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A bound on the Laguerre polynomials

by

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Abstract. We give the following bounds on Laguerre polynomials and their derivatives ($\alpha \ge 0$):

$$|t^k d^p (L_n^\alpha(t) e^{-t/2})| \le 2^{-\min(\alpha,k)} 4^k (n+1)\dots(n+k) \binom{n+p+\max(\alpha-k,0)}{n}$$

for all natural numbers $k, p, n \ge 0$ and $t \ge 0$. Also, we give (as the main result of this paper) a technique to estimate the order in k and p in bounds similar to the previous ones, which will be used to see that the estimate on k and p in the previous bounds is sharp and to give an estimate on k and p in other bounds on the Laguerre polynomials proved by Szegő.

Introduction and results. In [7, p. 239], the following estimate on the Laguerre polynomials is proved:

If $\alpha, k \in \mathbf{R}$ and $a > 0$, then there exist positive constants c_k, C_k (which depend on a) such that

$$c_k n^Q \le \max_{t \ge a} |e^{-t/2} t^k L_n^\alpha(t)| \le C_k n^Q \tag{1}$$

where $Q = \max(k - 1/3, \alpha/2 - 1/4)$.

In [1], the author, in order to characterize the Fourier–Laguerre coefficients in a space of Gel'fand–Shilov type, gave the following bounds on the Laguerre polynomials (see Corollary 1.4):

$$|t^k d^p (L_n^\alpha(t) e^{-t/2})| \le 4^k (n+1)\dots(n+k) \binom{n+p+\alpha}{n} \tag{2}$$

for all natural numbers $k, p, n \ge 0$ and $t, \alpha \ge 0$.

The bounds (1) cannot be used to prove that characterization because they do not give an estimate for C_k. Moreover, from Theorem 5 (of this paper) we can deduce that if an estimate on C_k were given, this would not be sufficient to prove the results which we prove using the bounds

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(2). Indeed, consider a sequence $(a_n)_n$ for which there exist two constants $\gamma > 0$ and $a > 1$ such that $|a_n| \le \gamma a^{-n}$ for all $n \ge 0$, and define $f(t) = \sum_n a_n L_n(t)e^{-t/2}$. From (1), we get

$$|t^k f(t)| \le C_k \sum_n a^{-n} n^k .$$

Since $\sum_n a^{-n} n^k \sim c_a^k k^k$ for a certain constant c_a which depends on a, from Theorem 5 of this paper we deduce that it is not possible to prove that there exists a constant $A > 0$ such that $|t^k f(t)| \le CA^k k^k$. However, using (2), we get

$$|t^k f(t)| \le 4^k k! \sum_n a^{-n} \binom{n+k}{k} \le \left(\frac{4a}{a-1}\right)^k k^k .$$

In this paper, we shall improve (2) as follows:

THEOREM 1. *If $t \ge 0$ then*

$$(3) \quad |t^k d^p(L_n^\alpha(t)e^{-t/2})| \le 2^{-\min(\alpha,k)} 4^k (n+1)\dots(n+k)\binom{n+\max(\alpha-k,0)+p}{n}$$

for all natural numbers $k, p, n \ge 0$ and $\alpha \ge 0$.

Although the order in n of the bound (1) ($\max(k-1/3, \alpha/2-1/4)$) is better than that of (3) ($\max(k,\alpha)$), it should be noticed that:

a) *The bounds* (1) *are proved for $t \ge a$, where a is a positive number*, and it is not possible to extend them to $(0,\infty)$ without increasing the order in n.

Indeed, if we denote by $x_0(\alpha)$ the first zero of $L_n^\alpha(t)$, it is well known that $x_0(\alpha) > c/(n+(\alpha+1)/2)$, where c is a constant which does not depend on α. Now, it is easy to prove that $|L_n^\alpha(t)| \ge y(x)$ if $0 \le x \le c/(n+(\alpha+1)/2)$ where $y(x)$ is the line through the points $(0, \binom{n+\alpha}{n})$ and $(c/(n+(\alpha+1)/2), 0)$. Therefore, if k is fixed, it is easy to prove that the bounds (1) cannot be extended to $(0,\infty)$ (without increasing the order in n) for $\alpha \ge M_k$, where M_k is a constant which depends on k. Also (proceeding in a similar way), if α is fixed, it is possible to find k's such that (1) cannot be extended to $(0,\infty)$.

b) *The bounds in* (1) *do not give an estimate on the constant C_k*, while in (3), we give an estimate on the order in k and p. Moreover, we shall prove (as the main result of this paper) that these estimates are the best possible in the following sense:

THEOREM 2. *Let $\alpha \ge 0$, $M, N \in \mathbb{N}$ be given. If there exists an infinite subset $X \subset \mathbb{N}$ such that*

$$|t^k d^p(L_n^\alpha(t)e^{-t/2})| \le CA^k B^p (n+1)\dots(n+k)\binom{n+\max(\alpha-k,0)+p}{n}$$

when $n \in X$ and either $k \ge M$ or $p \ge N$, then $A \ge 4$ and $B \ge 1$.

(It is surprising that in order to prove Theorem 2, we only need to use the following two properties of the Laguerre polynomials: (i) they generate an orthonormal system in $L^2((0,\infty))$, and (ii) the formula for the Fourier–Laplace transform of these orthonormal functions.)

We want to remark that the technique used in the proof of Theorem 2 can be used to estimate the order in k and p in bounds similar to (1) or (3). Indeed, using this technique, we shall extend Theorem 2 and give an estimate on the constants C_k which appear in (1) (see Theorem 5):

THEOREM 3. *Let $\alpha \ge 0$, $M \in \mathbb{N}$ be given. If there exists an infinite subset $X \subset \mathbb{N}$ such that*

$$|t^k L_n^\alpha(t)e^{-t/2}| \le CA^k(n+1)\dots(n+k)\binom{n+\max(\alpha-k,0)}{n}$$

when $k \ge M$ and $n \in X$, then $A \ge 4$.

THEOREM 4. *Let $\alpha \ge 0$, $N \in \mathbb{N}$ be given. If there exists an infinite subset $X \subset \mathbb{N}$ such that*

$$|d^p(L_n^\alpha(t)e^{-t/2})| \le CB^p\binom{n+p+\alpha}{n}$$

when $p \ge N$ and $n \in X$, then $B \ge 1$.

THEOREM 5. *Given $\alpha, \varepsilon \ge 0$ and $M \in \mathbb{N}$, if there exists an infinite subset $X \subset \mathbb{N}$ such that the constants C_k ($k \ge 0$) satisfy*

$$|t^k(L_n^\alpha(t)e^{-t/2})| \le C_k n^{k-\varepsilon} \quad \text{for } t \ge 0$$

when $k \ge M$ and $n \in X$, then for all $A > 0$ there exists an infinite subset $X_A \subset \mathbb{N}$ such that $C_k \ge A^k$ if $k \in X_A$.

From this theorem it follows that the order in k in the bounds (3) is better (for infinitely many k's) than in the bounds (1), although we would do the best choice for the constant C_k in the bounds (1). For instance, taking $n = k$, $p = 0$, we deduce from (1) that $|t^k L_k(t)e^{-t/2}| \le C_k k^k$ and from (3) that $|t^k L_k(t)e^{-t/2}| \le (4/e)^k k^k$, but $C_k \ge (4/e)^k$ for infinitely many k's.

Proofs of the results. The proof of Theorem 1 is similar to that of (2) (see [1, Th. 1.3]).

Proof of Theorem 1. First, we shall prove that

$$(4) \quad |t^k L_n^\alpha(t)e^{-t/2}| \le 2^{-\min(\alpha,k)} 4^k (n+1)\dots(n+k)\binom{n+\max(\alpha-k,0)}{n}.$$

Indeed, since

$$tL_n^\alpha(t) = (n+\alpha)L_n^{\alpha-1}(t) - (n+1)L_{n+1}^{\alpha-1}(t)$$

(see [5, (23), p. 190]; we take $L_{n-k}^\alpha = 0$ if $n < k$ and $\binom{n}{k} = 0$ if $k < 0$), proceeding by induction on k, we obtain

$$(5)\quad t^k L_n^\alpha(t)e^{-t/2} = \sum_{m=0}^{k}(-1)^m\binom{k}{m}(n+1)\dots(n+m) \times (n+\alpha)\dots(n+\alpha+m-k+1)L_{n+m}^{\alpha-k}(t)e^{-t/2}.$$

(Notice that the factors in the product $(n+1)\dots(n+m)$ (which appears in the above formula) are increasing, so when $m=0$, this product must be taken to be 1. Analogously, the factors in $(n+\alpha)\dots(n+\alpha+m-k+1)$ are decreasing, so for $m=k$, this product is 1.)

Now, we suppose that $\alpha \le k$. Since $|L_n^\alpha(t)e^{-t/2}| \le 2^{-\alpha}$ if $\alpha \le 0$ (see Lemma 2.1 of [3]), from (5) we get

$$(6)\quad |t^k L_n^\alpha(t)e^{-t/2}| \le \sum_{m=0}^{k}\binom{k}{m}(n+1)\dots(n+m) \times (n+\alpha)\dots(n+\alpha+m-k+1)2^{k-\alpha} \le 2^{-\alpha}4^k(n+1)\dots(n+k).$$

Now, if $k < \alpha$, since

$$L_n^\alpha(t) = \sum_{l=0}^{n}\binom{l+\alpha-\beta-1}{l}L_{n-l}^\beta(t)$$

(see [5, (39), p. 192]), from (6) we get

$$\begin{aligned}|t^k L_n^\alpha(t)e^{-t/2}| &= \left|\sum_{l=0}^{n}\binom{l+\alpha-k-1}{l}t^k L_{n-l}^k(t)\right| \\ &\le 2^k(n+1)\dots(n+k)\sum_{l=0}^{n}\binom{l+\alpha-k-1}{l} \\ &= 2^k(n+1)\dots(n+k)\sum_{l=0}^{n}\binom{l+\alpha-k-1}{l}L_{n-l}(0) \\ &= 2^k(n+1)\dots(n+k)L_n^{\alpha-k}(0) \\ &= 2^k(n+1)\dots(n+k)\binom{n+\alpha-k}{n}.\end{aligned}$$

So, (4) is proved.

Now, since $(d/dt)L_n^\alpha(t) = -L_{n-1}^{\alpha+1}(t)$, we have

$$\begin{aligned}(7)\quad |t^k d^p(L_n^\alpha(t)e^{-t/2})| &= \left|t^k\sum_{m=0}^{p}\binom{p}{m}d^m(L_n^\alpha(t))\left(\frac{-1}{2}\right)^{p-m}e^{-t/2}\right| \\ &\le \left(\frac{1}{2}\right)^p\sum_{m=0}^{p}\binom{p}{m}2^m|t^k L_{n-m}^{\alpha+m}(t)e^{-t/2}| \\ &\le \left(\frac{1}{2}\right)^p\sum_{m=0}^{n}\binom{p}{m}2^m 2^{-\min(\alpha+m,k)} \\ &\quad\times 4^k(n-m+1)\dots(n-m+k) \\ &\quad\times\binom{n-m+\max(\alpha+m-k,0)}{n-m}.\end{aligned}$$

If $\alpha < k$, (7) gives

$$\begin{aligned}|t^k d^p(L_n^\alpha(t)e^{-t/2})| &\le 2^{-\alpha}4^k(n+1)\dots(n+k)\left(\frac{1}{2}\right)^p\sum_{\alpha+m\le k}\binom{p}{m} \\ &\quad+ 2^{-k}4^k(n+1)\dots(n+k)\sum_{\alpha+m\ge k}\binom{p}{m}\binom{n+\alpha-k}{n-m} \\ &\le 2^{-\min(\alpha,k)}4^k(n+1)\dots(n+k)\sum_{m=0}^{n}\binom{p}{m}\binom{n}{n-m} \\ &= 2^{-\min(\alpha,k)}4^k(n+1)\dots(n+k)\binom{n+p}{n},\end{aligned}$$

and if $\alpha \ge k$, (7) gives

$$\begin{aligned}|t^k d^p(L_n^\alpha(t)e^{-t/2})| &\le 2^k(n+1)\dots(n+k)\sum_{m=0}^{n}\binom{p}{m}\binom{n+\alpha-k}{n-m} \\ &= 2^k(n+1)\dots(n+k)\binom{n+\alpha-k+p}{n},\end{aligned}$$

and the theorem is proved.

Preliminaries to proofs of Theorems 2–5. The proof of Theorem 2 is unexpected and gives a technique for estimating the order in k and p of bounds like those given in Theorem 1. We start by giving a short sketch of the proof, which we use, firstly, to motivate the definitions and lemmas prior to the proof of Theorem 2, and secondly, as a guide to the complete proof of the theorem.

Sketch of proof. Given $A, B > 0$, we suppose that there exists an infinite subset $X \subset \mathbf{N}$ such that

$$|t^k d^p(L_n^\alpha(t)e^{-t/2})| \le CA^k B^p (n+1)\dots(n+k)\binom{n+\max(\alpha-k,0)+p}{n}$$

when $n \in X$ and either $k \ge M$ or $p \ge N$.

First step. We construct a function f, and using the above bounds on the Laguerre polynomials and their derivatives, we prove that f belongs to a certain space of smooth functions $S_{1,A}^{+0,B}$, $A, B > 0$.

Second step. Given a function h in $S_{1,A}^{+0,B}$, we prove that a function closely related to its Fourier transform (more precisely, the function g_h defined by

$$g_h(w) = (1-w)^{-\alpha-1}\int_0^\infty t^\alpha h(t)\exp\Big(-\frac{1}{2}\frac{1+w}{1-w}t\Big)\,dt)$$

is analytic in a certain open set in the complex plane. This open set depends on A, B and will be denoted by $\mathcal{G}_{A,B}$.

Third step. On the other hand, we find a point $w_{A,B}$ (which depends on A, B) where the function g_f defined in the previous step (here f is the function constructed in the first step) is not analytic.

Conclusion. The second and third steps imply that $w_{A,B} \notin \mathcal{G}_{A,B}$. Hence we deduce that $A \ge 4$ and $B \ge 1$.

Now, we give the definitions of the spaces $S_{1,A}^{+0,B}$, $A, B > 0$, and the open sets $\mathcal{G}_{A,B}$.

In [1], we defined (in a similar form to Gel'fand–Shilov spaces) the following spaces:

$$S^{+0,B} = \{f \in C^\infty((0,\infty)) : \text{there exist constants } C_k > 0 \text{ such that } |t^k f^{(n)}(t)| \le C_k B^n \text{ for all } n, k \ge 0\},$$

$$S_{1,A}^{+} = \{f \in C^\infty((0,\infty)) : \text{there exist constants } C_n > 0 \text{ such that } |t^k f^{(n)}(t)| \le C_n A^k k^k \text{ for all } n, k \ge 0\},$$

$$S_{1,A}^{+0,B} = \{f \in C^\infty((0,\infty)) : \text{there exists a constant } C > 0 \text{ such that } |t^k f^{(n)}(t) \le CA^k B^n k^k \text{ for all } n, k \ge 0\}$$

for $A, B > 0$, and $S_1^{+0} = \bigcup_{A,B>0} S_{1,A}^{+0,B}$. (Notice that $S_{1,A}^{+0,B} \subset S_{1,A}^{+}$ and $S_{1,A}^{+0,B} \subset S^{+0,B}$.) In the space S_1^{+0}, we consider the topology of the inductive limit of the Banach spaces $S_{1,A}^{+0,B}$ with the norms

$$\|f\|_{A,B} = \sup_{t>0,k,n\in\mathbf{N}} \frac{|t^k f^{(n)}(t)|}{B^n A^k k^k}.$$

We also consider the following Fourier operators:

$$\mathcal{F}_\Pi : L^2((0,\infty)) \to H(\Pi^-), \quad \mathcal{F}_\Pi(f)(z) = \int_0^\infty f(t)e^{-2\pi izt}\,dt, \tag{8}$$

$$\mathcal{F}_D : L^2((0,\infty)) \to H(D), \quad \mathcal{F}_D(f)(w) = \int_0^\infty f(t)\exp\Big(-\frac{1}{2}\frac{1+w}{1-w}t\Big)\,dt. \tag{9}$$

(We denote by Π^- the lower half plane and by D the unit disc. Notice that $\mathcal{F}_D(f)(w) = \mathcal{F}_\Pi(f)(Z(w))$ where

$$Z(w) = \frac{1}{4\pi i}\frac{1+w}{1-w} \tag{10}$$

is a bilinear mapping which transforms the unit disc onto the lower half plane.)

Now, we define the open sets in the complex plane which appear in the second step.

Let $A, B > 0$. We denote by $G_{A,B}$ the following open set in the complex plane:

$$G_{A,B} = \Big\{z \in \mathbf{C} : \Im z < \frac{1}{2\pi Ae}\Big\} \cup \Big\{z \in \mathbf{C} : |z| > \frac{B}{2\pi}\Big\}.$$

The bilinear mapping

$$W(z) = \frac{-1/2 + 2\pi iz}{1/2 + 2\pi iz}$$

transforms the lower half plane onto the unit disc. (Notice that W is the inverse mapping of the mapping Z defined in (10).) Let $\mathcal{G}_{A,B} = W(G_{A,B})$. It is the clear that $\mathcal{G}_{A,B} = \mathcal{G}_A \cup \mathcal{K}_B$ where $\mathcal{G}_A$ and $\mathcal{K}_B$ are defined as follows: If $A > 2/e$, $\mathcal{G}_A$ is the interior of the disc symmetric with respect to the real axis and which cuts it at the points $(-(Ae+2)/(Ae-2), 0)$, $(1,0)$. If $A < 2/e$, $\mathcal{G}_A$ is the exterior of the same disc and if $A = 2/e$, $\mathcal{G}_A$ is the half plane $\{w \in \mathbf{C} : \Re w < 1\}$. If $B > 1/2$, $\mathcal{K}_B$ is the interior of the disc symmetric with respect to the real axis and which cuts it at the points $((1+2B)/(2B-1), 0)$, $((2B-1)/(1+2B), 0)$. If $B < 1/2$, $\mathcal{K}_B$ is the exterior of the same disc and if $B = 1/2$, $\mathcal{K}_B$ is the right half plane.

In order to prove the second step, we need the following lemmas. Their proofs will be given at the end of this paper.

LEMMA 1. *Let $\alpha \ge 0$ and $A > 0$. If a function f satisfies*

$$|t^k f(t)| \le CA^k k^k \quad \textit{for } t \ge 0 \textit{ and } k \in \mathbf{N}$$

then the function $\mathcal{F}_\Pi(t^\alpha f)(z) = \int_0^\infty t^\alpha f(t)e^{-2\pi izt}\,dt$ is defined for all complex numbers z satisfying $\Im z < 1/(2\pi eA)$.

LEMMA 2. *Let* $f\in S_1^{+0}$, $\alpha\ge 0$ *and* $B>0$. *If*

$$|f^{(p)}(t)|\le CB^p \quad \text{for } t\ge 0 \text{ and } p\in\mathbb{N}$$

then there exists a function h *analytic in* G_B *such that* $h(z)=(1/2+2\pi iz)^{\alpha+1}\mathcal{F}_\Pi(t^\alpha f)(z)$ *if* $\Im z<0$, *where* $G_B=\{z\in\mathbb{C}:|z|>B/(2\pi)\}$ *if* $B>1/2$ *and* $G_B=\{z\in\mathbb{C}:|z|>B/(2\pi)\}\setminus\{i/(4\pi)\}$ *if* $B\le 1/2$.

Finally, in order to prove the third step, we need the following formula, which relates the function $\mathcal{F}_D f$ to the sequence $a_n^\alpha=\int_0^\infty \tau_n^\alpha t^\alpha f(t)L_n^\alpha(t)\times e^{-t/2}\,dt$ (where $\tau_n^\alpha=(n!/\Gamma(n+\alpha+1))^{1/2}$), when $f\in L^2((0,\infty))$ (see [2, Th. 4.2]):

$$\mathcal{F}_D f(w)=(1-w)^{\alpha+1}\sum_n(\tau_n^\alpha)^{-1}a_n^\alpha w^n \quad \text{for } |w|<1\,. \tag{11}$$

Now, we are ready to prove Theorem 2:

Proof of Theorem 2. We suppose that there exists an infinite subset $X\subset\mathbb{N}$ such that

$$|t^k d^p(L_n^\alpha(t)e^{-t/2})|\le CA^kB^p(n+1)\dots(n+k)\binom{n+\max(\alpha-k,0)+p}{n}$$

when $n\in X$ and either $k\ge M$ or $p\ge N$. We put $X=\{n_m:m\in\mathbb{N}\}$.

First step (see the sketch of proof given at the beginning of this section). Let $a>1$ and

$$f(t)=\sum_m a^{-n_m}L_{n_m}^\alpha(t)e^{-t/2}\,. \tag{12}$$

By (3), if $0\le k\le\max(M,\alpha)$, $0\le p\le N$ then $|t^kf^{(p)}(t)|\le$ const. If $k>\max(M,\alpha)$ or $p>N$, taking $b,c>1$ such that $a/(bc)>1$, we find by the hypothesis that (if $0<x<1$ then $\sum_n\binom{n+\alpha-1}{n}x^n=1/(1-x)^\alpha$)

$$\begin{aligned}|t^kf^{(p)}(t)|&\le\sum_m a^{-n_m}|t^kd^p(L_{n_m}^\alpha(t)e^{-t/2})|\\ &\le CA^kB^p\sum_m a^{-n_m}(n_m+1)\dots(n_m+k)\binom{n_m+p}{n_m}\\ &\le CA^kB^pk!\sum_n a^{-n}\binom{n+k}{k}\binom{n+p}{n}\\ &\le C'\Big(\frac{Ab}{e(b-1)}\Big)^k\Big(\frac{Bc}{c-1}\Big)^pk^k\,.\end{aligned}$$

It follows that if $k,p\ge 0$, then

$$|t^kf^{(p)}(t)|\le C''\Big(\frac{Ab}{e(b-1)}\Big)^k\Big(\frac{Bc}{c-1}\Big)^pk^k$$

for a constant $C''>0$. So $f\in S_{1,A'}^{+0,B'}$ where $A'=Ab/(e(b-1))$ and $B'=Bc/(c-1)$.

Second and third steps (see the sketch of proof). By Lemmas 1 and 2, there exists an analytic function h in $G'_{A',B'}$ such that $(1/2+2\pi iz)^{\alpha+1}\times\mathcal{F}_\Pi(t^\alpha f)(z)=h(z)$ if $\Im z<0$ where $G'_{A',B'}=G_{A',B'}$ (see the definition of the sets $G_{A,B}$ before Lemma 1) if $B'>1/2$ and $A'>2$, and $G'_{A',B'}=G_{A',B'}\setminus\{i/(4\pi)\}$ if $B'\le 1/2$ or $A'\le 2$. If we consider the bilinear mapping defined in (10), we see that the function $h(Z(w))$ is analytic in $\mathcal{G}_{A',B'}$ and $h(Z(w))=(1-w)^{-\alpha-1}\mathcal{F}_D(t^\alpha f)(w)$ for $|w|<1$. Since

$$\mathcal{F}_D(t^\alpha f)(w)=(1-w)^{\alpha+1}\sum_m(\tau_{n_m}^\alpha)^{-2}a^{-n_m}w^{n_m}$$

(see (11)) it follows that the function

$$g(w)=h(Z(w))=\sum_m(\tau_{n_m}^\alpha)^{-2}a^{-n_m}w^{n_m}$$

is analytic in $\mathcal{G}_{A',B'}$.

By a classical Tauber Theorem (notice that the $(\tau_{n_m}^\alpha)^{-2}a^{-n_m}$ are positive), the function g has a singularity at $w=a$. So $a\notin\mathcal{G}_{A',B'}$.

Changing the function f (see (12)) to

$$\tilde f(t)=\sum_m(-a)^{-n_m}L_{n_m}^\alpha(t)e^{-t/2}$$

we find that $-a\notin\mathcal{G}_{A',B'}$.

Conclusion (see the sketch of proof). We define

$$s_a=\max\Big\{\frac{2+A'e}{A'e-2},\frac{1+2B'}{2B'-1}\Big\}$$

if $A'>2/e$ and $B'>1/2$, and $s_a=\infty$ otherwise (that is, s_a is the maximum of the "extreme" points of $\mathcal{G}_{A',B'}$ on the real axis). For $a>1$, as $a,-a\notin\mathcal{G}_{A',B'}$, it follows that $a\ge s_a$, hence s_a cannot be infinite.

So $A'>2/e$, $B'>1/2$, and

$$a\ge\frac{2+A'e}{A'e-2},\quad a\ge\frac{1+2B'}{2B'-1}\,.$$

As $A'=Ab/(e(b-1))>2/e$ and $B'=Bc/(c-1)>1/2$, taking the limit as a, b, c tend to ∞ we find that $A\ge 2$, $B\ge 1/2$.

Now, if $a,b,c>1$ and $a/(bc)>1$ then

$$a\ge\frac{2b-2+Ab}{Ab-2b+2},\quad a\ge\frac{c-1+2Bc}{2Bc-c+1}\,,$$

and so taking the limit as b, c tend to a, we get

$$a\ge\frac{2a-2+Aa}{Aa-2a+2},\quad a\ge\frac{a-1+2Ba}{2Ba-a+1}\,.$$

As $A \ge 2$ and $B \ge 1/2$, we deduce that

$$a^2(A-2) - Aa + 2 \ge 0, \quad a^2(2B-1) - 2Ba + 1 \ge 0,$$

for all $a > 1$. As

$$a^2(A-2) - Aa + 2 = (A-2)(a-1)\left(a - \frac{4}{2A-4}\right),$$

$$a^2(2B-1) - 2Ba + 1 = (2B-1)(a-1)\left(a - \frac{2}{4B-2}\right),$$

we deduce that $4/(2A-4) \le 1$ and $2/(4B-2) \le 1$, i.e. $A \ge 4$ and $B \ge 1$, and the theorem is proved.

The proofs of Theorems 3 and 4 are the same as that of Theorem 2. We just need to notice that the function $f(t)$ defined in (12) satisfies the conditions of Lemma 1 or Lemma 2. (Lemma 2 may be applied to f. Indeed, we must prove that $f \in S_1^{+0}$. If we compute the sequence $a_n^\alpha = \int_0^\infty \tau_n^\alpha t^\alpha f(t) L_n^\alpha(t) e^{-t/2}\,dt$, we obtain $a_n^\alpha = a^{-n}(\tau_n^\alpha)^{-1}$ if $n \in X$ and 0 otherwise, hence it is clear that $(a_n^\alpha)_n$ satisfies the hypothesis of Theorem 2.7 of [1], and so $f \in S_1^{+0}$.)

Proof of Theorem 5. Let $\alpha, \varepsilon \ge 0$ and $M \in \mathbb{N}$. Suppose that Theorem 5 is not true, that is, there exist a positive constant A and a positive integer k_A such that

$$|t^k(L_n^\alpha(t)e^{-t/2})| \le CA^k n^{k-\varepsilon} \quad \text{for } t \ge 0$$

when $k \ge \max(k_A, M)$ and $n \in X$.

In order to find a contradiction, we take $a > 1$ such that $A/\ln a < 2/e$ and the function

$$f(t) = \sum_{n\in X} a^{-n} L_n^\alpha(t) e^{-t/2}.$$

For $k \ge \max(k_A, M)$ we get

$$|t^k f(t)| \le CA^k \sum_{n\in X} a^{-n} n^{k-\varepsilon} \le CA^k \sum_{n\ge 1} a^{-n} n^{k-\varepsilon}$$
$$\le CA^k \Big(\int_0^\infty e^{-x\ln a} x^k\,dx + \max_{t\ge 0}(a^{-t}t^k)\Big) \le 2C\left(\frac{A}{\ln a}\right)^k k^k,$$

and for $k < \max(k_A, M)$, by (3), $|t^k f(t)| \le$ const., so f satisfies the conditions of Lemma 1 for $A' = A/\ln a < 2/\varepsilon$.

Proceeding as in the proof of Theorem 2, we find that the function $g(w) = \sum_{n\in X} (\tau_n^\alpha)^{-2} a^{-n} w^n$ is analytic in $\mathcal{G}_{A'}$.

As $A' < 2/e$, we deduce that $a \in \mathcal{G}_{A'}$ (see the definition of $\mathcal{G}_A$). But $g(w)$ has a singularity at $w = a$, so $a \notin \mathcal{G}_{A'}$. Hence the theorem is proved.

Proof of Lemma 1. It is well known that a function belongs to the Gel'fand–Shilov space S_α ($\alpha \ge 0$) if and only if $|f^{(n)}(t)| \le C_n \exp(-(\alpha/e) \times |t/A|^{1/\alpha})$ (see [6, p. 171]). In a similar way, we can prove that if a function f satisfies $|t^k f(t)| \le CA^k k^k$ for $t \ge 0$ and $k \in \mathbb{N}$, then $|f(t)| \le Ce^{-t/(eA)}$ for $t \ge 0$. Now, the lemma follows easily from (8).

In order to prove Lemma 2 we need some previous result. In [1, Th. 6.2], we proved that the following integral operators are isomorphisms of S_1^{+0} onto itself ($\alpha \ge 0$):

$$\mathcal{H}_\alpha(f)(x) = \tfrac{1}{2}\int_0^\infty f(t)(tx)^{-\alpha/2} t^\alpha J_\alpha(\sqrt{xt})\,dt.$$

Moreover, $\mathcal{H}_\alpha^2 = \mathrm{Id}$. So, we can extend these operators to the space $(S_1^{+0})'$ as follows: if $u \in (S_1^{+0})'$, $\mathcal{H}_\alpha(u)$ is defined by $\langle \mathcal{H}_\alpha(u), \varphi\rangle = \langle u, \mathcal{H}_\alpha(\varphi)\rangle$, and we also have $\mathcal{H}_\alpha^2 = \mathrm{Id}$. We need the following lemma (it is easy to prove—see [1]—that functions in S_1^{+0} are entire, so $\delta^{(n)} \in (S_1^{+0})'$):

LEMMA 3. *Let $\alpha \ge 0$. Then $\mathcal{H}_\alpha(t^{\alpha+n}) = \Gamma(\alpha+n+1)2^{\alpha+2n+1}\delta^{(n)}$ for all $n \ge 0$.*

Proof. As $\mathcal{H}_\alpha^2 = \mathrm{Id}$, it is sufficient to prove that

$$\mathcal{H}_\alpha(\delta^{(n)}) = \frac{1}{\Gamma(\alpha+n+1)}\left(\frac{1}{2}\right)^{\alpha+2n+1} t^{\alpha+n}.$$

In [1, Sect. 4], for $u \in (S_1^{+0})'$, we defined its Fourier transform $\mathcal{F}_\Pi(u)$ as the following analytic function in the lower half plane:

$$\mathcal{F}_\Pi(u)(z) = \langle u(t), e^{-2\pi izt}\rangle,$$

and we proved that the Fourier transform is an isomorphism of $(S_1^{+0})'$ onto $H(\Pi^-)$, the space of analytic functions in the lower half plane. We need to show that

$$\mathcal{F}_\Pi(\mathcal{H}_\alpha(\delta^{(n)}))(z) = \mathcal{F}_\Pi\left(\frac{1}{\Gamma(\alpha+n+1)}\left(\frac{1}{2}\right)^{\alpha+2n+1} t^{\alpha+n}\right)(z).$$

Indeed, by [4, p. 137, (1)], the right hand side equals $(\frac{1}{2})^{\alpha+2n+1}(2\pi iz)^{-\alpha-n-1}$. Now, using (30) of [4, p. 185], we get

$$\mathcal{F}_\Pi(\mathcal{H}_\alpha(\delta^{(n)}))(z) = \langle \mathcal{H}_\alpha(\delta^{(n)}), e^{-2\pi izt}\rangle = \langle \delta^{(n)}, \mathcal{H}_\alpha(e^{-2\pi izt})\rangle$$
$$= \langle \delta^{(n)}, \tfrac{1}{2}\int_0^\infty e^{-2\pi izt}(tx)^{-\alpha/2} t^\alpha J_\alpha(\sqrt{xt})\,dt\rangle$$
$$= \langle \delta^{(n)}, (\tfrac{1}{2})^{\alpha+1}(2\pi iz)^{-\alpha-1} e^{-x/(8\pi iz)}\rangle$$

$$= \left(\tfrac{1}{2}\right)^{\alpha+2n+1}(2\pi i z)^{-\alpha-n-1},$$

which finishes the proof.

Proof of Lemma 2. We consider the analytic function in the upper half plane $g(z) = \mathcal{F}_{\Pi}(t^{\alpha}f)(1/z)$. Using (32) of [4, p. 132], we obtain

$$g(z) = \int_0^\infty t^\alpha f(t) e^{-2\pi i t/z}\,dt$$

$$= \left(\frac{z}{2\pi i}\right)^{\alpha+1}\left(\frac{1}{2}\right)^{\alpha+2}\int_0^\infty t^{\alpha/2}\int_0^\infty u^{\alpha/2}J_\alpha(\sqrt{ut})f(u)\,du\,e^{-zt/(8\pi i)}\,dt$$

$$= \left(\frac{z}{2\pi i}\right)^{\alpha+1}\left(\frac{1}{2}\right)^{\alpha+1}\int_0^\infty t^\alpha \mathcal{H}_\alpha(f)(t)e^{-zt/(8\pi i)}\,dt.$$

Let

$$K(z) = g(z)\left(\frac{2\pi i}{z}\right)^{\alpha+1} = \left(\frac{1}{2}\right)^{\alpha+1}\int_0^\infty t^\alpha \mathcal{H}_\alpha(f)(t)e^{-zt/(8\pi i)}\,dt.$$

As $f \in S_1^{+0}$, it follows that $\mathcal{H}_\alpha(f) \in S_1^{+0}$ and so $\mathcal{H}_\alpha(f) \in S_{1,A}^{+}$ for some $A > 0$. By Lemma 1 it follows that $K(z)$ is analytic at $z = 0$. Now, by the Taylor formula, we obtain

$$K(z) = \sum_{n=0}^{\infty}\frac{K^{(n)}(0)}{n!}z^n \tag{13}$$

for $|z| < (\limsup_n \sqrt[n]{|K^{(n)}(0)/n!|})^{-1}$. Using Lemma 3, we get

$$K^{(n)}(0) = \left(\tfrac{1}{2}\right)^{\alpha+1}(-1)^n(8\pi i)^{-n}\int_0^\infty t^{\alpha+n}\mathcal{H}_\alpha(f)(t)\,dt$$

$$= \left(\tfrac{1}{2}\right)^{\alpha+1}(-1)^n(8\pi i)^{-n}\langle t^{\alpha+n}, \mathcal{H}_\alpha(f)(t)\rangle$$

$$= \left(\tfrac{1}{2}\right)^{\alpha+1}(-1)^n(8\pi i)^{-n}\langle \mathcal{H}_\alpha(t^{\alpha+n}), f\rangle$$

$$= (-1)^n(2\pi i)^{-n}\Gamma(\alpha+n+1)\langle\delta^{(n)}, f\rangle$$

$$= (2\pi i)^{-n}\Gamma(\alpha+n+1)f^{(n)}(0).$$

By the hypothesis on f, it follows that $|K^{(n)}(0)| \le C'(2\pi)^{-n}\Gamma(n+\alpha+1)B^n$, and so (13) converges for $|z| < 2\pi/B$.

Now $\mathcal{F}_{\Pi}(t^\alpha f)(z) = (2\pi i z)^{-\alpha-1}K(1/z)$, and the lemma is proved by taking

$$h(z) = \left(\frac{1/2 + 2\pi i z}{2\pi i z}\right)^{\alpha+1}K\left(\frac{1}{z}\right).$$

References

[1] A. J. Duran, *The analytic functionals in the lower half plane as a Gel'fand–Shilov space*, Math. Nachr. (to appear).

[2] —, *The Stieltjes moments problem for rapidly decreasing functions*, Proc. Amer. Math. Soc. 107 (1989), 731–741.

[3] —, *Laguerre expansions of tempered distributions and generalized functions*, J. Math. Anal. Appl. 150 (1990), 166–180.

[4] A. Erdélyi (ed.), *Tables of Integral Transforms*, Vol. 1, McGraw-Hill, New York 1953.

[5] A. Erdélyi (ed.), *Higher Transcendental Functions*, Vol. 2, McGraw-Hill, New York 1953.

[6] I. M. Gel'fand et G. E. Shilov, *Les distributions*, Vol. 2, Dunod, Paris 1964.

[7] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. 23, New York 1959.

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