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Two-weight weak type maximal inequalities in Orlicz classes

by

LUBOŠ PICK (Praha)

Abstract. Necessary and sufficient conditions are shown in order that the inequalities of the form

$$\varrho(\{M_\mu f > \lambda\})\Phi(\lambda) \le C \int_X \Psi(C|f(x)|)\sigma(x)\,d\mu\,,$$

or

$$\varrho(\{M_\mu f > \lambda\}) \le C \int_X \Phi(C\lambda^{-1}|f(x)|)\sigma(x)\,d\mu$$

hold with some positive C independent of $\lambda > 0$ and a μ-measurable function f, where (X, μ) is a space with a complete doubling measure μ, M_μ is the maximal operator with respect to μ, Φ, Ψ are arbitrary Young functions, and ϱ, σ are weights, not necessarily doubling.

1. Our aim is to study weighted weak type modular inequalities involving the maximal operator and Young functions. The classical *Hardy–Littlewood maximal operator* M defined for Lebesgue-measurable functions f on $\mathbf{R}^n$ by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\,dy\,,$$

where Q is a nondegenerate cube with sides parallel to the coordinate axes and $|Q|$ is the Lebesgue measure of Q, has been proved by Muckenhoupt [8] to satisfy the two-weight weak type inequality

$$\varrho(\{Mf > \lambda\}) \le C\lambda^{-p} \int_X |f(x)|^p \sigma(x)\,d\mu \tag{1}$$

with $1 \le p < \infty$ and C independent of f and λ if and only if the couple of weights (σ, ϱ) satisfies the A_p *condition*, i.e. for $p > 1$,

$$\left(\frac{1}{|Q|}\int_Q \varrho(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \sigma(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C \tag{2}$$

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for all Q. There exist at least two different types of inequalities analogous to (1) which involve a Young function $\Phi(t)$ instead of t^p; indeed, we may consider either

$$\varrho(\{Mf>\lambda\})\Phi(\lambda)\le C\int_X \Phi(C|f(x)|)\sigma(x)\,dx\,, \tag{3}$$

or

$$\varrho(\{Mf>\lambda\})\le C\int_X \Phi(C\lambda^{-1}|f(x)|)\sigma(x)\,dx\,, \tag{4}$$

with C independent of f and λ. As we shall see, (3) always implies (4). Therefore we introduce the following terminology: (3) will be called the two-weight *weak* type inequality, and (4) will be called the two-weight *extra-weak* type inequality.

A *Young function* Φ is given by $\Phi(t)=\int_0^t\varphi(s)\,ds$, $t\ge0$, where φ is a left-continuous nondecreasing function satisfying $\varphi(0+)=0$ and $\varphi(\infty)=\infty$. Dealing with arbitrary Young functions we cannot rely on homogeneity arguments applicable for powers or any "subhomogeneity" arguments applicable for functions satisfying the Δ_2 *condition* ($\Phi\in\Delta_2$ if there is $C\ge1$ such that $\Phi(2t)\le C\Phi(t)$, $t\ge0$). Introduction of appropriate constants seems to be a proper tool for compensation of this handicap. This is to justify the C in the argument of Φ.

Fortunately, (2) also offers two different candidates for a condition analogous to A_p, namely,

$$\sup_{\alpha>0}\sup_Q\left(\frac{1}{|Q|}\int_Q\alpha\varrho(x)\,dx\right)\varphi\left(\frac{\varepsilon}{|Q|}\int_Q\varphi^{-1}\left(\frac{1}{\alpha\sigma(x)}\right)dx\right)\le C\,, \tag{5}$$

where φ^{-1} is the usual generalized left-continuous inverse of φ, and

$$\sup_Q\int_Q\widetilde{\Phi}\left(\varepsilon\frac{\varrho_Q}{\sigma(x)}\right)\frac{\sigma(x)}{\varrho(Q)}\,dx\le C\,, \tag{6}$$

where $\widetilde{\Phi}$ is the *complementary Young function* of Φ given by $\widetilde{\Phi}(t)=\sup_{\tau>0}(t\tau-\Phi(\tau))=\int_0^t\varphi^{-1}(s)\,ds$, $\varrho(Q)=\int_Q\varrho$, and $\varrho_Q=\varrho(Q)/|Q|$. Observe that (5) and (6) coincide if $\Phi(t)=t^p$. The condition (5) was introduced by Kerman and Torchinsky ([4]) and recently treated by Gallardo ([2]), who generalized the Muckenhoupt theorem in the following sense:

If Φ, $\widetilde{\Phi}\in\Delta_2$, then each of the statements (3), (5) *is equivalent to the "modified two-weight Jensen inequality"*

$$\Phi(|f|_Q)\le C\frac{1}{\varrho(Q)}\int_Q\Phi(|f(x)|)\sigma(x)\,dx \tag{7}$$

with C independent of f and Q.

In the special case when Φ is essentially equal to $t(1+\log^+t)^K$ the condition (6) is known to describe good weights for the inequality (4) (see [1], [6], [9]). While sufficiency of (6) can be easily extended to the general case ([10]), there is no obvious way to prove its necessity. Since, moreover, (7) offers

$$\varrho(Q)\le C\int_Q\Phi(C|f(x)|/|f|_Q)\sigma(x)\,dx\,, \tag{8}$$

natural questions arise:

a) *Can the Gallardo theorem be extended to the general context* (*that means, without Δ_2 or any other assumptions on the growth of Φ or $\widetilde{\Phi}$*) *and how should then the characterizing condition look?*

b) *Are there any general relations linking* (4), (6), *and/or* (8)?

The purpose of the present note is to answer these questions. Theorem 1 below determines the pairs (σ,ϱ) for which the weak type inequality with two different Young functions Φ, Ψ holds, that is, in a certain sense, the maximal operator takes the Orlicz (Ψ,σ)-class into the weak Orlicz (Φ,ϱ)-class. The characterizing $A_{\Phi,\Psi}$-*condition* is a natural analogue of those due to Muckenhoupt and Kerman–Torchinsky and it is formulated by means of auxiliary functions R_Φ and S_Φ.

The couples (σ,ϱ) for which the extra-weak type inequality is true, i.e. those for which the maximal operator maps the Orlicz (Φ,σ)-class into weak $L_{1,\varrho}$, are characterized in Theorem 2. Saturation of the Hölder inequality and some properties of the Luxemburg norm in Orlicz space are the main tools in the proof of necessity of a condition like (6) (called E_Φ).

Some other equivalent statements which shed light on connection between the usual and the two-weight maximal operators are also added. Finally, we consider a bit more general situation than $\mathbf{R}^n$ with the Lebesgue measure (cf. [11]). A simple method employing centered maximal operators allows us not to require any doubling condition on ϱ or σ. Throughout we assume $0\cdot\infty=0$.

2. Let (X,μ) be a complete measure space and suppose that there is given a nonnegative finite real-valued function d in $X\times X$ that satisfies $d(x,x)=0$, $d(x,y)=d(y,x)$, $d(x,y)>0$ if $x\ne y$, and $d(x,y)\le C(d(x,z)+d(z,y))$ for all $x,y,z\in X$; that is, d is a *quasimetric* on X. Moreover, we shall assume that the following two conditions are satisfied:

(i) every d-ball $B=B(y,r)=\{x\in X:d(x,y)<r\}$, $y\in X$, $r>0$, is μ-measurable, $0<\mu(B)<\infty$, and μ is a *doubling measure* with respect to d, i.e. $\mu(B(y,2r))\le C\mu(B(y,r))$ for all y and r (in particular, μ is σ-finite);

(ii) the space (X, d, μ) possesses the *Besicovitch property*: for every d-bounded set A any family $\{B(y, r(y))\}_{y\in A}$ of balls contains a countable (or finite) subfamily $\{B_n\} = \{B(y_n, r(y_n))\}$, $n \in \mathbb{N}$, such that $A \subset \bigcup B_n$ and $\sum \chi_{B_n} \le C$, where χ_B is the characteristic function of the set B.

The *maximal operator* M_μ is defined for μ-measurable real functions f by

$$M_\mu f(x) = \sup_{B\ni x} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y), \quad x \in X.$$

Similarly, we define the *centered maximal operator*

$$M_\mu^c f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, d\mu(y), \quad x \in X.$$

The operators M_μ and M_μ^c are equivalent because μ is a doubling measure; precisely,

$$M_\mu^c f(x) \le M_\mu f(x) \le a M_\mu^c f(x) \tag{9}$$

for some $a \ge 1$ and all f and x. In what follows, ϱ and σ will be *weights*, i.e. nonnegative μ-measurable functions on X. The Besicovitch property guarantees that the *centered two-weight maximal operator*

$$M_{\varrho,\sigma}^c f(x) = \sup_{r>0} \frac{1}{\varrho(B(x,r))} \int_{B(x,r)} |f(y)|\sigma(y)\, d\mu(y)$$

is of *weak type* $(1, \sigma; 1, \varrho)$, that is, there exists $b \ge 1$ such that

$$\varrho(\{M_{\varrho,\sigma}^c f > \lambda\}) \le b\lambda^{-1} \int_X |f(y)|\sigma(y)\, d\mu(y) \tag{10}$$

for all f and $\lambda > 0$. To see this it suffices to realize that $M_{\varrho,\varrho}^c$ is of weak type $(1, \varrho; 1, \varrho)$ and that $M_{\varrho,\sigma}^c f = M_{\varrho,\varrho}^c (f \cdot (\sigma/\varrho))$. (As is customary, we write $\{g > \lambda\}$ for $\{x \in X : g(x) > \lambda\}$, and, given a set E, $\varrho(E)$ for $\int_E \varrho(y) d\mu(y)$ and ϱ_E for the integral mean $\varrho(E)/\mu(E)$.)

Throughout the paper, Φ and Ψ denote Young functions. Let us still recall the *Young inequality* $st \le \Phi(s) + \widetilde{\Phi}(t)$ and its important consequence (see [5])

$$t \le \Phi^{-1}(t)\widetilde{\Phi}^{-1}(t) \le 2t, \quad t \ge 0. \tag{11}$$

The *weighted modular* is defined for every f by

$$m_\varrho(f, \Phi) = \int_X \Phi(|f(x)|)\varrho(x)\, d\mu(x);$$

the *weighted Orlicz space* $L_{\Phi,\varrho}(X)$ is then the set of all functions f for which there is some $\lambda > 0$ such that $m_\varrho(f/\lambda, \Phi) < \infty$. This space is equipped with

the *Orlicz norm*

$$|||f|||_{\Phi,\varrho} = \sup\Big\{ \int_X fg\varrho\, d\mu : m_\varrho(g, \widetilde{\Phi}) \le 1 \Big\},$$

and with the *Luxemburg norm*

$$\|f\|_{\Phi,\varrho} = \inf\{\lambda > 0 : m_\varrho(f/\lambda, \Phi) \le 1\}.$$

These norms are equivalent, precisely,

$$\|f\|_{\Phi,\varrho} \le |||f|||_{\Phi,\varrho} \le 2\|f\|_{\Phi,\varrho}, \tag{12}$$

and they provide us with two important facts (see [7]):

a) the closed unit ball in $L_{\Phi,\varrho}$ with respect to the Luxemburg norm coincides with the closed unit ball with respect to the modular, i.e. $m_\varrho(f, \Phi) \le 1$ if and only if $\|f\|_{\Phi,\varrho} \le 1$;

b) the *Hölder inequality*

$$\int_X fg\varrho\, d\mu \le \|f\|_{\Phi,\varrho} |||g|||_{\widetilde{\Phi},\varrho}$$

holds for all μ-measurable functions f, g and is *saturated* in the sense that

$$\|f\|_{\Phi,\varrho} = \sup\Big\{ \int_X fg\varrho\, d\mu : |||g|||_{\widetilde{\Phi},\varrho} \le 1 \Big\},$$

and, as follows from a),

$$|||f|||_{\Phi,\varrho} = \sup\Big\{ \int_X fg\varrho\, d\mu : \|g\|_{\widetilde{\Phi},\varrho} \le 1 \Big\}.$$

Let us introduce

$$R_\Phi(t) = \Phi(t)/t, \quad S_\Phi(t) = \widetilde{\Phi}(t)/t, \quad t > 0, \quad R_\Phi(0) = S_\Phi(0) = 0.$$

$R_\Phi(t)$ and $S_\Phi(t)$ are substitutes for the useful expressions t^{p-1}, $t^{1/(p-1)}$ in the L_p case. Although they are not mutually inverse in general, some more subtle relations can be easily verified. We present a brief survey.

PROPOSITION. *If Φ is a Young function, then R_Φ and S_Φ are continuous and increasing functions which map $[0, \infty)$ onto itself and satisfy for every $t \ge 0$ the following estimates:*

$$R_\Phi(t) \le \varphi(t) \le 2R_\Phi(2t);$$

$$\widetilde{\Phi}(R_\Phi(t)) \le \Phi(t) \le \widetilde{\Phi}(2R_\Phi(t)); \tag{13}$$

$$S_\Phi(R_\Phi(t)) \le t \le 2S_\Phi(2R_\Phi(t)). \tag{14}$$

Moreover, if $\Phi \in \Delta_2$, then $R_\Phi^{-1}(t) \le S_\Phi(Ct)$ and $\varphi(t) \le CR_\Phi(t)$ for some $C \ge 1$ independent of $t \ge 0$.

Proof. Substituting $t \to \Phi(t)$ in (11) we get $\Phi(t) \le t\widetilde{\Phi}^{-1}(\Phi(t)) \le 2\Phi(t)$, which yields (13) and, on dividing by $R_\Phi(t)$ or by $2R_\Phi(t)$, in turn (14). If $\Phi \in \Delta_2$, then

$$C\Phi(S_\Phi(t)) \ge \Phi(2S_\Phi(t)) \ge \widetilde{\Phi}(t)$$

by the complementary version of the second inequality of (13). Dividing this by $S_\Phi(t)$ we infer that $CR_\Phi(S_\Phi(t)) \ge t$, or, which is the same, $R_\Phi^{-1}(t) \le S_\Phi(Ct)$. The remaining assertions follow directly from the definitions.

Remark. It is worth noticing that the Proposition is valid for *any* Young function; therefore, as $R_\Phi = S_{\widetilde{\Phi}}$, each statement of the Proposition has its complementary version. For example, (14) may be replaced by

$$R_\Phi(S_\Phi(t)) \le t \le 2R_\Phi(2S_\Phi(t)), \tag{14'}$$

and so on.

Definition. The couple (σ,ϱ) of weights satisfies the $A_{\Phi,\Psi}(\mu)$ *condition* ($(\sigma,\varrho) \in A_{\Phi,\Psi}(\mu)$) if there exist positive constants ε, A such that

$$\sup_B \sup_{\alpha>0} \alpha\varrho_B R_\Phi\left(\frac{\varepsilon}{\mu(B)} \int_B S_\Psi\left(\frac{1}{\alpha\sigma(x)}\right) d\mu(x)\right) \le A.$$

(We remind that the convention $0\cdot\infty = 0$ is used here.) Similarly, $(\sigma,\varrho) \in E_\Phi(\mu)$ if for some $\varepsilon, A > 0$,

$$\sup_B \int_B \widetilde{\Phi}\left(\varepsilon\frac{\varrho_B}{\sigma(x)}\right)\frac{\sigma(x)}{\varrho(B)}\, d\mu(x) \le A.$$

We use the symbol $A_{\Phi,\Psi}$ to indicate the analogy with A_p ([8]) and A_Φ ([4], [2]), and E_Φ to indicate its relevance to extra-weak type inequalities. Of course, if $\Phi(t) = \Psi(t) = t^p$, $p > 1$, then $A_{\Phi,\Psi} = E_\Phi = A_p$.

3. Now, we are in a position to formulate the main results.

Theorem 1. *The following conditions on σ,ϱ are equivalent.*

(i) *There exists $C \ge 1$ such that for all f and $\lambda > 0$,*

$$\varrho(\{M_\mu f > \lambda\})\Phi(\lambda) \le C \int_X \Psi(C|f(x)|)\sigma(x)\, d\mu(x);$$

(ii) *there exists $K \ge 1$ such that for all f and B,*

$$\varrho(B)\Phi(|f|_B) \le K \int_B \Psi(K|f(x)|)\sigma(x)\, d\mu(x);$$

(iii) $(\sigma,\varrho) \in A_{\Phi,\Psi}(\mu)$;

(iv) *there exists $D \ge 1$ such that for all f and $x \in X$,*

$$\Phi(M_\mu^c f(x)) \le DM_{\varrho,\sigma}^c[\Psi(D|f|)](x).$$

Theorem 2. *The following conditions on σ,ϱ are equivalent.*

(i) *There exists $C \ge 1$ such that for all f and $\lambda > 0$,*

$$\varrho(\{M_\mu f > \lambda\}) \le C \int_X \Phi(C\lambda^{-1}|f(x)|)\sigma(x)\, d\mu(x);$$

(ii) *there exists $K \ge 1$ such that for all f and B,*

$$\varrho(B) \le K \int_B \Phi(K|f(x)|/|f|_B)\sigma(x)\, d\mu(x);$$

(iii) $(\sigma,\varrho) \in E_\Phi(\mu)$;

(iv) *there exists a continuous increasing function h defined on $[0,\infty)$ such that for all $x \in X$,*

$$M_\mu^c f(x) \le h(M_{\varrho,\sigma}^c[\Phi(|f|)](x)).$$

Remark. When $\Phi = \Psi$, inserting $\lambda = 1$ in Theorem 1(i) leads to

$$\varrho(\{M_\mu f > 1\}) \le C \int_X \Phi(C|f(x)|)\sigma(x)\, d\mu(x),$$

which is obviously equivalent to (i) in Theorem 2 (simply take f/λ instead of f and use homogeneity of M_μ). Therefore, if $\Phi = \Psi$, then each statement of Theorem 1 implies any one of Theorem 2. In particular, (4) is weaker than (3) (this justifies our terminology), (6) follows from (5), (7) suffices for (8), and $A_{\Phi,\Phi}(\mu) \subset E_\Phi(\mu)$. Moreover, any statement of Theorem 2 obviously implies the *two-weight doubling condition* $\varrho(B(y,2r)) \le C\sigma(B(y,r))$.

Remark. Putting $\varrho \equiv \sigma \equiv 1$ we obtain the following corollary of Theorem 1: *The nonweighted weak type maximal inequality*

$$\mu(\{M_\mu f > \lambda\})\Phi(\lambda) \le C \int_X \Psi(C|f(x)|)\, d\mu(x) \tag{15}$$

holds with C independent of f, λ if and only if there exists $K \ge 1$ such that $\Phi(t) \le \Psi(Kt)$ for all $t \ge 0$.

Indeed, by Theorem 1, (15) is equivalent to the existence of some ε, A such that for all $\alpha > 0$,

$$\alpha R_\Phi(\varepsilon S_\Psi(1/\alpha)) \le A. \tag{16}$$

If $\Phi(t) \le \Psi(Kt)$, then $R_\Phi(t) \le KR_\Psi(Kt)$ and we may choose $\varepsilon = K^{-1}$ and apply the complementary version of the first inequality of (14) to the function Ψ to obtain

$$\alpha R_\Phi(\varepsilon S_\Psi(1/\alpha)) \le \alpha KR_\Psi(S_\Psi(1/\alpha)) \le K.$$

To prove the "only if" part, assume for contradiction that there exists a sequence $\{t_n\}$ such that $\Phi(t_n) > \Psi(nt_n)$, $n \in \mathbb{N}$, and define α_n by the equality $t_n = \varepsilon S_\Psi(1/\alpha_n)$. It then follows from (16) that

$$A\alpha_n^{-1} \geq R_\Phi(t_n) > nR_\Psi(nt_n) \geq nR_\Psi(2S_\Psi(\alpha_n^{-1})),$$

provided that $n \geq 2\varepsilon^{-1}$. Via (14), or rather (14′), this yields $A\alpha_n^{-1} \geq 2^{-1}n\alpha_n^{-1}$, $n > 2\varepsilon^{-1}$, which is absurd.

Remark. The preceding remark shows that, in particular, M_μ maps L_Φ into weak L_Φ without any restriction on the growth of Φ or $\widetilde{\Phi}$. This quite naturally corresponds to the more general result of Gogatishvili, Kokilashvili and Krbec ([3]): *The inequality*

$$\mu(\{M_\mu f > \lambda\})F(\lambda) \leq C \int_X F(C|f(x)|)\, d\mu(x)$$

holds for a continuous increasing F, $F(0) = 0$, $F(\infty) = \infty$, if and only if F is quasiconvex.

4. We now proceed to prove Theorems 1 and 2.

Proof of Theorem 1. We first show (i)⇒(ii)⇒(iii)⇒(i), and afterwards (ii)⇒(iv)⇒(i).

(i)⇒(ii). Clearly, given a fixed ball B, $M_\mu(f\chi_B)(x) \geq |f|_B\chi_B(x)$, hence B is a subset of $\{M_\mu(f\chi_B) > \lambda\}$ for any $\lambda < |f|_B$, and (ii) will follow by the continuity argument.

(ii)⇒(iii). If $\varrho \equiv 0$, there is nothing to prove. Otherwise $\sigma(x) > 0$ for μ-almost every $x \in X$. Given $\alpha > 0$ and $k \in \mathbb{N}$, we put $f(x) = K^{-1}S_\Psi(1/\alpha\sigma(x))$ and $f_k = f\chi_{E_k}$, where $E_k = \{x \in B : \sigma(x) > 1/k\}$. By (ii) and the complementary version of the first inequality of (13), applied to Ψ, we have

$$\varrho(B)\Phi((f_k)_B) \leq K \int_{E_k} \Psi(S_\Psi(1/\alpha\sigma(x)))\sigma(x)\, d\mu(x)$$
$$= K^2\alpha^{-1} \int_B f_k(x)\, d\mu(x) = K^2\alpha^{-1}\mu(B)(f_k)_B,$$

and, as the last quantity is finite,

$$\alpha\varrho_B R_\Phi\left(\frac{1}{K\mu(B)} \int_{E_k} S_\Psi(1/\alpha\sigma(x))\, d\mu(x)\right) \leq K^2.$$

Since the constant on the right does not depend on k and $\mu(B \setminus \bigcup E_k) = 0$, $(\varrho, \sigma) \in A_{\Phi,\Psi}(\mu)$.

(iii)⇒(i). The Young inequality and $A_{\Phi,\Psi}$ give

$$|f|_B \leq \frac{\alpha}{\mu(B)} \int_B \Psi(|f(x)|)\sigma(x)\, d\mu(x) + \frac{1}{\mu(B)} \int_B S_\Psi(1/\alpha\sigma(x))\, d\mu(x)$$
$$\leq \frac{\alpha}{\mu(B)} \int_B \Psi(|f(x)|)\sigma(x)\, d\mu(x) + \varepsilon^{-1}R_\Phi^{-1}(A/\alpha\varrho_B)$$

for every $\alpha > 0$ and B. Put $\alpha = \lambda\mu(B)(\int_B \Psi(|f(x)|)\sigma(x)\, d\mu(x))^{-1}$ for $\lambda > 0$. Then

$$|f|_B \leq \lambda + \varepsilon^{-1}R_\Phi^{-1}\left(\frac{A}{\lambda\varrho(B)} \int_B \Psi(|f(x)|)\sigma(x)\, d\mu(x)\right),$$

whence, passing to supremum,

$$M_\mu^c f(x) \leq \lambda + \varepsilon^{-1}R_\Phi^{-1}(\lambda^{-1}AM_{\varrho,\sigma}^c(\Psi(|f|))(x)),$$

which yields

$$\{M_\mu^c f > 2\lambda\} \subset \{AM_{\varrho,\sigma}^c[\Psi(|f|)] > \lambda R_\Phi(\lambda\varepsilon)\} = \{M_{\varrho,\sigma}^c[\Psi(|f|)] > \Phi(\lambda\varepsilon)/A\varepsilon\}.$$

Thus, making use of (9), (10) and an appropriate substitution we obtain

$$\varrho(\{M_\mu f > \lambda\})\Phi(\lambda) \leq A\varepsilon b \int_X \Psi(2a\varepsilon^{-1}|f(x)|)\sigma(x)\, d\mu(x).$$

(ii)⇒(iv). Divide by $\varrho(B)$ and pass to supremum.

(iv)⇒(i). It follows from (iv) that

$$\{M_\mu^c f > \lambda\} \subset \{M_{\varrho,\sigma}^c[\Psi(D|f|)] > D^{-1}\Phi(\lambda)\},$$

which via (9) and (10) yields

$$\varrho(\{M_\mu f > \lambda\})\Phi(\lambda) \leq Db \int_X \Phi(aD|f(x)|)\sigma(x)\, d\mu(x),$$

and the proof is thus complete.

Proof of Theorem 2. The proof of (i)⇒(ii) goes along the same lines as in Theorem 1.

(ii)⇒(iii). Assume for contradiction that (ii) holds and $(\varrho, \sigma) \notin E_\Phi(\mu)$. Then there is a sequence $\{B_n\}$ of balls such that

$$\int_{B_n} \widetilde{\Phi}\left(\frac{\varrho_{B_n}}{2n\sigma(x)}\right)\frac{\sigma(x)}{\varrho(B_n)}\, d\mu(x) > 1,$$

i.e.

$$\|\varrho_{B_n}\chi_{B_n}/\sigma\|_{\widetilde{\Phi},\sigma/\varrho(B_n)} > 2n.$$

The saturation of the Hölder inequality ensures the existence of a sequence $\{f_n\}$ such that

(17) $$|||f_n|||_{\Phi,\sigma/\varrho(B_n)} \leq 1,$$

(18) $$n < \int_X f_n(x)\varrho_{B_n} \frac{\chi_{B_n}(x)}{\sigma(x)} \frac{\sigma(x)}{\varrho(B_n)}\, d\mu(x) = (f_n)_{B_n}\,.$$

Now, (17), (12) and a) imply

$$\int_X \Phi(f_n(x)) \frac{\sigma(x)}{\varrho(B_n)}\, d\mu(x) \le 1\,,$$

that is, using (18) and the convexity of Φ,

$$\begin{aligned}\varrho(B_n) &\ge \int_X \Phi(nf_n(x)/(f_n)_{B_n})\sigma(x)\, d\mu(x)\\ &\ge \sqrt{n} \int_{B_n} \Phi(\sqrt{n} f_n(x)/(f_n)_{B_n})\sigma(x)\, d\mu(x)\,,\end{aligned}$$

which contradicts (ii).

(iii)⇒(iv). By the Young inequality,

$$|f|_B \le \frac{1}{\varepsilon} \int_B \Phi(|f(x)|) \frac{\sigma(x)}{\varrho(B)}\, d\mu(x) + \frac{1}{\varepsilon} \int_B \widetilde{\Phi}\left(\varepsilon \frac{\varrho_B}{\sigma(x)}\right) \frac{\sigma(x)}{\varrho(B)}\, d\mu(x)\,.$$

That is, (iv) holds with $h(t) = \varepsilon^{-1}(t + A)$.

(iv)⇒(i). We have

$$\{M_\mu^c f > h(1)\} \subset \{M_{\varrho,\sigma}^c [\Phi(|f|)] > 1\}\,,$$

and, by (9), (10) and an appropriate substitution,

$$\varrho(\{M_\mu f > \lambda\}) \le b \int_X \Phi(\lambda^{-1} a h(1) |f(x)|)\sigma(x)\, d\mu(x)\,.$$

The proof is complete.

Remark. In the proof of Theorem 1 no use is made of Orlicz spaces and their norms. However, using Hölder's inequality instead of Young's we can obtain an alternative direct proof of (iii)⇒(ii) (this method is due to Gallardo [2]). Indeed, Hölder's inequality together with (12) leads to

(19) $$\int_B |f|\, d\mu \le 2\|f\chi_B\|_{\Psi,\alpha\sigma} \|\chi_B/\alpha\sigma\|_{\widetilde{\Psi},\alpha\sigma}$$

for arbitrary B and $\alpha > 0$. We want to estimate $\|\chi_B/\alpha\sigma\|_{\widetilde{\Psi},\alpha\sigma}$. It follows from $A_{\Phi,\Psi}$ that for any $\lambda > 0$,

$$\int_B \Psi\left(\frac{1}{\lambda\alpha\sigma(x)}\right) \alpha\sigma(x)\, d\mu(x) \le \frac{\mu(B)}{\lambda\varepsilon} R_\Phi^{-1}\left(\frac{A}{\lambda\,\alpha\,\varrho_B}\right).$$

The right hand side does not exceed 1 provided that $\lambda \ge \varepsilon^{-1}\mu(B) \times \Phi^{-1}(A\varepsilon/\alpha\varrho(B))$. Therefore, by the definition of the Luxemburg norm,

$$\left\|\frac{\chi_B}{\alpha\sigma}\right\|_{\widetilde{\Psi},\alpha\sigma} \le \Phi^{-1}\left(\frac{A\varepsilon}{\alpha\varrho(B)}\right) \frac{\mu(B)}{\varepsilon}\,.$$

On choosing $\alpha^{-1} = m_\sigma(f\chi_B, \Psi)$ and inserting into (19) we get (ii).

Remark. The proof of (i)⇒(ii)⇒(iv)⇒(i) in Theorem 1 remains valid upon replacing the Young function Φ with any nonnegative, continuous and increasing function F on $[0, \infty)$. This observation provides a restricted version of Theorem 1 which may be of independent interest. Actually, it gives a slight generalization of Proposition 1 in [4].

We shall use E to denote a μ-measurable set in X.

COROLLARY. *If F is a nonnegative, continuous and increasing function defined on $[0, \infty)$, then the following statements are equivalent.*

(i) *There exists $C \ge 1$ such that for all $\lambda > 0$ and E,*

$$F(\lambda)\varrho(\{M_\mu \chi_E > \lambda\}) \le C\sigma(E)\,;$$

(ii) *there exists $K \ge 1$ such that for all B and E,*

$$F(\mu(E \cap B)/\mu(B))\varrho(B) \le K\sigma(E \cap B)\,;$$

(iii) *there exists $D \ge 1$ such that for all f and $x \in X$,*

$$F(M_\mu^c \chi_E(x)) \le D M_{\varrho,\sigma}^c \chi_E(x)\,.$$

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MATHEMATICAL INSTITUTE
CZECHOSLOVAK ACADEMY OF SCIENCES
ŽITNÁ 25
115 67 PRAHA 1, CZECHOSLOVAKIA

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Korovkin theory in normed algebras

by

FERDINAND BECKHOFF (Münster)

Abstract. If $\mathcal{A}$ is a normed power-associative complex algebra such that the selfadjoint part is normally ordered with respect to some order, then the Korovkin closure (see the introduction for definitions) of $T \cup \{t^* \circ t \mid t \in T\}$ contains $J^*(T)$ for any subset T of $\mathcal{A}$. This can be applied to C^*-algebras, minimal norm ideals on a Hilbert space, and to H^*-algebras. For bounded H^*-algebras and dual C^*-algebras there is even equality. This answers a question posed in [1].

§ 1. Introduction. Let us recall some definitions. Let $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ be normed power-associative complex algebras with a continuous involution $*$, and let $\mathcal{A}_s = \{x \in \mathcal{A} \mid x^* = x\}$ and $\widetilde{\mathcal{A}}_s$ be normally ordered, i.e. $\mathcal{A}_s$ (resp. $\widetilde{\mathcal{A}}_s$) is an ordered real vector space such that the norm $\|\cdot\|$ is equivalent to a monotone norm $\|\cdot\|_m$ ($0 \le x \le y$ implies $\|x\|_m \le \|y\|_m$). A continuous linear map $P : \mathcal{A} \to \widetilde{\mathcal{A}}$ which is selfadjoint (i.e. $P(\mathcal{A}_s) \subset \widetilde{\mathcal{A}}_s$) and which satisfies $(Px)^2 \le P(x^2)$ for all $x \in \mathcal{A}_s$ is called a *Jordan–Schwarz map*, since for such a P the Schwarz inequality holds with respect to the Jordan product: $P(x)^* \circ P(x) \le P(x^* \circ x)$ for all $x \in \mathcal{A}$. A *J^*-subalgebra* of $\mathcal{A}$ is by definition a $*$-closed and norm-closed subspace which is also closed with respect to the Jordan product. The J^*-subalgebra of $\mathcal{A}$ generated by $T \subset \mathcal{A}$ is denoted by $J^*(T)$.

1.1. THEOREM. *Let $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ be normed power-associative complex algebras with a continuous involution $*$, and let $\mathcal{A}_s$ and $\widetilde{\mathcal{A}}_s$ be normally ordered. If $S : \mathcal{A} \to \widetilde{\mathcal{A}}$ is a continuous $*$-homomorphism and $(P_\alpha)_{\alpha \in A}$ an equicontinuous net of Jordan–Schwarz maps $P_\alpha : \mathcal{A} \to \widetilde{\mathcal{A}}$, then*

$$\{x \in \mathcal{A} \mid P_\alpha x \underset{\alpha}{\to} Sx \text{ and } P_\alpha(x^* \circ x) \underset{\alpha}{\to} S(x^* \circ x)\}$$

is a J^*-subalgebra of $\mathcal{A}$.

Here convergence always means convergence in the norm topology. It is also possible, and other authors do so, to consider weaker topologies, see for

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