[1] P. J. Boland and S. Dineen, *Holomorphic functions on fully nuclear spaces*, Bull. Soc. Math. France 106 (1978), 311–336.
[2] P. A. Chalov and V. P. Zakharyuta, *A quasiequivalence criterion for absolute bases in an arbitrary (F)-space*, Izv. Severo-Kavkaz. Nauchn. Tsentra Vyssh. Shkoly Estestv. Nauki 1983 (2), 22–24 (in Russian).
[3] P. B. Djakov, *A short proof of the theorem of Crone and Robinson on quasi-equivalence of regular bases*, Studia Math. 53 (1975), 269–271.
[4] H. Jarchow, *Locally Convex Spaces*, Teubner, 1981.
[5] R. Meise and D. Vogt, *Structure of spaces of holomorphic functions on infinite dimensional polydiscs*, Studia Math. 75 (1983), 235–252.
[6] —, —, *Analytic isomorphisms of infinite dimensional polydiscs and an application*, Bull. Soc. Math. France 111 (1983), 3–20.
[7] —, —, *Holomorphic functions of uniformly bounded type on nuclear Fréchet spaces*, Studia Math. 83 (1986), 147–166.
[8] —, —, *Holomorphic Functions on Nuclear Sequence Spaces*, Departamento de Teoría de Funciones, Universidad Complutense, Madrid 1986.
[9] L. Mirsky, *Transversal Theory*, Academic Press, 1971.
[10] B. S. Mityagin, *The equivalence of bases in Hilbert scales*, Studia Math. 37 (1971), 111–137 (in Russian).
[11] A. Pietsch, *Nuclear Locally Convex Spaces*, Ergeb. Math. Grenzgeb. 66, Springer, 1972.
[12] H. H. Schaefer, *Topological Vector Spaces*, Springer, 1971.
[13] M. Scheve, *Räume holomorpher Funktionen auf unendlich-dimensionalen Polyzylindern*, Dissertation, Düsseldorf 1988.
[14] D. Vogt, *Fréchetraume, zwischen denen jede stetige lineare Abbildung beschränkt ist*, J. Reine Angew. Math. 345 (1983), 182–200.
[15] M. J. Wagner, *Unterräume und Quotienten von Potenzreihenräumen*, Dissertation, Wuppertal 1977.
[16] V. P. Zakharyuta, *Isomorphism and quasiequivalence of bases for Köthe power series spaces*, in: Mathematical Programming and Related Problems (Proc. 7th Winter School, Drogobych 1974), Theory of Operators in Linear Spaces, Akad. Nauk SSSR, Tsentr. Ekon.-Mat. Inst., Moscow 1976, 101–126 (in Russian); see also Dokl. Akad. Nauk SSSR 221 (1975), 772–774 (in Russian).

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Weighted weak type inequalities for certain maximal functions

by

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Abstract. We give an A_p type characterization for the pairs of weights (w,v) for which the maximal operator $Mf(y) = \sup \frac{1}{b-a}\int_a^b |f(x)|\,dx$, where the supremum is taken over all intervals $[a,b]$ such that $0 \le a \le y \le b/\psi(b-a)$, is of weak type (p,p) with weights (w,v). Here ψ is a nonincreasing function such that $\psi(0)=1$ and $\psi(\infty)=0$.

The Poisson integral for the Hermite expansion of a function f is given by

$$P_r f(y) = \int_{\mathbb{R}} P(r,y,z) f(z) e^{-z^2}\,dz \tag{1}$$

where

$$P(r,y,z) = \frac{1}{\sqrt{\pi(1-r^2)}} e^{-(r^2y^2-2ryz+r^2z^2)/(1-r^2)}.$$

C. Calderón [C] and B. Muckenhoupt [M1] proved that the maximal operator

$$P^* f(y) = \sup_{r\in(0,1)} |P_r f(y)|$$

is bounded in $L^p(e^{-x^2}dx)$ $(1 < p \le \infty)$ and of weak type $(1,1)$ with respect to the Gaussian measure $e^{-x^2}dx$. We can write (1) in the form

$$P_r f(y) = \int_{\mathbb{R}} K(r,y,z)\, f(z)\,dz$$

where

$$K(r,y,z) = \frac{1}{\sqrt{\pi\,(1-r^2)}} e^{-((ry-z)/\sqrt{1-r^2})^2}.$$

If we take $\varepsilon = \sqrt{1-r^2}$ and $\chi_{(-1,1)}$ instead of e^{-t^2}, we are led to the

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maximal operator

$$\sup_{\varepsilon>0}\frac{1}{\varepsilon}\int\limits_{\psi(\varepsilon)y-\varepsilon}^{\psi(\varepsilon)y+\varepsilon}|f(x)|\,dx\,, \tag{2}$$

with $\psi(\varepsilon)=\sqrt{1-\varepsilon^2}$.

In this note we consider weighted weak type inequalities for a maximal operator which is both larger than (2) and than the Hardy–Littlewood maximal operator.

Let $\psi:\mathbf{R}^+\cup\{0\}\longrightarrow[0,1]$ be a nonincreasing function such that $\psi(0)=1$ and $\lim_{t\to\infty}\psi(t)=0$. Given a locally integrable function f on $\mathbf{R}^+$ and $y\in\mathbf{R}^+$, define

$$Mf(y)=\sup\frac{1}{b-a}\int\limits_a^b|f(x)|\,dx\,, \tag{3}$$

where the supremum is taken over all intervals $[a,b]$ such that $0\le a\le y\le b/\psi(b-a)$; and

$$\widetilde{M}f(y)=\sup\frac{1}{b-a}\int\limits_a^{b/\psi(b-a)}|f(x)|\,dx \tag{4}$$

where the supremum is taken over all intervals $[a,b]$ such that $0\le a\le y\le b$.

Definition. We shall say that the pair (w,v) of nonnegative locally integrable functions on $\mathbf{R}^+$ satisfies *condition A'_p*, $1<p<\infty$, if there exists a positive constant C such that

$$\left(\frac{1}{b-a}\int\limits_a^{b/\psi(b-a)}w(x)\,dx\right)\left(\frac{1}{b-a}\int\limits_a^b v(x)^{-1/(p-1)}\,dx\right)^{p-1}\le C\,. \tag{5}$$

We shall say that (w,v) satisfies *condition A'_1* if there exists a positive constant C such that

$$\widetilde{M}w(y)\le Cv(y)\,.$$

Theorem.

(6) $(w,v)\in A'_p$, $1\le p<\infty$, *if and only if*

$$w\{Mf(y)>\lambda\}\le\frac{C}{\lambda^p}\int\limits_{\mathbf{R}}|f(x)|^p v(x)\,dx\,.$$

(7) *There are no (nontrivial) weights w for which $(w,w)\in A'_p$, $1\le p<\infty$.*

Remark. If $1\le p<q$, then $A'_p\subset A'_q$. If, for example, ψ has a finite right derivative at 0 and $w\in L^2(\mathbf{R}^+)$, we have $\widetilde{M}w\in L^1_{\mathrm{loc}}(\mathbf{R}^+)$, so that $(w,\widetilde{M}w)$ belongs to A'_p for $1\le p<\infty$.

The next extension of Riesz' Lemma is the main tool in the proof of (6).

Lemma. *Let f be a nonnegative function with compact support, λ a positive real number and $\Omega=\{Mf(y)>\lambda\}$. Then there exists a sequence of disjoint intervals $\{(a_k,b_k):k\in\mathbf{N}\}$ such that*

$$\Omega\subset\bigcup_{k\in\mathbf{N}}(a_k,B_k),\quad \textit{where}\quad B_k=\frac{b_k}{\psi(b_k-a_k)}\,, \tag{8}$$

$$\frac{\lambda}{2}\le\frac{1}{b_k-a_k}\int\limits_{a_k}^{b_k}f(x)\,dx\le 2\lambda\,. \tag{9}$$

Proof. Let

$$F(x)=\int\limits_{-\infty}^{x}f(t)\,dt-\lambda x\,,$$
$$O_1=\{x:F(z)>F(x)\text{ for some }z>x\}\,,$$
$$O_2=\{x:F(z)<F(x)\text{ for some }z<x\}\,,$$
$$O_3=O_1\cup O_2\,.$$

The sets O_1,O_2 and O_3 are open and bounded, so that, for $j=1,2,3$, we have $O_j=\bigcup_{k\in\mathbf{N}}(\alpha^j_k,\beta^j_k)$, with $(\alpha^j_k,\beta^j_k)\cap(\alpha^j_h,\beta^j_h)=\emptyset$ for $h\ne k$. Observe that, for $j=1,2$, we have

$$\frac{1}{\beta^j_k-\alpha^j_k}\int\limits_{\alpha^j_k}^{\beta^j_k}f(t)\,dt=\lambda,\qquad k\in\mathbf{N}\,. \tag{10}$$

We now take $a_k=\alpha^3_k$, $b_k=\beta^3_k$. Notice that if $z\in(a_k,b_k)$ and $x>z$ is such that $F(x)>F(z)$, then $x\in(a_k,b_k)$. Given a point $y\in\Omega$, there exist a and b such that $a<y<b/\psi(b-a)$ and $F(b)>F(a)$. Consequently, there is a $k\in\mathbf{N}$ for which a and b belong to (a_k,b_k). Therefore

$$a_k<a<y<\frac{b}{\psi(b-a)}<\frac{b_k}{\psi(b_k-a_k)},$$

since $b-a<b_k-a_k$ and ψ is nonincreasing. This proves (8).

Let us now prove (9). For each $k\in\mathbf{N}$ there exist two sets of integers I^j_k, $j=1,2$, such that

$$(a_k,b_k)=\bigcup_{i\in I^1_k}(\alpha^1_i,\beta^1_i)\cup\bigcup_{i\in I^2_k}(\alpha^2_i,\beta^2_i)\,. \tag{11}$$

In fact, the sets

$$I_k^j = \{i \in \mathbf{N} : (\alpha_i^j, \beta_i^j) \cap (a_k, b_k) \neq \emptyset\}, \quad j = 1, 2,$$

satisfy (11), since $(\alpha_i^j, \beta_i^j) \cap (a_k, b_k) \neq \emptyset$ implies $(\alpha_i^j, \beta_i^j) \subset (a_k, b_k)$. Inequalities (9) now follow from (10) and (11):

$$\frac{\lambda}{2}(b_k - a_k) \le \frac{\lambda}{2} \sum_{j=1}^{2} \sum_{i \in I_k^j} (\beta_i^j - \alpha_i^j) = \frac{1}{2} \sum_{j=1}^{2} \sum_{i \in I_k^j} \int_{\beta_i^j}^{\alpha_i^j} f(t)\, dt \le \int_{a_k}^{b_k} f(t)\, dt$$

$$\le \sum_{j=1}^{2} \sum_{i \in I_k^j} \int_{\beta_i^j}^{\alpha_i^j} f(t)\, dt = \lambda \sum_{i \in I_k^j} (\beta_i^j - \alpha_i^j) \le 2\lambda(b_k - a_k).$$

Proof of the Theorem. Let us show first that A_1' suffices for the weak type (1,1) of M. Let f, λ and ω be as in the Lemma. From (8), (9) and condition A_1', we have

$$w(\Omega) \le \sum_{k=1}^{\infty} \int_{a_k}^{B_k} w(x)\, dx \le \frac{2}{\lambda} \sum_{k=1}^{\infty} \frac{1}{b_k - a_k} \int_{a_k}^{B_k} w(x)\, dx \int_{a_k}^{b_k} f(y)\, dy$$

$$\le \frac{2}{\lambda} \sum_{k=1}^{\infty} \int_{a_k}^{b_k} f(y) \widetilde{M} w(y)\, dy \le \frac{2}{\lambda} \int_{\mathbf{R}} f(y) \widetilde{M} w(y)\, dy$$

$$\le \frac{C}{\lambda} \int_{\mathbf{R}} f(y) v(y)\, dy.$$

In order to prove the weak type (p,p) when the pair (w,v) belongs to A_p', first observe that from (9), Hölder's inequalities and A_p' we have

$$\frac{\lambda}{2} \le \frac{1}{b_k - a_k} \int_{a_k}^{b_k} f(x)\, dx$$

$$\le \left(\frac{1}{b_k - a_k} \int_{a_k}^{b_k} f(x)^p v(x)\, dx \right)^{1/p} \left(\frac{1}{b_k - a_k} \int_{a_k}^{b_k} v(x)^{-1/(p-1)}\, dx \right)^{(p-1)/p}$$

$$\le C \left(\frac{1}{w(a_k, B_k)} \int_{a_k}^{b_k} f(x)^p v(x)\, dx \right)^{1/p},$$

so that

$$w\left(a_k, \frac{b_k}{\psi(b_k - a_k)}\right) \le \frac{C}{\lambda^p} \int_{a_k}^{b_k} f(x)^p v(x)\, dx.$$

From (8) we finally get

$$w(\Omega) \le \sum_{k \in \mathbf{N}} w\left(a_k, \frac{b_k}{\psi(b_k - a_k)}\right) \le \frac{C}{\lambda^p} \int_{\mathbf{R}} f(x)^p v(x)\, dx.$$

Let us now show that A_1' is a necessary condition for the weak type (1,1) of M. If M is of weak type (1,1) for the pair of weights (w,v), then the same is true for the Hardy–Littlewood maximal operator, hence (w,v) satisfies the usual A_1 [M2]. Consequently, it is enough to prove that

$$\text{(12)} \qquad \frac{1}{b-a} \int_{b}^{b/\psi(b-a)} w(t)\, dt \le C v(y)$$

for every a, b and y such that $y \in (a,b)$. Take $f = \chi_{(y,y+h)}$, $y > 0$, $h > 0$, in the weak type inequality in (6) with $p = 1$. Since the set $\{Mf(y) > \lambda\}$ contains the interval (y,z) with $z = (h/\lambda + y)/\psi(h/\lambda)$, it follows that

$$\int_{y}^{z} w(x)\, dx \le \frac{C}{\lambda} \int_{y}^{y+h} v(x)\, dx,$$

for every $\lambda > 0$. Let (a,b) be a given interval, $y \in (a,b)$ and $h > 0$. Taking $\lambda = h/(b-a)$ in the preceding inequality we have

$$\frac{1}{b-a} \int_{b}^{b/\psi(b-a)} w(x)\, dx \le \int_{y}^{z} w(x)\, dx \le \frac{C}{\lambda} \int_{y}^{y+h} v(x)\, dx,$$

from which (12) follows by differentiation.

In order to prove that the weak type inequality implies A_p' for $1 < p < \infty$, observe that for $f = \chi_{(a,b)} v^{-1/(p-1)}$ and $\lambda = [2(b-a)]^{-1} \int f(x) dx$ we have

$$(a, b/\psi(b-a)) \subset \{Mf > \lambda\}.$$

Thus, the weighted weak type (p,p) implies

$$\int_{a}^{b/\psi(b-a)} w(x)\, dx$$

$$\le C \left(\frac{1}{2(b-a)} \int_{a}^{b} v(x)^{-1/(p-1)}\, dx \right)^{-p} \left(\int_{a}^{b} v(x)^{-p/(p-1)} v(x)\, dx \right),$$

which is equivalent to A_p'.

Let us finally prove (7). Since $A_1' \subset A_p'$ $(1 < p < \infty)$, it is enough to prove (7) for $1 < p < \infty$. Let $(w,w) \in A'_p$, $w \not\equiv 0$. We have

$$\left(\int_0^x w(t)^{-1/(p-1)}\,dt\right)^{p-1} \le Cx^p\left(\int_0^{x/\psi(x)} w(t)\,dt\right)^{-1} \le Cx^p\left(\int_0^x w(t)\,dt\right)^{-1}.$$

Since w satisfies Muckenhoupt's A_p condition [M2], it also satisfies a reverse Hölder inequality, i.e., there exist $\varepsilon > 0$ and a constant B such that

$$\left(\frac{1}{x}\int_0^x w(t)^{1+\varepsilon}\,dt\right)^{1/(1+\varepsilon)} \le \frac{B}{x}\int_0^x w(t)\,dt\,.$$

Thus, for $x \ge 1$ we have

$$\left(\int_0^x w(t)^{-1/(p-1)}\,dt\right)^{p-1} \le \frac{CB}{\left(\int_0^1 w(t)^{1+\varepsilon}\,dt\right)^{1/(1+\varepsilon)}} x^{p-1+1/(1+\varepsilon)}. \tag{13}$$

Now, from A'_p, Hölder's inequality and (13) it follows that

$$\begin{aligned}\left(\int_0^x w(t)^{-1/(p-1)}\,dt\right)^{p-1} &\le Cx^p\left(\int_0^{x/\psi(x)} w(t)\,dt\right)^{-1}\\ &\le C\psi(x)^p\left(\int_0^{x/\psi(x)} w(t)^{-1/(p-1)}\,dt\right)^{p-1}.\\ &\le C\psi(x)^{\varepsilon/1+\varepsilon}\frac{CB}{\left(\int_0^1 w(t)^{1+\varepsilon}\,dt\right)^{1/(1+\varepsilon)}} x^{p-1+1/(1+\varepsilon)},\end{aligned}$$

so by iteration we get

$$\left(\int_0^x w(t)^{-1/(p-1)}\,dt\right)^{p-1} \le A\left(C\psi(x)^{\varepsilon/(1+\varepsilon)}\right)^n x^{p-1+1/(1+\varepsilon)},$$

for some positive constant A, $x \ge 1$ and every $n \in \mathbb{N}$. Since $\lim_{t\to\infty}\psi(t) = 0$, we see that $\int_0^x w^{-1/(p-1)}\,dt = 0$ for x large enough. This finishes the proof of the Theorem.

References

[C] C. P. Calderón, *Some remarks on the multiple Weierstrass transform and Abel summability of multiple Fourier–Hermite Series*, Studia Math. 32 (1969), 119–148.

[M1] B. Muckenhoupt, *Poisson integrals for Hermite and Laguerre expansions*, Trans. Amer. Math. Soc. 139 (1969), 231–242.

[M2] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, ibid. 165 (1972), 207–226.

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