption that $K(x,y)$ also satisfies the smoothness condition (0.11) in the y variable, Theorem (0.13) had been obtained by Frazier, Han, Jawerth and Weiss [FHJW] using the Frazier–Jawerth theory of atoms and molecules. The stronger Frazier–Jawerth molecular condition seems to require smoothness in both the x and y variables. We also remark that Torres [T] had obtained results for $\alpha > 1$ by this method, without smoothness in y, but his technique does not seem to extend to $0 < \alpha < 1$, unless one compensates by imposing extra smoothness in x. One of the motivations for introducting the weak condition (0.8) is that we will now be able to recover the [HS], [HJTW] result (Theorem (0.13)) by means of an atomic-molecular approach. Perhaps this weaker condition may prove useful in other applications as well.

Our definition of molecules is justified because we are able to prove the following, which is the main result of this paper:

THEOREM (0.15). *Suppose $f = \sum s_Q m_Q$, where the sum runs over the dyadic cubes, and where each m_Q satisfies* (0.8) *and* (0.9). *Then, if $0 < \alpha < \delta \le \varepsilon$, and $1 < p, q < \infty$, we have*

$$\|f\|_{\dot F_p^{\alpha,q}} \le C\Big\|\Big(\sum_Q (|Q|^{-\alpha/n}|s_Q|\widetilde{\chi}_Q)^q\Big)^{1/q}\Big\|_p . \tag{0.16}$$

In the next section, we will deduce Theorem (0.13) as a relatively easy corollary of Theorem (0.15). Section 2 will contain the proof of Theorem (0.15).

1. Proof of Theorem (0.13) (modulo Theorem (0.15)).

We shall use the following lemma of M. Meyer [MM] which has become by now a rather standard fact in Calderón–Zygmund theory:

LEMMA (1.1). *Suppose $\eta \in C_0^\infty(Q)$. Then, under the hypotheses of Theorem* (0.13),

$$\|T\eta\|_\infty \le C(\|\eta\|_\infty + |Q|^{1/n}\|\nabla\eta\|_\infty),$$

with C independent of η.

To prove Theorem (0.13), it is enough to show that for a smooth atom a_Q satisfying (0.2)–(0.5) (actually, we will not need (0.3)), $m_Q \equiv Ta_Q$ satisfies (0.8) and (0.9), modulo a multiplicative constant not depending upon a or Q. The proof of (0.9) uses the smoothness in the x variable and is essentially the

same as the argument in [FHJW]. The reader is referred to that paper for details, although we sketch the idea briefly for the sake of self-containment. Suppose $l(Q) = 2^{-j}$, and $|x_1 - x_2| < 2^{-j}$ or $|x_1 - x_Q| \ge 10\sqrt{n}\,|x_1 - x_2|$. We want to show, for $a \equiv a_Q$,

$$\begin{aligned}|Ta(x_1) - Ta(x_2)| &\le C|Q|^{-1/2}(2^j|x_1 - x_2|)^\varepsilon \\ &\quad\times\{(1 + 2^j|x_1 - x_Q|)^{-n-\varepsilon} + (1 + 2^j|x_2 - x_Q|)^{-n-\varepsilon}\}.\end{aligned} \tag{1.2}$$

There are two cases:

Case 1: $|x_1 - x_Q| \ge 10\sqrt{n}\,2^{-j}$ (in which case $|x_2 - x_Q| \ge 9\sqrt{n}\,2^{-j}$). Here either $|x_1 - x_2| < 2^{-j}$ or $|x_1 - x_Q| \ge 10\sqrt{n}\,|x_1 - x_2|$. Write

$$Ta(x_2) - Ta(x_1) = \int [K(x_2, y) - K(x_1, y)]a(y)\,dy .$$

In this case (1.2) now follows by straightforward computation from the smoothness condition (0.11) and the atomic conditions (0.2) and (0.4).

Case 2: $|x_1 - x_Q| < 10\sqrt{n}\,2^{-j}$ (in which case $|x_2 - x_Q| \le 11\sqrt{n}\,2^{-j}$). Note that in this case $|x_1 - x_2| < 2^{-j}$. Let $r \equiv |x_1 - x_2|$ and let $\eta_r(u) \equiv \eta(u/r)$, where $\eta \in C_0^\infty(|u| \le 20)$ and $\eta(u) \equiv 1$ for $|u| \le 10$. We now write

$$\begin{aligned}Ta(x_1) - Ta(x_2) &\equiv \int [K(x_1, y) - K(x_2, y)][a(y) - a(x_1)][1 - \eta_r(y - x_1)]\,dy \\ &\quad+ \int K(x_1, y)[a(y) - a(x_1)]\eta_r(y - x_1)\,dy \\ &\quad- \int K(x_2, y)[a(y) - a(x_2)]\eta_r(y - x_1)\,dy \\ &\quad+ (a(x_1) - a(x_2))T\eta_r(\cdot - x_1)(x_2) \equiv \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.\end{aligned} \tag{1.3}$$

The representation (1.3) is due to Y. Meyer [M], and may be easily verified on a formal level by using the fact that $T1 = 0$.

To obtain (1.2) for IV, we just use the atomic condition (0.5) plus Lemma (1.1). To handle II and III, we use (0.5) and the size condition (0.10). Next, define

$$S \equiv \{y : 10|x_1 - x_2| \le |y - x_1| \le 11\sqrt{n}\,2^{-j}\},$$
$$T \equiv \{y : |y - x_1| > 11\sqrt{n}\,2^{-j}\}.$$

We split I into $\mathrm{I} = \int_S + \int_T$. Here we use (0.5) in $\int_S$, (0.4) in $\int_T$ and (0.11) in both. Now (1.2) follows by direct computation. (This argument must be modified slightly if $\varepsilon = 1$; see [FHJW].)

Next we verify that Ta satisfies (0.8). In [FHJW], smoothness in the y variable of $K(x,y)$ was used, along with the fact that $\int a = 0$ (a fact which we do not require in this paper), to obtain the stronger Frazier–Jawerth molecular condition. In the present case $K(x,y)$ need not be regular in the y variable, so we shall have to be content with (0.8). This is easy. First suppose $|x-x_Q| \le 10\sqrt{n}\,2^{-j}$, where again $2^{-j}=l(Q)$. In this case, to verify (0.8) it is enough that

$$|Ta(x)| \le C|Q|^{-1/2},$$

which is immediate from Lemma (1.1) and the atomic conditions (0.4) and (0.5). On the other hand, suppose $|x-x_Q| \ge 10\sqrt{n}\,2^{-j}$. Then for $y \in \operatorname{supp} a$, we have $|x-y| \approx |x-x_Q|$, so by (0.10), (0.2) and (0.4),

$$|Ta(x)| = \Big|\int K(x,y)a(y)\,dy\Big| \le C|x-x_Q|^{-n}|Q|^{-1/2}|Q|,$$

which is comparable to the right hand side of (0.8) in the present case.

2. Proof of Theorem (0.15). The proof is motivated in part by the approach in [FJ], and in part by some ideas in [DJS], [CDMS], [HH] and [HS]. We shall want to use a discrete version of the "Calderón reproducing formula" due to Frazier and Jawerth [FJ]. They have shown that one can select a φ as in the definition (0.1) and a $\psi \in C_0^\infty(|x|\le 1)$ with $\int\psi=0$, such that

$$\sum_{k=-\infty}^{\infty} \widehat{\psi}_k(\xi)\widehat{\varphi}_k(\xi) \equiv 1, \quad \xi \neq 0,$$

where again $\varphi_k(x) = 2^{kn}\varphi(2^k x)$ and the same for ψ, so

$$\sum_{k=-\infty}^{\infty} \psi_k * \varphi_k * f = f. \tag{2.1}$$

Now, let $f=\sum_Q s_Q m_Q$ with m_Q as in (0.8) and (0.9) and s_Q as in (0.7). We dualize; for $g \in C_0^\infty \cap \dot{F}_{p'}^{-\alpha,q'}$, it is enough to show that

$$|\langle f,g\rangle| \le C\|g\|_{\dot{F}_{p'}^{-\alpha,q'}} \Big\|\Big(\sum_Q (|Q|^{-\alpha/n}|s_Q|\widetilde{\chi}_Q)^q\Big)^{1/q}\Big\|_p. \tag{2.2}$$

Using the reproducing formula (2.1) we write $\langle f,g\rangle$ as

$$\sum_{k=-\infty}^{\infty}\sum_Q s_Q\langle \psi_k * m_Q, \widetilde{\varphi}_k * g\rangle, \tag{2.3}$$

where $\widetilde{\varphi}_k(x) = \overline{\varphi_k(-x)}$. (To make things rigorous, one could take finite sums and use a limiting argument; the details are left to the reader.) Next write (2.3) as

$$\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\sum_{l(Q)=2^{-j}} s_Q\langle \psi_k * m_Q, \widetilde{\varphi}_k * g\rangle \tag{2.4}$$
$$\equiv \sum_k\sum_{j\le k}\sum_{l(Q)=2^{-j}} + \sum_k\sum_{j>k}\sum_{l(Q)=2^{-j}} \equiv \mathrm{I} + \mathrm{II}.$$

We shall need the following modification of a lemma from [FJ], which the reader may easily verify by using the smoothness condition (0.9), and the fact that ψ_k is supported on a ball of radius 2^{-k} and has mean value zero.

LEMMA (2.5). *Let ψ_k be as above, and let m_Q satisfy (0.9), where $l(Q)=2^{-j}$. Suppose also that either $j \le k$ or $|x-x_Q| \ge 10\sqrt{n}\,2^{-k}$. Then*

$$|\psi_k * m_Q(x)| \le C|Q|^{-1/2}2^{(j-k)\delta}(1+2^j|x-x_Q|)^{-n-\varepsilon}.$$

We will also use another lemma from [FJ], whose proof we include for the sake of self-containment.

LEMMA (2.6) (Frazier–Jawerth).

$$\sum_{l(Q)=2^{-j}} |s_Q|(1+2^j|x-x_Q|)^{-n-\varepsilon} \le C_\varepsilon M\Big(\sum_{l(Q)=2^{-j}}|s_Q|\chi_Q\Big)(x). \tag{2.7}$$

By translation invariance, we may assume $x \in Q_j(0)$, i.e. the cube with $l(Q)=2^{-j}$ and "lower left" corner at 0. Set $A_i \equiv \{Q \text{ dyadic} : l(Q) = 2^{-j},\ 2^{i-j-1} < |x_Q| \le 2^{i-j}\}$ for $i=1,2,\dots$ and $A_0 \equiv \{Q : l(Q)=2^{-j}, |x_Q| \le 2^{-j}\}$. Then the left hand side of (2.7) is bounded by a constant times

$$\sum_{i=0}^{\infty}\sum_{Q\in A_i} 2^{-i(n+\varepsilon)}|s_Q| = \sum_{i=0}^{\infty} 2^{-i(n+\varepsilon)+jn}\int\sum_{Q\in A_i}|s_Q|\chi_Q$$
$$\le C\sum_{i=0}^{\infty}2^{-i\varepsilon}M\Big(\sum_{l(Q)=2^{-j}}|s_Q|\chi_Q\Big)(x),$$

where in the last inequality, we have used the fact that

$$\int\sum_{Q\in A_i}|s_Q|\chi_Q \le \int_{|x-y|\le C2^{i-j}}\ \sum_{l(Q)=2^{-j}}|s_Q|\chi_Q(y)\,dy$$

if $x \in Q_j(0)$. This concludes the proof of Lemma (2.6).

Now we consider I from the splitting of (2.4). By Lemmas (2.5) and (2.6), I is bounded in absolute value by

$$\sum_k\sum_{j\le k}2^{(j-k)\delta}\Big\langle M\Big(\sum_{l(Q)=2^{-j}}|s_Q|\widetilde{\chi}_Q\Big), |\widetilde{\varphi}_k * g|\Big\rangle, \tag{2.8}$$

where $\widetilde{\chi}_Q \equiv |Q|^{-1/2}\chi_Q$. Next, multiply and divide in (2.8) by $2^{k\alpha}$, and apply Hölder's inequality twice to deduce that (2.8) is bounded by

$$\Big\|\Big(\sum_k\sum_{j\le k}\Big(2^{(j-k)(\delta-\beta)}2^{k\alpha}M\Big(\sum_{l(Q)=2^{-j}}|s_Q|\widetilde{\chi}_Q\Big)\Big)^q\Big)^{1/q}\Big\|_p \tag{2.9}$$
$$\times\Big\|\Big(\sum_k\sum_{j\le k}\Big(2^{(j-k)\beta}2^{-k\alpha}|\widetilde{\varphi}_k * g|\Big)^{q'}\Big)^{1/q'}\Big\|_{p'},$$

where β is a small positive number to be chosen (in fact, take $\delta-\beta>\alpha$). If we sum in j first, the second factor is just $\|g\|_{\dot F_{p'}^{-\alpha,q'}}$. Now observe that for $\delta-\beta>\alpha$, $\sum_{k:k\ge j}2^{(j-k)(\delta-\beta)}2^{k\alpha}=C2^{j\alpha}=C|Q|^{-\alpha/n}$. Thus, by the vector-valued inequality for the maximal function, the first factor is bounded by (0.7), since $(\sum_{l(Q)=2^{-j}}|s_Q|\widetilde\chi_Q)^q=\sum_{l(Q)=2^{-j}}(|s_Q|\widetilde\chi_Q)^q$ for the disjoint cubes with fixed side length.

Now, we turn to II in the splitting of (2.4). We shall use the following:

LEMMA (2.10). *For $j>k$, $l(Q)=2^{-j}$, and for all γ, $0<\gamma<\delta$, we have*

$$|\psi_k*m_Q(x)|\le C_\gamma|Q|^{-1/2}2^{(j-k)\gamma}(1+2^j|x-x_Q|)^{-n-\gamma}+C(j-k)|Q|^{-1/2}2^{(k-j)n}\chi\{|x-x_Q|\le 10\sqrt n\,2^{-k}\}.$$

We shall also use the following analogue of Lemma (2.6):

LEMMA (2.11). *For $j>k$,*

$$\sum_{l(Q)=2^{-j}}|s_Q|2^{(k-j)n}\chi\{|x-x_Q|\le 10\sqrt n\,2^{-k}\}\le CM\Big(\sum_{l(Q)=2^{-j}}|s_Q|\chi_Q\Big)(x).$$

Let us assume these lemmas for the moment, and deal with II in (2.4). We apply Lemma (2.10), and then either Lemma (2.6) with γ in place of ε, or Lemma (2.11), as appropriate, so that

$$|\mathrm{II}|\le C_\gamma\sum_k\sum_{j>k}2^{(j-k)\gamma}\Big\langle M\Big(\sum_{l(Q)=2^{-j}}|s_Q|\widetilde\chi_Q\Big),|\widetilde\varphi_k*g|\Big\rangle,$$

where we have used the fact that $j-k\le C_\gamma 2^{(j-k)\gamma}$. Now, let $0<\beta<\alpha$, multiply and divide by $2^{j\alpha}2^{-(j-k)\beta}$, and apply Hölder's inequality twice to show that the last expression is no larger than a constant times

$$\Big\|\Big(\sum_j\sum_{k<j}\Big(2^{(j-k)(\gamma-\beta)}M\Big(\sum_{l(Q)=2^{-j}}|Q|^{-\alpha/n}|s_Q|\widetilde\chi_Q\Big)\Big)^q\Big)^{1/q}\Big\|_p \times\Big\|\Big(\sum_k\sum_{j>k}\Big(2^{-j\alpha}2^{(j-k)\beta}|\widetilde\varphi_k*g|\Big)^{q'}\Big)^{1/q'}\Big\|_{p'}. \tag{2.12}$$

The first factor can be handled by taking $\gamma<\beta$, and summing first in k. The bound (0.7) then follows exactly as it did for the analogous term in (2.9). To handle the second factor in (2.12), we sum first in j, with $\beta<\alpha$, to obtain the bound $\|g\|_{\dot F_{p'}^{-\alpha,q'}}$.

To conclude the proof of Theorem (0.15), it therefore remains to prove Lemmas (2.10) and (2.11). We prove (2.10) first. There are two cases.

Case 1: $|x-x_Q|>10\sqrt n\,2^{-k}$. By the molecular size condition (0.8), we have the crude estimate

$$|\psi_k*m_Q(x)|\le|Q|^{-1/2}\int|\psi_k(x-y)|(1+2^j|y-x_Q|)^{-n}\,dy\le C|Q|^{-1/2}(1+2^j|x-x_Q|)^{-n}\|\psi\|_1, \tag{2.13}$$

where in the last inequality we have used the fact that by the support of ψ_k, $|x-y|\le 2^{-k}\ll|x-x_Q|$, so that $|x-x_Q|\approx|y-x_Q|$.

On the other hand, by the smoothness condition (0.9), and the fact that ψ has mean value zero, we have

$$\begin{aligned}|\psi_k*m_Q(x)|&\le\int|\psi_k(x-y)||m_Q(y)-m_Q(x)|\,dy\\&\le|Q|^{-1/2}C\|\psi\|_1 2^{(j-k)\delta}(1+2^j|x-x_Q|)^{-n-\varepsilon}\\&\le|Q|^{-1/2}C\|\psi\|_1 2^{(j-k)\delta}(1+2^j|x-x_Q|)^{-n-\delta},\end{aligned} \tag{2.14}$$

for $\delta\le\varepsilon$. But Lemma (2.10) follows in the present case by interpolating between (2.13) and (2.14).

Case 2: $|x-x_Q|\le 10\sqrt n\,2^{-k}$. By (0.8), $|\psi_k*m_Q(x)|$ is less than or equal to $|Q|^{-1/2}$ times

$$\int|\psi_k(x-y)|(1+2^j|y-x_Q|)^{-n}\,dy\equiv\int_{|y-x_Q|\le 2^{-j}}+\int_{2^{-j}<|y-x_Q|\le C2^{-k}}\equiv A+B, \tag{2.15}$$

where in B we have used the fact that $|y-x_Q|\le|x-x_Q|+|x-y|\le C2^{-k}$. Now

$$A\le 2^{kn}\|\psi\|_\infty\int_{|y-x_Q|\le 2^{-j}}1\,dy,$$

$$B\le 2^{kn}\|\psi\|_\infty 2^{-jn}\int_{2^{-j}}^{C2^{-k}}\frac1\varrho\,d\varrho.$$

Lemma (2.10) follows easily.

Finally, we turn to the proof of Lemma (2.11).

As in the proof of Lemma (2.6), we may take $x\in Q_j(0)$. We recall that $j>k$. For all $x\in Q_j(0)$, the only cubes appearing in the sum on the left hand side of (2.11) are those for which $Q\in E_k\equiv\{Q\text{ dyadic}: l(Q)=2^{-j},|x_Q|\le C_n2^{-k}\}$. Thus, it is enough to consider

$$\sum_{Q\in E_k}|s_Q|2^{(k-j)n}=2^{kn}\int\sum_{Q\in E_k}|s_Q|\chi_Q\le 2^{kn}\int_{|x-y|\le C_n2^{-k}}\sum_{l(Q)=2^{-j}}|s_Q|\chi_Q(y)\,dy,$$

and (2.11) follows.

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A noncommutative version of a Theorem of Marczewski for submeasures

by

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Abstract. It is shown that every monocompact submeasure on an orthomodular poset is order continuous. From this generalization of the classical Marczewski Theorem, several results of commutative Measure Theory are derived and unified.

1. Introduction. According to the well known theorems of Aleksandrov [2], von Neumann [22] and Marczewski [19], a mild regularity condition is sufficient for the σ-additivity of a real-valued set function defined on a family of sets. One of the purposes of this paper is to unify these apparently unrelated results via an extension of the Marczewski Theorem to submeasures on an orthomodular poset. Incidentally, we indicate that one of the particular interests of the noncommutative Measure Theory is its relevance to the Hilbert space formulation of Quantum Mechanics (see [16], [23], [28] and [31]).

The paper is organized as follows: In Section 2 we give some elementary notions of orthoposets and uniform semigroups, and we define some pertinent classes of functions from an orthoposet into a uniform semigroup. Section 3 introduces the notion of an approximating paving for the aforesaid kind of functions, and this notion is illustrated with appropriate examples. In the next section we extend properly the notion of compact measure of Marczewski to the noncommutative setting, and we establish the first of the main results of this paper. We also deduce, as by-products, several results bearing the names of Aleksandrov [2], K. P. S. Bhaskara Rao and M. Bhaskara Rao [5], Glicksberg [13], Huneycutt [14], Kluvánek [17], Marczewski [19], Millington [20], von Neumann [22] and Topsøe [30]. In the last

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