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Ergodic properties of group extensions of dynamical systems with discrete spectra

by

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Abstract. Ergodic group extensions of a dynamical system with discrete spectrum are considered. The elements of the centralizer of such a system are described. The main result says that each invariant sub-σ-algebra is determined by a compact subgroup in the centralizer of a normal natural factor.

Introduction. In this paper, we shall be concerned with extending the results of [4] to ergodic isometric extensions of systems with discrete spectra. We will prove that each such system is a natural factor of an ergodic group extension and that other theorems of [4] are true in this more general case.

In [4], the structure of invariant sub-σ-algebras for ergodic abelian group extensions of transformations with discrete spectra has been described. Theorem 3 in [4] says that for any such sub-σ-algebra $\mathcal{C}$ there exists a compact subgroup $\mathcal{H}(\mathcal{C})$ in the centralizer of a natural factor of the original system such that $\mathcal{C}$ consists of exactly those subsets of this natural factor which are invariant with respect to all elements of $\mathcal{H}(\mathcal{C})$. The present paper includes a generalization of the above result to ergodic nonabelian group extensions of transformations with discrete spectra.

D. Newton in [5] has proved that each factor map of an ergodic abelian group extension of a transformation with discrete spectrum which preserves the base is of the form $\overline{S}(x,g) = (Sx, \theta_x(g))$, where $\theta : X \times G \to G$ splits into the product of a map from the base into the group and a continuous group epimorphism. We will prove that this result is also true in the nonabelian case. In particular, we will describe all elements in the centralizer of such a system.

All results in this paper follow from the specific structure of ergodic self-joinings of ergodic group extensions of transformations with discrete spectra. The joinings turn out to be natural, namely, each of them is the

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relatively independent extension of an isomorphism between two normal natural factors of the original group extension.

I. Definitions and notations. We will use "joinings" as the basic tools to prove our theorems (see for instance [1]). If $T_i : (X_i, \mathcal{B}_i, \mu_i) \to (X_i, \mathcal{B}_i, \mu_i)$, $i = 1, \ldots, n$, are ergodic automorphisms and λ is a $T_1 \times \ldots \times T_n$-invariant measure on $X_1 \times \ldots \times X_n$ such that for each $i = 1, \ldots, n$ and each $A_i \in \mathcal{B}_i$

$$\lambda(X_1 \times \ldots \times X_{i-1} \times A_i \times X_{i+1} \times \ldots \times X_n) = \mu_i(A)$$

then λ is called an *n-joining* of $T_1, \ldots, T_n$. The set of all n-joinings of $T_1, \ldots, T_n$ will be denoted by $J(T_1, \ldots, T_n)$. The subset of $J(T_1, \ldots, T_n)$ consisting of all ergodic measures will be denoted by $J^e(T_1, \ldots, T_n)$. Observe that if $\lambda \in J(T_1, \ldots, T_n)$ has the ergodic decomposition

$$\lambda = \int_{E(T_1, \ldots, T_n)} m \, d\tau(m)$$

with $E(T_1, \ldots, T_n)$ being the set of all ergodic measures on $\mathcal{B}_1 \otimes \ldots \otimes \mathcal{B}_n$, then $\tau(J^e(T_1, \ldots, T_n)) = 1$. Therefore the ergodic components of n-joinings are n-joinings. In particular, $J^e(T_1, \ldots, T_n)$ is nonempty since $\mu_1 \times \ldots \times \mu_n \in J(T_1, \ldots, T_n)$.

If $T_1 = \ldots = T_n = T$, then λ is called an *n-self-joining*. If $n = 2$ we speak (for short) about *joinings* (instead of 2-joinings).

Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic automorphism. By the *centralizer*, $C(T)$, of T we mean the set of all $S : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ commuting with T, i.e. $ST = TS$. This set is endowed with the *weak topology* given by

$$S_n \to S \quad \text{iff} \quad \mu(S_n^{-1}(A) \triangle S^{-1}(A)) \to 0 \text{ for each } A \in \mathcal{B}.$$

For $S \in C(T)$, we define the corresponding (ergodic) *graph joining* μ_S defined on rectangles as

$$\mu_S(A \times B) = \mu(A \cap S^{-1} B).$$

Let X be a compact metric monothetic group with the family $\mathcal{B}$ of Borel sets and with the normalized Haar measure μ. Let $T : X \to X$, $T(x) = ax$ for some $a \in X$. Then the dynamical system $(X, \mathcal{B}, \mu, T)$ has discrete spectrum, i.e. the set of all eigenfunctions of the unitary operator $U_T : L^2(X, \mu) \to L^2(X, \mu)$, $U_T(f) = f \circ T$, is linearly dense in $L^2(X, \mu)$. Assume that T is ergodic.

Let G be a compact metric group (not necessarily abelian) equipped with the normalized Haar measure $\nu = \nu_G$ on the family $\mathcal{D}$ of Borel subsets of G. Define $\widetilde{\mathcal{B}} = \mathcal{B} \otimes \mathcal{D}$. There is a natural right action of G on $\widetilde{\mathcal{B}}$ coming from the natural right action of G on $X \times G$ given by

$$(x, h)g = (x, hg).$$

Let $\widetilde{\mu} = \mu \times \nu$. For a measurable function $\varphi : X \to G$ we define a transformation $T_\varphi : (X \times G, \widetilde{\mathcal{B}}, \widetilde{\mu}) \to (X \times G, \widetilde{\mathcal{B}}, \widetilde{\mu})$ setting

$$T_\varphi(x, g) = (T(x), \varphi(x)g). \tag{1}$$

Then T_φ is a measure-preserving invertible transformation. Such a transformation is called a *group extension*, or, indicating the group, a *G-extension* of T.

If λ is a T_φ-invariant measure on $X \times G$ then for $g \in G$ we define (T_φ-invariant) measures λg and $g\lambda$ on $X \times G$ setting

$$\lambda g(A \times B) = \lambda(A \times Bg^{-1}), \quad g\lambda(A \times B) = \lambda(A \times g^{-1}B)$$

for $A \in \mathcal{B}$, $B \in \mathcal{D}$. An equivalent definition of λg and $g\lambda$ is the following: for each continuous function f on $X \times G$

$$\int f \, d(\lambda g) = \int f \circ g \, d\lambda, \qquad \int f \, d(g\lambda) = \int g \circ f \, d\lambda,$$

where $f \circ g(x, h) = f(x, hg)$, $g \circ f(x, h) = f(x, gh)$.

If F is a closed subgroup of G then we can consider the largest sub-σ-algebra $\widetilde{\mathcal{B}}_F$ of $\widetilde{\mathcal{B}}$ which is F-invariant, i.e. $\widetilde{\mathcal{B}}_F$ is the largest sub-σ-algebra of $\widetilde{\mathcal{B}}$ satisfying

$$A \in \widetilde{\mathcal{B}}_F \quad \text{implies} \quad Ag = A \text{ for all } g \in F.$$

It is clear that $\widetilde{\mathcal{B}}_F$ is equal (up to an obvious identification) to the family of all Borel subsets of $X \times G/F$. Obviously $\widetilde{\mathcal{B}}_F$ is T_φ-invariant. Therefore $\widetilde{\mathcal{B}}_F$ is a factor of $T_\varphi : (X \times G, \widetilde{\mathcal{B}}, \mu \times \nu) \to (X \times G, \widetilde{\mathcal{B}}, \mu \times \nu)$. The map $T_\varphi : (X \times G/F, \mu \times \nu) \to (X \times G/F, \mu \times \nu)$ will be denoted by $T_{\varphi,F}$ and called a *natural factor* of T_φ. The transformation $T_{\varphi,F}$ will also be called an *isometric extension* of T. If F is normal in G, then we call $\widetilde{\mathcal{B}}_F$, or $T_{\varphi,F}$, a *normal natural factor* of T_φ.

If $A \in \widetilde{\mathcal{B}}$ then we denote by $E(A \mid F)$ the conditional expectation of the characteristic function of A with respect to $\widetilde{\mathcal{B}}_F$.

II. Results

THEOREM 1. *Each ergodic isometric extension of an ergodic rotation on a compact monothetic group is a natural factor of some ergodic group extension of the rotation.*

Let G_i, $i = 1, 2$, be compact metric groups equipped with normalized Haar measures ν_i, $i = 1, 2$. Assume that $\varphi_i : X \to G_i$, $i = 1, 2$, are measurable maps such that T_{φ_i}, $i = 1, 2$, are ergodic. Assume that $\lambda \in J^e(T_{\varphi_1}, T_{\varphi_2})$.

THEOREM 2. *There exist closed normal subgroups $H_1 \subset G_1$, $H_2 \subset G_2$, a continuous group isomorphism $v : G_1/H_1 \to G_2/H_2$, an $S \in C(T)$ and a* *measurable function* $f : X \to G_2/H_2$ *such that*

$$\lambda(A\times B)=\int\limits_{X\times G_1/H_1} E(A\mid H_1)(x,gH_1)\times E(B\mid H_2)(S(x),f(x)v(gH_1))\,d(\mu\times\nu_1)(x,gH_1)$$

for all $A\in\mathcal{B}\otimes\mathcal{D}_1$, $B\in\mathcal{B}\otimes\mathcal{D}_2$.

THEOREM 3. *If* T_{φ_2} *is a factor of* T_{φ_1} *via a map* $\overline{S}$ *then there exist an* $S\in C(T)$, *a measurable map* $f : X\to G_2$ *and a continuous group epimorphism* $v : G_1\to G_2$ *such that*

$$\overline{S}=S_{f,v}\,,\quad i.e.\quad \overline{S}(x,g)=(Sx,f(x)v(g))\,.$$

Assume that $T_\varphi : (X\times G,\widetilde{\mu})\to(X\times G,\widetilde{\mu})$ is an ergodic automorphism.

THEOREM 4. *If* $\overline{S}\in C(T_\varphi)$ *then there exist an* $S\in C(T)$, *a measurable function* $f : X\to G$ *and a continuous group epimorphism* $v : G\to G$ *satisfying*

$$\overline{S}(x,g)=S_{f,v}(x,g)=(S(x),f(x)v(g))\,.$$

Let $\mathcal{C}\subset\widetilde{\mathcal{B}}$ be a T_φ-invariant sub-σ-algebra. For a closed subgroup F of G, define

$$F(\mathcal{C})=\{U\in C(T_{\varphi,F}) : U^{-1}(A)=A \text{ for each } A\in\mathcal{C}\}\,.$$

THEOREM 5. *There exists a closed normal subgroup* $F\subset G$ *such that*

$$\mathcal{C}=\{A\in\widetilde{\mathcal{B}}_F : U^{-1}(A)=A \textit{ for all } U\in F(\mathcal{C})\}$$

and $F(\mathcal{C})$ *is a compact subgroup of* $C(T_{\varphi,F})$.

III. Proofs. Assume that λ is a T_φ-ergodic component of $T_\varphi : (X\times G,\widetilde{\mathcal{B}},\widetilde{\mu})\to(X\times G,\widetilde{\mathcal{B}},\widetilde{\mu})$ where $T : (X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is an ergodic rotation. It follows from [2] that there exists a closed subgroup H of G such that $(X\times G,\widetilde{\mathcal{B}},\lambda,T_\varphi)$ is isomorphic to some H-extension of T. In other words, $(X\times G,\lambda,T_\varphi)$ is isomorphic to $(X\times H,\mu\times\nu_H,T_\psi)$ for some measurable $\psi : X\to H$. In what follows we will need more information about ψ so, in fact, we will give a new proof of Th. 2.3 of [2]. To this end, we recall three lemmata from [4]. Although, in [4], they are formulated for the abelian case, their proofs work in the nonabelian case as well.

Let H be the *stabilizer* of λ in G, i.e. $H=\{g\in G : \lambda g=\lambda\}$.

LEMMA 1 ([4]). (i) *H is a closed subgroup of G.*
(ii) *If* $(x,g),(x,h)\in H$ *then* $hH=gH$. ∎

Let us decompose λ over the factor $(X,\mathcal{B},\mu)$:

$$\lambda=\int\limits_X \lambda_x\,d\mu(x)\,. \tag{2}$$

LEMMA 2 ([4]). *For μ-almost each $x\in X$ there is a* $g=g_x\in G$ *such that*

$$\lambda_x=\delta_x\times g\nu_H\,,$$

where δ_x is the Dirac measure on X concentrated at x. ∎

Define a function $\tau : X\to G/H$ by

$$\tau(x)=g_xH\,. \tag{3}$$

The map τ is measurable because by Lemma 2, $(X\times G/H,\lambda,T_\varphi)$ is isomorphic to (X,μ,T) and this isomorphism has the form $j : (X,\mu,T)\to(X\times G/H,\lambda,T_\varphi)$, $j(x)=(x,\tau(x))$. This forces τ to be measurable.

Observe that the T_φ-invariance of λ implies

$$\tau(Tx)=\varphi(x)\tau(x)\,. \tag{4}$$

LEMMA 3 ([4]). *There is a measurable function* $t : X\to G$ *such that the system* $(X\times G,\lambda,T_\varphi)$ *is isomorphic to* $(X\times H,\mu\times\nu_H,T_\psi)$, *where* $\psi(x)=t(Tx)^{-1}\varphi(x)t(x)$. ∎

Proof of Theorem 1. Let $(X\times G,\widetilde{\mu},T_\varphi)$ be a G-extension of an ergodic rotation (X,μ,T). Assume that $(X\times G/F,\widetilde{\mu},T_{\varphi,F})$ is an ergodic natural factor of $(X\times G,\widetilde{\mu},T_\varphi)$. Let λ be a T_φ-ergodic component of $\widetilde{\mu}$ such that $(X\times G/F,\widetilde{\mu},T_{\varphi,F})$ is a factor of $(X\times G,\lambda,T_\varphi)$. Such a λ exists because almost all ergodic components of $\widetilde{\mu}$ enjoy this property. Then by Lemma 3, there exist a closed subgroup $H\subset G$ and a measurable function $\psi : X\to H$ such that $(X\times G,\lambda,T_\varphi)$ is isomorphic to $(X\times H,\mu\times\nu_H,T_\psi)$. We will prove that $T_{\varphi,F}$ is a natural factor of T_ψ. More precisely, we will show that $(X\times G/F,\widetilde{\mu},T_{\varphi,F})$ is isomorphic to $(X\times H/H\cap F,\mu\times\nu_H,T_{\psi,H\cap F})$.

Let $W_{F\cap H} : G/F\cap H\to G$ be a measurable selector for the natural map $p : G\to G/F\cap H$ (see [3]), i.e. $W_{F\cap H}$ satisfies

$$W_{F\cap H}(g(F\cap H))F\cap H=g(F\cap H)\,. \tag{5}$$

First, we prove the following:

(A) $\qquad gF\cap\tau(x)\neq\emptyset\quad$ for $\widetilde{\mu}$-a.e. $(x,y)\in X\times G$,

Indeed, let $A=\{(x,g) : gF\cap\tau(x)\neq\emptyset\}$. Define $\overline{A}=(\mathrm{Id}\times p)(A)$, $B=\bigcup_{x\in X}\{x\}\times\tau(x)$, $\overline{B}=(\mathrm{Id}\times p)(B)$. Then $\lambda(B)=1=\widetilde{\mu}(\overline{B})$. Moreover, $\overline{B}\subset\overline{A}$, because if $(x,g_xhF)\in\overline{B}$, where $\tau(x)=g_xH$, then $g_xh\in g_xhF\cap\tau(x)$. Thus $\widetilde{\mu}(\overline{A})=1$ and for $\widetilde{\mu}$-a.e. $(x,g)\in X\times G$, $(x,gF)\in\overline{A}$. This implies that $\widetilde{\mu}(A)=1$, and (A) is proved.

Let $W : X\times G/F\to X\times G$ be given by

$$W(x,gF)=(x,W_{F\cap H}(gF\cap\tau(x)))\,. \tag{6}$$

By (A), W is well defined. Consider the diagram

By (A), W is well defined. Consider the diagram

$$\begin{array}{ccc} (X\times G,\lambda,T_\varphi) & \xleftarrow{W} & (X\times G/F,\widetilde{\mu},T_{\varphi,F}) \\ \big\downarrow{\scriptstyle J^{-1}} & & \big\downarrow{\scriptstyle R} \\ (X\times H,\mu\times\nu_H,T_\psi) & \xrightarrow{\mathrm{Id}\times\pi} & (X\times H/F\cap H,\mu\times\nu_H,T_{\psi,F\cap H}) \end{array}$$

where $\pi: H\to H/F\cap H$ is the natural projection while $R=(\mathrm{Id}\times\pi)J^{-1}W$. Then for any set of the form $\{x\}\times h(F\cap H)\subset X\times H$, the inverse image $(J^{-1}W)^{-1}(\{x\}\times h(F\cap H))\subset X\times G/F$ consists of a single point, namely, $(x,t(x)hF)$. Therefore R is a one-to-one map.

Now, we prove that

(B) $$R^*\widetilde{\mu}=\mu\times\nu_H\,.$$

To do this we show two other formulas. First, for each $x\in X$ and for any set $B\subset H$

(C) $$W^{-1}(\{x\}\times t(x)B(F\cap H))=\{x\}\times t(x)BF\,.$$

Indeed, take $(x,gF)\in W^{-1}(\{x\}\times t(x)B(F\cap H))$. Then there are $b\in B$ and $s\in F\cap H$ such that $W(x,gF)=(x,t(x)bs)$. By (6), $W(x,gF)=(x,W_{F\cap H}(gF\cap g_xH))=(x,gf)$ for some $f\in F$, where $g_xH=\tau(x)$. Thus $t(x)bs=gf$, which implies $t(x)bF=gF$. Thus $(x,gF)\in\{x\}\times t(x)BF$.

On the other hand, if $(x,t(x)bF)\in\{x\}\times t(x)BF$ then

$$W(x,t(x)bF)=(x,W_{F\cap H}(t(x)bF\cap g_xH))=(x,t(x)bf)$$

for some $f\in F$ and $W(x,t(x)bF)=(x,g_xh)$ for some $h\in H$. Thus $t(x)bf=g_xh$. By the definition of the function t (see [4]), $t(x)=g_xh_0$ for some $h_0\in H$. Therefore $f=b^{-1}h_0^{-1}h\in F\cap H$, i.e. $t(x)bf\in t(x)B(F\cap H)$. We have shown that $(x,t(x)bF)\in W^{-1}(\{x\}\times t(x)B(F\cap H))$, which proves (C).

Now, we will show that

(D) for each $g\in G$ there exist $f_g\in F$, $h_g\in H$ such that $g=h_gf_g$.

Indeed, because $(X\times G/F,\widetilde{\mu},T_{\varphi,F})$ is a factor of $(X\times G,\lambda,T_\varphi)$ and $\lambda(\bigcup_{x\in X}\{x\}\times g_xH)=1$, we have (denoting by $\overline{p}$ the natural projection $G\to G/F$)

$$\begin{aligned}1&=\widetilde{\mu}\Big((\mathrm{Id}\times\overline{p})\Big(\bigcup_{x\in X}\{x\}\times g_xH\Big)\Big)=\widetilde{\mu}\Big(\bigcup_{x\in X}\{x\}\times g_xHF\Big)\\&=\int_X(\delta_x\times\nu)\Big(\bigcup_{x\in X}\{x\}\times g_xHF\Big)\,d\mu(x)=\int_X\nu\Big(\bigcup_{x\in X}g_xHF\Big)\,d\mu(x)\\&=\int_X\nu(HF)\,d\mu(x)=\nu(HF)\,.\end{aligned}$$

Thus $HF=G$, which proves (D).

Now, we can prove (B). Let $A\subset X$, $B\subset H$. Using (C), we have

$$\begin{aligned}R^*\widetilde{\mu}(A\times B(F\cap H))&=\widetilde{\mu}(R^{-1}(A\times B(F\cap H)))\\&=\int_X(\delta_x\times\nu)\Big(W^{-1}J(A\times B(F\cap H))\Big)\,d\mu(x)\\&=\int_X(\delta_x\times\nu)\Big(W^{-1}J\Big(\bigcup_{a\in A}\{a\}\times B(F\cap H)\Big)\Big)\,d\mu(x)\\&=\int_X(\delta_x\times\nu)\Big(\bigcup_{a\in A}W^{-1}(\{a\}\times t(a)B(F\cap H))\Big)\,d\mu(x)\\&=\int_X(\delta_x\times\nu)\Big(\bigcup_{a\in A}\{a\}\times t(a)BF\Big)\,d\mu(x)\\&=\int_A\nu(t(a)BF)\,d\mu(a)=\int_A\nu(BF)\,d\mu(x)=\mu(A)\nu(BF)\,.\end{aligned}$$

By (D),

$$\begin{aligned}\nu(BF)&=\int_G\nu_H(BFg^{-1})\,d\nu(g)=\int_G\nu_H(BFf_g^{-1}h_g^{-1})\,d\nu(g)\\&=\int_G\nu_H(BFh_g^{-1})\,d\nu(g)=\int_G\nu_H(BF)\,d\nu(g)\\&=\nu_H(BF)=\nu_H((BF)\cap H)=\nu_H(B(F\cap H))\end{aligned}$$

because $B\subset H$. Thus

$$R^*\widetilde{\mu}(A\times B(F\cap H))=\mu(A)\nu_H(B(F\cap H))=(\mu\times\nu_H)(A\times B(F\cap H))\,,$$

which proves (B).

To finish the proof of Theorem 1 we show that

$$R\circ T_{\varphi,F}=T_{\psi,F\cap H}\circ R\,.$$

By Lemma 3, it is enough to prove that

(*) $$t(Tx)^{-1}\varphi(x)W_{F\cap H}(gF\cap\tau(x))F\cap H=t(Tx)^{-1}W_{F\cap H}(\varphi(x)gF\cap\tau(Tx))F\cap H\,.$$

Obviously $t(Tx)^{-1}W_{F\cap H}(\varphi(x)gF\cap\tau(Tx))\in H$ and $t(Tx)^{-1}\varphi(x)\cdot W_{F\cap H}(gF\cap\tau(x))\in H$. Moreover, by (4),

$$t(Tx)^{-1}W_{F\cap H}(\varphi(x)gF\cap\tau(Tx))F\cap H=t(Tx)^{-1}\varphi(x)W_{F\cap H}(gF\cap\tau(x))F\cap H$$

and (*) is proved.

We have shown that R is an isomorphism and therefore $T_{\varphi,F}$ is a natural factor of T_ψ. The proof of Theorem is complete. ∎

To prove Theorem 2 we will need some lemmata.

LEMMA 4 ([4]). *If* $T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ *is an ergodic automorphism with discrete spectrum then* $C(T)$ *is a group and*

$$J^e(T,T)=\{\mu_S : S\in C(T)\}\,. \ \blacksquare$$

Assume that G_1 and G_2 are compact metric groups equipped with the normalized Haar measures ν_1 and ν_2 respectively. Let $\varphi_i : X\to G_i$ be a measurable map such that T_{φ_i} is ergodic, $i=1,2$. Assume that $\lambda\in J^e(T_{\varphi_1},T_{\varphi_2})$. Denote by π the map $\pi : X\times G_1\times X\times G_2\to X\times X$, $\pi(x,g,y,h)=(x,y)$. Then, by Lemma 4, $\pi^*\lambda=\mu_S$ for some $S\in C(T)$. Therefore

$$\lambda\Big(\bigcup_{x\in X}\{x\}\times G_1\times\{Sx\}\times G_2\Big)=1\,.$$

We define a measure $\widetilde{\lambda}$ on $X\times G_1\times G_2$ setting

$$\widetilde{\lambda}(A\times B\times C)=\lambda(A\times B\times SA\times C)\,. \tag{7}$$

Then $(X\times G_1\times X\times G_2,\lambda,T_{\varphi_1}\times T_{\varphi_2})$ is isomorphic to $(X\times G_1\times G_2,\widetilde{\lambda},T_{\varphi_1\times\varphi_2\circ S})$. In what follows, we will consider $T_{\varphi_1\times\varphi_2\circ S}$ and the measure $\widetilde{\lambda}$ on $X\times G_1\times G_2$. Let $H\subset G_1\times G_2$ be the stabilizer of $\widetilde{\lambda}$: $H=\{(g_1,g_2)\in G_1\times G_2 : \widetilde{\lambda}(g_1,g_2)=\widetilde{\lambda}\}$. By (2) and Lemma 2,

$$\widetilde{\lambda}=\int_X \delta_x\times(g_x^1,g_x^2)\nu_H\,d\mu(x)\,,$$

where $(g_x^1,g_x^2)H=\tau(x)$.

Let

$$H_1=\{g_1\in G_1:(g_1,e_2)\in H\}\,,\qquad H_2=\{g_2\in G_2:(e_1,g_2)\in H\}\,,$$

where e_i denotes the unit element of the group G_i, $i=1,2$.

Let $\pi_i : G_1\times G_2\to G_i$, $\pi_i(g_1,g_2)=g_i$, $i=1,2$.

LEMMA 5. $\pi_1(H)=G_1$, $\pi_2(H)=G_2$.

Proof. We will prove that $\pi_1^*\nu_H=\nu_1$. Indeed, let $A\subset G_1$. Then

$$\begin{aligned}\nu_1(A)&=\lambda(X\times A\times X\times G_2)\\ &=\widetilde{\lambda}(X\times A\times G_2)=\int_X(g_x,\overline{g}_x)\nu_H(A\times G_2)\,d\mu(x)\\ &=\int_X\nu_H(g_x^{-1}A\times G_2)\,d\mu(x)=\int_X\pi_1^*\nu_H(g_x^{-1}A)\,d\mu(x)\,.\end{aligned}$$

Define $M=\{x:\pi_1^*\nu_H(g_x^{-1}A)<\nu_1(A)\}$. Assume that $\mu(M)>0$. Then

$$\mu(M)\cdot\nu_1(A)=\widetilde{\lambda}(M\times A\times G_2)=\int_M\pi_1^*\nu_H(g_x^{-1}A)\,d\mu(x)<\mu(M)\cdot\nu_1(A)\,,$$

a contradiction. In the same way we prove that $\mu(\{x:\pi_1^*\nu_H(g_x^{-1}A)>\nu_1(A)\})=0$. Therefore for μ-a.e. $x\in X$ we have $\pi_1^*\nu_H(g_x^{-1}A)=\nu_1(A)$. This is equivalent to $(g_x\pi_1^*\nu_H)(A)=\nu_1(A)$, but this is the same as $\pi_1^*\nu_H(A)=(g_x^{-1}\nu_1)(A)=\nu_1(A)$.

Therefore $\pi_1^*\nu_H=\nu_1$ and this implies that $\pi_1H=G_1$. The proof of the equality $\pi_2H=G_2$ is similar. ■

The following lemma is an immediate consequence of Lemma 5.

LEMMA 6. *The subgroup* H_1 (H_2) *is normal in* G_1 (G_2). ■

LEMMA 7. (a) *If* $(g_1,g_2)\in H$, $(g_1,\widetilde{g}_2)\in H$, *then* $\widetilde{g}_2^{-1}g_2\in H_2$.

(b) *If* $(g_1,g_2)\in H$, $(\widetilde{g}_1,g_2)\in H$, *then* $\widetilde{g}_1^{-1}g_1\in H_1$.

(c) $(g_1,g_2)\in H$ *iff* $g_1H_1\times g_2H_2\subset H$.

Proof. (a) Assume that $(g_1,g_2),(g_1,\widetilde{g}_2)\in H$. Then $(g_1^{-1},\widetilde{g}_2^{-1})\in H$ and $H\ni(g_1^{-1},\widetilde{g}_2^{-1})(g_1,g_2)=(e_1,\widetilde{g}_2^{-1}g_2)$. Therefore $\widetilde{g}_2^{-1}g_2\in H_2$.

The proof of (b) is similar.

(c) Assume that $(g_1,g_2)\in H$. Take $h_1\in H_1$, $h_2\in H_2$. Then $(h_1,e_2)\in H$, $(e_1,h_2)\in H$ and $(h_1,h_2)=(h_1,e_2)(e_1,h_2)\in H$. Therefore $H\ni(g_1,g_2)(h_1,h_2)=(g_1h_1,g_2h_2)$. Thus $g_1H_1\times g_2H_2\subset H$. ■

We define a map $v:G_1/H_1\to G_2/H_2$ by

$$v(g_1H_1)=\pi_2((g_1H_1\times G_2)\cap H)\,.$$

LEMMA 8. *The map* v *is a continuous group isomorphism.*

Proof. By Lemma 7, v is well defined. The continuity of v is evident. Obviously v is bijective. We will prove that v is a group homomorphism.

Since $H_1\times H_2\subset H$, $v(H_1)=H_2$. Take $gH_1,\overline{g}H_1\in G_1/H_1$. Set $v(gH_1\overline{g}H_1)=\widetilde{g}H_2$, $v(gH_1)=g_1H_2$, $v(\overline{g}H_1)=\overline{g}_1H_2$. Then $g\overline{g}H_1\times\widetilde{g}H_2\subset H$. Moreover, $gH_1\times g_1H_2\subset H$, $\overline{g}H_1\times\overline{g}_1H_2\subset H$, which implies $gH_1\overline{g}H_1\times g_1H_2\overline{g}_1H_2\subset H$. Thus $g_1\overline{g}_1H_2=\widetilde{g}H_2$, i.e. $v(gH_1\overline{g}H_1)=v(gH_1)v(\overline{g}H_1)$.

As an immediate consequence of Lemmas 7 and 8 we have

LEMMA 9. $H=\bigcup_{g\in G}gH_1\times v(gH_1)$. ■

Our next aim is to define an isomorphism $\overline{S}$ of T_{φ_1,H_1} and T_{φ_2,H_2}. It will have the form

$$\overline{S}=S_{f,v}:X\times G_1/H_1\to X\times G_2/H_2\,,$$
$$S_{f,v}(x,gH_1)=(Sx,f(x)v(gH_1))\,,$$

where $S\in C(T)$ and $f:X\to G_2/H_2$ is a measurable map.

We will need the following

LEMMA 10. *If* $(h_1,h_2)\in H$ *then* $h_2v(h_1^{-1}H_1)=H_2$.

Proof. By assumption and Lemma 7, $h_1H_1 \times h_2H_2 \subset H$. Therefore $v(h_1H_1) = h_2H_2$ or, which is the same, $h_2 v(h_1^{-1}H_1) = H_2$. ∎

Proof of Theorem 2. Let $\alpha : (G_1 \times G_2)/H \to G_2/H_2$ be given by $\alpha((g_1, g_2)H) = g_2 v(g_1^{-1}H_1)$. By Lemma 10, α is well defined. Let $f(x) = \alpha(\tau(x))$, where τ satisfies (4) for $\varphi = \varphi_1 \times \varphi_2 \circ S$.

Put $\overline{S} = S_{f,v}$. It is clear that $\overline{S}^*(\widetilde{\mu}) = \mu \times \nu_2$ and $\overline{S}$ is $(\mu \times \nu_1)$-a.e. one-to-one. The only fact we have to prove is that $\overline{S} \circ T_{\varphi_1, H_1} = T_{\varphi_2, H_2} \circ \overline{S}$.

Take $(x, gH_1) \in X \times G_1/H_1$. By (4), applied to $\varphi = \varphi_1 \times \varphi_2 \circ S$, we have $\tau(Tx) = (\varphi_1(x), \varphi_2(Sx))\tau(x)$ and therefore

$$\begin{aligned} S_{f,v} \circ T_{\varphi_1,H_1}(x, gH_1) &= S_{f,v}(Tx, \varphi_1(x)gH_1) = (STx, f(Tx)v(\varphi_1(x)gH_1)) \\ &= (STx, f(Tx)v(\varphi_1(x)H_1)v(gH_1)) \\ &= (STx, \alpha(\tau(Tx))v(\varphi_1(x)H_1)v(gH_1)) \\ &= (STx, \alpha((\varphi_1(x), \varphi_2(Sx))\tau(x))v(\varphi_1(x)H_1)v(gH_1)) . \end{aligned}$$

Set $\tau(x) = (g_1, g_2)H$. Then

$$\begin{aligned} &S_{f,v} \circ T_{\varphi_1,H1}(x, gH_1) \\ &= (STx, \alpha((\varphi_1(x), \varphi_2(Sx))(g_1, g_2)H)v(\varphi_1(x)H_1)v(gH_1)) \\ &= (STx, \alpha((\varphi_1(x)g_1, \varphi_2(Sx)g_2)H)v(\varphi_1(x)H_1)v(gH_1)) \\ &= (STx, \varphi_2(Sx)g_2v(g_1^{-1}\varphi_1(x)^{-1}H_1)v(\varphi_1(x)H_1)v(gH_1)) \\ &= (STx, \varphi_2(Sx)g_2v(g_1^{-1}H_1)v(gH_1)) \\ &= (STx, \varphi_2(Sx)\alpha(\tau(x))v(gH_1)) \\ &= (STx, \varphi_2(Sx)f(x)v(gH_1)) . \end{aligned}$$

On the other hand,

$$\begin{aligned} T_{\varphi_2,H_2} \circ S_{f,v}(x, gH_1) &= T_{\varphi_2,H_2}(Sx, f(x)v(gH_1)) \\ &= (TSx, \varphi_2(Sx)f(x)v(gH_1)) \\ &= (STx, \varphi_2(Sx)f(x)v(gH_1)) \\ &= S_{f,v} \circ T_{\varphi_1,H_1}(x, gH_1) . \end{aligned}$$

Therefore $S_{f,v}$ is an isomorphism.

By (7),

$$\widetilde{\lambda} = \int_X \delta_x \times (g_x^1, g_x^2)\nu_H \, d\mu(x) ,$$

where $(g_x^1, g_x^2)H = \tau(x)$ and $S_{f,v}(x, g_x^1H_1) = (Sx, g_x^2H_2)$. From Lemma 9,

$$\Pi^*_{H_1,H_2}\widetilde{\lambda} = \int_X \delta_x \times ((g_x^1, g_x^2)\nu_1)_v \, d\mu(x) ,$$

where $\Pi_{H_1,H_2} : X \times G_1 \times G_2 \to X \times G_1/H_1 \times G_2/H_2$ is the natural factor map and $((g_x^1, g_x^2)\nu_1)_v$ is the graph measure of v on $G_1/H_1 \times G_2/H_2$, i.e.

$$((g_x^1, g_x^2)\nu_1)_v(AH_1 \times BH_2) = \nu_1((g_x^1)^{-1}AH_1 \cap v^{-1}((g_x^2)^{-1}BH_2)) .$$

By this and Lemma 2,

$$\lambda(A \times B) = \int_{X \times G_1/H_1} E(A \mid H_1)(x, gH_1) \times E(B \mid H_2)(S_{f,v}(x, gH_1)) \, d(\mu \times \nu_1)(x, gH_1)$$

for measurable $A \subset X \times G_1$, $B \subset X \times G_2$, which finishes the proof of Theorem 2. ∎

Proof of Theorem 3. Assume that $T_{\varphi_i} : (X \times G_i, \mu \times \nu_1) \to (X \times G_i, \mu \times \nu_2)$ is an ergodic automorphism, $i = 1, 2$. Let $\overline{S} : X \times G_1 \to X \times G_2$ be a factor map. Define a measure λ on $X \times G_1 \times X \times G_2$ by

$$\lambda(A \times B) = (\mu \times \nu_1)(A \cap \overline{S}^{-1}(B))$$

for measurable $A \subset X \times G_1$, $B \subset X \times G_2$. In other words, $\lambda = (\mu \times \nu_1)_{\overline{S}}$. Then $\lambda \in J^e(T_{\varphi_1}, T_{\varphi_2})$ and by Theorem 2, there are normal subgroups $H_1 \subset G_1$, $H_2 \subset G_2$, an $S \in C(T)$, a measurable map $f : X \to G_2/H_2$ and a continuous group isomorphism $w : G_1/H_1 \to G_2/H_2$ such that

$$\lambda(A \times B) = \int_{X \times G_1/H_1} E(A \mid H_1)(x, gH_1) \times E(B \mid H_2)S_{f,w}(x, gH_1) \, d(\mu \times \nu_1)(x, gH_1) .$$

Since $\lambda = (\mu \times \nu_1)_{\overline{S}}$, $H_2 = \{e_2\}$.

Let $p : G_1 \to G_1/H_1$ be the natural projection. Put $v = w \circ p$. Then for measurable $A \subset X \times G_1$, $B \subset X \times G_2$

$$\begin{aligned} \lambda(A \times B) &= \int_{X \times G_1} E(A \mid H_1)(\mathrm{Id} \times p)(x, g) \cdot \chi_B \circ S_{f,v}(x, g) \, d(\mu \times \nu_1) \\ &= \int_{X \times G_1} E(A \mid H_1)(\mathrm{Id} \times p)(x, g) \circ \chi_{S_{f,v}^{-1}(B)}(x, g) \, d(\mu \times \nu_1)(x, g) \\ &= \int_{S_{f,v}^{-1}(B)} E(A \mid H_1)(\mathrm{Id} \times p)(x, g) \, d(\mu \times \nu_1)(x, g) \\ &= \int_{S_{f,v}^{-1}(B)} E(A \mid H_1)(x, g) \, d(\mu \times \nu_1)(x, g) \\ &= \int_{S_{f,v}^{-1}(B)} \chi_A(x, g) \, d(\mu \times \nu_1)(x, g) = (\mu \times \nu_1)(A \cap S_{f,v}^{-1}(B)) \end{aligned}$$

where χ_B denotes the characteristic function of B. Since A and B are arbitrary, $\overline{S} = S_{f,v}$. ∎

Proof of Theorem 4. This is a simple consequence of Theorem 3, where $G_2 = G_1 = G$ and $\varphi_2 = \varphi_1 = \varphi$. ■

Let $T_\varphi : (X \times G, \widetilde{\mathcal{B}}, \widetilde{\mu}) \to (X \times G, \widetilde{\mathcal{B}}, \widetilde{\mu})$ be an ergodic group extension of a transformation with discrete spectrum $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$, where $\widetilde{\mu} = \mu \times \nu_G$ and $\widetilde{\mathcal{B}}$ is the corresponding product σ-algebra.

Let $\mathcal{C} \subset \widetilde{\mathcal{B}}$ be a T_φ-invariant sub-σ-algebra. Then $\mathcal{C}$ gives rise to a self-joining $\widetilde{\mu} \times_{\mathcal{C}} \widetilde{\mu}$ of T_φ by

$$(\widetilde{\mu} \times_{\mathcal{C}} \widetilde{\mu})(A \times B) = \int_{\overline{X}} E(X \mid \mathcal{C})(\overline{x}) \cdot E(B \mid \mathcal{C})(\overline{x})\, d\widetilde{\mu}(\overline{x}),$$

where $\overline{X}$ is the quotient space corresponding to $\mathcal{C}$.

Define $\lambda = \widetilde{\mu} \times_{\mathcal{C}} \widetilde{\mu}$. To prove Theorem 5 we will need some lemmata.

LEMMA 11 ([1, 6]). *Let A be a Borel subset of $X \times G$. Then*

$$A \in \mathcal{C} \quad \textit{iff} \quad \lambda(A \times A^c \cup A^c \times A) = 0. \ ■$$

The measure λ is not necessarily ergodic. Let

$$\lambda = \int_{J^e(T_\varphi, T_\varphi)} m\, d\gamma(m) \tag{8}$$

be its ergodic decomposition, where γ is a probability measure on $J^e(T_\varphi, T_\varphi)$. Let

$$E = \{m \in J^e(T_\varphi, T_\varphi) : m(A \times A^c \cup A^c \times A) = 0 \text{ for each } A \in \mathcal{C}\}. \tag{9}$$

Lemma 11 and (8) yield

LEMMA 12. $\gamma(E) = 1$. ■

Let $m \in J^e(T_\varphi, T_\varphi)$. By Theorem 2,

$$m = \int_{X \times G_1/H_1} E(\cdot \mid H_1)(x, gH_1) \cdot E(\cdot \mid H_2)(S_{f,v}(x, gH_1))\, d\widetilde{\mu}(x, gH_1),$$

where $S \in C(T)$, $v : G/H_1 \to G/H_2$ is a continuous group isomorphism for some closed normal subgroups H_1 and H_2 of G, and $f : X \to G/H_2$ is a measurable function. The following three lemmata have the same proofs as the corresponding lemmata in [4].

LEMMA 13. *$m \in E$ iff $\mathcal{C} \subset \widetilde{\mathcal{B}}_{H_1 H_2}$ and for each $A \in \mathcal{C}$, $S_{f,v}^{-1}(A) = A$.* ■

Denote by F the largest closed normal subgroup of G such that $\mathcal{C} \subset \widetilde{\mathcal{B}}_F$ (F is taken as the closed subgroup generated by all normal subgroups F_β such that $\mathcal{C} \subset \widetilde{\mathcal{B}}_{F_\beta}$). In other words, $\widetilde{\mathcal{B}}_F$ is the smallest normal natural factor of T_φ such that $\mathcal{C}$ is a factor of $T_{\varphi,F}$. We will consider this factor as a group extension of T for which $\mathcal{C}$ is a factor. In particular, let E_F denote the set given by (9) for $T_{\varphi,F}$.

Put $F(\mathcal{C}) = \{\overline{S} \in C(T_{\varphi,F}) : \overline{S}^{-1}(A) = A \text{ for each } A \in \mathcal{C}\}$.

LEMMA 14. *If $\overline{S} \in F(\mathcal{C})$ then $\overline{S}$ is invertible.* ■

LEMMA 15. *For each $m \in E_F$ there exists an invertible $\overline{S} \in C(T_{\varphi,F})$ such that $m = (\widetilde{\mu})_{\overline{S}}$, i.e. m is the graph measure of $\overline{S}$.* ■

Proof of Theorem 5. Let $\mathcal{C}$ be a factor of an ergodic group extension $T_\varphi : (X \times G, \widetilde{\mathcal{B}}, \widetilde{\mu}) \to (X \times G, \widetilde{\mathcal{B}}, \widetilde{\mu})$ of a system with discrete spectrum T. Let F be the largest (closed) normal subgroup of G such that $\mathcal{C}$ is a factor of $T_{\varphi,F}$. By Lemmata 12–15 the measure $\widetilde{\mu} \times_{\mathcal{C}} \widetilde{\mu}$ on $(X \times G/F) \times (X \times G/F)$ has the ergodic decomposition which consists of graph measures of the form $(\widetilde{\mu})_{\overline{S}}$ where $\overline{S}$'s are invertible elements of the centralizer of $T_{\varphi,F}$. Therefore by Theorem 1.8.2 in [1], Theorem 5 holds. ■

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