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Spherical functions and uniformly bounded representations of free groups

by

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Abstract. We give a construction of an analytic series of uniformly bounded representations of a free group G, through the action of G on its Poisson boundary. These representations are irreducible and give as their coefficients all the spherical functions on G which tend to zero at infinity. The principal and the complementary series of unitary representations are included. We also prove that this construction and the other known constructions lead to equivalent representations.

§1. Introduction. Let G be a free group with r generators. In this paper we study an analytic series of uniformly bounded representations of G defined through the action of G on its Poisson boundary with respect to the simple random walk. These representations are indexed by complex numbers from the ellipse

$$E = \left\{ z \in \mathbf{C} : \left| z - \frac{\sqrt{2r-1}}{r} \right| + \left| z + \frac{\sqrt{2r-1}}{r} \right| < 2 \right\}$$

and give as their coefficients all the spherical functions which tend to zero at infinity. In particular, for real indices we obtain the principal and the complementary series of unitary representations of G. Moreover, all these representations are irreducible and two different representations are inequivalent.

The principal and complementary series of unitary representations were studied by Cartier [1], [2], Figà-Talamanca and Picardello [3], [4], Pytlik [9] and Sawyer [12]. Later Mantero and Zappa [6] constructed for any spherical function in $C_0(G)$ a uniformly bounded representation of G on a Hilbert space such that the spherical function was one of its coefficients. This construction was improved in [7]. Quite another series of representations of

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G was constructed in [11]. Later Szwarc [13] extracted from them irreducible subrepresentations as direct factors and obtained the same spherical functions as in [6] as coefficients. An important part of the constructions presented in both [7] and [13] is a transfer of the obtained representations as analytic series on a fixed Hilbert space. However, in both cases the final formulas become rather complicated. It turns out that the constructions mentioned above lead to equivalent representations. This was recently shown in [5].

The idea of our construction bases on an unpublished paper [10] where a transfer of representations alternative to that in [13] was obtained. We repeat here the definition of these representations but give a direct, very short proof of the two major properties: uniform boundedness and irreducibility. There is an additional profit when using our construction. First of all the formula (4) for any complex z, also for those outside E, produces representations in bounded operators (perhaps not uniformly bounded or not irreducible). These representations give all the spherical functions as coefficients. The second advantage is that our construction is very much in the spirit of cocycle representations in the sense of Pimsner [8] and following Valette [15] one can repeat the construction with minor changes for some other groups which act isometrically on a tree.

§2. Notations. We denote by $|x|$ the *length* of the word $x \in G$, i.e. the number of letters of the word x in its reduced form. A complex-valued function f on G will be called *radial* if it depends only on the length of the word, that is, $f(x) = f(y)$ whenever $|x| = |y|$. Any radial function φ can be uniquely expressed in the form

$$\varphi = \sum_{m=0}^{\infty} \alpha_m \mu_m ,$$

where $\mu_0 = \delta_e$ is the Dirac function at e and μ_m the probability measure uniformly distributed on all words of length m.

Let us consider the operator of convolution with μ_1. For the free group it plays a role analogous to that of the Laplace–Beltrami operator on semisimple Lie groups. The formula

$$\mu_1 * \mu_n = \frac{1}{2r}\mu_{n-1} + \frac{2r-1}{2r}\mu_{n+1} \tag{1}$$

implies that radial functions can also be expressed in terms of convolution powers of μ_1. Thus convolution of radial functions is a commutative operation (whenever it makes sense) and it gives a radial function.

A radial function φ on G will be called *spherical* if $\varphi(e) = 1$ and φ is an eigenfunction of the operator of convolution with μ_1. It is an easy consequence of (1) that to any complex number z there corresponds exactly

one spherical function φ_z such that $\mu_1 * \varphi_z = z\varphi_z$. It can be given by the formula

$$\varphi_z = \sum_{m=0}^{\infty} Q_m(z)\chi_m ,$$

where $\chi_m = 2r(2r-1)^{m-1}\mu_m$ is the characteristic function of the set of all words of length m and $Q_0(z), Q_1(z), \ldots$ are polynomials in z satisfying the recurrence formula $Q_0(z) = 1$, $Q_1(z) = z$ and

$$zQ_m(z) = \frac{1}{2r}Q_{m-1} + \frac{2r-1}{2r}Q_{m+1}(z).$$

To have a more handy realization of the polynomials $Q_m(z)$ choose a nonzero complex number ζ so that

$$z = \frac{\sqrt{2r-1}}{2r}\left(\zeta + \zeta^{-1}\right) .$$

Then (cf. [9], Theorem 2.1)

$$Q_m(z) = \frac{(2r-1)^{1-m/2}}{2r}\left[\zeta^m + \zeta^{-m} + \frac{2r-2}{2r-1}\Big(\sum_{k=0}^{m-2}\zeta^{2k}\Big)\zeta^{2-m}\right], \tag{2}$$

which for $\zeta \neq \pm 1$ gives

$$Q_m(z) = \frac{1}{(2r-1)^{m/2}}[A(\zeta)\zeta^m + A(\zeta^{-1})\zeta^{-m}]$$

with

$$A(\zeta) = \frac{\frac{2r-1}{2r}\zeta - \frac{1}{2r}\zeta^{-1}}{\zeta - \zeta^{-1}} .$$

We are only concerned with those spherical functions which vanish at infinity. Clearly it is the case of $\lim_{m\to\infty} Q_m(z) = 0$. By (2) this is equivalent to $(2r-1)^{-1/2} < |\zeta| < (2r-1)^{1/2}$, which means that z belongs to the ellipse E.

For any $z \in E$ we want to construct a uniformly bounded representation of G for which φ_z will be a matrix coefficient. The common Hilbert space for the action of all these representations will be $L^2(\Omega, \mu)$, where (Ω, μ) is the Poisson boundary relative to μ_1 defined below.

Let Ω be the space of all infinite reduced words in the generators of G. The group G acts on Ω in a natural way by multiplication on the left. For a word ω in Ω and a natural number n let ω_n be the word in G which consists of the first n letters of ω. For a given $x \in G$ with $|x| = n$, let Ω_x denote the set of those $\omega \in \Omega$ for which $\omega_n = x$. By the *usual topology* in Ω we mean the compact topology with $\{\Omega_x\}_{x\in G}$ as the base of open sets. Take $x \in G$

with $|x| = n > 0$ and put

$$\mu(\Omega_x) = \frac{1}{2r(2r-1)^{n-1}}$$

(note that $2r(2r-1)^{n-1}$ is the cardinality of the set of all words of length n). Then μ extends uniquely to a Borel probability measure on Ω. This measure is quasi-invariant for the action of G. Namely,

$$\int_\Omega f(x\omega)\,d\mu(\omega) = \int_\Omega f(\omega)P(x,\omega)\,d\mu(\omega),$$

where $P(x,\omega) = (2r-1)^{n-|x^{-1}\omega_n|}$, with $n = |x|$, is the so-called *Poisson kernel* (see [3] for details). The regular representation λ of G on Ω is the unitary representation of G on $L^2(\Omega,\mu)$ defined by

$$\lambda(x)f(\omega) = f(x^{-1}\omega)P^{1/2}(x,\omega).$$

§3. The main result. Let x be a word in G of length 1. Consider the two-dimensional subspace H_x in $L^2(\Omega,\mu)$ spanned by the characteristic function χ_x of the set Ω_x and the constant function 1. The vectors $\xi_e = 1$ and $\xi_x = (2r-1)^{-1/2}(2r\chi_x - 1)$ form an orthogonal basis for H_x. Let a be a free generator (or its inverse) in G. Then the operator $\lambda(a)$ maps $H_{a^{-1}}$ onto H_a. Namely, we have

$$\lambda(a)\xi_e = \frac{\sqrt{2r-1}}{r}\xi_e + \frac{r-1}{r}\xi_a, \qquad (3)$$
$$\lambda(a)\xi_{a^{-1}} = \frac{r-1}{r}\xi_e - \frac{\sqrt{2r-1}}{r}\xi_a.$$

For a complex number $z \in E$ we define a representation π_z of G on $L^2(\Omega,\mu)$ by changing slightly the representation λ. Put

$$\pi_z(a)\xi_e = z\xi_e + \sqrt{1-z^2}\,\xi_a, \quad \pi_z(a)\xi_{a^{-1}} = \sqrt{1-z^2}\,\xi_e - z\xi_a \qquad (4)$$

instead of (3) but $\pi_z(a) = \lambda(a)$ on the orthogonal complement $H_{a^{-1}}^\perp$ of $H_{a^{-1}}$.

THEOREM 1. *Let G be a free group on r generators. The representations π_z, $z \in E$, defined above form an analytic family of uniformly bounded representations of G on the Hilbert space $L^2(\Omega,\mu)$. Moreover:*

(i) *Each π_z is irreducible (i.e. has no nontrivial closed invariant subspaces) and two different π_z's are topologically inequivalent.*

(ii) $\pi_z^*(x) = \pi_{\bar z}(x^{-1})$. *In particular, π_z is a unitary representation if z is real.*

(iii) $\pi_z(x) - \lambda(x)$ *is a finite rank operator.*

(iv) $\langle \pi_z(x)1, 1\rangle = \varphi_z(x)$ *for $x \in G$.*

Properties (ii) and (iii) are obvious. Also (iv) is rather simple to show. The major part of the proof is to show that π_z's are uniformly bounded and irreducible. The case when z is a real number is much simpler than the other one. The representation π_z is unitary in this case and we can recognize π_z among known unitary representations by looking at the spherical function φ_z which is positive definite if z is real. It was shown in [3] and [10] that if z is in the interval $[-\sqrt{2r-1}/r, \sqrt{2r-1}/r]$ then φ_z is a coefficient of an irreducible representation from the principal series. Since the constant function 1 is a cyclic vector for π_z, φ_z determines π_z uniquely. In particular, this implies that π_z must be irreducible. The argument of spherical functions also shows that one may use π_z's instead of the representations of [3] or of [10] to get the decomposition formula for the regular representation of G on $\ell^2(G)$. A similar consideration can be made in the case when z is real but outside the interval $[-\sqrt{2r-1}/r, \sqrt{2r-1}/r]$. As was shown in [3], Theorem 5, the spherical function φ_z is then a coefficient of an irreducible representation from the complementary series.

The situation $z \notin \mathbf{R}$ is much more complicated. Although there are known uniformly bounded representations having φ_z as coefficients, we cannot use arguments which we used before. The representations π_z are not unitarizable and φ_z's do not determine π_z's completely. For this reason we give the full proof here. However, it can be shown, and we do this in §6, that the representations are equivalent via [5] to those of [6], [7] and [13], but the proof is complicated.

§4. π_z's are uniformly bounded. Given two words x, y in G we write $x \perp y$ when there is no cancellation in the product xy. We also write $y \le x$ when y is a left hand part of the reduced form of x, i.e. when $y \perp y^{-1}x$. For the empty word e it is always assumed that $e \le x$.

Let $x \in G$. Denote by H_x the finite-dimensional subspace of the Hilbert space $L^2(\Omega,\mu)$, generated by the functions χ_y, $y \le x$. Clearly $\dim H_x = |x| + 1$. Note that all the functions $\lambda(y)1 = P^{1/2}(y,\cdot)$, $y \le x$, are in H_x. Moreover, since they are linearly independent, they generate H_x.

LEMMA 1. *Let $x \in G$. If $f \in H_x^\perp$ then $\pi_z(x^{-1})f = \lambda(x^{-1})f$.*

Proof. Indeed, when $|x| \le 1$ the statement is obvious just from the definition of π_z. Otherwise write $x = ya$ with $|a| = 1$ and $|y| < |x|$. Since the operator $\lambda(y)$ maps H_a into H_x (as was observed above), the adjoint operator $\lambda(y)^* = \lambda(y^{-1})$ maps $H_x^\perp$ into $H_a^\perp$. Therefore

$$\lambda(x^{-1})f = \lambda(a^{-1})\lambda(y^{-1})f = \pi_z(a^{-1})\lambda(y^{-1})f,$$

and the proof can go by induction on the length of x. ■

To prove that for each $z \in E$ the representation π_z is uniformly bounded it suffices by Lemma 1 to consider the restrictions of $\pi_z(x^{-1})$ to H_x and to find for each of these operators a norm estimate which will not depend on the particular choice of x in G.

So let $x \in G$. Put $n = |x|$ and for $k = 0, 1, \ldots, n$ denote by x_k the word in G which consists of the first k letters of x. In the space H_x we consider two natural bases of vectors. The first one is orthonormal and consists of the vectors

$$\xi_{x_0} = \xi_e = 1, \quad \xi_{x_1} = (2r-1)^{-1/2}(2r\chi_{x_1} - 1) \tag{5a}$$

as before and

$$\xi_{x_k} = \sqrt{r/(r-1)}(2r-1)^{k/2}\left(\chi_{x_k} - \frac{1}{2r-1}\chi_{x_{k-1}}\right) \tag{5b}$$

for $k = 2, \ldots, n$. The second basis is convenient for the action of π_z and consists of the vectors $\pi_z(x_k)1$, $k = 0, 1, \ldots, n$ (a priori it is not obvious that the $\pi_z(x_k)1$ are linearly independent or that they all belong to H_x, but both the facts will follow from the formula (6) below). The first basis can be expressed in terms of the second in the following way:

$$\xi_{x_0} = 1 = \pi_z(x_0)1, \quad \xi_{x_1} = \frac{1}{\sqrt{1-z^2}}\pi_z(x_1)1 - \frac{z}{\sqrt{1-z^2}}\pi_z(x_0)1 \tag{6a}$$

as follows directly from (4), and since for $k \geq 2$ the vector ξ_{x_k} is orthogonal to $H_{x_{k-1}}$ we have

$$\begin{aligned}\xi_{x_k} &= \pi_z(x_{k-1})\pi_z(x_{k-1}^{-1})\xi_{x_k} = \pi_z(x_{k-1})\lambda(x_{k-1}^{-1})\xi_{x_k} \\ &= \frac{\sqrt{r/(r-1)}}{2r}\pi_z(x_{k-1})((2r-1)\xi_{a_k} + \xi_{a_{k-1}^{-1}})\end{aligned}$$

where $a_k = x_{k-1}^{-1}x_k$ is the kth letter of the word x. Applying (6a) we get

$$\xi_{x_k} = \frac{\sqrt{r/(r-1)}}{\sqrt{1-z^2}}\left[\frac{(2r-1)}{2r}\pi_z(x_k)1 - z\pi_z(x_{k-1})1 + \frac{1}{2r}\pi_z(x_{k-2})1\right]. \tag{6b}$$

Note here that the norm of the transition matrix A_x from the first basis to the second one is dominated by the number

$$\sqrt{r/(r-1)}\frac{1+|z|}{\sqrt{|1-z^2|}}, \tag{7}$$

which is independent of x.

Now we want to compute the inverse matrix $A_{x^{-1}}$.

LEMMA 2. *Let $x \in G$ with $|x| = n > 0$. If $\xi_{x_0}, \ldots, \xi_{x_n}$ are defined by* (5) *then*

$$\pi_z(x)1 = Q_n(z)\xi_{x_0} + \sum_{k=1}^{n} c_k \frac{Q_{n-k}(z) - zQ_{n+1-k}(z)}{\sqrt{1-z^2}}\xi_{x_k} \tag{8}$$

where $c_1 = 1$ and $c_k = \dfrac{2\sqrt{r}\sqrt{r-1}}{2r-1}$ for $k > 1$. In particular,

$$\langle \pi_z(x)1, 1\rangle = Q_n(z) = \varphi_z(x).$$

Proof. Taking the scalar product of ξ_{x_k} and $\xi_{x_0} = 1$ in (6) we observe that the numbers $\langle \pi_z(x_k)1, 1\rangle$, $k = 0, 1, \ldots, n$, satisfy exactly the same recurrence formula as the $Q_k(z)$ do. Thus $\langle \pi_z(x_k)1, 1\rangle = Q_k(z)$. This proves the second part of the lemma. Now we have

$$\pi_z(x)1 = \sum_{k=1}^{n}\langle \pi_z(x)1, \xi_{x_k}\rangle \xi_{x_k}.$$

If we express ξ_{x_k} in terms of $\pi_z(x_j) = \pi_z^*(x_j^{-1})$ via (6) and use the fact that $\langle \pi_z(x_j^{-1}x)1, 1\rangle = Q_{n-j}(z)$ we get (8). ∎

The norm of the inverse matrix A_x^{-1} is easily estimated by

$$\Big(\sum_{k=0}^{\infty}|Q_k(z)|^2\Big)^{1/2} + \sum_{k=1}^{\infty}\left|\frac{Q_k(z) - zQ_{k+1}(z)}{\sqrt{1-z^2}}\right|, \tag{9}$$

which is convergent for $z \in E$ (see (2)).

Now observe that

$$\pi_z(x^{-1})(\pi_z(x_k)1) = \pi_z(x^{-1}x_k)1 = \pi_z((x^{-1})_{n-k})1$$

for $k = 0, 1, \ldots, n$, i.e. $\pi_z(x^{-1})$ is just the reflection when both spaces H_x and $H_{x^{-1}}$ are equipped with the second basis. Thus $\|\pi_z(x^{-1})\| \leq \|A_x^{-1}\|\cdot\|A_{x^{-1}}\|$, which by (7) and (9) is uniformly bounded in $x \in G$.

§5. **π_z's are irreducible.** We refer to [3] or [9] for the proof of irreducibility of the representations π_z from the principal series, i.e. when $z \in [-\sqrt{2r-1}/r, \sqrt{2r-1}/r]$. For all the other z in E we will show that the Banach algebra generated by $\pi_z(x)$, $x \in G$, contains a one-dimensional projection on the cyclic vector 1 in $L^2(\Omega, \mu)$. This will imply irreducibility by some general arguments from group representation theory.

On $L^2(\Omega, \mu)$ consider the linear operator Q defined by

$$Qf(\omega) = f(\overline{\omega}),$$

where the word $\overline{\omega}$ is obtained from ω by deleting the first letter. Observe that

$$\int_{\Omega} f(\overline{\omega})\,d\mu(\omega) = \int_{\Omega} f(\omega)\,d\mu(\omega),$$

which implies that Q is an isometry. The adjoint operator Q^* has the form

$$Q^*f(\omega) = \frac{1}{2r}\sum_{a\perp\omega, |a|=1} f(a\omega), \tag{10}$$

where $a \perp \omega$ means that there is no cancellation in the product $a\omega$.

As in §2, let μ_1 denote the probability measure on G uniformly distributed on all words of length 1. Then

$$\lambda(\mu_1)f(\omega) = \frac{1}{2r}\sum_{|a|=1} f(a^{-1}\omega)(2r-1)^{(1-|a^{-1}\omega_1|)/2} = \frac{\sqrt{2r-1}}{2r}(Q^*+Q)f(\omega).$$

Denote by E_1 the conditional expectation with respect the σ-field generated by Ω_x, $|x| \le 1$, i.e.

$$E_1 f(\omega) = 2r \int_{(\omega,\omega')\ge 1} f(\omega')\, d\mu(\omega')$$

where $(\omega,\omega') = \sup\{|x| : \omega, \omega' \in \Omega_x\} = \lim_{n\to\infty}(n - \frac{1}{2}|\omega_n^{-1}\omega_n'|)$ denotes the number of common letters of ω and ω'. For a complex number ζ let T_ζ be the operator on $L^2(\Omega,\mu)$ defined by

$$T_\zeta = (I - E_1) + \zeta E_1 .$$

LEMMA 3. *Let $z \in E$. Then the operator $\pi_z(\mu_1)$ is normal. Moreover:*

(i) *If ζ is a complex number in the annulus $(2r-1)^{-1/2} < |\zeta| < (2r-1)^{1/2}$ so that $z = \frac{\sqrt{2r-1}}{2r}(\zeta+\zeta^{-1})$ then*

$$\pi_z(\mu_1) = \frac{\sqrt{2r-1}}{2r}(Q^*T_\zeta + T_{\zeta^{-1}}Q).$$

(ii) *If t is a nonzero complex number then*

$$\frac{\sqrt{2r-1}}{2r}(t+t^{-1})I - \pi_z(\mu_1) = \frac{\sqrt{2r-1}}{2rt}(tI - Q^*T_\zeta)(tI - T_{\zeta^{-1}}Q).$$

(iii) *The spectrum in $L^2(\Omega,\mu)$ of the operator $\pi_z(\mu_1)$ is contained in the set $\{z\} \cup [-\sqrt{2r-1}/r, \sqrt{2r-1}/r]$. The eigenspace corresponding to $\{z\}$ is one-dimensional and consists of constant functions.*

Proof. Let $f \in L^2(\Omega,\mu)$ and let $a \in G$ with $|a| = 1$. Then

$$\pi_z(a)f - \lambda(a)f = \langle f, \xi_e\rangle(\pi_z(a) - \lambda(a))\xi_e + \langle f, \xi_{a^{-1}}\rangle(\pi_z(a) - \lambda(a))\xi_a .$$

Applying (4) to the average of the above over all words of length 1 and remembering that $\sum_{|a|=1}\xi_a = 0$ we get

$$\pi_z(\mu_1)f - \lambda(\mu_1)f = \left(z - \frac{\sqrt{2r-1}}{2r}\right)\left(\langle f,\xi_e\rangle\xi_e - \frac{1}{2r}\sum_{|a|=1}\langle f, \xi_{a^{-1}}\rangle\xi_a\right).$$

On the other hand,

$$Q^*\chi_a = \frac{1}{2r}(1 - \chi_{a^{-1}})$$

by (10). Thus

$$(11) \qquad Q^*E_1 f = Q^*\Big(2r\sum_{|a|=1}\langle f,\chi_a\rangle\chi_a\Big) = \langle f,\xi_e\rangle\xi_e - \frac{1}{2r}\sum_{|a|=1}\langle f,\xi_a\rangle\xi_{a^{-1}} .$$

Furthermore,

$$\pi_z(a)f - \lambda(a)f = \left(z - \frac{\sqrt{2r-1}}{2r}\right)Q^*E_1 .$$

But (11) shows that Q^*E_1 is a self-adjoint operator, so that $Q^*E_1 = E_1Q$. All this together gives (i). It also proves that $\pi_z(\mu_1)$ is a normal operator.

The identity (ii) is an immediate consequence of (i). To get (iii) observe that $\pi_z(\mu_1)1 = z \cdot 1$ and that the subspace of constant functions reduces the operators Q, Q^*, T_ζ and $\pi_z(\mu_1)$. The lemma will follow if we prove that the operator $uI - \pi_z(\mu_1)$ is invertible on $(\mathbf{C}\cdot 1)^\perp$ for any complex number u outside the interval $[-\sqrt{2r-1}/r, \sqrt{2r-1}/r]$. This in view of (ii) is satisfied when $tI - Q^*T_\zeta$ is invertible on $(\mathbf{C}\cdot 1)^\perp$ whenever $|t| > 1$ and $(2r-1)^{-1/2} < |\zeta| < (2r-1)^{1/2}$ since then $tI - T_{\zeta^{-1}}Q = (\bar{t}I - Q^*T_{\bar\zeta^{-1}})^*$ is also invertible and we can choose $t \in \mathbf{C}$, $|t| > 1$, so that $u = \frac{\sqrt{2r-1}}{2r}(t+t^{-1})$. In other words, we require

$$\sup_n \|(Q^*T_\zeta)^n\|^2 = \sup_n \|(Q^*T_\zeta)^n(T_{\bar\zeta}Q)^n\| < \infty .$$

According to (11) on $(\mathbf{C}\cdot 1)^\perp$ we have $Q^*T_\zeta Q = T_{\zeta'}$ with $\zeta' = 1 + (\zeta - 1) \times (2r-1)^{-2}$. Thus by induction $(Q^*T_\zeta)^n(T_{\bar\zeta}Q)^n = T_{\zeta_n}$, where ζ_n is defined by the recurrence

$$\zeta_0 = 1, \qquad \zeta_n = 1 + \frac{|\zeta|^2\zeta_{n-1} - 1}{(2r-1)^2} \qquad \text{for } n = 1, 2, \ldots$$

But the above formula gives a bounded sequence. ∎

Now we are ready to complete the proof that π_z's are irreducible. Let z be in E but outside the interval $[-\sqrt{2r-1}/r, \sqrt{2r-1}/r]$ and let P be the orthogonal projection in $L^2(\Omega,\mu)$ which corresponds to $\{z\}$ in the spectral decomposition of the operator $\pi_z(\mu_1)1$. By Lemma 3, P has the form $Pf = \langle f, 1\rangle 1$ and since z is an isolated point in the spectrum of $\pi_z(\mu_1)1$ it belongs to the Banach algebra generated by all $\pi_z(x)$, $x \in G$. Suppose that M is a closed subspace in $L^2(\Omega,\mu)$ invariant under the action of π_z. Then $P(M) \subset M$. If $\langle f, 1\rangle \ne 0$ for an f in M, then $1 \in M$ and so $M = L^2(\Omega,\mu)$. If not then since 1 is also a cyclic vector for the representation π_z, the only function f in $L^2(\Omega,\mu)$ such that $0 = \langle \pi_z(x)f, 1\rangle = \langle f, \pi_z(x^{-1})1\rangle$ for all x in G is $f = 0$, so $M = \{0\}$ in that case.

§6. Relation to other constructions. For a complex number u in the open unit disc $\{u \in \mathbf{C} : |u| < 1\}$ denote by Π_u the uniformly bounded representation of G on $\ell^2(G)$ from [11], which on the space $\mathcal{F}(G)$ of finitely supported functions has the form

$$\Pi_u(x) = T_u^{-1}(I-uP)^{-1}\lambda(x)(I-uP)T_u\,. \tag{12}$$

Here $\lambda(x)$, T_u and P are the operators: $\lambda(x)\delta_y = \delta_{xy}$, $T_u\delta_e = \sqrt{1-u^2}\,\delta_e$, $P\delta_e = 0$ but $T_u\delta_y = \delta_y$, $P\delta_y = \delta_{\overline{y}}$ when $y \neq e$ and $\overline{y}$ denotes the word in G obtained from y by deleting the last letter.

When $|u| > (2r-1)^{-1/2}$ the space $\ell^2(G)$ splits into a direct sum as

$$\ell^2(G) = \operatorname{Im} T_u(I-uP^*) \oplus \operatorname{Ker}(I-uP)T_u \tag{13}$$

and both subspaces are Π_u-invariant (see [13]).

In this section we want to prove that for any complex number u in the annulus $E' = \{u \in \mathbf{C} : (2r-1)^{-1/2} < |u| < 1\}$ there exists an isomorphism $\mathcal{R}_u$ from $L^2(\Omega,\mu)$ onto $\operatorname{Ker}(I-uP)T_u$ which intertwines the representations Π_u and π_z, where

$$z = \frac{(2r-1)u + u^{-1}}{2r}$$

(then $z \in E$). We refer to [5] for the proof that also other series of spherical representations are equivalent to $\Pi_u|_{\operatorname{Ker}(I-uP)T_u}$, $u \in E'$.

The operator I_u and the decomposition formula in Lemma 4 below can be found in [7] but with a different parametrization, so we give the proof for completeness.

For $n = 0,1,2,\ldots$ denote by M_n the finite-dimensional subspace in $L^2(\Omega,\mu)$ generated by all ξ_x with $|x| = n$ (see (5) for definition). We then have

$$L^2(\Omega,\mu) = \oplus\sum_{n=0}^{\infty} M_n\,. \tag{14}$$

Fix a complex number u in the annulus $(2r-1)^{-1/2} < |u| < 1$ and define a bounded operator I_u on $L^2(\Omega,\mu)$ by

$$I_u f(\omega) = \int_\Omega f(\omega')u^{-2(\omega,\omega')}\,d\mu(\omega')\,,$$

where, as in §4, (ω,ω') denotes the number of common letters of ω and ω'.

LEMMA 4. *In the decomposition* (14) *of* $L^2(\Omega,\mu)$ *the operator* I_u *has the diagonal form*

$$I_u = \oplus\sum_{n=0}^{\infty} d_n\ \mathrm{id}_{M_n}\,,$$

where id_{M_n} *denotes the identity operator on* M_n *and* $d_0, d_1, \ldots$ *is the sequence of complex numbers given by*

$$d_0 = (1-v^2)\alpha, \quad d_n = (1-u^2)\alpha(v/u)^n, \quad n = 1,2,\ldots,$$

with $v^{-1} = (2r-1)u$ *and* $\alpha^{-1} = 2rv(u-v)$.

Proof. We have

$$I_u\xi_e(\omega) = \int_\Omega u^{-2(\omega,\omega')}\,d\mu(\omega') = \frac{2r-1}{2r} + \frac{2r-2}{2r(2r-1)}u^{-2} + \frac{2r-2}{2r(2r-1)^2}u^{-4} + \ldots = d_0\,.$$

For $x \in G$, $x \neq e$, put $n = |x|$ and $k = (x,\omega)$. Then

$$I_u\chi_x(\omega) = \int_{\Omega_x} u^{-2(\omega,\omega')}d\mu(\omega') = \begin{cases} \dfrac{1}{2r(2r-1)^{n-1}u^{2k}} & \text{if } k < n,\\[2mm] \dfrac{2r-2}{2r(2r-1)^{n-1}u^{2n-2}[(2r-1)u^2-1]} & \text{if } k = n. \end{cases}$$

This gives immediately that $I_u\xi_x = d_{|x|}\xi_x$ and so the lemma follows. ∎

Let u be in the annulus $(2r-1)^{-1/2} < |u| < 1$ as before. The operator $I_u^{-1/2}$ is well defined on the dense subspace $\operatorname{lin}\{\xi_x : x \in G\}$ in $L^2(\Omega,\mu)$. To any function f in this subspace we associate the function $\widehat{f}_u$ on G defined by

$$\begin{aligned} \widehat{f}_u(e) &= \int_\Omega I_u^{-1/2}f(\omega)\,d\mu(\omega)\,,\\ \widehat{f}_u(x) &= \sqrt{1-u^2}\,u^{-|x|}\int_{\Omega_x} I_u^{-1/2}f(\omega)\,d\mu(\omega), \quad x \neq e\,. \end{aligned} \tag{15}$$

LEMMA 5. *The map* $f \to \widehat{f}_u$ *extends uniquely to an isometric embedding of* $L^2(\Omega,\mu)$ *into* $\ell^2(G)$.

Proof. Let $f,g \in \operatorname{lin}\{\xi_x : x \in G\}$. Then

$$\begin{aligned} \sum_{x\in G}\widehat{f}_u(x)\widehat{g}_u(x) &= \int_\Omega\int_\Omega I_u^{-1/2}f(\omega)I_u^{-1/2}g(\omega')\\ &\quad\times\Big[1-(1-u^2)\sum_{x\neq e}u^{-2|x|}\chi_x(\omega)\chi_x(\omega')\Big]\,d\mu(\omega)\,d\mu(\omega')\\ &= \int_\Omega\int_\Omega I_u^{-1/2}f(\omega)I_u^{-1/2}g(\omega')u^{-2(\omega,\omega')}\,d\mu(\omega)\,d\mu(\omega')\\ &= \int_\Omega\int_\Omega I_u^{1/2}f(\omega)I_u^{-1/2}g(\omega)\,d\mu(\omega) = \int_\Omega f(\omega)g(\omega)\,d\mu(\omega)\,. \end{aligned}$$

If u is real then ^ commutes with complex conjugation. Thus the map $f \to \widehat{f}_u$ is an isometry. For nonreal u, $f \to \widehat{f}_u$ is the composition of the isomorphism $f \to I_{|u|}^{1/2} I_u^{-1/2} f$ of $L^2(\Omega,\mu)$ followed by the isometry $f \to \widehat{f}_{|u|}$ from $L^2(\Omega,\mu)$ into $\ell^2(G)$ and finally the isomorphism A (say) of $\ell^2(G)$ defined by $A\delta_e = \delta_e$, $A\delta_x = (1-u^2)^{1/2}(1-|u|^2)^{-1/2}(u/|u|)^{|x|}\delta_x$ for $x \neq e$. ■

Observe that since all the ξ_x are eigenfunctions of $I_u^{-1/2}$, we can write

$$\int_\Omega I_u^{-1/2} f(\omega)\xi_x(\omega)\,d\mu(\omega) = \int_\Omega f(\omega) I_u^{-1/2}\xi_x(\omega)\,d\mu(\omega) = d_{|x|}^{-1/2}\langle f,\xi_x\rangle .$$

This gives a formula for computing $\widehat{f}_u$, simpler than (15). Namely,

$$\widehat{f}_u(e) = d_0^{-1/2}\langle f,\xi_e\rangle,$$
$$\widehat{f}_u(x) - \frac{\sqrt{1-u^2}}{2ru}\widehat{f}_u(e) = \frac{2r-1}{2r}\alpha^{-1/2}\langle f,\xi_x\rangle$$

for $|x| = 1$ and

$$\widehat{f}_u(x) - \frac{1}{(2r-1)u}\widehat{f}_u(\overline{x}) = \sqrt{(r-1)/r}\,\alpha^{-1/2}\langle f,\xi_x\rangle$$

for all other $x \in G$. It follows in particular that $\widehat{f}_u \in \operatorname{Ker}(I-uP)T_u$. Moreover, we can write

$$(\xi_e)_u^\wedge = d_0^{-1/2}\left(\delta_e + \frac{2r-1}{2r}\sqrt{1-u^2}\,(\eta_e-\delta_e)\right),$$
$$(\xi_x)_u^\wedge = \frac{2r-1}{2r}\alpha^{-1/2}\left(\frac{2r}{2r-1}\eta_x - u\eta_e + u\delta_e\right), \qquad |x| = 1,$$
$$(\xi_x)_u^\wedge = \frac{2r-1}{2\sqrt{r(r-1)}}\alpha^{-1/2}\,(\eta_x - u\eta_{\overline{x}} + u\delta_{\overline{x}})\,, \qquad |x| > 1,$$

where $\eta_x = (I-vP^*)^{-1}\delta_x = \sum_{y\geq x} v^{|y|-|x|}\delta_y$ and $v = 1/((2r-1)u)$. But since $T_u(I-uP^*)\eta_x = (u/v)\delta_x + (1-u/v)\eta_x$ for $x \neq e$ we get immediately

LEMMA 6. *Let u be in E' and let $^\wedge$ denote the transformation defined by* (15). *Denote by $\mathcal{R}$ the projection onto* $\operatorname{Ker}(I-uP)T_u$ *in the decomposition* (13) *of $\ell^2(G)$. Then*

$$(\xi_e)_u^\wedge = \frac{\sqrt{1-v^2}}{2ru^2}\alpha^{1/2}\mathcal{R}(\delta_e),$$
$$(\xi_x)_u^\wedge = \frac{2r}{2r-1}\alpha^{1/2}\mathcal{R}\left(\delta_x - \frac{\sqrt{1-u^2}}{2ru}\delta_e\right)$$

if $|x| = 1$ *and*

$$(\xi_x)_u^\wedge = \sqrt{(r-1)/r}\,\alpha^{1/2}\mathcal{R}\left(\delta_x - \frac{1}{(2r-1)u}\delta_{\overline{x}}\right)$$

for other $x \in G$. Here α is the constant taken from Lemma 4. This implies that

$$[L^2(\Omega,\mu)]_u^\wedge = \operatorname{Ker}(I-uP)T_u\,. \ ■$$

Let now a be a free generator in G. Then from (12)

$$\Pi_u(a)\delta_e = u\delta_e + \sqrt{1-u^2}\,\delta_a, \quad \Pi_u(a)\delta_{a^{-1}} = \sqrt{1-u^2}\,\delta_e - u\delta_e$$

and $\Pi_u(a)\delta_x = \delta_{ax}$ for all other $x \in G$. Since the projection $\mathcal{R}$ commutes with Π_u, using (4) and Lemma 6 one can easily prove that

$$(\pi_z(a)\xi_x)_u^\wedge = \Pi_u(a)(\xi_x)_u^\wedge .$$

All this together gives

THEOREM 2. *Let u be a complex number in the annulus $(2r-1)^{-1/2} < |u| < 1$ and let*

$$z = \frac{(2r-1)u + u^{-1}}{2r}$$

be the corresponding number in the ellipse E. The transformation $f \to \widehat{f}_u$ defined by (15) *extends to an isomorphism from $L^2(\Omega,\mu)$ onto the subspace* $\operatorname{Ker}(I-uP)T_u$ *in $\ell^2(G)$ and*

$$\Pi_u(x)\widehat{f}_u = (\pi_z(x)f)_u^\wedge$$

for any $f \in L^2(\Omega,\mu)$ and $x \in G$. ■

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Pseudocomplémentation dans les espaces de Banach

par

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Abstract. This paper introduces the following definition: a closed subspace Z of a Banach space E is *pseudocomplemented* in E if for every linear continuous operator u from Z to Z there is a linear continuous extension $\overline{u}$ of u from E to E. For instance, every subspace complemented in E is pseudocomplemented in E. First, the pseudocomplemented hilbertian subspaces of L^1 are characterized and, in L^p with p in $[1,+\infty[$, classes of closed subspaces in which the notions of complementation and pseudocomplementation are equivalent are pointed out. Then, for Banach spaces with the uniform approximation property, Dvoretzky's theorem is strengthened by proving that they contain uniformly pseudocomplemented ℓ_n^2's. Finally, the study of Banach spaces in which every closed subspace is pseudocomplemented is started.

Introduction. Cet article, qui a pour origine des notes non publiées de S. Massonnet, introduit la notion, plus large que celle de complémentation, de pseudocomplémentation d'un sous-espace fermé d'un espace de Banach. Ainsi, un sous-espace fermé Z d'un espace de Banach E est dit pseudo-complémenté dans E si et seulement si tout opérateur de Z s'étend en un opérateur de E.

La question résolue suivante de J. Lindenstrauss sur la complémentation a motivé l'introduction de cette nouvelle notion : si E est un espace de Banach de dimension infinie, existe-t-il p dans $[1,+\infty]$ tel que E contienne une suite de sous-espaces uniformément isomorphe à la suite $(\ell_n^p)_{n\in\mathbb{N}^*}$ et uniformément complémentée dans E? Dans [PIS 1], G. Pisier répond par la négative en construisant un espace de Banach E qui ne contient aucune telle suite.

En revanche, le problème précédent posé dans le cadre plus général de la pseudocomplémentation admet une réponse plus positive. Notons d'abord que, si un espace de Banach E contient une suite de sous-espaces uniformément isomorphe à une suite $(\ell_n^p)_{n\in\mathbb{N}^*}$, avec p dans $[1,+\infty]$, et uniformément pseudocomplémentée dans E, alors E contient nécessairement

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