Hölder continuity
of proper holomorphic mappings*

by
FRANÇOIS BERTELoot (Villeneuve d'Ascq)

Abstract. We prove the Hölder continuity for proper holomorphic mappings onto
certain piecewise smooth pseudoconvex domains with "good" plurisubharmonic peak
functions at each point of their boundaries. We directly obtain a quite precise estimate for
the exponent from an attraction property for analytic disks. Moreover, this way does not
require any consideration of infinitesimal metric.

1. Introduction. The theorem of Carathéodory states that every
biholomorphic map $F : D_1 \to D_2$ between bounded and simply connected
domains in $\mathbb{C}$ extends to an homeomorphism $\overline{F} : \overline{D}_1 \to \overline{D}_2$ if both domains
satisfy the Schrödinger condition at each point of their boundaries (see [9],
p. 209).

For domains in $\mathbb{C}^n$ ($n > 1$) the known generalizations require more precise
assumptions. The basic result is due to Henkin [8]: if $D_1$ is bounded, defined
by a plurisubharmonic function and if $D_2$ is bounded with $C^2$ strongly
pseudoconvex boundary, then every proper holomorphic map $F : D_1 \to D_2$
extends to a Hölder continuous map $\overline{F} : \overline{D}_1 \to \overline{D}_2$ with exponent 1/2. (See
also [10] and [13]; [12] for piecewise smooth strongly pseudoconvex boundary; [5] for a local version of this theorem.)

This was generalized by Bedford–Fornaess and Diederich–Fornaess to
the case where $D_1$ is bounded pseudoconvex with $C^2$ boundary and $D_2$
is bounded pseudoconvex with real-analytic boundary; they proved that every
proper holomorphic map $F : D_1 \to D_2$ extends to a Hölder continuous
map $\overline{F} : \overline{D}_1 \to \overline{D}_2$ for some exponent $\varepsilon \in [0, 1]$ (see [2], [4]).

This paper is mainly motivated by the following observation: a proper
holomorphic map is easily seen to be Hölder continuous if the image domain
satisfies a simple attraction property for analytic disks and if the distance to the boundary behaves correctly under the map.

This leads us to introduce the following two properties:

- The boundary $\partial D$ of a domain $D \subset \mathbb{C}^n$ satisfies the attraction property of order $\alpha$ ($0 < \alpha < 1$) if for each $r \in [0, 1]$ there is a positive constant $C(r)$ such that the following estimate holds for any analytic disk $g : \Delta \to D$ and any $\eta \in \partial D$:

$$|u| \leq r \Rightarrow |g(u) - \eta| \leq C(r)|g(0) - \eta|^\alpha$$

($\Delta$ is the open unit disk in $\mathbb{C}$).

- A pair of domains $D_1, D_2$ satisfies the property $(D_1, D_2)_{\beta}$ ($0 < \beta < 1$) if for every proper holomorphic map $F : D_1 \to D_2$ one has

$$\forall z \in D_1 : \ d(F(z), \partial D_2) \leq C(d(z, \partial D_1)^\beta)$$

for some positive constant $C$.

Our basic result is:

**Theorem 1.** Let $D_1$ and $D_2$ be bounded domains in $\mathbb{C}^n$ with piecewise smooth boundaries. Assume that $\partial D_2$ satisfies the attraction property of order $\alpha$ and that $D_1, D_2$ satisfy $(D_1, D_2)_{\beta}$ with $0 < \alpha, \beta < 1$. Then every proper holomorphic map $F : D_1 \to D_2$ extends to a Hölder continuous map $\tilde{F} : \tilde{D}_1 \to \tilde{D}_2$ with exponent $\alpha\beta$.

(See [12], p. 206, for a precise definition of “piecewise smooth”.)

Our first goal is to obtain a quite precise control of $\alpha\beta$ for domains with “good” plurisubharmonic peak functions at each point of their boundaries. This is done in Propositions 3.1 and 3.2.

In Theorems 2 and 3, we use recent results of Forcstat–Sibony ([6]) in order to apply Theorem 1 to intersections of domains in $\mathbb{C}^n$ (convex if $n > 2$) having finite types and lying in general position.

2. Results. If $A(z)$ and $B(z)$ depend on a variable $z$, $A(z) \lesssim B(z)$ means that there is a constant $K$, $0 < K < \infty$, such that $A(z) \leq KB(z)$ for all $z$.

Any finite intersection of bounded pseudoconvex domains in $\mathbb{C}^n$, with at least $C^2$ boundaries and lying in general position, will be called an elementary pseudoconvex domain.

**Remark 2.1.** One can check that every elementary pseudoconvex domain satisfies the following property, which we call the cone property of order $\gamma$, written $C(\gamma)$ (actually this only follows from the “regularity” of the boundary).

$D \subset \mathbb{C}^n$ satisfies $C(\gamma)$, $0 < \gamma < 1$, if there is a continuous map $N : \partial D \to \{ \overline{\gamma} \in \mathbb{C}^n : \| \overline{\gamma} \| = 1 \}$ and a constant $\lambda$, $0 < \lambda < \infty$, such that:

$C_1$: the cone of vertex $p$, height $\lambda$, angle $\alpha \pi$ ($0 < \alpha < 1$) and directed by $N(p)$ is contained in $D$.

$C_2$: $\forall p, p' \in \partial D, \forall x, x' \in [0, \lambda] : p + xN(p) = p' + x'N(p') \Rightarrow p = p'$ and $x = x'$.

**Remark 2.2.** It directly follows from the Diederich–Forcstat exhaustion theorem ([3]) that every elementary pseudoconvex domain $D$ admits a plurisubharmonic function $\varphi$ satisfying $0 < -\varphi(z) \leq d(z, \partial D)^{\alpha}$ for all $z \in D$ and some $\alpha \in [0, 1]$. We shall say that $D$ is hyperconvex of order $\alpha$.

We are now able to state precisely our main results.

**Theorem 2.** Let $D_1$ and $D_2$ be elementary pseudoconvex domains in $\mathbb{C}^n$.

Assume moreover that $D_2$ is a finite intersection of bounded domains in $\mathbb{C}^n$ (convex if $n > 2$) having types less than $2\theta$ and lying in general position. Then every proper holomorphic map $F : D_1 \to D_2$ extends to a Hölder continuous map $\tilde{F} : \tilde{D}_1 \to \tilde{D}_2$ with exponent $(\alpha\gamma - \gamma)(2k)/2k$, where $\alpha$ and $\gamma$ are such that $D_1$ is hyperconvex of order $\alpha$ and $D_2$ satisfies the cone property of order $\gamma$.

Here and below, $(b - 0)$ (resp. $(b+0)$) denotes a constant which is strictly smaller (resp. larger) but arbitrarily close to $b$.

**Theorem 3.** Let $D_1$ (resp. $D_2$) be a finite intersection of bounded pseudoconvex domains in $\mathbb{C}^n$ (convex if $n > 2$) having types less than $2\theta$ (resp. $2k$) and lying in general position. Then every biholomorphism $F : D_1 \to D_2$ extends to a homeomorphism $\tilde{F} : \tilde{D}_1 \to \tilde{D}_2$ which satisfies

$$\forall z, z' \in \tilde{D}_1 : \ |z - z'|^{(1+\alpha\gamma)/(\alpha\gamma)} \preceq |\tilde{F}(z) - \tilde{F}(z')| \preceq |z - z'|^{(1+\gamma)/(\alpha\gamma)}/k$$

where $D_1$ and $D_2$ satisfy the cone property of order $\gamma_1$ and $\gamma_2$ respectively.

3. Two technical propositions. We first show that a domain admitting “good” plurisubharmonic peak functions satisfies the attraction property.

**Proposition 3.1.** Let $D$ be a bounded domain in $\mathbb{C}^n$ with neighbourhood $V$. Assume that for each $\eta \in \partial D$ there is a function $\varphi_\eta : V \to \mathbb{R}$, plurisubharmonic on $D$, peaking at $\eta$ and satisfying

$P_1$: $\forall z, z' \in V : \ |\varphi_\eta(z) - \varphi_\eta(z')| \leq |z - z'|$,

$P_2$: $\forall \eta \in \partial D : \ \varphi_\eta(z) \leq 1 - B|z - \eta|^{2k}$ and $\varphi_\eta(\eta) = 1$,

where $A, B$ and $k$ are strictly positive and independent of $\eta$. Then $\partial D$ satisfies the attraction property of order $(1 - 0)/2k$.

**Proof.** Let $g : \Delta \to D$ be an analytic disk. Let $\eta$ be any point in $\partial D$ and $\varphi$ a plurisubharmonic peak function at $\eta$ which satisfies conditions $P_1$.
and \( P_2 \). We shall assume without any loss of generality that \( \eta = 0 \). We write \( \varepsilon \) for \( |g(0) - \eta| = |g(0)| \).

For \( r \in [0, 1] \) and \( \lambda \in [0, +\infty] \), we consider

\[
M(r) = \sup_{\theta \in [0, 2\pi]} |g(re^{i\theta})|^{2k},
\]

\[
\mu(r, \lambda) = \max \{ \theta \in [0, 2\pi] : |g(re^{i\theta})|^{2k} \leq (A/B)\lambda \varepsilon \},
\]

\[
\bar{\mu}(r, \lambda) = \min \{ \theta \in [0, 2\pi] : \varphi \circ g(re^{i\theta}) \geq 1 - A\lambda \varepsilon \}.
\]

Condition \( P_2 \) implies that \( \bar{\mu}(r, \lambda) \leq \mu(r, \lambda) \). Therefore, since \( \varphi \circ g \) is subharmonic and bounded by 1, we get

\[
(1) \quad \varphi \circ g(0) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi \circ g(re^{i\theta}) \, d\theta \leq \frac{1}{2\pi} \left[ \bar{\mu}(r, \lambda) + (1 - A\lambda \varepsilon)/(2\pi - \bar{\mu}(r, \lambda)) \right].
\]

Condition \( P_1 \) implies that \( \varphi \circ g(0) \geq 1 - A \varepsilon \lambda \) and therefore

\[
(2) \quad \mu(r, \lambda) \geq 2\pi (1 - 1/\lambda).
\]

On the other hand, we have

\[
(3) \quad |g(u)|^{2k} \leq \frac{1}{r_1 - r_2} \cdot \frac{1}{2\pi} \int_0^{2\pi} |g(r_1 e^{i\theta})|^{2k} \, d\theta \quad \text{for} \quad |u| \leq r_2 < r_1 < 1.
\]

Hence

\[
(4) \quad M(r_1) \leq \frac{2}{r_1 - r_2} \left[ 2\pi \frac{A}{B} \lambda \varepsilon + (2\pi - \mu(r, \lambda)) M(r_1) \right]
\]

and by (2)

\[
(5) \quad M(r_2) \leq \frac{C}{r_1 - r_2} \frac{\lambda \varepsilon + M(r_1)}{\lambda} \quad \text{where} \quad C = 2 \max \{A/B, 1\}.
\]

On choosing \( \lambda = (M(r_1)/\varepsilon)^{1/2} \), (5) becomes

\[
(6) \quad M(r_2) \leq \frac{C}{r_1 - r_2} (M(r_1) \varepsilon)^{1/2}.
\]

Now we take a decreasing sequence \( (r_p)_{p \geq 1} \) with \( r_p < 1 \) and \( \lim r_p = r \). An obvious induction on (6) provides \( M(r_{p+1}) \leq C_p (2^{p-1})^{1/2} M(r_1)^{1/2p} \) where \( C_p \) only depends on \( r_1, \ldots, r_{p+1} \). The proposition follows immediately.

The second proposition provides an estimate for the distance to the boundary under proper holomorphic mappings.

**Proposition 3.2.** Let \( D_1 \) and \( D_2 \) be bounded domains in \( \mathbb{C}^n \). Assume that \( D_1 \) is hyperconvex of order \( \alpha \) and that \( D_2 \) satisfies the cone property of order \( \gamma \). Then \((D_1, D_2)_{\gamma, \alpha} \) is satisfied.

It is well known that this proposition remains true for \( \gamma = 1 \) if \( D_2 \) has at least \( C^2 \) boundary. This is actually an easy consequence of the classical Hopf lemma (see [11], p. 177).

In our case, Proposition 3.2 will follow from a suitable version of this lemma for domains satisfying the cone property:

**Lemma 3.1.** Let \( D \) be a bounded domain in \( \mathbb{C}^n \) which satisfies the cone property of order \( \gamma \). Assume \( \varphi : \overline{D} \to [-\infty, 0] \) is a plurisubharmonic continuous function on \( D \) which vanishes on \( \partial D \). Then \( \varphi(z) \leq -d(z, \partial D)^{1/\gamma} \) for all \( z \in D \).

**Proof.** Let \( T \) be the triangle with vertices \( b_1 = 0, b_2 = \lambda(1 + it \tan \gamma \pi/2) \) and \( b_3 = \lambda(1 - it \tan \gamma \pi/2) \). Let \( f : \overline{A} \to \overline{T} \) be a conformal map. One can take

\[
f(u) = C \int_0^u (a_1 - t)^{-\gamma-1} (a_2 - t)^{-\gamma+1/2} (a_3 - t)^{-\gamma+1/2} \, dt + C'
\]

where \( f(a_j) = b_j \) for \( j = 1, 2, 3 \).

For each \( p \in \partial D \) one defines \( \psi_p : \overline{T} \to [-\infty, 0] \) by \( \psi_p(u) = \varphi(p + uN(p)) \).

This is possible because of condition \( C_1 \). By applying the Poisson integral formula to \( \psi_p \) and using obvious estimates for \( f \) one easily gets

\[
(1) \quad \varphi(p + z N(p)) \leq z^{1/\gamma} \left[ \frac{1}{2\pi} \int_0^{2\pi} \varphi(f(e^{i\theta} N(p))) \, d\theta \right] = M(p) |z|^{1/\gamma}
\]

for \( z > 0 \). Since \( \varphi \) is continuous on \( \overline{D} \), \( M \) is continuous on \( \partial D \). So \( M \leq -1 \) and (1) becomes

\[
(2) \quad \forall p \in \partial D : \varphi(p + z N(p)) \leq -z^{1/\gamma} \quad \text{for} \quad z \in [0, \lambda].
\]

Finally, for each \( z \in D \) sufficiently close to \( \partial D \) condition \( C_2 \) provides a unique \( \gamma \in [0, \lambda] \) and a unique \( p \in \partial D \) such that \( z = p + x N(p) \). Then \( d(z, \partial D) \geq (\tan \gamma \pi/2)x \) and the lemma directly follows from (2).

We now sketch the proof of Proposition 3.2. Let \( g_1 \) be a plurisubharmonic function on \( D_1 \) which satisfies \( 0 < -g_1(w) \leq d(w, \partial D_1)^{\alpha} \). Assume \( F : D_1 \to D_2 \) is a proper holomorphic map. By applying Lemma 3.1 to the plurisubharmonic function \( S(z) = \sup \{ g_1(w) : F(w) = z \} \), one gets \( S(z) \leq -d(z, \partial D_2)^{1/\gamma} \). On the other hand, if \( F(w) = z \) one has \( g_1(w) \geq -d(w, \partial D_1)^{\alpha} \). Comparing these inequalities one finds \( d(F(w), \partial D_2) \leq d(w, \partial D_1)^{\alpha} \) and the proposition is proved (see [1], p. 140, for more details).
4. Proofs of Theorems 1–3. We first prove Theorem 1.

Let $D_1, D_2$ be domains in $\mathbb{C}^n$ with piecewise smooth boundaries. Assume that $\partial D_2$ satisfies the attraction property of order $\alpha$ and that $(D_1, D_2)_0$ is fulfilled. Let $F : D_1 \to D_2$ be a proper holomorphic map. Let $z_0 \in D_1$. We shall write $\varepsilon$ for $d(z_0, \partial D_1)$ and we choose $\eta \in \partial D_2$ such that $|F(z_0) - \eta| = d(F(z_0), \partial D_2)$.

Now we define an analytic disk $g : \Delta \to D_2$ by $g(u) = F(z_0 + \frac{1}{2} \varepsilon u \bar{\varepsilon})$ where $\bar{\varepsilon}$ is a fixed unit vector in $\mathbb{C}^n$.

It follows from $(D_1, D_2)_0$ that

$$d(F(z_0), \partial D_2) \leq \varepsilon^\alpha.$$  

Since $\partial D_2$ satisfies the attraction property of order $\alpha$ one gets

$$|u| \leq 1/2 \Rightarrow |g(u) - \eta| \leq |g(0) - \eta| \leq \varepsilon^\alpha.$$  

We have $\alpha \beta < 1$ and so

$$|u| \leq 1/2 \Rightarrow |g(u) - g(0)| \leq |g(u) - \eta| + |\eta - g(0)| \leq \varepsilon^\alpha.$$  

Hence by Cauchy's inequality

$$|g'(0)| \leq \frac{1}{\varepsilon} \varepsilon^\alpha.$$  

Denote the holomorphic tangent map of $F$ by $F'_z$. We have $F'_z \cdot \bar{\varepsilon} = g'(0) \bar{\varepsilon}$ and therefore

$$|F'_z| = \sup_{|z_0| = 1} \|F'_z \cdot \bar{\varepsilon}\| \leq d(z_0, \partial D_1)^{\alpha - 1}$$  

for all $z \in D_1$.

Finally, the Hölder continuity of $F$ with exponent $\alpha \beta$ is obtained from (5) by a classical integration argument for which we refer to [8] or [7], p. 62. Theorem 1 is proved.

We now explain how Theorems 2 and 3 are obtained from Theorem 1.

Proof of Theorem 2. Assume that $D_1$ and $D_2$ are elementary-pseudoconvex domains, $D_1$ is hyperconvex of order $\alpha$ and $D_2$ satisfies the cone property of order $\gamma$. Then, by Proposition 3.2, $(D_1, D_2)_0$ is fulfilled.

On the other hand, it follows from a result of Fornaess and Sibony ([6]) that if $D_2$ is an intersection of domains of finite type (convex if $n > 2$) then the assumptions of Proposition 3.1 are satisfied for some integer $k$. Therefore Theorem 2 is directly obtained by applying Proposition 3.1 and Theorem 1.

(In the case $n > 2$ the peak functions of Fornaess–Sibony satisfy slightly different conditions than the conditions $P_1$ and $P_2$ described in Proposition 3.1 (see [6], p. 651). However, one easily sees that this does not modify our result.)

Proof of Theorem 3. Under the assumptions of Theorem 3 one can apply Theorem 2 to $F$ and $F^{-1}$. So $F$ extends to a homeomorphism $\hat{F} : \hat{D}_1 \to \hat{D}_2$. Then it is possible to use a suitable version of Proposition 3.2 based on local plurisubharmonic exhausting functions instead of global ones, and to take the constant $\alpha$ arbitrarily close to 1. This leads to the estimates of Theorem 3.

References


U.F.R. DE MATHEMATIQUES PURES ET APPLIQUEES
UNIVERSITÉ DES SCIENCES ET TECHNIQUES DE LILLE FLANDRES ARTOIS
U.R.A. C.N.R.S. D 0751
59658 VILLENEUVE D'ASCQ CEDEX, FRANCE

Received October 5, 1990 (2724)