[9] L. Pick, *Two weights weak type inequality for the maximal function in* $L(\log^+ L)^K$, in: Constructive Theory of Functions, Proc. Conf. Varna 1987, B. Sendov *et al.* (eds.), Publ. House Bulgar. Acad. Sci., Sofia 1988, 377–381.

[10] —, *Weighted inequalities for the Hardy–Littlewood maximal operators in Orlicz spaces*, preprint, Math. Inst. Czech. Acad. Sci. 46 (1989), 1–22.

[11] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Math. 1381, Springer, Berlin 1989.

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Korovkin theory in normed algebras

by

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Abstract. If $\mathcal{A}$ is a normed power-associative complex algebra such that the selfadjoint part is normally ordered with respect to some order, then the Korovkin closure (see the introduction for definitions) of $T \cup \{t^* \circ t \mid t \in T\}$ contains $J^*(T)$ for any subset T of $\mathcal{A}$. This can be applied to C^*-algebras, minimal norm ideals on a Hilbert space, and to H^*-algebras. For bounded H^*-algebras and dual C^*-algebras there is even equality. This answers a question posed in [1].

§ 1. **Introduction.** Let us recall some definitions. Let $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ be normed power-associative complex algebras with a continuous involution $*$, and let $\mathcal{A}_s = \{x \in \mathcal{A} \mid x^* = x\}$ and $\widetilde{\mathcal{A}}_s$ be normally ordered, i.e. $\mathcal{A}_s$ (resp. $\widetilde{\mathcal{A}}_s$) is an ordered real vector space such that the norm $\|\cdot\|$ is equivalent to a monotone norm $\|\cdot\|_m$ ($0 \le x \le y$ implies $\|x\|_m \le \|y\|_m$). A continuous linear map $P : \mathcal{A} \to \widetilde{\mathcal{A}}$ which is selfadjoint (i.e. $P(\mathcal{A}_s) \subset \widetilde{\mathcal{A}}_s$) and which satisfies $(Px)^2 \le P(x^2)$ for all $x \in \mathcal{A}_s$ is called a *Jordan–Schwarz map*, since for such a P the Schwarz inequality holds with respect to the Jordan product: $P(x)^* \circ P(x) \le P(x^* \circ x)$ for all $x \in \mathcal{A}$. A *J^*-subalgebra* of $\mathcal{A}$ is by definition a $*$-closed and norm-closed subspace which is also closed with respect to the Jordan product. The J^*-subalgebra of $\mathcal{A}$ generated by $T \subset \mathcal{A}$ is denoted by $J^*(T)$.

1.1. THEOREM. *Let $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ be normed power-associative complex algebras with a continuous involution $*$, and let $\mathcal{A}_s$ and $\widetilde{\mathcal{A}}_s$ be normally ordered. If $S : \mathcal{A} \to \widetilde{\mathcal{A}}$ is a continuous $*$-homomorphism and $(P_\alpha)_{\alpha \in A}$ an equicontinuous net of Jordan–Schwarz maps $P_\alpha : \mathcal{A} \to \widetilde{\mathcal{A}}$, then*

$$\{x \in \mathcal{A} \mid P_\alpha x \underset{\alpha}{\to} Sx \text{ and } P_\alpha(x^* \circ x) \underset{\alpha}{\to} S(x^* \circ x)\}$$

is a J^*-subalgebra of $\mathcal{A}$.

Here convergence always means convergence in the norm topology. It is also possible, and other authors do so, to consider weaker topologies, see for

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example [11], [12], [15],...

The proof of Theorem 1.1 is very similar to W. M. Priestley's proof in [15]. The use of the identity element in [15] can be replaced by using the normality of the positive cone. This shows that $\{x \in \mathcal{A} \mid P_\alpha x \to Sx,\ P_\alpha(x^2) \to S(x^2)$ and $P_\alpha(x^* \circ x) \to S(x^* \circ x)\}$ is a J^*-subalgebra. Now use the trick of B. V. Limaye and M. N. N. Namboodiri in [10] to get rid of the condition $P_\alpha(x^2) \to S(x^2)$.

Now let us define the *Korovkin closure* $\mathrm{Kor}_{\mathcal{A}}(T)$ of a subset $T \subset \mathcal{A}$. $\mathrm{Kor}_{\mathcal{A}}(T)$ is the set of all $x \in \mathcal{A}$ that satisfy the following condition: If $(P_\alpha)_{\alpha\in A}$ is an equicontinuous net of Jordan–Schwarz maps $P_\alpha : \mathcal{A} \to \mathcal{A}$ and if $P_\alpha t \underset{\alpha}{\to} t$ for all $t \in T$, then also $P_\alpha x \underset{\alpha}{\to} x$.

The *universal Korovkin closure* of T, denoted by $\mathrm{Kor}^u_{\mathcal{A}}(T)$, is defined to be the set of all $x \in \mathcal{A}$ which satisfy: If $\widetilde{\mathcal{A}}$ is a C^*-algebra, $S : \mathcal{A} \to \widetilde{\mathcal{A}}$ a continuous $*$-homomorphism, $(P_\alpha)_{\alpha\in A}$ an equicontinuous net of Jordan–Schwarz maps $P_\alpha : \mathcal{A} \to \widetilde{\mathcal{A}}$, and if $P_\alpha t \underset{\alpha}{\to} St$ for all $t \in T$ then also $P_\alpha x \underset{\alpha}{\to} Sx$.

Obviously we have $T \subset \mathrm{Kor}_{\mathcal{A}}(T)$, and $T \subset \mathrm{Kor}^u_{\mathcal{A}}(T)$, and $\mathrm{Kor}^u_{\mathcal{A}}(T) \subset \mathrm{Kor}_{\mathcal{A}}(T)$ if $\mathcal{A}$ is a C^*-algebra.

By some well-known theorems it is easy to show that the Jordan–Schwarz maps between C^*-algebras are precisely the positive linear maps having norm less than or equal to 1. So in the special case of a C^*-algebra we obtain the usual definition of a Korovkin closure.

A simple application of Theorem 1.1 gives:

1.2. THEOREM. *Let $\mathcal{A}$ be a normed and power-associative complex algebra with a continuous involution such that $\mathcal{A}_s$ is normally ordered. Then for $T \subset \mathcal{A}$ we have $J^*(T) \subset \mathrm{Kor}_{\mathcal{A}}(T \cup \{t^* \circ t \mid t \in T\})$. If $\mathcal{A}$ is a C^*-algebra, then $J^*(T) \subset \mathrm{Kor}^u_{\mathcal{A}}(T \cup \{t^* \circ t \mid t \in T\})$.*

This will be applied in §2 to

— C^*-algebras. The purpose of that section is merely to show that the known results are covered by Theorems 1.1 and 1.2.

— minimal norm ideals on a Hilbert space. W. M. Priestley proved a Korovkin theorem for the trace class on a Hilbert space [15]. Here a somewhat stronger result is proved not only for the trace class but for all minimal norm ideals on a Hilbert space.

— H^*-algebras. The order will be defined by the left regular representation.

— bounded H^*-algebras. If $\mathcal{S}(H)$ is the Hilbert–Schmidt class on the Hilbert space H, then the orthogonal projection onto a J^*-subalgebra is a Jordan–Schwarz map. This result, that might be interesting in its own right, is the key for proving $J^*(T) = \mathrm{Kor}_{\mathcal{A}}(T \cup \{t^* \circ t \mid t \in T\})$.

This theorem also enables us to prove the same equality for dual C^*-algebras in §3; this gives a partial answer to a question raised by F. Altomare in [1].

This paper is a part of the author's doctoral dissertation. I am indebted to Prof. G. Maltese for supervision and many useful suggestions.

§ 2. Applications of the theorems. As already mentioned in the introduction there is a more common characterization of Jordan–Schwarz maps between C^*-algebras. By a well-known theorem of R. V. Kadison [8] (also used in [15]), a positive linear map $P : \mathcal{A} \to \widetilde{\mathcal{A}}$ with $\|P\| \le 1$ is a Jordan–Schwarz map. Conversely, if P is a Jordan–Schwarz map, then P is obviously positive and therefore continuous (see [18], p. 194). It is a simple matter to show that the norm of P does not exceed 1.

The following is a simple reformulation of 1.1 and 1.2:

2.1. THEOREM. *Let $\mathcal{A}, \widetilde{\mathcal{A}}$ be C^*-algebras. If $S : \mathcal{A} \to \widetilde{\mathcal{A}}$ is a $*$-homomorphism and if $(P_\alpha)_{\alpha\in A}$ is a net of positive maps $P_\alpha : \mathcal{A} \to \widetilde{\mathcal{A}}$ with $\|P_\alpha\| \le 1$ for all $\alpha \in A$, then*

$$\{x \in \mathcal{A} \mid P_\alpha x \underset{\alpha}{\to} Sx \text{ and } P_\alpha(x^* \circ x) \underset{\alpha}{\to} S(x^* \circ x)\}$$

is a J^*-subalgebra of $\mathcal{A}$. For all subsets $T \subset \mathcal{A}$ the following inclusions hold:

$$J^*(T) \subset \mathrm{Kor}^u_{\mathcal{A}}(T \cup \{t^* \circ t \mid t \in T\}) \subset \mathrm{Kor}_{\mathcal{A}}(T \cup \{t^* \circ t \mid t \in T\}).$$

Let (M, q) be a minimal norm ideal on a Hilbert space H (see [17] for definitions and basic properties). It is well known that (M, q) is a normed associative complex algebra with an isometric involution, which is the restriction of the involution of the operator algebra $\mathcal{L}(H)$ on H (isometric with respect to the norm q !). In order to apply Theorems 1.1 and 1.2 one has to show that M_s is normally ordered, where the order again is the restriction of the natural order on $\mathcal{L}(H)_s$. In fact, the norm is monotone. As I could not find a reference for this, a proof will be given now. So let $x, y \in M$ such that $0 \le x \le y$. Let $(\lambda_k)_k$ and $(\mu_k)_k$ be the nonincreasing sequences of eigenvalues of x and y, with multiplicities taken into account. By the formula for the nth eigenvalue [7] we have $\lambda_k \le \mu_k$ for all $k \in \mathbb{N}$. Define ν_k to be $\lambda_k\mu_k^{-1}$ if $\mu_k \ne 0$, and 0 otherwise. Then $\mu_k\nu_k = \lambda_k$ for all $k \in \mathbb{N}$. Let $(\varphi_k)_{k\in K}$ be an orthonormal base of H such that $x = \sum_{k\in K} \lambda_k \varphi_k \otimes \varphi_k$ (w.l.o.g. $\mathbb{N} \subset K$ and $\lambda_k = 0$ for all $k \in K \setminus \mathbb{N}$, and for $k \in \mathbb{N}$, λ_k is the λ_k defined above), and let $(\psi_k)_{k\in K}$ be an analogous base for y. Define $a = \sum_{k\in K} \nu_k \varphi_k \otimes \varphi_k \in \mathcal{L}(H)$, where $\nu_k = 0$ for all $k \in K \setminus \mathbb{N}$, and $v = \sum_{k\in K} \psi_k \otimes \varphi_k$. Then v is unitary, $\|a\| \le 1$, and obviously $x = av^*yv$, which finally implies $q(x) = q(av^*yv) \le \|av^*\|\|v\|q(y) \le q(y)$.

So the results of §1 are applicable:

2.2. THEOREM. *Let (M, q) be a minimal norm ideal on a Hilbert space. If $S: M \to M$ is a continuous $*$-homomorphism and $(P_\alpha)_{\alpha \in A}$ an equicontinuous net of Jordan–Schwarz maps $P_\alpha : M \to M$, then*

$$\{x \in M \mid P_\alpha x \underset{\alpha}{\to} Sx \text{ and } P_\alpha(x^* \circ x) \underset{\alpha}{\to} S(x^* \circ x)\}$$

is a J^*-subalgebra of M. For all subsets $T \subset M$ the following inclusion holds: $J^*(T) \subset \mathrm{Kor}_M(T \cup \{t^* \circ t \mid t \in T\})$.

A very important minimal norm ideal is the Hilbert–Schmidt class $\mathcal{S}(H)$ on a Hilbert space H.

An *H^*-algebra* is an involutive complex Banach algebra $(\mathcal{A}, \|\cdot\|)$ such that $(\mathcal{A}, \|\cdot\|)$ is a Hilbert space and for all $x, y, z \in \mathcal{A}$ we have $\langle xy, z\rangle = \langle y, x^*z\rangle$, $\langle yx, z\rangle = \langle y, zx^*\rangle$, and $x\mathcal{A} = \{0\}$ only for $x = 0$. Examples of H^*-algebras are $\mathcal{S}(H)$ and $L^2(G)$, where G is a compact group.

W. Ambrose [2] gives a structure theorem which is used in the sequel (see also [6], Part I, Ch. 8.5, Prop. 7). An idempotent $e \in \mathcal{A}$ ($e \neq 0$ and $e^2 = e$) is said to be *primitive* if there are no idempotents e_1, e_2 in $\mathcal{A}$ such that $\langle e_1, e_2\rangle = 0$ and $e = e_1 + e_2$. The following results are taken from [2].

(I) An H^*-algebra is an orthogonal sum of simple H^*-algebras, each of them being a closed two-sided ideal in $\mathcal{A}$.

(II) A simple H^*-algebra is up to a constant multiple α isometrically $*$-isomorphic to $\mathcal{S}(H)$ for a suitable Hilbert space H; more precisely: There is a $*$-isomorphism $\varphi : \mathcal{A} \to \mathcal{S}(H)$ and a constant $\alpha > 0$ such that $\|x\| = \alpha\|\varphi(x)\|$ for all $x \in \mathcal{A}$. We then write $\mathcal{A} = (\mathcal{S}(H), \alpha)$. α is the norm of an arbitrary primitive idempotent in the simple H^*-algebra $\mathcal{A}$, so $\alpha \geq 1$.

So H^*-algebras are "made up" of Hilbert–Schmidt classes: we have $\mathcal{A} \cong \sum_{k\in K}(\mathcal{S}(H_k), \alpha_k)$. This decomposition is unique in the usual sense, so the α_k's are uniquely determined by $\mathcal{A}$. For later purposes, call $\mathcal{A}$ *bounded* iff $\{\alpha_k \mid k \in K\}$ is a bounded set.

Since Korovkin theory requires order structures, let us call $x \in \mathcal{A}$ *positive* iff the left regular representation $L_x : \mathcal{A} \to \mathcal{A}$ is a positive Hilbert space operator. Now we have two order structures on $\mathcal{S}(H)$, but they turn out to be the same. Moreover, this order structure respects the above decomposition.

2.3. LEMMA. *Let $\mathcal{A}$ be an H^*-algebra. Then the norm restricted to $\mathcal{A}_s$ is monotone, hence normal.*

P r o o f. Let $0 \leq x \leq y$ in $\mathcal{A}$ and let $\mathcal{A} = \sum_{k\in K} \mathcal{A}_k$ be the decomposition described in (I) and (II). If $x = \sum_{k\in K} x_k$, $y = \sum_{k\in K} y_k$, where $x_k, y_k \in \mathcal{A}_k$, then for all k we have $0 \leq x_k \leq y_k$. Since $\mathcal{A}_k \cong (\mathcal{S}(H_k), \alpha_k)$, we have the same inequalities for the corresponding elements in $\mathcal{S}(H_k)$, which again are denoted by x_k and y_k. Then we are in the situation of a minimal norm ideal, so we may conclude $\|x\|^2 = \sum_{k\in K} \alpha_k^2 \|x_k\|_2^2 \leq \sum_{k\in K} \alpha_k^2\|y_k\|_2^2 = \|y\|^2$, and we are done.

Hence a theorem similar to those above may be stated. The continuity of the homomorphism S is automatic by a result of B. Yood [19, Prop. 5.1].

Recall that if $\mathcal{A} = \sum_{k\in K}(\mathcal{S}(H_k), \alpha_k)$ is the decomposition of the H^*-algebra $\mathcal{A}$ then $\mathcal{A}$ is said to be bounded iff $\{\alpha_k \mid k \in K\}$ is a bounded set. As explained earlier, the α_k's are the norms of primitive idempotents in simple two-sided ideals of $\mathcal{A}$. Since a selfadjoint idempotent is primitive iff it is already contained in one of those simple two-sided ideals and is primitive there, we see that $\mathcal{A}$ is bounded iff $\{\|e\| \mid e \in \mathcal{A}$ is a selfadjoint primitive idempotent$\}$ is a bounded set.

$\mathcal{S}(H)$ is bounded, because it is simple (another argument: primitive projections in $\mathcal{S}(H)$ are one-dimensional, and so have norm 1). Other examples of bounded H^*-algebras are obtained by

2.4. R e m a r k. *Let G be a compact group. Then the H^*-algebra $L^2(G)$ is bounded iff the supremum of $\dim(\pi)$, where π runs through the irreducible representations of G, is bounded.* (*Here only unitary representations need to be considered.*)

P r o o f. By [13], 39A, B, it is simple to show that $\sup\{\dim(\pi) \mid \pi$ irreducible$\} = \sup\{\|e\|^2 \mid e$ primitive$\}$. This clearly proves the claim.

So $L^2(G)$ is bounded if G is finite or commutative. By I. Kaplansky [9], Cor. to Th. 3, a connected compact group is already abelian if $\sup\{\dim(\pi) \mid \pi$ irreducible representation$\}$ is finite. In order to give an example of a noncommutative infinite bounded group let S^1 be the torus group and let P_n be the permutation group of order n. Then by [9], Th. 1, $L^2(G \times P_n)$ is bounded.

The following theorem is the key step to what will follow.

2.5. THEOREM. *Let H be a Hilbert space and $\mathcal{B} \subset \mathcal{S}(H)$ a J^*-subalgebra. Then the orthogonal projection onto $\mathcal{B}$ is a Jordan–Schwarz map. Moreover, it maps positive elements to positive elements.*

Since $\mathcal{S}(H)$ is a Hilbert space, the orthogonal projection P onto $\mathcal{B}$ makes sense. There are two notions of positivity:

(i) positivity induced by $\mathcal{L}(\mathcal{S}(H))$,

(ii) positivity in the sense that positive elements are mapped to positive elements.

These two notions do not coincide as can be seen by simple examples, so the last claim of the theorem deserves a proof.

P r o o f. The proof is divided into seven steps:

1. Px is selfadjoint whenever $x \in \mathcal{S}(H)$ is selfadjoint.
2. Let $x \in \mathcal{B}$ be selfadjoint, $\operatorname{spec}(x) \cap (0,\infty) \neq \emptyset$, λ the largest eigenvalue of x, and e the orthogonal projection onto the corresponding eigenspace. Then $e \in \mathcal{B}$.
3. Let $x \in \mathcal{B}$ be selfadjoint, $\lambda \neq 0$ an eigenvalue of x, and e the orthogonal projection onto the corresponding eigenspace. Then $e \in \mathcal{B}$.
4. $P(x^2) \geq 0$ for all selfadjoint elements $x \in \mathcal{S}(H)$.
5. $\mathcal{B} \circ \mathcal{B}^\perp \subset \mathcal{B}^\perp$, where "$\circ$" is the Jordan product.
6. $(Px)^2 \leq P(x^2)$ for all selfadjoint elements $x \in \mathcal{S}(H)$.
7. P maps positive elements to positive elements.

Since P is obviously continuous and linear, this will finish the proof.

Step 1. Simple.

Step 2. Let $\lambda_1, \lambda_2, \ldots \neq 0$ be the different eigenvalues of x, say $\lambda = \lambda_1$. Let $(\varphi_k)_{k\in K}$ be an orthonormal base of H consisting of eigenvalues of $x : x\varphi_k = \mu_k \varphi_k$ (in this case the eigenvalues are denoted by μ_k, since here multiplicities are taken into account). Let $\varepsilon > 0$ be given.

Since we have $\sum_{k\in K} \mu_k^2 < \infty$, there is a finite subset $I \subset K$ such that $\sum_{k \in K\setminus I} \mu_k^2 < \varepsilon^2$ and $\mu_k \neq 0$ for all $k \in I$. Then there is $n \in \mathbb{N}$ with $\{\mu_k \mid k \in I\} \subset \{\lambda_1, \ldots, \lambda_n\}$, and we may assume equality of these sets by enlarging I (all eigenvalues have finite multiplicities!).

By the Stone–Weierstraß approximation theorem there is a polynomial p having the properties $p(\lambda_1) = 1$, $p(\lambda_k) = 0$ for all $k = 2, \ldots, n$ and $|p(t)| \leq |t|$ for all t in the closure of the convex hull of $\operatorname{spec}(x) \setminus \{\lambda\}$ (drawing a picture of this might be helpful). Then we conclude

$$\|p(x) - e\|_2^2 = \sum_{k\in K} \|(p(x) - e)\varphi_k\|^2 = \sum_{k \in K, \mu_k \neq \lambda_1} \|p(\mu_k)\varphi_k\|^2$$

(since if $\mu_k = \lambda_1$, then $p(\mu_k)\varphi_k - e\varphi_k = \varphi_k - \varphi_k = 0$, and if $\mu_k \neq \lambda_1$, then $e\varphi_k = 0$)

$$= \sum_{k \in K\setminus I} |p(\mu_k)|^2 \quad (\text{since } p(\mu_k) = 0 \text{ for } k \in I \text{ and } \|\varphi_k\| = 1)$$

$$\leq \sum_{k \in K \setminus I} |\mu_k|^2 < \varepsilon^2 .$$

This gives $\|p(x) - e\|_2 < \varepsilon$. But $p(x) \in \mathcal{B}$ since $\mathcal{B}$ is a J^*-subalgebra. So by the closedness of $\mathcal{B}$ we may conclude $e \in \mathcal{B}$.

Step 3. W.l.o.g. $\lambda > 0$, otherwise consider $-x$. Let $\lambda_1 > \lambda_2 > \ldots$ be the different positive eigenvalues of x, and let e_k be the orthogonal projection onto the eigenspace of λ_k. By induction we show that

(*) If the sequence $\lambda_1 > \lambda_2 > \ldots$ does not end with λ_{k-1}, then $e_k \in \mathcal{B}$.

This is true for $k = 1$ by Step 2. So assume that (*) is proved for $1, \ldots, k-1$, and that the sequence does not end with λ_{k-1}. By the induction hypothesis we have $\widetilde{x} = x - \sum_{m=1}^{k-1} \lambda_m e_m \in \mathcal{B}$. Since λ_k is the largest eigenvalue of $\widetilde{x}$, again Step 2 implies $e_k \in \mathcal{B}$ and the induction is complete.

But this proves Step 3, since λ must be one of the λ_k.

Step 4. Let $\lambda \neq 0$ be an eigenvalue of $P(x^2)$ and let us show $\lambda \geq 0$. Let e be the orthogonal projection onto the eigenspace E_λ of λ. By Step 3 we have $e \in \mathcal{B}$, which implies $Pe = e$, and therefore $\lambda\langle e, e\rangle = \langle e, \lambda e\rangle = \langle e, P(x^2)e\rangle = \langle ee^*, P(x^2)\rangle = \langle Pe, x^2\rangle = \langle e, x^2\rangle = \operatorname{trace}(x^2 e) = \operatorname{trace}((ex)^* ex) \geq 0$. Since $\langle e, e\rangle = \operatorname{trace}(e) = \dim(E_\lambda) > 0$, we must have $\lambda \geq 0$.

Step 5. Let $x \in \mathcal{B}$ and $y \in \mathcal{B}^\perp$. For all $z \in \mathcal{B}$ we compute $2\langle x \circ y, z\rangle = \langle xy, x\rangle + \langle yx, z\rangle = \langle y, 2x^* \circ z - zx^*\rangle + \langle yx, z\rangle = -\langle y, zx^*\rangle + \langle yx, z\rangle = 0$, because $2x^* \circ z \in \mathcal{B}$. This implies $x \circ y \in \mathcal{B}^\perp$.

Step 6. If $x \in \mathcal{S}(H)$ is selfadjoint, there are $y \in \mathcal{B}$ and $z \in \mathcal{B}^\perp$ with $x = y + z$; z and y are selfadjoint, because $\mathcal{B}$ and $\mathcal{B}^\perp$ are $*$-closed. Then

$$\begin{aligned} P(x^2) &= P(y^2 + z^2 + 2y \circ z) = P(y^2) + P(z^2) \\ &\geq P(y^2) \quad (\text{by Steps 5 and 4}) \\ &= y^2 = (Px)^2 . \end{aligned}$$

So $(Px)^2 \leq P(x^2)$ for all selfadjoint elements $x \in \mathcal{S}(H)$.

Step 7. If $x \in \mathcal{S}(H)$ is positive, the spectral theorem tells us

$$x = \lim_{n\to\infty} \sum_{k=1}^{n} \lambda_k \varphi_k \otimes \varphi_k ,$$

where the λ_k's are nonnegative, and $(\varphi_k)_{k\in K}$ is an orthonormal base of H. Define $y_n = \sum_{k=1}^n \sqrt{\lambda_k}\varphi_k \otimes \varphi_k$. Then $y_n^2 \underset{n}{\to} x$ and therefore $P(x) = \lim_{n\to\infty} P(y_n^2) \geq 0$. This completes the proof.

This result can be generalized to bounded H^*-algebras.

2.6. Theorem. *Let $\mathcal{A}$ be a bounded H^*-algebra and $\mathcal{B} \subset \mathcal{A}$ a J^*-subalgebra. Then there is a Jordan–Schwarz map $P : \mathcal{A} \to \mathcal{A}$ with the following properties:*

(i) $P^2 = P$,

(ii) $\mathcal{B} = \{x \in \mathcal{A} \mid Px = x\}$,

(iii) *P maps positive elements to positive elements.*

Proof. We know $\mathcal{A} \cong \sum_{k\in K}(\mathcal{S}(H_k), \alpha_k)$ and by hypothesis $\alpha := \sup\{\alpha_k \mid k \in K\}$ is finite. Consider the embedding

$$\Phi : \mathcal{A} \simeq \sum_{k\in K}(\mathcal{S}(H_k), \alpha_k) \xrightarrow{\varphi} \sum_{k\in K} \mathcal{S}(H_k) \xrightarrow{\psi} \mathcal{S}\Big(\sum_{k\in K} H_k\Big) .$$

It is easy to see that ψ is isometric. Since $\|x\| \leq \alpha\|\varphi(x)\|$, this embedding is topological. So $\Phi(\mathcal{B}) \subset \mathcal{S}(\sum_{k\in K} H_k)$ is closed and therefore is a J^*-subalgebra of $\mathcal{S}(\sum_{k\in K} H_k)$. By Theorem 2.5 the orthogonal projection $\widetilde{P} : \mathcal{S}(\sum_{k\in K} H_k) \to \Phi(\mathcal{B})$ is a Jordan–Schwarz map which sends positive elements to positive elements. Then $P = \Phi^{-1} \circ \widetilde{P} \circ \Phi : \mathcal{A} \to \mathcal{A}$ does the job.

2.7. THEOREM. *Let $\mathcal{A}$ be a bounded H^*-algebra and $T \subset \mathcal{A}$. Then*

$$J^*(T) = \mathrm{Kor}_{\mathcal{A}}(T \cup \{t^* \circ t \mid t \in T\}).$$

Proof. The inclusion "$\subset$" clearly holds. Conversely, let $x \notin J^*(T)$. By Theorem 2.5 there is a Jordan–Schwarz map $P : \mathcal{A} \to \mathcal{A}$ with $Py = y$ iff $y \in J^*(T)$. Consider the constant net $(P_\alpha)_\alpha$ where $P_\alpha = P$ for all α. Then $P_\alpha y \underset{\alpha}{\to} y$ for all $y \in T \cup \{t^* \circ t \mid t \in T\}$, but $P_\alpha x \underset{\alpha}{\to} x$ is not true. This implies $x \notin \mathrm{Kor}_{\mathcal{A}}(T \cup \{t^* \circ t \mid t \in T\})$.

It is an open problem whether Theorem 2.7 holds without the boundedness condition. In general it can be shown that the Korovkin closure of any set is of the form $\{x \in \mathcal{A} \mid P_\alpha x \underset{\alpha}{\to} x\}$ for one fixed net $(P_\alpha)_{\alpha\in A}$ of Jordan–Schwarz maps, and that precisely those sets can occur as Korovkin closures [3]. In the above theorem it seems to be remarkable that we can arrange this by means of a constant net.

§3. Dual C^*-algebras. There are many equivalent descriptions of dual C^*-algebras. We will use the following one: A dual C^*-algebra is a C^*-algebra $\mathcal{A}$ which is isomorphic to a direct sum of the ideals of compact operators on Hilbert spaces H_i, i.e. $\mathcal{A} \cong \sum_{i\in I} \mathcal{C}(H_i)$. The main result is

3.1. THEOREM. *Let $\mathcal{A}$ be a dual C^*-algebra and T a subset of $\mathcal{A}$. Then*

$$J^*(T) = \mathrm{Kor}_{\mathcal{A}}(T \cup \{t^* \circ t \mid t \in T\}) = \mathrm{Kor}^u_{\mathcal{A}}(T \cup \{t^* \circ t \mid t \in T\}).$$

Let $\mathcal{A}$ be a direct sum of C^*-algebras $\mathcal{C}(H_i)$, as indicated above. Note that $\mathcal{S} := \sum_{i\in I} \mathcal{S}(H_i)$ is a subalgebra of $\mathcal{A}$, where the sum means Hilbert space sum.

3.2. LEMMA. *Let $\mathcal{B}$ be a J^*-subalgebra of $\mathcal{A}$. Then $\mathcal{B} \cap \mathcal{S}$ is dense in $\mathcal{B}$ with respect to the norm topology on $\mathcal{A}$.*

Proof. For a given x in $\mathcal{B}$ and positive ε we must produce an $x_\varepsilon \in \mathcal{S} \cap \mathcal{B}$ such that $\|x - x_\varepsilon\| < \varepsilon$; w.l.o.g. x is selfadjoint. Since $\mathcal{A}$ is the direct sum of the $\mathcal{C}(H_i)$, there is a finite set $I_\varepsilon \subset I$ so that $\alpha_0 = \sup\{\|x_i\| \mid i \in I \setminus I_\varepsilon\} < \varepsilon$, where $x = \sum_{i\in I} x_i$.

Define $x_{\varepsilon,i} = 0$ for all $i \in I \setminus I_\varepsilon$. For $i \in I_\varepsilon$ let $x_i = \sum_{j\in J_i} \lambda_{i,j} \varphi_{i,j} \otimes \varphi_{i,j}$ be the spectral representation of x_i. Let $x_{\varepsilon,i} = \sum_{j\in J_{i,\varepsilon}} \lambda_{i,j} \varphi_{i,j} \otimes \varphi_{i,j}$, where $J_{i,\varepsilon} = \{j \in J_i \mid |\lambda_{i,j}| \geq \varepsilon\}$. This defines x_ε, which is obviously in $\mathcal{S}$. To see that x_ε is in $\mathcal{B}$, observe that

$$\alpha := \max\{\max\{|\lambda| \mid |\lambda| < \varepsilon,\ \lambda \in \mathrm{spec}(x_i),\ i \in I_\varepsilon\}, \alpha_0\} < \varepsilon.$$

Define the continuous function $f : \mathbf{R} \to \mathbf{R}$ to be zero on $[-\alpha, \alpha]$, $f(\lambda) = \lambda$ on the complement of $[-\varepsilon, \varepsilon]$ and linear elsewhere. Then using spectral calculus we see that $x_\varepsilon = f(x)$ and so $x_\varepsilon \in \mathcal{B}$, since $f(0) = 0$, and $\mathcal{B}$ is a J^*-algebra. It is also easy to verify the required norm estimate. This proves the lemma.

Now let $\mathcal{E}$ be the set of all $E = (E_i)_{i\in I}$ where E_i is a finite-dimensional subspace of H_i and $E_i = 0$ for all but finitely many indices. With the obvious order $\mathcal{E}$ becomes a directed set which will serve as an index set. Let q_{E_i} be the orthogonal projection onto E_i and $q_E = \sum_{i\in I} q_{E_i}$. Then it is not difficult to see that $(q_E)_{E\in\mathcal{E}}$ is an (unbounded) approximate unit for $\mathcal{S}$.

Now consider $\mathcal{B}\cap\mathcal{S}$, which is obviously a J^*-subalgebra of $\mathcal{S}$. By Theorem 2.5 the orthogonal projection $P : \mathcal{S} \to \mathcal{B} \cap \mathcal{S}$ is a positive Jordan–Schwarz map. For $E \in \mathcal{E}$ define $P_E : \mathcal{A} \to \mathcal{A}$ by $P_E(x) = P(q_E x q_E)$. So P_E is a positive, hence continuous linear map. We even have $\|P_E\| \leq 1$ since for any selfadjoint x we have $(q_E x q_E)^2 = q_E x q_E x q_E \leq q_E x^2 q_E$ and therefore $(P_E x)^2 = (P(q_E x q_E))^2 \leq P((q_E x q_E)^2) \leq P(q_E x^2 q_E) = P_E(x^2)$, which implies $\|P_E\| \leq 1$ as mentioned in §2.

3.3. LEMMA. *In this situation we have $\mathcal{B} = \{x \in \mathcal{A} \mid P_E x \underset{E}{\to} x\}$.*

Proof. If $x \in \mathcal{S} \cap \mathcal{B}$ then $Px = x$ and therefore $P_E x \to x$, since $(q_E)_E$ is an approximate unit for $\mathcal{S}$. Therefore $\mathcal{S} \cap \mathcal{B}$ is contained in $\{x \in \mathcal{A} \mid P_E x \to x\}$, and so is $\mathcal{B}$ by Lemma 3.2.

If conversely $P_E x \to x$ we see that x is contained in the norm closure of $\{P_E x \mid E \in \mathcal{E}\} \subset \mathcal{S} \cap \mathcal{B}$, and so x must be in $\mathcal{B}$.

From this we have at once $\mathrm{Kor}_{\mathcal{A}}(\mathcal{B}) = \mathcal{B}$ and this is now proved for all J^*-subalgebras of $\mathcal{A}$. This clearly finishes the proof of the theorem.

3.4. Remark. Instead of Jordan–Schwarz maps one may also consider Schwarz maps (see [16]). Then all the theorems remain true if "J^*-subalgebra generated by T" is replaced by "norm-closed $*$-subalgebra generated by T".

References

[1] F. Altomare, *Korovkin closures in Banach algebras*, in: Advances in Invariant Subspaces and Other Results of Operator Theory, Proc. 9th Internat. Conf. on Operator Theory, Timişoara and Herculane 1984, Oper. Theory: Adv. Appl. 17, Birkhäuser, Basel 1986, 35–42.

[2] W. Ambrose, *Structure theorems for a special class of Banach algebras*, Trans. Amer. Math. Soc. 57 (1945), 364–386.

[3] F. Beckhoff, *Korovkin-Theory in Algebren*, Schriftenreihe Math. Inst. Univ. Münster, Ser. 2, Heft 45, 1987.
[4] —, *A counterexample in Korovkin theory*, Rend. Circ. Mat. Palermo (2) 37 (1988), 469–473.
[5] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Springer, Berlin 1979.
[6] J. Dixmier, *Von Neumann Algebras*, North-Holland Math. Library 27, 1981.
[7] N. Dunford and J. T. Schwartz, *Linear Operators II*, Interscience Publ., 1963.
[8] R. V. Kadison, *A generalized Schwarz inequality and algebraic invariants of operator algebras*, Ann. of Math. 56 (1952), 494–503.
[9] I. Kaplansky, *Groups with representations of bounded degree*, Canad. J. Math. 1 (1949), 105–112.
[10] B. V. Limaye and M. N. N. Namboodiri, *Korovkin approximation on C^*-algebras*, J. Approx. Theory 34 (1982), 237–246.
[11] —, —, *Weak Korovkin approximation by completely positive linear maps on $\beta(H)$*, ibid. 42 (1984), 201–211.
[12] —, —, *Weak approximation by positive maps on C^*-algebras*, to appear.
[13] L. H. Loomis, *An Introduction to Abstract Harmonic Analysis*, D. van Nostrand, New York 1953.
[14] M. Pannenberg, *Korovkin approximation in Waelbroeck algebras*, Math. Ann. 274 (1986), 423–437.
[15] W. M. Priestley, *A noncommutative Korovkin theorem*, J. Approx. Theory 16 (1976), 251–260.
[16] A. G. Robertson, *A Korovkin theorem for Schwarz maps on C^*-algebras*, Math. Z. 56 (1977), 205–207.
[17] R. Schatten, *Norm Ideals of Completely Continuous Operators*, Springer, Berlin 1960.
[18] M. Takesaki, *Theory of Operator Algebras I*, Springer, New York 1979.
[19] B. Yood, *Hilbert algebras as topological algebras*, Ark. Mat. 12 (1974), 131–151.

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Hölder continuity of proper holomorphic mappings*

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Abstract. We prove the Hölder continuity for proper holomorphic mappings onto certain piecewise smooth pseudoconvex domains with "good" plurisubharmonic peak functions at each point of their boundaries. We directly obtain a quite precise estimate for the exponent from an attraction property for analytic disks. Moreover, this way does not require any consideration of infinitesimal metric.

1. Introduction. The theorem of Carathéodory states that every biholomorphic map $F : D_1 \to D_2$ between bounded and simply connected domains in $\mathbf{C}$ extends to an homeomorphism $\widetilde{F} : \overline{D}_1 \to \overline{D}_2$ if both domains satisfy the Schönflies condition at each point of their boundaries (see [9], p. 209).

For domains in $\mathbf{C}^n$ $(n > 1)$ the known generalizations require more precise assumptions. The basic result is due to Henkin [8]: if D_1 is bounded, defined by a plurisubharmonic function and if D_2 is bounded with C^2 strongly pseudoconvex boundary, then every proper holomorphic map $F : D_1 \to D_2$ extends to a Hölder continuous map $\widetilde{F} : \overline{D}_1 \to \overline{D}_2$ with exponent $1/2$. (See also [10] and [13]; [12] for piecewise smooth strongly pseudoconvex boundary; [5] for a local version of this theorem.)

This was generalized by Bedford–Fornæss and Diederich–Fornæss to the case where D_1 is bounded pseudoconvex with C^2 boundary and D_2 is bounded pseudoconvex with real-analytic boundary; they proved that every proper holomorphic map $F : D_1 \to D_2$ extends to a Hölder continuous map $\widetilde{F} : \overline{D}_1 \to \overline{D}_2$ for some exponent $\varepsilon \in [0, 1]$ (see [2], [4]).

This paper is mainly motivated by the following observation: a proper holomorphic map is easily seen to be Hölder continuous if the image domain

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