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# Maximal functions related to subelliptic operators invariant under an action of a solvable Lie group

by

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**Abstract.** On the domain $S_a = \{(x, e^b) : x \in N, b \in \mathbf{R}, b > a\}$ where $N$ is a simply connected nilpotent Lie group, a certain $N$-left-invariant, second order, degenerate elliptic operator $L$ is considered. $N \times \{e^a\}$ is the Poisson boundary for $L$-harmonic functions $F$, i.e. $F$ is the Poisson integral

$$F(xe^b) = \int_N f(xy)\, d\mu_a^b(x),$$

for an $f$ in $L^\infty(N)$. The main theorem of the paper asserts that the maximal function

$$M^a f(x) = \sup\left\{\left|\int f(xy)\, d\mu_a^b(y)\right| : b > a\right\}$$

is of weak type (1,1).

**0. Introduction.** Let $N$ be a nilpotent Lie group on which the multiplicative group

$$A = \{e^r : r \in \mathbf{R}\}$$

acts as automorphic dilations $\{\delta_r\}_{r\in\mathbf{R}}$ (cf. Section 1 for the definition). We form the split extension

$$S = NA = \{xe^r : x \in N, r \in \mathbf{R}\}.$$

The fundamental example of $NA$ is the $NA$ part of the Iwasawa decomposition $G = NAK$ of a rank one semisimple Lie group $G$ with finite centre. Then $NA$ is identified with the symmetric space $G/K$. There is a very well developed theory of harmonic functions on $NA = G/K$, i.e. of functions $F$ such that $LF = 0$, where $L$ is a $G$-invariant elliptic operator on $G/K$. Harmonic functions with respect to various elliptic and degenerate elliptic operators on $S$ as defined above and the corresponding Poisson integrals

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have been studied in [D] and [DH] (also in the case when $A$ is multidimensional).

The space of our interest here is

$$S_a = \{xe^r : x \in N, r > a\}$$

with the topological boundary

$$\partial S_a = \{xe^a : x \in N\}.$$

Although the operator $L$ which we consider here is left $S$-invariant but, of course, $S_a$, $a > -\infty$, is not. So some of the methods used in [D] and [DH] are not applicable here. However, one of the crucial facts for our analysis is that the space $S_a$ is foliated by the action of $N$ on the left and $\partial S_a$ can be identified with $N$.

Let $X_1, \ldots, X_n$ be a basis of the Lie algebra $\mathfrak{n}$ of $N$ and suppose that the elements $X_1, \ldots, X_k$ generate $\mathfrak{n}$ as a Lie algebra.

On $S_a$ we consider a degenerate elliptic operator

$$L = \sum_{i,j \le k} \alpha_{ij}(a) X_i X_j + \sum_{j \le n} \alpha_j(a) X_j + \partial_a^2 - \kappa \partial_a, \tag{0.1}$$

where the matrix $[\alpha_{ij}(a)]$ is strictly positive definite for every $a \in \mathbf{R}$. In the case when

(0.2) $X_1, \ldots, X_n$ *are homogeneous with respect to the dilations, i.e.*

$$\delta_{e^r} X_j = e^{d_j r} X_j, \quad d_j > 0,$$

*and*

$$\alpha_{ij}(a) = \alpha_{ij} e^{(d_i + d_j)a},$$

$[\alpha_{ij}]$ *being strictly positive definite,*

$L$ is a left-invariant operator on the whole group $S$, and, in fact, every left-invariant degenerate elliptic second order operator on $S$ is of this form (cf. Section 1).

It has been shown in [D] that then $\kappa > 0$ is a necessary and sufficient condition for the existence of nonconstant bounded harmonic functions on $S$.

The aim of the present paper is to study bounded $L$-harmonic functions ($\kappa \ge 0$) on $S_a$, and the existence of the harmonic measures $\mu_a^b$ on $\partial S_a$ identified with $N$. We prove that every bounded harmonic function $F$ on $S_a$ is the Poisson integral of an $L^\infty$ function $f$ on $N$, i.e.

$$F(xe^b) = \int_N f(xy)\, d\mu_a^b(x), \tag{0.3}$$

where $f \in L^\infty(N)$.



Let

$$M'' f(x) = \sup\Big\{ \Big| \int f(xy)\, d\mu_a^b(y) \Big| : b > a + 1 \Big\},$$

$$M' f(x) = \sup\Big\{ \Big| \int f(xy)\, d\mu_a^b(y) \Big| : a < b < a + 1 \Big\}.$$

We are going to prove that if $\kappa > 0$, then $M''$ is of weak type $(1,1)$ and if

(0.4) *the $X_j$'s for which $\alpha_j \ne 0$ are expressible as linear combinations of $X_1, \ldots, X_k$ and $[X_i, X_j]$, $i, j \le k$,*

then also $M'$ is of weak type $(1,1)$.

*It can be verified that for the parabolic operator*

$$\sum_{i,j \le k} \alpha_{ij}(a) X_i X_j + \sum_{j \le n} \alpha_j(a) X_j - \partial_a,$$

*without condition* (0.4), *$M'$ may be unbounded on $L^2$* (cf. [Z]).

Thus we arrive at the main result of this paper.

(0.5) MAIN THEOREM. *Suppose $L$ is as in* (0.1), $\kappa > 0$, (0.2) *and* (0.4) *are satisfied. Let $\mu_a^b$ be the harmonic measures on $\partial S_a$. Then if*

$$M^a f(x) = \sup\Big\{ \Big| \int f(xy)\, d\mu_a^b(y) \Big| : b > a \Big\},$$

*then $M$ is of weak type* $(1,1)$.

The proof requires a number of methods. The Lie group techniques together with homogeneity of $N$ are heavily used together with a number of classical methods in proving the maximum principles. These often require less stringent assumptions on $L$, the full strength of the assumptions imposed being only used in Section 4. Some of the crucial estimates for the harmonic measures $\mu_a^b$ are obtained using probabilistic methods, especially the decomposition of the diffusion generated by $L$ into the "vertical component" $a(t)$ generated by $\partial_t^2 - \kappa \partial_t$ and the "horizontal component" for which the transition probabilities conditioned on a trajectory $a(t)$ of the vertical component satisfy the evolution equation

$$\partial_t u(t,x) = \Big( \sum_{i,j \le k} \alpha_{ij}(a(t)) X_i X_j + \sum_{j \le n} \alpha_j(a(t)) X_j \Big) u(t,x) \tag{0.6}$$

(cf. e.g. [T]).

One should perhaps mention that without any group invariance *boundedness on $L^2$ of the maximal functions related to Poisson integrals on $C^\infty$ boundaries of unbounded domains for even most regular elliptic operators seems to be an open question* [K].

The work on this paper lasted on and off for a number of years. During this time we have benefited a lot from conversations with many mathematicians. We would like to express our deep gratitude to those in particular

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**1. Preliminaries.** We start with some general facts concerning Lie groups. Let $G$ be a connected Lie group with a right-invariant Haar measure $dx$. A nonnegative Borel function $\Psi$ on $G$ is called *subadditive* if it is bounded on compact sets and

$$\Psi(xy) \le \Psi(x) + \Psi(y) \quad \text{for } x, y \in G, \tag{1.1}$$

$$\Psi(x^{-1}) = \Psi(x), \quad x \in G. \tag{1.2}$$

If instead of (1.1) we have

$$\Psi(xy) \le \Psi(x)\Psi(y) \quad \text{for } x, y \in G \tag{1.3}$$

and also $\Psi(x) \ge 1$ we say that $\Psi$ is *submultiplicative*. If $\Psi$ is subadditive, then $1 + \Psi$ is submultiplicative.

Let $\| \ \|$ be a euclidean norm in the Lie algebra $\mathfrak{g}$ of $G$ and $\tau_G$ the corresponding left-invariant distance (from the identity), i.e.

$$\tau_G(x) = \inf \int_0^1 \|\dot{\gamma}(t)\|\, dt,$$

where the infimum is over all $C^1$ curves $\gamma$ in $G$ such that $\gamma(0) = e$, $\gamma(1) = x$. Then $\tau_G$ is subadditive and for every nonnegative function $\Psi$ on $G$ which is bounded on compact sets and satisfies (1.1) there is a constant $C$ such that (see [H1])

$$\Psi(x) \le C(\tau_G(x) + 1), \quad x \in G. \tag{1.4}$$

Let $B_r(x)$ denote the ball of radius $r$ and centre $x$, i.e.

$$B_r(x) = \{y : \tau_G(x^{-1}y) < r\}.$$

We often write $B_r$ or $B(r)$ for $B_r(e)$. Let

$$\tau_m = \min(\tau_G, m), \quad m = 1, 2, \ldots, \infty.$$

Clearly $\tau_m$ is subadditive. Let $\varphi \in C_c^\infty(B_r)$ be a nonnegative function such that $\int \varphi\, dx = 1$. Then for any left-invariant vector fields $X$ and $Y$ and all $m$ we have [H1]

$$\tau_m(x) - r \le \tau_m * \varphi(x) \le \tau_m(x) + r, \tag{1.5}$$

$$|X(\tau_m * \varphi)(x)| \le \int |\varphi(y)| \|\operatorname{Ad}_y X\|\, dy, \tag{1.6}$$



$$|XY(\tau_m * \varphi)(x)| \le \int |Y\varphi(y)| \|\operatorname{Ad}_y X\|\, dy \tag{1.7}$$

for all $x \in G$, where $\tau_m * \varphi(x) = \int \tau_m(xy^{-1})\varphi(y)\, dy$, $x \in G$. Moreover, $1 + \tau_m * \varphi$ is submultiplicative and for every $m$ and all $x, y \in G$

$$1 + \tau_m * \varphi(xy) \le (1 + 2r)(1 + \tau_m * \varphi(x))(1 + \tau_m * \varphi(y)). \tag{1.8}$$

Let $N$ be a simply connected nilpotent Lie group and let $\mathfrak{n}$ be its Lie algebra. $N$ is called *homogeneous* [FS] if there is a basis $X_1, \ldots, X_n$ of the Lie algebra $\mathfrak{n}$ of $N$ and numbers $1 = d_1 \le d_2 \le \ldots \le d_n$ such that for $a \in \mathbf{R}$ the mapping

$$X_j \to e^{d_j a} X_j, \quad j = 1, \ldots, n,$$

extends to an automorphism $\delta_a$ of $\mathfrak{n}$. For $x = \exp X$ in $N$ we write

$$\delta_a x = \exp[\delta_a X].$$

Of course $\delta_a$ is an automorphism of $N$. It is called a *dilation*. Moreover,

$$Q = \sum_{j=1}^n d_j \tag{1.9}$$

is the *homogeneous dimension* of $N$. A *homogeneous norm* on $N$ is a function

$$N \ni x \to |x| \in \mathbf{R}^+$$

which is $C^\infty$ outside $x = e$, satisfies $|\delta_a x| = e^a|x|$, and $|x| = 0$ if and only if $x = e$. We have $|xy| \le \beta(|x| + |y|)$ for some $\beta \ge 1$.

*There is always a subadditive homogeneous norm, i.e. one with* $\beta = 1$ [HS]. Let $r > 0$ be such that in appropriate coordinates $\tau_G(x) = (\sum x_i^2)^{1/2}$ whenever $x \in B_r(e)$. There is a constant $C$ (cf. e.g. [FS]) such that

$$\tau_G(x) \le C \max(|x|^{d_n}, |x|^{d_1}), \quad x \in G, \tag{1.10}$$

$$|x| \le C\tau_G(x), \quad x \notin B_r(e), \tag{1.11}$$

$$|x|^{d_n} \le C\tau_G(x), \quad x \in B_r(e). \tag{1.12}$$

In this paper we study the solvable group $S = NA$ which is the semidirect product of $N$ and the group of dilations $A = \mathbf{R}^+$ with $e^a x e^{-a} = \delta_a x$, $a \in \mathfrak{a}$, $x \in N$, $\mathfrak{a}$ being the Lie algebra of $A$. Let

$$E_0 \in \mathfrak{a} \tag{1.13}$$

be the infinitesimal generator of the one-parameter subgroup $e^t$. Then $[E_0, X_j] = d_j X_j$.

We have the following simple

(1.14) Lemma. *Let $c_1 < \ldots < c_p$ be such that $\{c_1, \ldots, c_p\} = \{d_1, \ldots, d_n\}$ and $\mathfrak{n} = \bigoplus_{j=1}^p V_j$ with $V_j = \{X : [E_0, X] = c_j X\}$. Then for every $X \in \mathfrak{n}$*

*there is a decomposition* $\mathfrak{n} = \bigoplus_{j=1}^p V_j'$ *such that*

$$\bigoplus_{j=m}^p V_j = \bigoplus_{j=m}^p V_j' \quad \text{and} \quad [E_0 + X, X_j'] = c_j X_j'$$

*for* $X_j' \in V_j'$ *and* $m = 1, \ldots, p$.

Proof. Since $[X, V_p] = 0$ we put $V_p' = V_p$ and proceed by induction. Suppose $V_{m+1}', \ldots, V_p'$ are already defined. Let $X_m \in V_m$ and

$$[X, X_m] = \sum_{j=m+1}^p X_j' .$$

We write

$$X_j' = X_m + \sum_{j=m+1}^p (c_m - c_j)^{-1} X_j' .$$

Then

$$[E_0 + X, X_m'] = c_m X_m + \sum_{j=m+1}^p (X_j' + c_j(c_m - c_j)^{-1} X_j')$$

$$= c_m\Big(X_m + \sum_{j=m+1}^p (c_m - c_j)^{-1} X_j\Big) = c_m X_m' . \quad \blacksquare$$

By the last lemma for any linear complement $\mathfrak{a}'$ of $\mathfrak{n}$ in the Lie algebra $\mathfrak{s}$ of $S$ there is a decomposition $n = \bigoplus_{j=1}^p V_j$ such that

$$[a(E_0 + X_0), X_j] = a c_j X_j$$

if $a(E_0 + X_0) \in \mathfrak{a}'$, $X_j \in V_j$. Therefore $S$ is a semidirect product

$$S = NA' \tag{1.15}$$

of $N$ and $A' = \exp \mathfrak{a}'$ with

$$\exp a(E_0 + X_0) \exp\Big(\sum_{j=1}^p X_j\Big) \exp(-a(E_0 + X_0)) = \exp\Big(\sum_{j=1}^p e^{c_j a} X_j\Big) .$$

A decomposition of type (1.15) of $S$ will be called *admissible*.

**2. Maximum principles.** In this section we study general left-invariant degenerate elliptic operators $L$ on $S$. We are going to prove a maximum principle for them on the domains of the form

$$S_a = \{xe^b : x \in N,\ b > a\}$$

where $S = NA$ is a given admissible decomposition of $S$. Obviously, by (1.15), $S_a$ does not depend on the decomposition . The topological boundary $\{xe^a : x \in N\}$ of $S_a$ is denoted by $N_a$.



First, we rewrite $L$ in a more convenient form. To do this we distinguish between left-invariant vector fields on $S$ and on $N$ corresponding to the same element of $\mathfrak{n}$. If $X$ is a left-invariant field on $N$, let $X'$ denote the left-invariant vector field on $S$ such that $X_e = X_e'$. Moreover, $E_0'$ is the left-invariant vector field on $S$ corresponding in the same way to $E_0$ defined in (1.13). If $X \in V_j$ then

$$X' f(xa) = e^{d_j a} X f_a(x) , \quad \text{where } f_a(x) = f(xa) .$$

(2.1) Proposition. *Let $L$ be a left-invariant elliptic degenerate operator on $S$. There is an admissible decomposition of $S$ and bases $\partial_a$ of $\mathfrak{a}$ and $X_1, \ldots, X_n$ of $\mathfrak{n}$ such that*

$$[\partial_a, X_i] = d_i X_i , \quad i = 1, \ldots, n ,$$

*and*

$$L\varphi(xe^a) \tag{2.2}$$

$$= \Big(\alpha \partial_a^2 - \kappa \partial_a + \sum_{i,j=1}^n \alpha_{ij} e^{(d_i + d_j)a} X_i X_j + \sum_{j=1}^n \alpha_j e^{d_j a} X_j\Big) \varphi(xe^a)$$

*for $\varphi \in C_c^\infty(S)$, $x \in N$, $a \in \mathfrak{a}$. Moreover, $\alpha \geq 0$ and $[\alpha_{ij}]$ is positive semidefinite.*

Proof. Using simple algebra we find $X_0$, $X$ in $\mathfrak{n}$ such that $L$ can be written in the form

$$L = \alpha(E_0' + X_0')^2 + \kappa E_0' + \sum_{i,j=1}^n \beta_{ij} E_i' E_j' + X'$$

where $\alpha \geq 0$ and $[\beta_{ij}]$ is positive semidefinite. Now by Lemma (1.14) we choose a basis $X_1, \ldots, X_n$ of $\mathfrak{n}$ such that $[E_0 + X_0, X_i] = d_i X_i$. Then

$$L = \alpha(E_0' + X_0')^2 + \kappa(E_0' + X_0') + \sum_{i,j=1}^n \alpha_{ij} X_i' X_j' + \sum_{j=1}^n \alpha_j X_j'$$

where $[\alpha_{ij}]$ is positive semidefinite, and in the coordinates $x \cdot \exp t(E_0 + X_0)$ we obtain (2.2). $\blacksquare$

Let

$$L_0 = \sum_{i,j=1}^n \alpha_{ij} e^{(d_i + d_j)a} X_i X_j , \quad L_1 = \sum_{j=1}^n \alpha_j e^{d_j a} X_j .$$

We consider

$$L = \alpha \partial_a^2 - \kappa \partial_a + L_0 + L_1 \tag{2.3}$$

where $\alpha, \kappa \geq 0$. Define

$$D(a_0, a_1, R) = \{xe^a : x \in B(R),\ a_0 < a < a_1\} .$$

Let $\tau_N * \varphi$ be as in (1.5) and

$$C = \sum_{i,j=n}^{n} |\alpha_{ij}| \|X_i X_j(\tau_N * \varphi)\| + \sum_{j=1}^{n} |\alpha_j| \|X_j(\tau_N * \varphi)\| + \sum_{i,j=1}^{n} |\alpha_{ij}| \|X_i(\tau_N * \varphi)\| \|X_j(\tau_N * \varphi)\| \tag{2.4}$$

where $\|f\| = \sup_{n\in N} |f(x)|$. By (1.6) and (1.7), $C < \infty$.

(2.5) THEOREM. *Assume $\kappa > 0$. Let $a_0 < a_1$, $0 < \varepsilon < 1$ and let $\sigma, \gamma, R$ be constants satisfying*

$$0 < \sigma \le d_1\,, \quad \kappa - \alpha\sigma > 0\,, \quad 1 \le \gamma\,, \quad \gamma\alpha\sigma < \kappa\,,$$
$$R > \max\{C\gamma \max(e^{2d_n a_1}, e^{d_1 a_1})/\sigma(\kappa - \alpha\sigma\gamma), 2\}\,.$$

*Suppose that $F$ is a twice continuously differentiable function in*

$$D(a_0, a_1 + \sigma^{-1}\log(2\varepsilon^{-1}), R\varepsilon^{-2d_n/\sigma}) = D$$

*and $LF \ge 0$, $F$ is continuous in $\overline{D}$ and $|F| \le 1$. If $F(xe^{a_0}) \le 0$ for $x \in B(R\varepsilon^{-2d_n/\sigma})$, then $F(e^{a_1}) \le \varepsilon^\gamma$.*

Proof. Let

$$G_0(xe^a) = \tfrac{1}{2}\varepsilon e^{\sigma(a-a_1)} + (\varepsilon/2)^{2d_n/\sigma} R^{-1}(\tau_N * \Phi(x) + 1)$$

and let $G = -G_0^\gamma$. First we are going to prove that $LG(xe^a) > 0$ whenever $x \in N$ and $a_0 < a < a_1 + \sigma^{-1}\log(2\varepsilon^{-1})$. Let

$$I_1 = (L_0 + L_1)G_0\,,$$
$$I_2 = (\gamma - 1)G_0^{-1} \sum_{i,j=1}^{n} \alpha_{ij} e^{(d_i+d_j)a}(X_i G_0)(X_j G_0)\,,$$
$$I_3 = \alpha(\gamma-1)G_0^{-1}(\partial_a G_0)^2\,, \qquad I_4 = (\kappa\partial_a - \alpha\partial_a^2)G_0\,.$$

Then $LG = \gamma G_0^{\gamma-1}(I_4 - I_1 - I_2 - I_3)$. For every $d$ between $d_1$ and $2d_n$ we have

$$e^{da} \le \max\{e^{2d_n a_1}, e^{d_1 a_1}\} \begin{cases} e^{2d_n(a-a_1)} & \text{if } a \ge a_1\,, \\ e^{d_1(a-a_1)} & \text{if } a < a_1\,. \end{cases}$$

Therefore

$$|I_1| < (\varepsilon/2)^{2d_n/\sigma}\sigma(\kappa - \sigma\alpha\gamma)\gamma^{-1} \begin{cases} e^{2d_n(a-a_1)} & \text{if } a \ge a_1\,, \\ e^{d_1(a-a_1)} & \text{if } a < a_1\,. \end{cases}$$

Similarly,

$$|I_2| < (\gamma-1)(\varepsilon/2)^{2d_n/\sigma}\sigma(\kappa - \sigma\alpha\gamma)\gamma^{-1} \begin{cases} e^{2d_n(a-a_1)} & \text{if } a \ge a_1\,, \\ e^{d_1(a-a_1)} & \text{if } a < a_1\,. \end{cases}$$



Moreover,

$$|I_3| \le \tfrac{1}{2}(\gamma-1)\alpha\varepsilon\sigma^2 e^{\sigma(a-a_1)}$$

and

$$I_4 = \tfrac{1}{2}\varepsilon\sigma(\kappa - \alpha\sigma)e^{\sigma(a-a_1)}\,.$$

Consequently, if $a \le a_1$, then $I_4 - I_1 - I_2 - I_3$ is positive because $\sigma \le d_1$ and $2d_n/\sigma \ge 1$. If $a > a_1$, we have

$$I_4 - I_1 - I_2 - I_3 > \sigma(\kappa - \sigma\alpha\gamma)e^{2d_n(a-a_1)}\{(\varepsilon/2)e^{(\sigma-2d_n)(a-a_1)} - (\varepsilon/2)^{2d_n/\sigma}\}\,.$$

But for such $a$

$$e^{(\sigma-2d_n)(a-a_1)} \ge (\varepsilon/2)^{-1+2d_n/\sigma}\,,$$

so

$$I_4 - I_1 - I_2 - I_3 > 0\,.$$

Moreover, $G(xe^{a_0}) \le 0$ for $x \in N$ and by (1.5), $G \le -1$ on the remaining part of the boundary $\partial D$ of $D$. Hence $F + G \le 0$ on $\partial D$. The weak maximum principle for degenerate elliptic operators (Proposition 1.1 in [B]) implies $F + G \le 0$ in $D$ and the proof is complete. ■

Now we pass to the case $\kappa = 0$.

(2.6) THEOREM. *Assume $\kappa = 0$ and $C$ is given by (2.4). Let*

$$a_0 < a_1\,, \quad 0 < \varepsilon < 1\,, \quad 0 < \gamma < 1\,,$$
$$R > \max\{1, C(a_1 - a_0)^2 \max\{e^{2d_n a_0}, e^{d_1 a_0}\}/\gamma(1-\gamma)\}$$

*and*

$$D = D(a_0, a_0 + \varepsilon^{-1/\gamma}(a_1 - a_0), R\varepsilon^{-2/\gamma}\exp(2d_n\varepsilon^{-1/\gamma}(a_1 - a_0)))\,.$$

*If $F \in C^2(D) \cap C(\overline{D})$, $|F| \le 1$ in $\overline{D}$, $LF \ge 0$ in $D$ and $F(xe^{a_0}) \le 0$ for $x \in B(R\varepsilon^{-2/\gamma}\exp(2d_n\varepsilon^{-1/\gamma}(a_1 - a_0)))$ then $F(e^{a_1}) \le 3\varepsilon$.*

Proof. We consider the function

$$G(xe^a) = -\varepsilon(a - a_0)^\gamma/(a_1 - a_0)^\gamma - R^{-1}\varepsilon^{2/\gamma}\exp(-2d_n\varepsilon^{-1/\gamma}(a_1 - a_0))(\tau_N * \varphi(x) + 1)$$

and we show that $LG(xe^a) > 0$ for $x \in N$, $a_0 < a < a_0 + \varepsilon^{-1/\gamma}(a_1 - a_0)$. Since for every $0 \le d \le 2d_n$ and $a \ge a_0$

$$e^{da} \le \max\{e^{2d_n a_0}, e^{d_1 a_0}\}e^{2d_n(a-a_0)}$$

we have

$$\begin{aligned} &|(L_0 + L_1)G(xe^a)| \\ &\le \gamma(1-\gamma)(a_1 - a_0)^{-2}\varepsilon^{2/\gamma}\exp(-2d_n\varepsilon^{-1/\gamma}(a_1 - a_0))\exp(2d_n(a - a_0)) \\ &< \gamma(1-\gamma)(a_1 - a_0)^{-2}\varepsilon^{2/\gamma} \\ &< \varepsilon\gamma(1-\gamma)(a - a_0)^{\gamma-2}/(a_1 - a_0)^\gamma = \partial_a^2 G(xe^a)\,. \end{aligned}$$

The rest of the proof is as in Theorem (2.5). ■

(2.7) COROLLARY. *If $F \in C^2(S_a) \cap C(\overline{S}_a)$, $LF \geq 0$ and $F$ is bounded then for every $b > a$, $x \in N$ we have*

$$F(xe^b) \leq \sup_{y \in N} F(ye^a). \quad \blacksquare$$

(2.8) COROLLARY. *For every $a$, $b > a$ and $\varepsilon > 0$ there is*

$$R = \begin{cases} R(\varepsilon, b, a) & \text{if } \kappa = 0, \\ R(\varepsilon, b) & \text{if } \kappa > 0, \end{cases}$$

*and $\delta > 0$ such that if $F \in C^2(S_a) \cap C(\overline{S}_a)$, $LF = 0$, $F$ is bounded and $|F(xe^a)| \leq \delta$ for $x \in B(R)$ then $|F(e^b)| \leq \varepsilon$.* ■

**3. Harmonic measures.** From now on we shall assume that there are $Y_1, \ldots, Y_k$ generating $\mathfrak{n}$ as a Lie algebra such that

$$\sum_{i,j=1}^{n} \alpha_{ij} X_i X_j = Y_1^2 + \ldots + Y_k^2 \tag{3.0}$$

Then obviously we have the same for every $a$, i.e.

$$L = Y_1(a)^2 + \ldots + Y_k(a)^2 + L_1 + \alpha \partial_a^2 - \kappa \partial_a$$

where $Y_1(a), \ldots, Y_k(a)$ generate $\mathfrak{n}$. For such operators Bony's version of Harnack's inequality [B] is available. Our first goal is to show that for every $a$ the Dirichlet problem for $S_a$ has a solution:

(3.1) THEOREM. *For every bounded continuous function $f$ on $N$ there exists a bounded harmonic function $F$ on $S_a$ which is continuous on $\overline{S}_a$ such that $F(xe^a) = f(x)$ for $x \in N$.*

Proof. Since by Corollary 5.2 of [B] the Dirichlet problem can be solved in every set from a basis $\mathcal{R}$ of open sets in $S$ we can apply Perron's method [GT].

(3.2) DEFINITION. Let $U$ be an open set in $S$. An upper semicontinuous function $F : U \to [-\infty, \infty)$ is called *subharmonic* if for every $V$ in $\mathcal{R}$ such that $\overline{V} \subseteq U$ and $s \in V$ we have

$$F(s) \leq \int_{\partial V} F(y)\, d\mu_s^V(y),$$

where $\mu_s^V$ is the harmonic measure on $\partial V$ corresponding to $L$.

The following facts enable us to apply Perron's method.

(3.3) THE MAXIMUM PRINCIPLE. *Let $U$ be an open set in $S$ with compact closure, and $F$ a subharmonic function in $U$, upper semicontinuous in $\overline{U}$.*



*Then*

$$\sup_{s \in U} F(s) \leq \sup_{s \in \partial U} F(s).$$

For harmonic functions (3.3) follows from Theorem 3.2 in [B]. Generalization to subharmonic functions is standard.

(3.4) THE MAXIMUM PRINCIPLE FOR $S_a$. *Let $F : S_a \to [-\infty, \infty)$ be a subharmonic function in $S_a$, upper semicontinuous and bounded on $S_a$. Then*

$$\sup_{s \in S_a} F(s) \leq \sup_{s \in N_a} F(s).$$

(3.4) can be proved in the same way as Theorems (2.5) and (2.6). We notice that $F + G$ is subharmonic (in the sense of the definition above) as the sum of two subharmonic functions. Then we apply (3.3) to $F + G$ in $D$.

(3.5) THE UNIFORM CONVERGENCE PROPERTY. *Every monotonic sequence of harmonic functions which is bounded from above or below is almost uniformly convergent to a harmonic function.*

(3.5) follows from Harnack's inequality [B].

Let $f \in C_b(N_a)$ and let $\mathcal{SH}_a(f)$ be the set of functions $g$ subharmonic in $S_a$, upper semicontinuous and bounded in the closure of $S_a$ and such that $g(xe^a) \leq f(xe^a)$, $x \in N$.

(3.6) LEMMA. *If $f \in C_b(N_a)$ and $F(s) = \sup\{v(s) : v \in \mathcal{SH}_a(f)\}$, $s \in S$, then $LF = 0$ in $S_a$ and*

$$\lim_{s \to xe^a} F(s) = f(xe^a), \quad x \in N. \tag{3.7}$$

Proof. Clearly $F$ satisfies the mean value property, i.e.

$$F(s) = \int_{\partial V} F(y)\, d\mu_s^V(y), \quad s \in S.$$

By (3.4), $\inf f \leq F \leq \sup f$. It follows from (3.4) and (3.5) that an upper semicontinuous function satisfying the mean value property is harmonic.

To prove the second statement we have to construct a barrier function [GT]. Let $L$ be as in (2.3), $\Phi$ the Hunt function on $N$, i.e. $\Phi$, $X_j\Phi$, $X_iX_j\Phi$ are bounded, $\Phi(e) = 0$ and $\Phi(x) > 0$ for $x \neq e$ (cf. e.g. [H2]), $0 < \gamma < 1$ and

$$C = \sum_{i,j=1}^{n} |\alpha_{ij}| \|X_iX_j\Phi\| + \sum_{j=1}^{n} |\alpha_j| \|X_j\Phi\|,$$

where $\|\cdot\|$ is defined after (2.4). Set

$$\begin{aligned} W'(xe^a) = (a - a_0)^\gamma + C^{-1}(\kappa\gamma(a_1 - a_0)^{\gamma-1} \\ + \alpha\gamma(1-\gamma)(a_1 - a_0)^{\gamma-2}) e^{-2d_n a_1} \Phi(x_0^{-1}x) \end{aligned}$$

in the domain $a_0 < a < a_1$ with $a_1 > \max(a_0, 0)$. Then $W' \in C(\overline{S}_{a_0})$, $W'(xe^a) > 0$ if $x \neq x_0$ or $a \neq a_0$, $W'(x_0 e^{a_0}) = 0$ and $LW' \leq 0$ in $\{xe^a : a_0 < a < a_1\}$.

Now the construction of the barrier is easy. We let $a_0 < a_2 < a_1$ and

$$K = \inf\{W'(xe^a) : x \in N,\ a_2 < a < a_1\}.$$

Then

$$W(xe^a) = \begin{cases} \min\{K, W(xe^a)\} & \text{if } a_0 < a < a_1, \\ K & \text{if } a \geq a_1, \end{cases}$$

is a barrier function and the proof of (3.7) is now routine. ∎

Let $\mathcal{H}_a$ be the space of bounded harmonic functions on $S_a$ continuous on $\overline{S}_a$. By the previous theorem and Corollary (2.7) for every $s$ in $S_a$ the mapping

$$m_s(f) = F(s), \quad F \in \mathcal{H}_a,\ F|_{N_a} = f,$$

is a well defined continuous functional on $C_b(N)$ with $\|m_s\| = 1$ and by Corollary (2.8),

$$\sup\{|m_s(f)| : f \in C_c(N),\ \|f\| = 1\} = 1.$$

Hence there exists a probability measure $\mu_a^{x,b}$ on $N$ such that

$$(3.8) \qquad F(xe^b) = \langle f, \mu_a^{x,b} \rangle, \quad x \in N,\ a, b \in \mathbf{R},\ a < b,$$

for $f \in C_b(N)$. Since $L$ commutes with left translations we see that

$$(3.9) \qquad F(xe^b) = \langle f, \mu_a^{x,b} \rangle = \int_N f(xy)\, d\mu_a^b(y) = f \# \mu_a^b(x),$$

where $\mu_a^b = \mu_a^{e,b}$.

Let $f \in C_b(N)$ and $F(xe^b) = f \# \mu_{a+c}^b(x)$. Then $F \in \mathcal{H}_{a+c}$. Put $G(s) = F(e^a s)$. Then $G \in \mathcal{H}_c$ and $G(xe^b) = g \# \mu_c^b$ for some $g \in C_b(N)$. On the other hand,

$$G(xe^b) = F(e^a xe^b) = F(\delta_a(x) e^{a+b}) = f \# \mu_{a+c}^{a+b}(\delta_a(x))$$

and so putting $b = c$ we see that $g = f \circ \delta_a$. This implies

$$(3.10) \qquad \langle f \circ \delta_a, \mu_c^b \rangle = \langle f, \mu_{c+a}^{b+a} \rangle.$$

Let $d\check{\mu}(x) = d\mu(x^{-1})$. Then it follows immediately from (3.9) that

$$(3.11) \qquad \check{\mu}_a^c = \check{\mu}_a^b * \check{\mu}_b^c \quad \text{for } a < b < c.$$

(3.12) PROPOSITION. *For every right-invariant differential operator $\partial$ on $N$, $\partial \mu_a^b \in L^2(N)$ and consequently $\mu_a^b$ is smooth and $\partial \mu_a^b$ is bounded.*

The proof follows from Sobolev's lemma (cf. [D]).



We denote the density of $\mu_a^b$ also by $\mu_a^b$. Proposition (3.12) and (3.10) imply

$$(3.13) \qquad \mu_{b+a}^{c+a}(x) = e^{-aQ} \mu_b^c(\delta_{-a}(x)),$$

where $Q$ is defined in (1.9).

Another consequence of the maximum principle is the existence of fractional moments of the measures $\mu_a^b$ when $\kappa > 0$ and of a logarithmic moment when $\kappa = 0$.

(3.14) PROPOSITION. *Let $L$ be as in (2.3) with $\kappa > 0$. Let $\eta < \kappa/2d_n\alpha$. Then there exists a constant $c = c(\eta)$ independent of $a$ and $b$ such that*

$$\langle \tau_N^\eta, \mu_a^b \rangle \leq c e^{2d_n|b|}.$$

Proof. By Theorem (2.5) for every $\zeta < \kappa/\alpha$ there is a constant $c$ which depends on the group, the operator $L$ and $\zeta$ such that

$$\mu_a^b(B_u(e)^c) \leq c \exp(2d_n|b|) u^{-\zeta/2d_n}.$$

This implies the assertion. ∎

(3.15) PROPOSITION. *Let $L$ be as in (2.3) with $\kappa = 0$. For every $a < b$ and every $0 < \gamma < 1$*

$$(3.16) \qquad \int (\log(1 + \tau_N(x)))^\gamma\, d\mu_a^b(x) < \infty.$$

Proof. Let $a < b$, $0 < \gamma < 1$ and let $R$ be as in Theorem (2.6). Then

$$\mu_a^b(B_u(e)^c) \leq (\log(u/R))^{-\gamma} (2d_n(b-a) + 1)^\gamma$$

for $u$ sufficiently large, which gives (3.16). ∎

Remark. It can be proved that in the case when $\kappa > 0$ the family of measures $\{\mu_a^b\}_{a<b}$ is uniformly tight and $\mu_a^b$ converges weak* to a probability measure $\mu^b$ as $a \to -\infty$. $\mu^b$ is the Poisson kernel for $L$ as described in [D], [DH].

**4. Parabolic operators.** In this section we consider parabolic operators on $G \times \mathbf{R}^+$ where $G$ is an arbitrary Lie group. Let $X_1, \ldots, X_N$ be a fixed basis of the Lie algebra $\mathfrak{g}$ of $G$ and

$$(4.1) \qquad L = L_t - \partial_t, \quad \text{where} \quad L_t = \sum_{i,j=1}^k \alpha_{ij}(t) X_i X_j + \sum_{i=1}^N \alpha_i(t) X_i$$

and $[\alpha_{ij}(t)]$ is positive semidefinite.

We are going to write down some properties of the diffusion associated with $L$. Most of the proofs are standard and for operators with coefficients bounded on $G \times (0, T)$ can be found in [SV]. Our operators do not have this property, but are invariant with respect to $x$ and so using $\tau_G * \varphi$ (see (1.5)) instead of a euclidean norm we can rewrite the proofs from [SV].

Let $C^{d,1}(G\times(0,T))$ denote the set of functions on $G\times(0,T)$ $d$ times continuously differentiable in $x$ and once in $t$. If all right-invariant spatial derivatives of $F$ up to order $d$ are bounded on $G\times(0,T)$ we write $F\in C_{\mathrm{b}}^{d,1}(G\times(0,T))$. Moreover, $C_{\mathrm{b}}^{d}(G)$ is the set of functions on $G$ with continuous and bounded right-invariant derivatives up to order $d$.

(4.2) THEOREM [SV]. *Let $s<T<\infty$ and*

$$M=\sup\{|\alpha_{ij}(t)|,|\alpha_l(t)|:s<t<T,\ i,j=1,\ldots,k,\ l=1,\ldots,N\}.$$

*We fix positive numbers $\varepsilon$, $C$, $r$. There is $R=R(\varepsilon,M,C,r)$ such that if $f\in C^{2,1}(G\times(s,T))$, $f\geq -C$,*

$$Lf\leq 0 \quad \text{in } B_R(x)\times(s,T),$$
$$\liminf f(z,t)\geq 0 \quad \text{as } t\to s,\ z\to y,\ y\in B_R(x),$$

*then*

$$f(y,t)\geq-\varepsilon \quad \text{for } s<t<T \text{ and } y\in B_r(x).\ \blacksquare$$

(4.3) COROLLARY. *Let $\varepsilon,M,C,r,R=R(\varepsilon,M,2C,r)$ be as in the previous theorem. If $f\in C^{2,1}(G\times(s,T))\cap C(G\times[s,T))$, $Lf=0$, $|f|\leq C$ then for every $x\in G$*

$$\min_{y\in B_R(x)}f(y,s)-\varepsilon\leq f(z,t)\leq\max_{y\in B_R(x)}f(y,s)+\varepsilon$$

*where $z\in B_r(x)$ and $s<t<T$.* $\blacksquare$.

For the rest of this section we assume that $L$ can be written in the form

$$L=Y_1(t)^2+\ldots+Y_p(t)^2+Y_0(t)-\partial_t$$

where

$$Y_i(t)=\sum_{i,j=1}^{N}\beta_{ij}(t)X_j$$

with $\beta_{ij}\in C(\mathbb{R}^+)$ and for every $t$, $Y_1(t),\ldots,Y_p(t)$ generate $\mathfrak{g}$ as a Lie algebra. This is true for example if $X_1,\ldots,X_k$ in (4.1) generate $\mathfrak{g}$ and $[\alpha_{ij}(t)]$ is strictly positive definite.

(4.4) THEOREM. *Let $0<s<T$ and $\varphi\in C_{\mathrm{b}}^2(G)$. There is exactly one function $F\in C_{\mathrm{b}}(G\times[s,T))$ such that*

(4.5) $\qquad LF=0 \quad$ *on* $G\times(s,T)$, $\quad F(x,s)=\varphi(x) \quad$ *for* $x\in G$.

*Moreover, if $\varphi\in C_{\mathrm{b}}^d(G)$ then $F\in C_{\mathrm{b}}^{d,1}(G\times(s,T))$.*

Proof. If the $\beta_{ij}$ are smooth this follows in a standard way (described for example in Section 3 of this paper).



Assume now that the $\beta_{ij}$ are continuous and approximate them almost uniformly by smooth functions, i.e. we write $\beta_{ij}^n=\beta_{ij}*\varphi_n$, where $\varphi_n$ is an approximate identity in $L^1(\mathbb{R})$. We then obtain the family of operators

$$L_n=\sum_{i,j=1}^{k}\alpha_{ij}^n(t)X_iX_j+\sum_{j=1}^{N}\alpha_j^n(t)X_i-\partial_t$$

and let $F_n$ be the solution of the corresponding Dirichlet problem for $L_n$, i.e.

$$L_nF_n=0 \quad \text{on } G\times(s,T), \qquad F_n(x,s)=\varphi(x) \quad \text{for } x\in G.$$

We will show that $F_n$ converges almost uniformly to $F$ satisfying (4.5). We have

$$F_n(x,t)=\varphi*\mu_{s,t}^n(x), \qquad s\leq t<T,$$

where the $\mu_{s,t}^n$ are probability measures. Therefore for every $n$ and $I$ such that $|I|\leq 2$

$$\|F_n\|\leq\|\varphi\| \quad \text{and} \quad \|X^IF_n\|\leq\|X^I\varphi\|$$

where $\|f\|=\sup_{x\in G}|f(x)|$. Let

$$\varepsilon_{n,m}=\max\{|\alpha_{ij}^n(t)-\alpha_{ij}^m(t)|,|\alpha_l^n(t)-\alpha_l^m(t)|:$$
$$i,j=1,\ldots,k,\ l=1,\ldots,N,\ s\leq t<T\}.$$

There are $C$, $M$ independent of $n$, $m$ such that

$$|(L_m-L_n)F_m(x,t)|\leq C(1+\tau_G(x))^M\varepsilon_{n,m}, \qquad x\in G.$$

Let now $\varepsilon,M,r$ be as in Theorem (4.2) and $R=R(\varepsilon,M,2\|\varphi\|,r)$. We consider the function

$$G(x,t)=F_n(x,t)-F_m(x,t)+2Ct(1+R)^M\varepsilon_{n,m}$$

on $B_R(e)\times[s,T)$. Since $L_n(F_n-F_m)=(L_m-L_n)F_m$ we have

$$L_nG(x,t)=(L_m-L_n)F_m(x,t)-2C(1+R)^M\varepsilon_{n,m}\leq 0$$
$$\text{for } x\in B_r(e),\ s<t<T,$$

and so by Theorem (4.2)

$$F_n(x,t)-F_m(x,t)+2t(1+R)^M\varepsilon_{n,m}\geq-\varepsilon$$

for $x\in B_r(e)$ and $s\leq t<T$. Interchanging $n$ and $m$ we obtain

$$F_n(x,t)-F_m(x,t)-2t(1+R)^M\varepsilon_{n,m}\leq\varepsilon$$

and the uniform convergence of $F_n$ and also of $X^IF_n$ is proved.

Now, on the one hand $LF_n\to LF$ in the sense of distributions, and on the other hand, for $\psi\in C_c^\infty(B_R(e))$ we have

$$|\langle LF_n,\psi\rangle|=|\langle(L-L_n)F_n,\psi\rangle|\leq C(1+R)^M\varepsilon_n\|\psi\|_{L^1}$$

with $\varepsilon_n\to 0$ so $LF=0$. Uniqueness of $F$ follows from Corollary (4.3).

Moreover, we prove as above that if $\varphi \in C_b^d(G)$ and $X^I$ ($|I| \le d$) is a right-invariant differential operator then $X^I F_n$ converges almost uniformly to $X^I F$. This means that $F \in C_b^{d,1}(G \times (s,T))$ because $\partial_t F = L_t F$. ■

(4.6) COROLLARY. *Let $F$ be as in Theorem (4.4). There is a family of probability measures $P(s,t,dx) = P(s,t)$ (transition probability function) such that*

$$F(x,t) = \varphi * P(s,t)(x), \tag{4.7}$$

$$P(s,t) = P(s,u) * P(u,t) \quad \text{for } s < u < t. \ ■ \tag{4.8}$$

(4.9) PROPOSITION [SV]. *If $f \in C_b(G \times [s,t)) \cap C_b^{2,1}(G \times (s,t))$ then*

$$\int_G f(xy^{-1},s)\, P(s,t,dy) - f(x,t) = \int_s^t du \int_G ((L_u - \partial_u) f(xy^{-1},u))\, P(u,t,dy). \ ■$$

(4.10) THEOREM. *Let $s < T$ and $M = \max\{|\alpha_{ij}(t)|, |\alpha_i(t)| : s \le t \le T\}$. There is a constant $C$ independent of $s$, $T$, $M$ such that for $0 < t < T$ and $r \ge CM(t-s)$*

$$P(s,t,B_r(x)^c) \le 2 \dim G \exp(-r^2/C^2 M(t-s)).$$

Proof. Let $r$ be such that in appropriate coordinates $x_1, \ldots, x_N$

$$\tau_G(x) = \Big(\sum_{j=1}^N x_j^2\Big)^{1/2}$$

when $\tau_G(x) < r$. We consider the functions

$$\Phi_m^K = \psi\eta_K + (1-\psi)\tau_m * \varphi, \quad m = 1,2,\ldots,\infty, \ K = 1,\ldots,2N,$$

where

$$\eta_K = \begin{cases} x_K & \text{if } K = 1,\ldots,N, \\ -x_{K-N} & \text{if } K = N+1,\ldots,2N, \end{cases}$$

$\psi \in C_c^\infty(B_r)$, $0 \le \psi \le 1$, $\psi(x) = 1$ for $x \in B_{r/2}$ and $\varphi$ in (1.5) is such that $\tau_m - r/4 \le \tau_m * \varphi \le \tau_m + r/4$. Then $\Phi_m^K(e) = 0$, right-invariant derivatives of $\Phi_m^K$ are bounded and if $X^I$ is left-invariant then the $\|X^I \Phi_m^K\|$ are bounded independently of $m$, $K$. Let $\lambda \ge 1$ and

$$C = 2 \max_{m,K}\Big(\sum_{i,j=1}^N \|X_i X_j \Phi_m^K\| + \sum_{i=1}^N \|X_i \Phi_m^K\| + \sum_{i,j=1}^N \|X_i \Phi_m^K\| \|X_j \Phi_m^K\| + 1\Big).$$

The function $G(x,u) = \exp(\lambda^2 M(u-s) + \lambda \Phi_m^K(x)/C)$ satisfies $(L_u - \partial_u)G < 0$ in $G \times (s,t)$ so by the previous proposition

$$\int_G \exp(\lambda \Phi_m^K(x^{-1})/C)\, P(s,t,dx) \le \exp(\lambda^2 M(t-s)).$$

Passing with $m$ to infinity we obtain

$$\int_G \exp(\lambda \Phi_\infty^K(x^{-1})/C)\, P(s,t,dx) \le \exp(\lambda^2 M(t-s))$$

for every $K$. Now proceeding as in [S] we have

$$P(s,t,B_r(e)^c) \le 2N \exp(-\lambda r/2C\sqrt{N} + \lambda^2 M(t-s)).$$

If $\lambda = r/4C\sqrt{N} M(t-s)$ this means

$$P(s,t,B_r(e)^c) \le 2N \exp(-r^2/16C^2 N M(t-s)). \ ■$$



(4.11) COROLLARY. *Let $C$, $M$ be as in the previous theorem. For every $q \in [1,\infty)$ there is a constant $C_1$ depending only on $C$, $q$ such that*

$$\int_G \tau_G(x)^q\, P(s,t,dx) \le C_1 \max(M^q, M^{q/2}) \begin{cases} (t-s)^{q/2} & \text{if } t-s \le 1, \\ (t-s)^q & \text{if } t-s \ge 1, \end{cases}$$

*for $0 < s < t$.* ■

Let $\mathcal{B}(G)$ be the Borel $\sigma$-field on $G$ and $\Omega(G)$ the set of continuous functions $x(\,)$ on $[0,\infty)$ with values in $G$. $\mathcal{F}_t$ denotes the $\sigma$-field on $\Omega(G)$ generated by the sets

$$\{x(\,) : x(s) \in \Gamma\}, \quad 0 \le s \le t, \ \Gamma \in \mathcal{B}(G),$$

and

$$\mathcal{F} = \sigma\Big(\bigcup_{0 \le t < \infty} \mathcal{F}_t\Big).$$

(4.12) COROLLARY. *For every $s > 0$ and $x \in G$ there is a unique probability measure $P_{s,x}$ on $\Omega(G)$ such that $(x(t), \mathcal{F}_t, P_{s,x})$ is a continuous Markov process with the transition probability function*

$$P(s,x,t,V) = P(s,t,x^{-1}V) \tag{4.13}$$

*and initial distribution $\delta_x$.* ■

For more details about the diffusion $P_{s,x}$ see [SV].

**5. Estimates.** Let $N$ be a homogeneous group and let $X_1, \ldots, X_n$ be a homogeneous basis in the Lie algebra $\mathfrak{n}$ of $N$, i.e. for the group of dilations $\delta_r$, $r > 0$, we have

$$\delta_r X_j = r^{d_j} X_j, \quad j = 1,\ldots,n,$$

where $1 = d_1 = \ldots = d_k < d_{k+1} \le \ldots \le d_n$.

Suppose that the vector fields

(5.1) $$Y_0(t), Y_1(t), \ldots, Y_k(t)$$

satisfy the following conditions:

(a) $Y_i(t) = \sum_{j=1}^k \beta_{ij}(t)X_j$ for $i = 1, \ldots, k$ and $Y_0(t) = \sum_{j=1}^n \beta_{0j}(t)X_j$ with $\beta_{ij} \in C(0,\infty)$ for all $i$ and $j$,

(b) for every $t$, $Y_1(t), \ldots, Y_k(t)$ generate $\mathfrak{n}$,

(c) there are $\lambda > 0$, $\Lambda > 1$ such that for every $\xi \in \mathbf{R}^k$ and $t > 0$

$$\lambda|\xi|^2 \le \sum_{i,j=1}^k \Big(\sum_{l=1}^k \beta_{li}(t)\beta_{lj}(t)\Big)\xi_i\xi_j \le \Lambda|\xi|^2$$

and also

$$|\beta_{0j}(t)| \le \Lambda \quad \text{for } j = 1, \ldots, n.$$

We consider an operator on $N \times \mathbf{R}^+$ given by

(5.2) $$Lu(x,t) = L_0u(x,t) + L_1u(x,t) - \partial_t u(x,t)$$

where

$$L_0u(x,t) = \sum_{j=1}^k Y_j(t)^2 u(x,t), \quad L_1u(x,t) = Y_0(t)u(x,t).$$

First for the transition probability function $P(s,t)$ given by Corollary (4.6) we prove the following estimate:

(5.3) THEOREM. *Let $L$ be as in* (5.2), *$P(s,t)$ the transition probability function corresponding to $L$ and $f \in C_c^\infty(N)$. Then for every multiindex $I$ there are constants $K = K(I)$, $C = C(\lambda, I)$ such that*

(5.4) $$\|f * X^I P(s,t)\|_{L^\infty} \le C\Lambda^K (t-s)^{-Q/4-|I|/2} c(t-s)^K \|f\|_{L^2}$$

*for all $f \in C_c^\infty(N)$ and $0 \le s < t$ where $c(t) = \max\{\max\{t^{(2-d_j)/2} : j = 1, \ldots, n\}, 1\}$.*

Remark. The main idea of the proof is due to Waldemar Hebisch.

Proof. Let

$$B = \{(x,t) : |x| < 1, 1/8 < t < 1\}$$

where $|\ |$ is a homogeneous norm in $N$ and let $\varphi \in C_c^\infty(B)$. Suppose $u$ satisfies

(5.5) $$(L - \partial_t)u(x,t) = 0$$

in a neighbourhood of $\operatorname{supp}\varphi$. Then

$$0 = \langle \partial_t(\varphi u), \varphi u\rangle = \langle (\partial_t\varphi)u, \varphi u\rangle + \langle \varphi Lu, \varphi u\rangle.$$



Hence

$$\begin{aligned}|\langle \varphi L_0u, \varphi u\rangle| &\le |\langle L_1(\varphi u), \varphi u\rangle| + |\langle (L_1\varphi)u, \varphi u\rangle| + |\langle (\partial_t\varphi)u, \varphi u\rangle| \\ &= |\langle (L_1\varphi)u, \varphi u\rangle| + |\langle (\partial_t\varphi)u, \varphi u\rangle| \le C\|u\|^2_{L^2(B)}\end{aligned}$$

where $C = C(\varphi)\Lambda$. We have

$$\langle L_0(\varphi u), \varphi u\rangle = \langle \varphi L_0u, \varphi u\rangle + \langle [L_0, \varphi]u, \varphi u\rangle.$$

Now we fix $t$ and we abbreviate $Y_j(t) = Y_j$. Let $\eta_j = Y_j\varphi$. Then

$$[L_0, \varphi] = 2\sum_{j=1}^k (\eta_j Y_j + Y_j^2\varphi),$$

whence

$$|\langle [L_0, \varphi]u, \varphi u\rangle| \le \Big|\sum_{j=1}^k \langle \varphi Y_j u, \eta_j u\rangle\Big| + C\|u\|^2_{L^2(B)}$$

where $C = C(\varphi)\Lambda$. But

$$\Big|\sum_{j=1}^k \langle \varphi Y_j u, \eta_j u\rangle\Big| \le 3^{-1}\sum_{j=1}^k \langle \varphi Y_j u, \varphi Y_j u\rangle + 3C\|u\|^2_{L^2(B)}$$

where $C = C(\varphi)\Lambda$. On the other hand,

$$\begin{aligned}3^{-1}\sum_{j=1}^k \langle \varphi Y_j u, \varphi Y_j u\rangle &= -3^{-1}\sum_{j=1}^k \langle \varphi Y_j^2 u, \varphi u\rangle - (2/3)\sum_{j=1}^k \langle \eta_j Y_j u, \varphi u\rangle \\ &= -3^{-1}\langle \varphi L_0 u, \varphi u\rangle - (2/3)\sum_{j=1}^k \langle \eta_j Y_j u, \varphi u\rangle,\end{aligned}$$

whence

$$\Big|\sum_{j=1}^k \langle \varphi Y_j u, \eta_j u\rangle\Big| \le \langle \varphi L_0 u, \varphi u\rangle + 9C(\varphi)\Lambda\|u\|^2_{L^2(B)},$$

i.e.

$$|\langle L_0(\varphi u), \varphi u\rangle| \le 2\langle \varphi L_0 u, \varphi u\rangle + C(\varphi)\Lambda\|u\|^2_{L^2(B)}$$

and, finally,

(5.6) $$|\langle L_0(\varphi u), \varphi u\rangle| \le C(\varphi)\Lambda\|u\|^2_{L^2(B)}.$$

Let $\check{X}_1^2 + \ldots + \check{X}_n^2 = -\Delta$, where $\check{X}$ denotes the right-invariant field defined by $X$ in $\mathfrak{n}$. For $\varepsilon \ge 0$ we define a Sobolev norm on functions supported in $B$ putting

$$\|\Delta^{\varepsilon/2}f\|^2_{L^2} + \|f\|^2_{L^2} = \|f\|^2_{H(\varepsilon)}.$$

We need the following

LEMMA (J. J. Kohn). *There is an $\varepsilon > 0$ and a constant $C$ such that*

$$\|\varphi u\|^2_{H(\varepsilon)} \le C\Big(\sum_{j=1}^{k} \|X_j(\varphi u)\|^2_{L^2} + \|\varphi u\|^2_{L^2}\Big),$$

*for all $u$ in $C^1$ and $\varphi$ in $C_c^\infty$, where $\varepsilon$ depends only on the length of commutators in $X_1, \ldots, X_k$ needed to span $\mathfrak{n}$. Therefore*

$$\|\varphi u\|^2_{H(\varepsilon)} \le c\Big(k^2\lambda^{-1} \int_B \sum_{j=1}^{k} |Y_j(t)(\varphi u)(x,t)|^2\,dx\,dt + \|\varphi u\|^2_{L^2}\Big).$$

Hence for $u$ which satisfies (5.5) in a neighbourhood of $\operatorname{supp}\varphi$, by (5.6), we then have

$$\|\varphi u\|_{H(\varepsilon)} \le C(\varphi,\lambda)\Lambda\|u\|_{L^2}. \tag{5.7}$$

Let $\varphi, \psi \in C_c^\infty(B)$ and $\psi(x,t) = 1$ for $(x,t) \in \operatorname{supp}\varphi$. Suppose $u$ satisfies (5.5) in a neighbourhood of $\operatorname{supp}\psi$. Then, since $\Delta$ commutes with $L$, $\Delta^\varepsilon(\psi u)$ satisfies (5.5) in a neighbourhood of $\operatorname{supp}\varphi$. Hence, by (5.6) and (5.7), since $\Delta^\varepsilon[\Delta^\varepsilon, \varphi]$ is a pseudodifferential operator of order $\le \varepsilon$ (of course we may assume $\varepsilon \le 1$),

$$\begin{aligned}\|\Delta^\varepsilon(\varphi u)\|_{L^2} &= \|\Delta^\varepsilon(\varphi\psi u)\|_{L^2}\\ &\le \|\Delta^{\varepsilon/2}(\varphi\Delta^{\varepsilon/2}\psi u)\|_{L^2} + \|\Delta^{\varepsilon/2}([\varphi,\Delta^{\varepsilon/2}](\psi u))\|_{L^2}\\ &\le \|\varphi\Delta^{\varepsilon/2}(\psi u)\|_{H(\varepsilon)} + C(\varphi,\varepsilon)\|\psi u\|_{H(\varepsilon)}\\ &\le (C(\varphi,\varepsilon) + C(\varphi)\Lambda)\|\psi u\|_{H(\varepsilon)}.\end{aligned}$$

Hence

$$\|\varphi u\|_{H(2\varepsilon)} \le (C(\varphi,\varepsilon) + C(\varphi)\Lambda)C(\psi,\lambda)\Lambda\|u\|_{L^2(B)},$$

and continuing in this way we obtain

$$\|\varphi u\|_{H(r)} \le C(r,\varphi,\lambda)\Lambda^K\|u\|_{L^2(B)}$$

for arbitrary $r$, where $K = K(r)$.

Consequently, for every multiindex $I$ and $X^I = X_1^{i_1}\ldots X_n^{i_n}$

$$\|\varphi X^I u\|^2_{H(r)} \le C\Lambda^K\|u\|^2_{L^2(B)} \tag{5.8}$$

where $K = K(r,I)$, $C = C(\varphi,\lambda,r,I)$, and the same with $\check{X}^I$ in place of $X^I$. Now let $u$ satisfy (5.5) in a neighbourhood of $\operatorname{supp}\varphi$. Hence also $\check{X}^I u$ satisfies (5.5) in a neighbourhood of $\operatorname{supp}\varphi$. By the Sobolev lemma, since the domain is bounded, this yields

$$\int_{1/8}^{1} \sup_{|x|<1/2} |\varphi X^I u(x,t)|^2\,dt \le C\Lambda^K\|u\|_{L^2(B)}, \tag{5.9}$$



where $K = K(n/2+1, I)$ and $C = C(\varphi,\lambda,I)$. Putting $X^I(L_0+L_1)u = \partial_t X^I u$ in place of $X^I u$ in (5.9) we obtain

$$\sup_{1/4\le t\le 1/2,\ |x|<1/2} |X^I u(x,t)| \le C\Lambda^K\|u\|_{L^2(B)}$$

with $K = K(I)$ and $C = C(\lambda, I)$ for an appropriate choice of $\varphi$ in (5.9).

Let $D_r(x,t) = (\delta_r x, r^2 t)$. Then by (5.1) and (5.2) we have

$$r^2((\partial_t - L^1)u)\cdot D_r = (\partial_t - L^r)(u\cdot D_r), \tag{5.10}$$

where

$$L^r u(x,t) = \sum_{j=1}^{k} Y_j(r^2t)^2 u(x,t) + \sum_{j=1}^{n} \beta_{0j}(r^2t) r^{2-d_j} X_j u(x,t).$$

It follows immediately from (5.10) that $(\partial_t - L^r)(u\cdot D_r) = 0$ iff $Lu = 0$. Consequently,

$$\sup_{1/4\le t\le 1/2,\ |x|\le 1/2} |X^I(u\cdot D_r)| \le C(\lambda,I)c(r^2)^K\Lambda^K\|u\cdot D_r\|_{L^2(B)} \tag{5.11}$$

where $c(r) = \max\{\max\{r^{(2-d_j)/2} : j = 1,\ldots,n\}, 1\}$. Let $|I| = d_1 i_1 + \ldots + d_n i_n$. Then $X^I(u\cdot D_r) = r^{|I|}X^I u\cdot D_r$, and by (5.11),

$$\begin{aligned} r^{|I|} &\sup_{1/4\le t\le 1/2,\ |x|\le 1/2} |X^I u\cdot D_r|\\ &\le C(\lambda,I)\Lambda^K c(r^2)^K r^{-Q/2}\Big(r^{-2}\int_{r^2/8}^{r^2}\int_{|x|<r} |u(x,t)|^2\,dx\,dt\Big)^{1/2}.\end{aligned}$$

Therefore

$$\begin{aligned}&\sup_{|x|<(2r)^{1/2}/2} |X^I u(x,r)|\\ &\le C'(\lambda,I)\Lambda^K c(r)^K r^{-|I|/2-Q/4}\Big(r^{-1}\int_{r/8}^{r}\int_{|x|<\sqrt{r}} |u(x,t)|^2\,dx\,dt\Big)^{1/2}.\end{aligned} \tag{5.12}$$

Now let $p_t = P(0,t)$ where $P(0,t)$ is given by Corollary (4.6) and

$$u(x,t) = f * p_t(x), \quad f \in C_c^\infty(N).$$

Then by (5.12)

$$\sup_{|x|<(2r)^{1/2}/2} |f * X^I p_t(x)| \le C'(\lambda,I)\Lambda^K c(t)^K t^{-|I|/2-Q/4}\|f\|_{L^2}.$$

Using ${}_x f(y) = f(xy)$ instead of $f$ we get

$$\|f * X^I p_t\|_{L^\infty} \le C'(\lambda,I)\Lambda^K c(t)^K t^{-|I|/2-Q/4}\|f\|_{L^2}. \tag{5.13}$$

If we now consider the operator $L'$ with $Y_i'(t) = Y_i(s+t)$, $i = 0,\ldots,k$, then we obtain (5.4). ■

(5.14) THEOREM. *For every multiindex $I$ there are constants $C = C(\lambda, I)$ and $K = K(I)$ such that*

$$\|X^I P(s,t)\|_{L^\infty} \le C\Lambda^K c(t-s)^K (t-s)^{-|I|/2-Q/2} \tag{5.15}$$

*for $0 \le s < t$ where $c(t)$ is as in Theorem* (5.3).

Proof. Let $f$ be an $L^1$ function with compact support and $u$ the $L$-harmonic function given by $u(x,t) = f * p_t(x)$. In view of (5.12) and (5.4) we have

$$|f * X^I p_t(e)| \le C\Lambda^K c(t)^K t^{-|I|/2-Q/4}\Big(t^{-1}\int_{t/8}^{t} \|f * p_s\|_{L^2}^2\,ds\Big)^{1/2}$$

and

$$\|f * p_t\|_{L^2} \le C\Lambda^K c(t)^K t^{-Q/4}\|f\|_{L^1}$$

for some constants $C = C(\lambda, I)$ and $K = K(I)$. Replacing $f$ by ${}_x f$ we obtain

$$\|f * X^I p_t\|_{L^\infty} \le C'\Lambda^{K'} c(t)^{K'} t^{-|I|/2-Q/4}\|f\|_{L^1}\,.$$

Taking an approximate identity in $L^1$ we arrive at (5.15) for $s = 0$ and, changing the operator as at the end of the previous proof, for arbitrary $s$. ∎

(5.16) THEOREM. *For every multiindex $I$ and every positive integer $\xi$ there exist $C = C(I, \lambda, \xi)$ and $K = K(I, \xi)$ such that*

$$\langle |X^I P(s,t)|, (1+\tau_N)^\xi\rangle \le C\Lambda^K c(t)^K (t-s)^{-|I|/2} \tag{5.17}$$

*when $t - s \le 1$. Here $X^I$ may be either left- or right-invariant.*

Proof. If $|I| = 0$, (5.17) follows from Corollary (4.11). Let $\{\varphi_m\}_{m=1,2,\ldots}$, $0 \le \varphi_m \le 1$, be a family of smooth functions with compact support such that $\varphi_m = 1$ on $B_m(e)$. We write $(1+\tau_N)^\xi = \Phi$ and consider the harmonic functions $u_m(x,t) = \Phi_m * p_t(x)$ where $\Phi_m = \varphi_m\Phi\,\mathrm{sgn}(X^I p_t)$. Then by (5.12) for $t \le 1$ we have

$$\begin{aligned}\langle \check\varphi_m\Phi, |X^I p_t|\rangle &= \langle \check\Phi_m, X^I p_t\rangle = \Phi_m * X^I p_t(e)\\ &\le C'(\lambda, I)\Lambda^K c(t)^K t^{-|I|/2-Q/4}\Big(t^{-1}\int_0^t \int_{|x|\le\sqrt t} |\Phi_m * p_s(x)|^2\,dx\,ds\Big)^{1/2}.\end{aligned}$$

But by the fact that $\Phi$ is submultiplicative and Corollary (4.11)

$$\begin{aligned}&\Big(t^{-1}\int_0^t \int_{|x|\le\sqrt t}\Big|\int_N \Phi(xy^{-1})p_s(y)\,dy\Big|^2\,dx\,ds\Big)^{1/2}\\ &\quad\le \Big(\Big(\int_{|x|\le\sqrt t} |\Phi(x)|^2\,dx\Big)t^{-1}\int_0^t \langle\check\Phi, p_s\rangle\,ds\Big)^{1/2} \le C(k)\Lambda^{\xi/2}t^{Q/4}\end{aligned}$$



and (5.17) follows for $s = 0$.

To obtain (5.17) for arbitrary $s$ we consider the operator $L'$ with $Y_i'(t) = Y_i(s+t)$, $i = 0, \ldots, k$. ∎

Now we are going to prove some pointwise estimates for $P(s,t)$ and its derivatives. We start with some inequalities in terms of $\tau_N$ but later we pass to a homogeneous norm to obtain estimates which we really need. For the rest of this section we assume that $Y_0$ is at most of order 2, i.e. $\beta_{0j} = 0$ when $d_j > 2$.

(5.18) THEOREM. *Given a multiindex $I$ and a positive integer $\xi$ there are constants $C = C(I, \lambda, \xi)$ and $K = K(I, \xi)$ such that for $t - s \le 1$*

$$|X^I P(s,t,x)| \le C\Lambda^K (t-s)^{-|I|/2-Q/2}(1+\tau_N(x))^{-\xi}\,. \tag{5.19}$$

Proof. Let $W = |X^I P(s,t,x)|(1+\tau_N(x))^\xi$. By Corollary (4.6) and by the fact that $1+\tau_N(x)$ is submultiplicative

$$W \le \int_N (1+\tau_N(xy^{-1}))^\xi (1+\tau_N(y))^\xi P(s,u,xy^{-1})|X^I P(u,t,y)|\,dy\,.$$

Applying now the Schwarz inequality and (5.15),(5.17) we obtain

$$\begin{aligned}W &\le \Big(\int_N (1+\tau_N(xy^{-1}))^{2\xi}(P(s,u,xy^{-1}))^2\,dy\Big)^{1/2}\\ &\quad\times\Big(\int_N (1+\tau_N(y))^{2\xi}(X^I P(u,t,y))^2\,dy\Big)^{1/2}\\ &\le C\Lambda^K (u-s)^{-Q/4}(t-u)^{-Q/4-|I|/2}\end{aligned}$$

for some constants $C = C(\lambda, I, \xi)$, $K = K(I, \xi)$. Finally, taking minimum over $u$ we arrive at (5.19). ∎

When $|I| = 0$ we need an estimate of type (5.19) for $t - s \ge 1$. But in view of Corollary (4.11), proceeding as in the previous proof we obtain

$$P(s,t,x) \le C\Lambda^{K_1}(t-s)^{K_2}(1+\tau_N(x))^{-\xi} \tag{5.20}$$

for $t - s \ge 1$. Here $C$ is a constant depending on $\lambda$, $\xi$ and $K_1$, $K_2$ depend on $\xi$.

(5.21) THEOREM. *Let $|\ |$ be a subadditive homogeneous norm in $N$* [HS]. *For every multiindex $I$ there are constants $C = C(I, \lambda)$ and $K = K(I)$ such that*

$$|X^I P(s,t,x)| \le C\Lambda^K (t-s)^{1/2}|x|^{-|I|-Q-1}\,. \tag{5.22}$$

Proof. Let $L^r$ be as in (5.10) and $u(x,t) = \varphi * P(0,t)(x)$, $\varphi \in C_c^\infty$. Then $v = u \cdot D_r$ is the solution of the Dirichlet problem for $L^r - \partial_t$ with boundary value $\varphi \cdot \delta_r$. Therefore on the one hand

$$v(x,t) = \varphi * P(0, r^2 t)(\delta_r x)\,, \tag{5.23}$$

and on the other hand,

$$v(x,t) = (\varphi \cdot \delta_r) * P^r(0,t)(x) \tag{5.24}$$

where $P^r(0,t)$ is the transition probability function corresponding to $L^r$. Comparing (5.23) and (5.24) we obtain

$$P^r(0,t,x) = r^Q P(0, r^2 t, \delta_r x)\,.$$

Assume now that $r \le 1$. Then by (5.19)

$$|X^I P^r(0,1,x)| \le C\Lambda^K (1 + \tau_N(x))^{-Q-|I|-1}$$

with $C = C(I,\lambda,Q)$, $K = K(I,Q)$ and $\Lambda$ independent of $r$. This means

$$|X^I P(0,r^2,\delta_r x)| \le C\Lambda^K r^{-Q-|I|}|x|^{-Q-|I|-1}\,.$$

Now writing $r^{1/2}$ instead of $r$ and $x = \delta_r y$ we obtain (5.22) for $s = 0$. To get (5.22) for an arbitrary $s$ we consider, as before, the operator $L'$ with $Y_i'(t) = Y_i(t+s)$. ■

**6. Diffusion.** Let $L$ be an operator of the form (2.3) on the group $NA$ and assume that $L$ satisfies (3.0). Without loss of generality we may assume that $\alpha = 1$. Let $\Omega_T(S)$ be the set of continuous mappings from $[0,T]$ into $S$ and

$$\{\mathcal{P}_{s,xe^b} : s > 0, xe^b \in S\}$$

the diffusion associated to $L$ considered on $\Omega_T(S)$, i.e. the $\mathcal{P}_{s,xe^b}$ are defined on the $\sigma$-field $\mathcal{F}_T(S)$ generated by the sets $\{x(\cdot) \in \Omega_T(S) : x(s) \in \Gamma\}$, $0 \le s \le T$, $\Gamma \in \mathcal{B}(G)$. Given $\underline{a} \in \Omega_T(A)$ we look at the operator

$$L^{\underline{a}} = \sum_{i,j=1}^{n} \alpha_{ij} e^{a(t)(d_i+d_j)} X_i X_j + \sum_{j=1}^{n} \alpha_j e^{a(t)d_j} X_j\,. \tag{6.1}$$

Let $\{\mathcal{P}^{\underline{a}}_{s,x} : s > 0, x \in N\}$ be the diffusion on $\Omega_T(N)$ given for $L^{\underline{a}}$ by Corollary (4.12). If $\{\mathcal{W}_{s,b} : s > 0, b \in A\}$ is the Wiener measure on $\Omega_T(A)$ associated to $\partial_a^2 - \kappa\partial_a$ then (see e.g. [T])

$$\mathcal{P}_{s,xe^b} = \int \mathcal{P}^{\underline{a}}_{s,xe^b} \mathcal{W}_{s,b}(d\underline{a})\,, \tag{6.2}$$

i.e. if $Z \in \mathcal{F}_T(S)$ and $Z_{\underline{a}} = \{\underline{x} \in \Omega_T(N) : (\underline{x},\underline{a}) \in Z\}$ then

$$\mathcal{P}_{s,xe^b}(Z) = \int \mathcal{P}^{\underline{a}}_{s,xe^b}(Z_{\underline{a}}) \mathcal{W}_{s,b}(d\underline{a})\,.$$

Let $a < b$ and $T_a = \inf\{t : a(t) \le a\}$. Then in view of (6.2) for $M \subset N$ we have

$$\check{\mu}^b_a(M) = \int P(\underline{a}; 0, T_a, M)\, d\mathcal{W}_{0,b} = E_b P(\underline{a}; 0, T_a, M)\,. \tag{6.3}$$



where $P(\underline{a}; s,t,M) = \mathcal{P}^{\underline{a}}_{s,e}\{x(t) \in M\}$.

The following immediate corollary of (4.8) and (6.3) will be used: for $t < T_a$

$$P(\underline{a}; 0, T_a, M) = P(\underline{a}; 0, t, \cdot) * P(\underline{a}; t, T_a, M)\,. \tag{6.4}$$

Let us also formulate an easy

(6.5) Proposition. *Let $G$, $H$ be two Lie groups. Let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ be elements in the Lie algebra $\mathfrak{g}$ of $G$ and $X_1, \ldots, X_n$ elements in the Lie algebra $\mathfrak{h}$ of $H$ and let $\sigma$ be a homomorphism of $G$ onto $H$ such that*

$$\sigma_* \mathcal{X}_j = X_j\,, \quad j = 1, \ldots, n\,.$$

*Assume that $\mathcal{X}_1, \ldots, \mathcal{X}_k$ generate $\mathfrak{g}$. Let $\alpha_{ij}$ and $\alpha_j$ be continuous functions on $\mathbf{R}^+$ such that the matrix $[\alpha_{ij}(t)]$ is positive definite for each $t$. Let*

$$\mathcal{L} = \sum_{i,j=1}^{n} \alpha_{ij}(t) \mathcal{X}_i \mathcal{X}_j + \sum_{j=1}^{n} \alpha_j(t) \mathcal{X}_j - \partial_t\,,$$

$$L = \sum_{i,j=1}^{n} \alpha_{ij}(t) X_i X_j + \sum_{j=1}^{n} \alpha_j(t) X_j - \partial_t\,.$$

*Then the transition probability functions $P^G(s,t)$ and $P^H(s,t)$ corresponding to $\mathcal{L}$ and $L$ in view of Corollary (4.6) satisfy*

$$P^H(s,t,V) = P^G(s,t,\sigma^{-1}(V))\,. \ ■$$

The following fact is well known and not difficult to prove by standard methods.

(6.6) Proposition. *Let $T_a = \inf\{t : a(t) \le a\}$. Then*

$$\mathcal{W}_b\{T_a < t\} = \int_0^t (4\pi)^{-1/2}(b-a)s^{-3/2} \exp[-(b-a-\kappa s)^2(4s)^{-1}]\, ds\,. \ ■$$

To construct an appropriate free group for the operator $L$ we must assume that there is a subset $\Pi$ of $\{1, \ldots, n\}$ such that

$$L_0 = \sum_{i,j\in\Pi} \alpha_{ij} e^{(d_i+d_j)t} X_i X_j\,, \tag{6.7}$$

$\{X_i\}_{i\in\Pi}$ generate $\mathfrak{n}$ and the matrix $[\alpha_{ij}]$ is strictly positive definite.

Now let $L^{\underline{a}}$ be the operator

$$L^{\underline{a}} = \sum_{i,j\in\Pi} \alpha_{ij}(t) X_i X_j + \sum_{j=1}^{n} \alpha_j(t) X_j \tag{6.8}$$

with

$$\alpha_{ij}(t)=\begin{cases}\alpha_{ij}e^{a(t)(d_i+d_j)} & \text{when } t\le 1\,,\\ \alpha_{ij}e^{a(1)(d_i+d_j)} & \text{when } t>1\,,\end{cases}$$

$$\alpha_{i}(t)=\begin{cases}\alpha_{i}e^{a(t)d_i} & \text{when } t\le 1\,,\\ \alpha_{i}e^{a(1)d_i} & \text{when } t>1\,.\end{cases}$$

Then for $\Lambda$ and $\lambda$ in (5.1)(c) we have

$$\Lambda=C_1\exp[C_2\max\{a(t):0\le t\le 1\}]\,,$$
$$\lambda=c_1\exp[c_2\min\{a(t):0\le t\le 1\}]$$

for some constants $C_1$, $C_2$, $c_1$, $c_2$ and we can assume that $\Lambda\ge 1$.

Passing to the free nilpotent Lie algebra generated by $\mathcal{X}_j$, $j\in\Pi$, on which dilations are defined by $\delta_r\mathcal{X}_j=r\mathcal{X}_j$ we apply Proposition (6.5) to derive from (5.17) the following

(6.9) THEOREM. *Let $L^{\underline{a}}$ be as in* (6.8) *and let $P(\underline{a};0,t,\cdot)$ be the transition probability function associated to $L^{\underline{a}}-\partial_t$. Then for every $\xi$ and every left- or right-invariant differential operator $\partial$ on $N$ there exist a constant $C=C(\partial,\xi)$ and an exponent $K=K(\partial,\xi)$ such that for $t\le 1$ we have*

$$\int|\partial P(\underline{a};0,t,x)|(1+\tau_N(x))^{\xi}\,dx\le C\Lambda^K t^{-K}\,.\ \blacksquare$$

Now let $\mu_a^b$ be a harmonic measure as described in Section 3.

(6.10) THEOREM. *Let $\eta>0$ be the exponent as in Proposition* (3.14). *Then for every left- (right-) invariant differential operator $\partial$ ($\partial^\vee$) on $N$ we have*

$$\int|\partial\check\mu_a^b(x)|(1+\tau_N(x))^{\eta}\,dx<\infty\,,\tag{6.11}$$

$$\int|\partial^\vee\check\mu_a^b(x)|(1+\tau_N(x))^{\eta}\,dx<\infty\,.\tag{6.12}$$

Proof. Since $\check\mu_a^b(x)$ is harmonic as a function of $x$, $b$, (6.11) follows from Harnack's inequality and Proposition (3.14). Let $T_a=\min\{t:a(t)=a\}$ and $T_a'=\min\{1,T_a\}$. By (6.3) and (6.4) we have

$$\partial^\vee\check\mu_a^b(x)=E_b\partial^\vee P(\underline{a};0,T_a',\cdot)*P(\underline{a};T_a',T_a,x)\,.$$

Hence by the strong Markov property

$$\begin{aligned}\int|\partial^\vee\check\mu_a^b(x)|(1+\tau_N(x))^{\eta}\,dx\\
\le E_b\Big\{\int|\partial^\vee P(\underline{a};0,T_a',x)|(1+\tau_N(x))^{\eta}\,dx\\
\times\int P(\underline{a};T_a',T_a,x)(1+\tau_N(x))^{\eta}\,dx\Big\}\\
=E_b\Big\{\int|\partial^\vee P(\underline{a};0,T_a',x)|(1+\tau_N(x))^{\eta}\,dx\\
\times E_{a(T_a')}\int P(\underline{a};0,T_a,x)(1+\tau_N(x))^{\eta}\,dx\Big\}\\
=E_b\Big\{\int|\partial^\vee P(\underline{a};0,1,x)|(1+\tau_N(x))^{\eta}\,dx\\
\times\int E_{a(1)}P(\underline{a};0,T_a,x)(1+\tau_N(x))^{\eta}\,dx;T_a\ge 1\Big\}\\
+E_b\Big\{\int|\partial^\vee P(\underline{a};0,T_a,x)|(1+\tau_N(x))^{\eta}\,dx;T_a<1\Big\}.\end{aligned}$$



By (6.3) and Proposition (3.14), the second factor in the first summand is equal to

$$\langle\check\mu_a^{a(1)},(1+\tau_N)^{\eta}\rangle\le C(\eta)e^{2d_n|a(1)|}\le C(\eta)\Lambda^2$$

and so, by Theorem (6.9), for $K$ large enough the first summand is less than or equal to

$$CC(\eta)E_b\Lambda^{K+2}\le C'(\eta)E_be^{2C_2(K+2)a(1)}\,,$$

which is finite. By Theorem (6.9), the second summand is less than or equal to

$$CE_b\{\Lambda^KT_a^{-K};T_a<1\}\le C'(E_b\{e^{4C_2Ka(1)}\})^{1/2}(E_b\{T_a^{-2K};T_a<1\})^{1/2}\,,$$

which, by Proposition (6.6), is finite. $\blacksquare$

Remark. If $\kappa$ in (2.3) is sufficiently large then (6.12) follows (with an exponent smaller than $\eta$) from (6.11) and since this is enough to prove that the maximal function $M''$ (see (7.4)) is of weak type (1, 1) we do not have to consider the operators $L^{\underline{a}}$ and their transition probability functions in this case.

**7. Maximal functions.** Let $S_a=\{xe^b:x\in N,b>a\}$, and let $L$ be of the form (2.3) with $\alpha=1$ and $\kappa>0$. We assume that $L_0$ satisfies (6.7).

Let $\mu_a^b$, $b>a$, be harmonic measures on $N$. For a function $f\in L^p(N)$, $1\le p\le\infty$, we are going to study the harmonic functions $F(xe^b)=f\#\mu_a^b(x)$ and the maximal function

$$M_af(x)=\sup\{F(ye^b):b\ge a,|x^{-1}y|\le e^b\}\,.\tag{7.1}$$

Of course,

$$M_af(x)\le M_a'f(x)+M_a''f(x)\,,\tag{7.2}$$

where

(7.3) $$M'_a f(x) = \sup\{F(ye^b) : a+1 \geq b \geq a, |x^{-1}y| \leq e^b\},$$

(7.4) $$M''_a f(x) = \sup\{F(ye^b) : b > a+1, |x^{-1}y| \leq e^b\}.$$

First we prove

(7.5) THEOREM. *$M''_a$ is of weak type* (1, 1) *uniformly in* $a$.

Remark. If $\kappa$ in (2.3) is sufficiently large (6.12) follows only from Harnack's inequality and $M''$ is of weak type (1, 1) under a weaker assumption on $L$: for every $a$, $Y_1(a), \ldots, Y_m(a)$ in (3.0) and $L_1$ generate $\mathfrak{n}$.

Proof. First we observe that by (3.10)

$$M''_0(f \circ \delta_a) \circ \delta_{-a} = M''_a f.$$

Hence

$$|\{x : M''_a f(x) > \xi\}| \leq c\xi^{-1}\|f\|_{L^1} \quad \text{iff} \quad |\{x : M''_0 f(x) > \xi\}| \leq c\xi^{-1}\|f\|_{L^1}$$

with the same constant $c$. Thus we restrict our attention to $M''_0 = M''$.

Our next reduction is the following. By Harnack's inequality [B],

$$\sup\{F(ye^b) : 1 \leq b \leq 2, |y| \leq e^b\} \leq CF(e^1).$$

Putting ${}_{xe^n}F$ in place of $F$ we obtain

$$\sup\{F(ye^{n+b}) : 1 \leq b \leq 2, |x^{-1}y| \leq e^{n+b}\} \leq CF(xe^{n+1}).$$

Consequently,

(7.6) $$M''f(x) \leq C\sup\{f\#\mu_0^n(x) : n = 1, 2, \ldots\}.$$

Let us write $\nu = \mu_0^1$.

(7.7) LEMMA. *If $\eta > 0$ is as in Proposition* (3.14), *then*

(7.8) $$\int |\nu(hx) - \nu(x)|(1+\tau_N(x))^\eta\,dx \leq C\tau_N(h)^\eta,$$

(7.9) $$\int |\nu(xh) - \nu(x)|(1+\tau_N(x))^\eta\,dx \leq C\tau_N(h)^\eta.$$

Proof. Since $\check{\nu}$ is a harmonic function (7.8) follows immediately from Harnack's inequality and Proposition (3.14) while (7.9) follows from Theorem (6.9) and Proposition (3.14). ∎

For a function $f$ on $N$ we write $\delta_n f(x) = e^{-nQ}f(\delta_{-n}x)$.

(7.10) LEMMA. *Let $\nu_{-n} = \delta_{-n}\nu$ and*

$$\varphi(x) = \sup\Big\{\int \nu * \nu_{-1} * \ldots * \nu_{-n}(x) : n = 0, 1, \ldots\Big\}.$$

*Then*

(7.11) $$M''f(x) \leq \sup\Big\{\int f(xy)\delta_n\varphi(y)\,dy : n = 0, 1, \ldots\Big\}.$$



Proof. By (3.11) and (3.13),

$$\begin{aligned}\mu_0^n &= \mu_{n-1}^n * \mu_{n-2}^{n-1} * \ldots * \mu_0^1 = \delta_n\mu_0^1 * \delta_{n-1}\mu_0^1 * \ldots * \mu_0^1\\ &= \delta_n(\nu * \nu_{-1} * \ldots * \nu_{-n})\end{aligned}$$

and the proof follows from (7.6). ∎

The idea of the proof of the following lemma in the case when $N$ is abelian is due to Jarosław Wróblewski.

(7.12) LEMMA. *There exists $\varrho > 0$ such that*

$$\int \varphi(x)(1+\tau_N(x))^\varrho\,dx < \infty.$$

Proof. We write

$$\varphi(x) \leq \nu(x) + \sum_{k=1}^\infty |\nu * \ldots * \nu_{-k}(x) - \nu * \ldots * \nu_{-k+1}(x)|,$$

whence for $0 < \varepsilon < \eta$ of Proposition (3.14),

$$\begin{aligned}I = &\int \varphi(x)(1+\tau_N(x))^\varepsilon\,dx \leq \int \nu(x)(1+\tau_N(x))^\varepsilon\,dx\\ &+ \sum_{k=1}^\infty \int |\nu * \ldots * \nu_{-k}(x) - \nu * \ldots * \nu_{-k+1}(x)|(1+\tau_N(x))^\varepsilon\,dx.\end{aligned}$$

Let

(7.13) $$\varphi_k(x) = \nu_{-1} * \ldots * \nu_{-k+1}(x).$$

We estimate

$$\begin{aligned}I_k &= \int |\nu * \ldots * \nu_{-k}(x) - \nu * \ldots * \nu_{-k+1}(x)|(1+\tau_N(x))^\varepsilon\,dx\\ &\leq \int \Big|\int [\nu * \varphi_k(xy^{-1}) - \nu * \varphi_k(x)]\nu_{-k}(y)\,dy\Big|(1+\tau_N(x))^\varepsilon\,dx\\ &\leq \int\int\int |\nu(xy^{-1}z^{-1})\varphi_k(z) - \nu(xz^{-1})\varphi_k(z)|\,dz\,\nu_{-k}(y)\,dy(1+\tau_N(x))^\varepsilon\,dx.\end{aligned}$$

Replacing $x$ by $xz$ we obtain

$$\begin{aligned}I_k \leq \int\int\int |\nu(xzy^{-1}z^{-1}) - \nu(x)|(1+\tau_N(x))^\varepsilon\,dx&\\ \times\varphi_k(z)(1+\tau_N(z))^\varepsilon\,dz\,\nu_{-k}(y)dy&,\end{aligned}$$

whence, by (7.9),

$$I_k \leq \int\int \tau_N(zy^{-1}z^{-1})^\varepsilon\varphi_k(z)(1+\tau_N(z))^\varepsilon\,dz\,\nu_{-k}(y)\,dy.$$

But (cf. e.g. [D]),

$$\tau_N(zy^{-1}z^{-1}) \leq \tau_N(y^{-1})\|\,\mathrm{Ad}_z\,\| \leq \tau_N(y)(1+\tau_N(z))^q$$

for some $q$. Thus

(7.14) $$I_k \leq \int \varphi_k(z)(1+\tau_N(z))^{(q+1)\varepsilon}\,dz\int \tau_N(y)^\varepsilon\nu_{-k}(y)\,dy.$$

But if $\varepsilon$ is small enough, then for some $\zeta < \eta$

(7.15) $$\int \tau_N(y)^\varepsilon \nu_{-k}(y)\,dy \le e^{-k\zeta} \int \nu(y)|y|^\zeta\,dy\,,$$

where $|\,|$ is a subadditive homogeneous norm. Also there is a constant $C$ such that for every $k$

(7.16) $$\int \varphi_k(z)(1+\tau_N(z))^{(q+1)\varepsilon}\,dz \le C\,.$$

To prove (7.16) we note first that for some $0<\zeta<\eta$

$$\begin{aligned}\int \nu_{-r}(z)(1+\tau_N(z))^{(q+1)\varepsilon}\,dz &\le \int \nu_{-r}(z)(1+|z|^\zeta)\,dz\\ &\le \int \nu(z)(1+e^{-r\zeta}|z|^\zeta)\,dz \le 1+e^{-r\zeta}\int \nu(z)|z|^\zeta\,dz\,.\end{aligned}$$

Hence

$$\begin{aligned}\int \varphi_k(x)(1+\tau_N(x))^{(q+1)\varepsilon}\,dx &\le \int \nu_{-1}*\ldots*\nu_{-k+1}(x)(1+\tau_N(x))^{(q+1)\varepsilon}\,dx\\ &\le \prod_{r=1}^{\infty}\Big(1+e^{-r\zeta}\int \nu(z)|z|^\zeta\,dz\Big) \le C\,.\end{aligned}$$

Thus for $\varepsilon>0$ small enough and for appropriate $0<\zeta<\varepsilon$, by (7.14) and (7.16),

$$\sum_{k=1}^{\infty} I_k \le C\sum_{k=1}^{\infty} e^{-k\zeta} < \infty\,,$$

which completes the proof of Lemma 7.12. ∎

(7.17) Lemma. *There is a constant $C$ such that*

$$\begin{aligned}I &= \int |\nu(xhy)-\nu(xh)-\nu(xy)+\nu(x)|\,dx\\ &\le \min\{C, C\tau_N(h)\tau_N(y)(1+\tau_N(y))^q\}\end{aligned}$$

*for some $q$. Hence, for every $0<\alpha<1$,*

$$I \le C\tau_N(h)^\alpha \tau_N(y)^\alpha (1+\tau_N(y))^{\alpha q}\,.$$

Proof. Let $\|\cdot\|$ denote a euclidean norm in $\mathfrak{n}$. We write $y=\exp(\|y\|Y)$, $h=\exp(\|h\|H)$. Of course, $\|\cdot\|$ and $\tau_N$ are equivalent for small elements in $N$. We have

$$\begin{aligned}I &= \int \Big|\int_0^{\|y\|} Y\nu(xh\exp tY)\,dt - \int_0^{\|y\|} Y\nu(x\exp tY)\,dt\Big|\,dx\\ &\le \int_0^{\|y\|}\int |Y\nu(xh\exp tY)-Y\nu(x\exp tY)|\,dx\,dt\\ &= \int_0^{\|y\|}\int |Y\nu(x\exp tY\exp(\|h\|\operatorname{Ad}_{-tY}H)) - Y\nu(x\exp tY)|\,dx\,dt\\ &= \int_0^{\|y\|}\int\Big|\int_0^{\|h\|}(\operatorname{Ad}_{-tY}H)Y\nu(x\exp tY\exp(s\operatorname{Ad}_{-tY}H))\,ds\Big|\,dx\,dt\\ &\le \int_0^{\|y\|}\int_0^{\|h\|}\int |(\operatorname{Ad}_{-tY}H)Y\nu(x\exp tY\exp(s\operatorname{Ad}_{-tY}H))|\,dx\,ds\,dt\\ &\le \int_0^{\|y\|}\int_0^{\|h\|} |(\operatorname{Ad}_{-tY}H)Y\nu(x)|\,ds\,dx\,dt\\ &\le C\int_0^{\|y\|}\|(\operatorname{Ad}_{-tY}H)\|\|h\|\,dt \le C\|y\|(1+\|y\|)^q\|h\|\,.\end{aligned}$$

(7.18) Lemma. *Let $\varphi$ be as in Lemma (7.10). Then there exists $\varepsilon>0$ such that*

$$\int |\varphi(xh)-\varphi(x)|\,dx \le C\tau_N(h)^\varepsilon\,.$$

Proof. Let $\varphi_k$ be as in (7.13). Then

$$\begin{aligned}\int |\varphi(xh)-\varphi(x)|dx &\le \int \sup_k |\nu*\varphi_k*\nu_{-k}(xh)-\nu*\varphi_k*\nu_{-k}(x)|\,dx\\ &\le \int |\nu(xh)-\nu(x)|\,dx\\ &\quad+\sum_{k=1}^{\infty}\int |\nu*\varphi_k*\nu_{-k}(xh)-\nu*\varphi_k*\nu_{-k}(x)-\nu*\varphi_k(xh)+\nu*\varphi_k(x)|\,dx.\end{aligned}$$

But

$$\begin{aligned}&\int |\nu*\varphi_k*\nu_{-k}(xh)-\nu*\varphi_k*\nu_{-k}(x)-\nu*\varphi_k(xh)+\nu*\varphi_k(x)|\,dx\\ &\le \int |\nu*\varphi_k(xhy^{-1})-\nu*\varphi_k(xy^{-1})-\nu*\varphi_k(xh)+\nu*\varphi_k(x)|\nu_{-k}(y)\,dy\,dx\\ &\le \int |\nu(xhy^{-1}z^{-1})-\nu(xy^{-1}z^{-1})-\nu(xhz^{-1})+\nu(xz^{-1})|\\ &\qquad\qquad\times\varphi_k(z)\nu_{-k}(y)\,dz\,dy\,dx\\ &\le \int |\nu(xh^z(y^{-1})^z)-\nu(x(y^{-1})^z)-\nu(xh^z)+\nu(x)|\,dx\,\varphi_k(z)\nu_{-k}(y)\,dz\,dy\,,\end{aligned}$$

where $y^z = zyz^{-1}$. By Lemma (7.17),

$$\begin{aligned}&\int |\nu(xh^z(y^{-1})^z)-\nu(x(y^{-1})^z)-\nu(xh^z)+\nu(x)|\,dx\,\varphi_k(z)\nu_{-k}(y)\,dz\,dy\\ &\le \int C\tau_N(h^z)^\alpha\tau_N((y^{-1})^z)^\alpha(1+\tau_N((y^{-1})^z))^{\alpha q}\varphi_k(z)\nu_{-k}(y)\,dz\,dy\\ &\le C\int \tau_N(h)^\alpha\tau_N(y)^\alpha(1+\tau_N(y))^{\alpha q}(1+\|\operatorname{Ad}_z\|)^{\alpha(q+2)}\varphi_k(z)\nu_{-k}(y)\,dz\,dy\\ &\le C\tau_N(h)^\alpha\int(1+\|\operatorname{Ad}_z\|)^{\alpha(q+2)}\varphi_k(z)\,dz\\ &\qquad\times\int \tau_N(y)^\alpha(1+\tau_N(y))^{\alpha(q+2)}\nu_{-k}(y)\,dy\end{aligned}$$

$$\leq C\tau_N(h)^\alpha e^{-k\alpha(q+2)} \int (1+\|\operatorname{Ad}_z\|)^{\alpha(q+2)}\varphi_k(z)\,dz \cdot \int \tau_N(y)^\alpha \nu(y)\,dy\,,$$

which, for $\alpha > 0$ small enough, by (7.15) and (7.16) completes the proof of Lemma (7.18). ■

To complete the proof of Theorem (7.5) we recall Zo's lemma in the form which has been used by E. M. Stein and W. Hebisch (cf. [St], [He]).

(7.19) LEMMA (Zo). *Suppose that on a space of homogeneous type (with the metric $d(\cdot,\cdot)$) a family of kernels $\{K_n\}_{n\in\mathbf{Z}}$ is given. Suppose that*

$$\sup_{n,x} \int |K_n(x,y)|\,dy < \infty\,,$$

$$\sup_{z,y} \int_{d(x,y)>2d(y,z)} \sup_n |K_n(x,y)-K_n(x,z)|\,dx < \infty\,.$$

*Then the operator*

$$Kf(x) = \sup_n \Big| \int K_n(x,y)f(y)\,dy\Big|$$

*is of weak type* $(1,1)$.

The following lemma has been used by E. M. Stein [St] and is not difficult to prove.

(7.20) LEMMA. *Suppose a function $\varphi$ on a homogeneous group $N$ satisfies for some positive $\varrho$, $\varepsilon$ the following conditions*:

$$\int |\varphi(x)|(1+\tau_N(x))^\varrho\,dx < \infty\,, \qquad \int |\varphi(xh)-\varphi(x)|\,dx \leq C\tau_N(h)^\varepsilon\,.$$

*Then the kernels $K_n(x,y) = \delta_n\varphi(x^{-1}y)$ satisfy the conditions of Zo's lemma.*

Thus, by (7.11), (7.12) and (7.18) Theorem (7.5) follows. ■

To complete our study of the maximal function (7.1) we are going to assume that the operator (2.3) with $\alpha = 1$ has the property that $L_1$ is at most of order 2, i.e. $L_1$ is a linear combination of $X_j$, $j \in \Pi$, and of the commutators $[X_i, X_j]$, $i,j\in\Pi$. Under this assumption we prove

(7.21) THEOREM. *The maximal function $M'$ as defined in (7.3) is of weak type* $(1,1)$.

Proof. Let $G$ be the free group with the Lie algebra generated by $\mathcal{X}_j$, $j\in\Pi$, on which dilations are defined by $\delta_r\mathcal{X}_j = r\mathcal{X}_j$. Let $\sigma: G\to N$ be the homomorphism of $G$ onto $N$ such that

$$\sigma_*\mathcal{X}_j = \mathcal{X}_j\,, \quad j\in\Pi\,,$$

and $P^G(\underline{a};s,t)$ the transition probability function associated to

$$\mathcal{L} = \sum_{i,j\in\Pi} \alpha_{ij}e^{(d_i+d_j)\underline{a}(t)}\mathcal{X}_i\mathcal{X}_j + \sum_{j=1}^n \alpha_j e^{d_j\underline{a}(t)}\mathcal{X}_j - \partial_t\,.$$



For a function $f$ on $N$ and $x\in N$ we define

$$f * P^G(\underline{a};s,t)(x) = \int_G f(x\sigma(y))P^G(\underline{a};s,t,dy)\,.$$

We fix $a<b$ and we would like to estimate

$$M_a^b f(x) = \sup\Big\{\int |f(xy)|\mu_a^c(y)\,dy = f * \check{\mu}_a^c(x) : a\leq c\leq b\Big\}\,.$$

For a fixed trajectory $\underline{a}$ of the diffusion on $\mathbf{R}$ generated by $\partial_a^2 - \kappa\partial_a$ we let

$$T_a = T_a(\underline{a}) = \inf\{t : a(t) = a\}\,, \quad T_b = T_b(\underline{a}) = \inf\{t : a(t)=b\}\,.$$

For $f\geq 0$ we have

$$M_a^b f(x) = \sup_{a\leq c\leq b} E_c\{f * P(\underline{a};0,T_a,x)\} \leq \mathcal{M}_a^b f(x) + \mathcal{N}_a^b f(x)$$

where

$$\mathcal{M}_a^b f(x) = \sup_{a\leq c\leq b} E_c\{f*P(\underline{a};0,T_a,x); T_a<T_b\}\,,$$

$$\mathcal{N}_a^b f(x) = \sup_{a\leq c\leq b} E_c\{f*P(\underline{a};0,T_a,x); T_a>T_b\}\,.$$

Assume first that $T_a<T_b$. For $n=1,2,\ldots$ let

$${}^nM_a^b f(x) = \sup_{a\leq c\leq b} E_c\{f*P(\underline{a};0,T_a,x); n-1<T_a\leq n, T_a<T_b\}\,.$$

Then obviously

$$\mathcal{M}_a^b f(x) \leq \sum_{n=1}^\infty {}^nM_a^b f(x)\,.$$

Since by Theorems (5.14) and (5.21) for $t\leq 1$

$$P^G(\underline{a};s,t,x) \leq C(a,b)\min((t-s)^{-Q/2},(t-s)^{1/2}|x|^{-Q-1})\,,$$

we have

$$P^G(\underline{a};0,t,x)\leq k_t(x) \quad \text{for } t\leq 1\,,$$

where

$$k_1(x) = C(a,b)2^{Q+1}(1+|x|)^{-Q-1}\,,$$
$$k_t(x) = t^{-Q/2}k_1(\delta_{t^{-1/2}}(x))\,.$$

On the other hand, in view of [HJ]

$$\sup_{t\leq 1} f*k_t(x) \leq m^*(f)$$

where $m^*(f)$ is the Hardy–Littlewood maximal function, which is of weak type $(1,1)$. Therefore

$${}^1M_a^b f(x) \leq \sup_{a\leq c\leq b} E_c\{\sup_{t\leq 1} f*k_t(x); T_a\leq 1\} \leq Cm^*(f)$$

is of weak type (1, 1).

Let $\beta$ be such that $\int_G (1+\tau_G(y))^{-\beta}\,dy < \infty$. For $f \in L^1(N)$ we define

$$Rf(x) = \int\limits_G f(x\sigma(y)^{-1})(1+\tau_G(x))^{-\beta}\,dy\,, \quad x \in N\,.$$

Then

$$\|Rf\|_{L^1} \le \Big(\int\limits_G (1+\tau_G(y))^{-\beta}\,dy\Big)\|f\|_{L^1}$$

and by (5.20) for $t \ge 1$ and a $K$

$$f * P(\underline{a}; 0, t, x) \le C(a, b, \beta) t^K Rf(x)\,.$$

Hence

$${}^n M_a^b f(x) \le C(a, b, \beta) n^K Rf(x) P_c\{n-1 < T_a \le n\}\,.$$

But there is $q < 1$ such that for all $a \le c \le b$

$$P_c\{n-1 < T_a \le n\} \le q^n\,.$$

Therefore finally

$$\mathcal{M}_a^b f(x) \le C(m^*(f)(x) + Rf(x))$$

for a constant $C$ depending on $a$ and $b$.

On the other hand, if $T_b \le T_a$, then

$$\begin{aligned}
&\sup_{a\le c\le b} E_c\{f * P(\underline{a}; 0, T_b, y) * P(\underline{a}; T_b, T_a, x)\}\\
&\quad = \sup_{a\le c\le b} E_c\{E_c\{f * P(\underline{a}; 0, T_b, \cdot) * P(\underline{a}; T_b, T_a, x) \mid \mathcal{F}_{T_b}\}\}\\
&\quad = \sup_{a\le c\le b} E_c\{f * P(\underline{a}; 0, T_b, \cdot) * E_c\{P(\underline{a}; T_b, T_a, x) \mid \mathcal{F}_{T_b}\}\}\\
&\quad = \sup_{a\le c\le b} E_c\{f * P(\underline{a}; 0, T_b, \cdot) * E_b\{P(\underline{a}; 0, T_a, x)\}\}\\
&\quad = \sup_{a\le c\le b} E_c\{f * P(\underline{a}; 0, T_b, \cdot) * \check{\mu}_a^b(x)\}\,.
\end{aligned}$$

Again we write $\mathcal{N}_a^b f(x) \le \sum_{n=1}^{\infty} {}^n M_a^b f(x)$ but now

$${}^n M_a^b f(x) = \sup_{a\le c\le b} E_c\{f * P(\underline{a}; 0, T_b, \cdot) * \check{\mu}_a^b(x); n \ge T_b > n-1\}\,.$$

Therefore by Proposition (6.5)

$${}^1 M_a^b f(x) \le \sup_{a\le c\le b} E_c\{f * P^G(\underline{a}; 0, T_b, \cdot) * {}^G\check{\mu}_a^b(x); 1 \ge T_b\}$$

where ${}^G\check{\mu}_a^b$ is the harmonic measure (3.8) corresponding to $\mathcal{L}$. As before for $t \le T_b \le 1$

$$P^G(\underline{a}; 0, T_b, x) \le k_t(x)$$



and

$${}^1 M_a^b f(x) \le f * \sup_{t\le 1} k_t * {}^G\check{\mu}_a^b(x)\,.$$

We are going to show that

$$\psi(x) = \sup_{t\le 1} k_t * {}^G\check{\mu}_a^b(x) \in L^1(G)\,. \tag{7.22}$$

But for $|x| \ge 1$ and $t \le 1$

$$k_t(x) \le 2^{Q+1} C(a, b)|x|^{-Q-1}$$

so

$$\sup_{t\le 1} \int\limits_{|y|\ge 1} k_t(y)\, {}^G\check{\mu}_a^b(yx)\,dy$$

belongs to $L^1(G)$. On the other hand,

$$\sup_{t\le 1} \int\limits_{|y|\le 1} k_t(y)\, {}^G\check{\mu}_a^b(yx)\,dy \le \sup_{|y|\le 1} {}^G\check{\mu}_a^b(yx)\,.$$

Let $\mu_x(y) = {}^G\check{\mu}_a^b(yx)$. Then

$$\begin{aligned}
\sup_{|y|\le 1} \mu_x(y) &\le \int\limits_{|y|\le 2} |\partial_{x_1} \ldots \partial_{x_n} \mu_x(y)|\,dy\\
&\le \sum_{\beta} a_\beta \int\limits_{|y|\le 2} |\check{\partial}^\beta \mu_x(y)| \le \sum_{\beta} a_\beta 1_{|y|\le 2} * \check{\partial}^\beta\, {}^G\check{\mu}_a^b(x)
\end{aligned}$$

where the summation is over all multiindices such that $|\beta| \le n$, $\check{\partial}^\beta = \check{X}_1^{\beta_1} \ldots \check{X}_n^{\beta_n}$ with $\check{X}_1, \ldots, \check{X}_n$ right-invariant and $a_\beta$ are constants depending only on the group. Since $\check{\partial}^\beta\, {}^G\check{\mu}_a^b$ is integrable, (7.22) follows. As before

$${}^n M_a^b f(x) \le C(a, b, \beta) n^K Rf * \check{\mu}_a^b(x) P_c\{n < T_b \le n+1\}\,,$$

and the rest of the proof is as in the first case. ∎

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STUDIA MATHEMATICA 101 (1) (1991)

# On a dual locally uniformly rotund norm on a dual Vašák space

by

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**Abstract.** We transfer a renorming method of transfer, due to G. Godefroy, from weakly compactly generated Banach spaces to Vašák, i.e., weakly $\mathcal{K}$-countably determined Banach spaces. Thus we obtain a new construction of a locally uniformly rotund norm on a Vašák space. A further cultivation of this method yields the new result that every dual Vašák space admits a dual locally uniformly rotund norm.


**0. Introduction.** Let $V$ be a (subspace of a) weakly compactly generated Banach space. Then, according to Troyanski [10] modulo Amir and Lindenstrauss [1], $V$ has an equivalent locally uniformly rotund (LUR) norm. If $V$ is moreover a dual space, then it even admits a dual LUR norm [6]. However, the proof of the last fact is quite different; in fact, starting from [1], then a method of transfer due to Godefroy [5] is used.

Let us consider a more general situation when $V$ is a Vašák space, that is, $V$, provided with the weak topology, is countably $\mathcal{K}$-determined; see below for an exact definition. Then, replacing [1] by a result of Vašák [11], Troyanski's theorem [10] also yields a LUR norm on $V$. In this paper we show that *a Vašák space which is, moreover, dual admits an equivalent dual LUR norm*; thus a question raised in [4] is settled affirmatively. This assertion really extends the theorem from [6] mentioned above because Mercourakis has constructed a dual Vašák space which is not a subspace of a weakly compactly generated space [8].

Of course, a hopeful candidate for a proof of our result is the method of transfer. Indeed, it does work but we have to refine this approach in accordance with the more complicated structure of the Vašák spaces.

In the paper we consider three stages of complexity: from weakly compactly generated space through Vašák space to dual Vašák space. In the second section we reprove the well known facts that *a (dual) weakly compactly generated Banach space admits a (dual) LUR norm* [2, p. 164] ([6, Corollary 2.2]). We present here the method of transfer but we translate

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