There seems to be no simple way to obtain this as a direct consequence of Theorem 5.2.

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Almost everywhere summability of Laguerre series

by

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Abstract. We apply a construction of generalized twisted convolution to investigate almost everywhere summability of expansions with respect to the orthonormal system of functions $\ell_n^a(x) = (n!/\Gamma(n+a+1))^{1/2} e^{-x/2} L_n^a(x)$, $n = 0, 1, 2, \ldots$, in $L^2(\mathbb{R}_+, x^a\,dx)$, $a \geq 0$. We prove that the Cesàro means of order $\delta > a+2/3$ of any function $f \in L^p(x^a\,dx)$, $1 \leq p \leq \infty$, converge to f a.e. The main tool we use is a Hardy–Littlewood type maximal operator associated with a generalized Euclidean convolution.

1. Introduction. The problem of mean convergence of Laguerre expansions has attracted considerable attention in the last thirty years or so. The articles by Askey and Wainger [2] and Muckenhoupt [12, 13] are fundamental in the subject but also papers by Freud and Knapowski [6], Poiani [15] and Długosz [5] brought interesting results. A new impulse was given to the field in the '80s by Görlich and Markett in a series of papers [7–11]. Their method of investigation of the mean convergence problem was based on a convolution structure for Laguerre polynomials defined first by McCully and extended by Askey. An underlying device there is Watson's product formula for Laguerre polynomials.

In contrast with mean convergence surprisingly little is known for almost everywhere convergence of Laguerre series. The first result in this direction was obtained by Muckenhoupt for expansions with respect to the Laguerre polynomials.

Let

$$L_n^a(x) = (n!)^{-1} e^x x^{-a} (d/dx)^n (e^{-x} x^{n+a})$$

denote the nth Laguerre polynomial of order $a > -1$. Then the normalized polynomials

$$\widetilde{L}_n^a(x) = (n!/\Gamma(n+a+1))^{1/2} L_n^a(x), \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

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form an orthonormal basis in $L^2(\mathbf{R}_+, x^a e^{-x}\,dx)$. With any $f \in L^1(x^a e^{-e}\,dx)$, we associate its formal series expansion $f \sim \sum_{n=0}^{\infty} d_n \widetilde{L}_n^a, d_n = d_n(f) = \langle f, \widetilde{L}_n^a \rangle_{L^2(x^a e^{-x}\,dx)}$, provided such an expansion exists (there is no problem when $f \in L^p(x^a e^{-x}\,dx)$, $p > 1$). In 1969 Muckenhoupt [12] proved the almost everywhere convergence of Poisson integrals for expansions with respect to the system $\widetilde{L}_n^a$: for any $a > -1$ and $f \in L^1(x^a e^{-x}\,dx)$

$$g(r,x) \to f(x) \tag{1.2}$$

almost everywhere as $r \to 1^-$. Here, the Poisson integral

$$g(r,x) = \int_0^{\infty} K(r,x,y) f(y) y^a e^{-y}\,dy$$

with $K(r,x,y) = \sum_{n=0}^{\infty} r^n \widetilde{L}_n^a(x) \widetilde{L}_n^a(y)$, $0 \le r < 1$, is well defined for any f in $L^1(x^a e^{-x}\,dx)$ whether or not it has a Laguerre expansion. Moreover, if $f(x)$ has the Laguerre expansion $\sum_{n=0}^{\infty} d_n \widetilde{L}_n^a(x)$ then for every $0 \le r < 1$, $g(r,x)$ has the Laguerre expansion $\sum_{n=0}^{\infty} r^n d_n \widetilde{L}_n^a(x)$. However, we cannot simply expect that the almost everywhere convergence (1.2) still holds for every f in $L^p(x^a e^{-x}\,dx)$, $p > 1$, with $g(r,x)$ replaced by $\sum_{n=0}^{\infty} r^n d_n \widetilde{L}_n^a(x)$. As pointed out by Muckenhoupt in [12], for every $1 \le p < 2$, $f(x) = e^{cx}$ with $1/2 < c < 1/p$ (cf. [16], p. 367) is an example of a function in $L^p(x^a e^{-x}\,dx)$ such that for all $r < 1$ sufficiently close to 1 the series $\sum_{n=0}^{\infty} r^n d_n \widetilde{L}_n^a(x)$ diverges for every x.

The second kind of expansion deals with the Laguerre functions

$$\mathcal{L}_n^a(x) = x^{a/2} e^{-x/2} \widetilde{L}_n^a(x), \quad n = 0, 1, 2, \ldots, \tag{1.3}$$

which form an orthonormal basis in $L^2(\mathbf{R}_+, dx)$. Now, for any $f \in L^p(dx)$, $1 \le p \le \infty$, we write its formal series expansion $f \sim \sum_{n=0}^{\infty} c_n \mathcal{L}_n^a$, where $c_n = c_n(f) = \langle f, \mathcal{L}_n^a \rangle_{L^2(dx)}$. In [5] Długosz proved the a.e. convergence of the Riesz means: let $a = 0, 1, 2, \ldots$ and $\delta \ge 10$; then for any $f \in L^p(dx)$, $1 \le p \le \infty$,

$$\sum_{n=0}^{N} (1 - n/N)^{\delta} c_n \mathcal{L}_n^a(x) \to f(x) \tag{1.4}$$

almost everywhere as $N \to \infty$. The restriction on a is due to her method, which is based on a group-theoretic approach.

Still another kind of expansion is considered if the functions

$$\ell_n^a(x) = e^{-x/2} \widetilde{L}_n^a(x), \quad n = 0, 1, 2, \ldots, \tag{1.5}$$

forming an orthonormal basis in $L^2(\mathbf{R}_+, x^a\,dx)$, are taken into account. Then, for any $f \in L^p(x^a\,dx)$, $1 \le p \le \infty$, the corresponding formal expansion is $f \sim \sum_{n=0}^{\infty} b_n \ell_n^a$ where $b_n = b_n(f) = \langle f, \ell_n^a \rangle_{L^2(x^a\,dx)}$. The mean convergence problem for such expansions has been undertaken by the author in [18] and, among other things, the L^p-convergence of the Riesz means of order $\delta > a + 5$ was proved for any $f \in L^p(x^a\,dx)$, $1 \le p < \infty$, $a \ge 0$.

Recently, Thangavelu [20] proved the almost everywhere convergence of the Riesz means of such expansions: let $a = 0, 1, 2, \ldots$ and $\delta > a + 1/2$; then for any $f \in L^p(x^a\,dx)$, $2 \le p \le \infty$,

$$\sum_{n=0}^{N} (1 - n/N)^{\delta} b_n \ell_n^a(x) \to f(x)$$

almost everywhere as $N \to \infty$.

As in the case of Długosz' result the restriction on a is motivated by the group-theoretic method used in [20], while the restriction on p is caused, roughly, by a technique applied by Thangavelu depending on L^2-estimates of Riesz kernels.

In this paper we prove the following result concerning the Cesàro means.

THEOREM 1.1. *Let $a \ge 0$ and $\delta > a + 2/3$. Then for any $f \in L^p(x^a\,dx)$, $1 \le p \le \infty$,*

$$(A_N^{\delta})^{-1} \sum_{n=0}^{N} A_{N-n}^{\delta} b_n \ell_n^a(x) \to f(x) \tag{1.6}$$

almost everywhere as $N \to \infty$.

Observe at this point that Theorem 1.1 immediately gives the almost everywhere convergence of the Cesàro means of expansions of functions in $L^2(x^a e^{-x}\,dx)$, and therefore in $L^p(x^a e^{-x}\,dx)$, $p > 2$, with respect to the system $\widetilde{L}_n^a(x)$. More specifically, we have

COROLLARY 1.2. *Let $a \ge 0$ and $\delta > a + 2/3$. Then for every $f \in L^2(x^a e^{-x}\,dx)$*

$$(A_N^{\delta})^{-1} \sum_{n=0}^{N} A_{N-n}^{\delta} d_n \widetilde{L}_n^a(x) \to f(x) \tag{1.7}$$

almost everywhere as $N \to \infty$. For every $1 \le p \le 2$ there exists a function f in $L^p(x^a e^{-x}\,dx)$ for which the left-hand side of (1.7) diverges for every x.

Proof. The mapping $f(x) \mapsto f(x)e^{-x/2}$ is an isometry from $L^2(x^a e^{-x}\,dx)$ onto $L^2(x^a\,dx)$ and thus

$$(A_N^{\delta})^{-1} \sum_{n=0}^{N} A_{N-n}^{\delta} \langle f e^{-x/2}, \widetilde{L}_n^a(x) e^{-x/2} \rangle_{L^2(x^a\,dx)} \widetilde{L}_n^a(x) e^{-x/2} \to f(x) e^{-x/2}$$

almost everywhere with $N \to \infty$. But

$$\langle f e^{-x/2}, \widetilde{L}_n^a(x) e^{-x/2} \rangle_{L^2(x^a\,dx)} = \langle f, \widetilde{L}_n^a \rangle_{L^2(x^a e^{-x}\,dx)}$$

and therefore (1.7) follows. Since the convergence of the Cesàro means of any order implies the convergence of the Abel–Poisson means the counterexample mentioned above works in case $1 \le p \le 2$.

It is interesting to note here that despite the a.e. convergence given in (1.7) for $p > 2$ the mean convergence of these expansions fails to hold. More precisely, Askey and Hirschman [1] proved that for any $\delta > 0$ the mean convergence of the δ-Cesàro means of the expansions of functions in $L^p(x^a e^{-x}\,dx)$, $p > 1$, with respect to the system $\widetilde{L}_n^a(x)$ is restricted to $p = 2$ only.

The present paper is a natural continuation of [18] where the author investigated mean summability of Laguerre expansions with respect to the system ℓ_n^a, $n \ge 0$. The main tool used there was a convolution structure in $L^1(\mathbf{R}_+ \times \mathbf{R}, x^{2\alpha-1}\,dx\,dt)$, $\alpha \ge 1$, coming, when $\alpha = n$, $n \ge 1$, from convolution of radial functions on the Heisenberg group $\mathbf{H}_n$, identified with $\mathbf{C}^n \times \mathbf{R}$; while in the present context we use a convolution structure in $L^1(\mathbf{R}_+, x^{2\alpha-1}\,dx)$, $\alpha \ge 1$, now coming when $\alpha = n$, $n \ge 1$, from radial twisted convolution on $\mathbf{C}^n$ (cf. Section 4 for a detailed discussion). The idea of working with the radial twisted convolution, which is much simpler and natural than the radial Heisenberg convolution, became clear to the author after reading a series of papers by Görlich and Markett [7,8] and Markett [9–11]. Also the recent papers by Thangavelu [19, 20] confirm the right choice of the method. The present approach also allows us to improve a result of [18] concerning the L^p-convergence of the Riesz means and provides more straightforward proofs of convergence in L^p-norms for both the Abel–Poisson and Riesz means. Specifically, we prove

THEOREM 1.3. *Let $\delta > a + 1/2$, $a \ge 0$. Then for every $f \in L^p(x^a\,dx)$, $1 \le p < \infty$, the Cesàro means*

$$(1.9)\qquad (A_n^\delta)^{-1} \sum_{n=0}^{N} A_{N-n}^\delta \langle f, \ell_n^a \rangle_{L^2(x^a\,dx)} \ell_n^a$$

converge in L^p-norm to f as $N \to \infty$.

It is known that the convergence in L^p-norms of the Riesz means of an order $\delta \ge 0$ is equivalent to the L^p-convergence of the Cesàro means of the same order. The equivalence is also true for almost everywhere convergence. For technical reasons we will consider the Cesàro means rather than the Riesz means. Recall that the nth Cesàro mean of order $\delta \ge 0$ of a function $f \in L^p(x^a\,dx)$ is given by

$$C_N^\delta f(x) = (A_N^\delta)^{-1} \sum_{n=0}^{N} A_{N-n}^\delta \langle f, \ell_n^a \rangle_{L^2(x^a\,dx)} \ell_n^a(x),$$

where $A_m^\delta = \binom{m+\delta}{m} = L_m^\delta(0)$. Further, instead of working with the orthonormal sequence $\ell_n^\alpha(x)$, $n = 0, 1, 2, \ldots$, we will use the functions

$$\varphi_n^*(x) = (2n!/\Gamma(n+\alpha))^{1/2} e^{-x^2/2} L_n^{\alpha-1}(x^2), \qquad \alpha \ge 1,$$

forming an orthonormal sequence in $L^2(\mathbf{R}_+, x^{2\alpha-1}\,dx)$.

For any fixed p, $1 \le p \le \infty$, the mapping $A : L^p(x^{2\alpha-1}\,dx) \to L^p(x^{\alpha-1}\,dx)$, $Af(x) = 2^{-1/2} f(x^{1/2})$, is a bijection satisfying:

a) $\|Af\|_{L^p(x^{\alpha-1}dx)} = 2^{1/p-1/2} \|f\|_{L^p(x^{2\alpha-1}dx)}$,
b) $\langle Af, Ag \rangle_{L^2(x^{\alpha-1}dx)} = \langle f, g \rangle_{L^2(x^{2\alpha-1}dx)}$,
c) $A\varphi_n^* = \ell_n^{\alpha-1}$.

Thus, any convergence result proved for the Cesàro means

$$C_N^\delta f = (A_N^\delta)^{-1} \sum_{n=0}^{N} A_{N-n}^\delta \langle f, \varphi_n^* \rangle_{L^2(x^{2\alpha-1}dx)} \varphi_n^*(x)$$

with $f \in L^p(x^{2\alpha-1}\,dx)$ will also imply the corresponding result for the Cesàro means $C_N^\delta f$ with $f \in L^p(x^a\,dx)$, $a = \alpha - 1 \ge 0$.

The paper is organized as follows. In the next section we discuss a product in $L^1(\mathbf{R}_+, x^{2\alpha-1}\,dx)$ turning this space into a commutative Banach algebra. In the third section the weak type $(1,1)$ result is established for a maximal operator connected with still another product in $L^1(x^{2\alpha-1}\,dx)$, which, roughly speaking, majorizes the previous one. The connection of both products and the Heisenberg convolution will be discussed in the fourth section. Finally, we prove almost everywhere as well as L^p-convergence results.

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2. Twisted generalized convolution. Let $F, G \in L^1(\mathbf{C}^n, dw)$, $n \ge 1$. Then the *twisted convolution* $F \times G$ is defined by

$$(2.1)\qquad F \times G(z) = \int_{\mathbf{C}^n} F(z-w) G(w) e^{i\,\mathrm{Im}\langle z,w\rangle}\,dw.$$

Here dw denotes the Lebesgue measure on $\mathbf{C}^n$ and $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$, $|z| = \langle z, z\rangle^{1/2}$ are the scalar product and the norm in $\mathbf{C}^n$ respectively. The space $L^1(\mathbf{C}^n, dw)$ equipped with the product (2.1) is a noncommutative Banach algebra. However, if F and G are radial functions then $F \times G = G \times F$, that is, the algebra $L_r^1(\mathbf{C}^n, dw)$ of radial integrable functions is commutative.

Suppose now f and g are the radial parts of radial functions $F, G \in L_r^1(\mathbf{C}^n, dw)$, i.e. $F(z) = f(|z|)$, $G(z) = g(|z|)$, and denote by $f \times g$ the radial part of the twisted convolution $F \times G$. As one can verify

$$(2.2)\qquad f \times g(t) = \frac{2\pi^{n-1/2}}{\Gamma(n-1/2)} \int_0^\infty g(r) \int_0^\pi f((t^2+r^2-2tr\cos\theta)^{1/2}) \times \mathcal{J}_{n-3/2}(tr\sin\theta)(\sin\theta)^{2n-2}\,d\theta\, r^{2n-1}\,dr\,.$$

Here $\mathcal{J}_s(x) = \Gamma(s+1)J_s(x)/(x/2)^s$, and $J_s(x)$ denotes the Bessel function of order $s > -1$. An important well-known estimate we will use is

$$(2.3)\qquad |\mathcal{J}_s(x)| \le 1, \quad x \ge 0,$$

which is valid for any $s \ge -1/2$.

From now on we fix an arbitrary real parameter $\alpha \ge 1$ and equip the half-axis $[0,\infty)$ with the measure $d\mu(x) = x^{2\alpha-1}\,dx$. Next by $L^p(\mu)$ we denote the corresponding Lebesgue spaces endowed with the norms

$$\|f\|_p = \Big(\int_0^\infty |f|^p\,d\mu\Big)^{1/p}, \quad 1 \le p \le \infty, \quad \|f\|_\infty = \operatorname*{ess\,sup}_{x\in[0,\infty)} |f(x)|\,.$$

Also, we define the probability measure $d\nu$ on $[0,\pi)$ by

$$(2.4)\qquad d\nu(\theta) = \frac{\Gamma(\alpha)}{\pi^{1/2}\Gamma(\alpha-1/2)}(\sin\theta)^{2(\alpha-1)}\,d\theta\,.$$

Next according to (2.2) we define the *twisted generalized convolution* (cf. also [8])

$$(2.5)\qquad f \times g(x) = \int_0^\infty T^x f(y) g(y)\,d\mu(y)$$

where T^x, $x \ge 0$, are the *twisted generalized translation operators*

$$(2.6)\qquad T^x f(y) = \int_0^\pi f((x,y)_\theta)\mathcal{J}_{\alpha-3/2}(xy\sin\theta)\,d\nu(\theta)$$

with *θ-product* $(x,y)_\theta$ given by

$$(2.7)\qquad (x,y)_\theta = (x^2+y^2-2xy\cos\theta)^{1/2}, \quad x, y \ge 0,\ \theta\in[0,\pi]\,.$$

The operator T^x can also be described by

$$(2.8)\qquad T^x f(y) = \int_{|x-y|}^{x+y} f(t)\mathcal{J}_{\alpha-3/2}(\Delta(x,y,t))\,dW_{x,y}(t)$$

where the probability measure $dW_{x,y}(t)$ is supported on $[|x-y|, x+y]$ and given by

$$(2.9)\qquad dW_{x,y}(t) = c(\alpha)\frac{\Delta(x,y,t)^{2\alpha-3}}{(xyt)^{2\alpha-2}}\,d\mu(t)$$

with $c(\alpha) = 2^{2\alpha-3}\Gamma(\alpha)\Gamma(\alpha-1/2)^{-1}\pi^{-1/2}$. In the above formula $x,y,t \ge 0$, $|x-y| \le t \le x+y$ and $\Delta(x,y,t)$ denotes the area of a triangle with sides x, y, t. It is quite straightforward to go from (2.6) to (2.8) by a change of variables. Also, as one can immediately remark the measure

$$\mathcal{J}_{\alpha-3/2}(\Delta(x,y,t))\,dW_{x,y}(t)\,d\mu(y)$$

is symmetric with respect to y and t, i.e.

$$(2.10)\qquad \mathcal{J}_{\alpha-3/2}(\Delta(x,y,t))\,dW_{x,y}(t)\,d\mu(y) = \mathcal{J}_{\alpha-3/2}(\Delta(x,t,y))\,dW_{x,t}(y)\,d\mu(t)\,.$$

LEMMA 2.1. *For every $1 \le p \le \infty$ and $x \ge 0$ the twisted generalized translation T^x is a contraction, i.e. $\|T^x\| \le 1$.*

Proof. The above is an easy consequence of the estimate (2.3), the definition (2.8), the symmetry property (2.10) and the fact that the $W_{x,y}$ are probability measures.

Furthermore, we have

LEMMA 2.2. *The operator T^x, $x \ge 0$, is selfadjoint on $L^2(\mu)$. Moreover,*

$$\int_0^\infty T^x f(y) g(y)\,d\mu(y) = \int_0^\infty f(y) T^x g(y)\,d\mu(y)$$

for any reasonable pair of functions f and g, e.g. for $f \in L^p(\mu)$, $1 \le p \le \infty$, and $g \in L^1(\mu)$.

Proof. Straightforward consequence of (2.10).

THEOREM 2.3. *The space $L^1(\mu)$ with the product given by (2.5) is a commutative Banach algebra. Moreover,*

$$(2.11)\qquad \|f \times g\|_p \le \|f\|_p \|g\|_1$$

for any $f \in L^p(\mu)$, $1 \le p \le \infty$, and $g \in L^1(\mu)$.

Proof. The inequality (2.11) is an easy consequence of the fact that T^x, $x \ge 0$, are contractions. Commutativity is implied by Lemma 2.2.

Having defined the twisted generalized convolution structure in $L^1(\mu)$ it is remarkable to note that an underlying differential operator is

$$(2.12)\qquad L = -\Big(\frac{d^2}{dx^2} + \frac{2\alpha-1}{x}\frac{d}{dx} - x^2\Big)\,,$$

which is a positive symmetric operator in $L^2(\mu)$. Moreover, L has a discrete spectrum and the functions

$$(2.13)\qquad \varphi_n(x) = \frac{n!\,\Gamma(\alpha)}{\Gamma(n+\alpha)} e^{-x^2/2} L_n^{\alpha-1}(x^2)$$

are eigenfunctions of L. More precisely,

$$L\varphi_n(x) = 4(n+\alpha/2)\varphi_n(x)\,, \tag{2.14}$$

which can be easily verified by using the well-known differential identities for the Laguerre polynomials. Note also that the normalization constant in (2.13) is such that $|\varphi_n(x)| \le 1$, $0 \le x \le \infty$ (cf. [21]). We use the eigenfunctions as kernels in a transform defined on $L^p(\mu)$: for any $f \in L^p(\mu)$, $1 \le p \le \infty$, we put

$$\widehat{f}(n) = \int_0^\infty f(x)\varphi_n(x)\,d\mu(x), \quad n = 0,1,2,\ldots \tag{2.15}$$

Then

$$(Lf)^\wedge(n) = 4(n+\alpha/2)\widehat{f}(n)$$

provided Lf is also integrable with some power. In 1939 Watson [21] gave the following integral representation for the product $L_n^{\alpha-1}(x^2)L_n^{\alpha-1}(y^2)$:

$$\begin{aligned} L_n^{\alpha-1}(x^2)L_n^{\alpha-1}(y^2) &= \frac{\Gamma(n+\alpha)}{n!\pi^{1/2}\Gamma(\alpha-1/2)} \\ &\times \int_0^\pi L_n^{\alpha-1}((x,y)_\theta^2)\mathcal{J}_{\alpha-3/2}(xy\sin\theta)e^{xy\cos\theta}(\sin\theta)^{2\alpha-2}\,d\theta\,. \end{aligned} \tag{2.16}$$

In terms of the generalized translation operator (2.6) this is nothing else but the following identity:

$$T^x\varphi_n(y) = \varphi_n(x)\varphi_n(y)\,. \tag{2.17}$$

Note also that (2.17) immediately implies

$$f \times \varphi_n = \widehat{f}(n)\varphi_n$$

for any function $f \in L^p(\mu)$. This formula can be thought of as an analogue of the spherical functions formula in Gelfand pairs theory. Though we will not use this fact the product formula (2.17) also yields an identification of the Gelfand space of the Banach algebra $L^1(\mu)$ with the set $\mathbb{N} = \{0,1,2,\ldots\}$. Specifically, we have

THEOREM 2.4. *For any $n \in \mathbb{N}$ the mapping $f \mapsto \widehat{f}(n)$ is a multiplicative functional on $L^1(\mu)$ and every multiplicative functional is of this form.*

3. Euclidean generalized convolution and maximal functions.

Consider now the *Eulidean convolution*

$$F * G(z) = \int_{\mathbb{C}^n} F(z-w)G(w)\,dw$$

of two radial integrable functions on $\mathbb{C}^n$, $n \ge 1$. If $F(z) = f(|z|)$, $G(z) = g(|z|)$ then $F * G(z) = f * g(|z|)$ where $f * g$ is a function on $[0,\infty)$ given by a formula like (2.2) with the factor $\mathcal{J}_{n-3/2}(tr\sin\theta)$ dropped.

By allowing n to be an arbitrary real parameter ≥ 1 we then come to the well-known generalized convolution structure

$$f * g(x) = \int_0^\infty T_E^x f(y)g(y)\,d\mu(y) \tag{3.1}$$

where T_E^x, $x \ge 0$, are the (*Euclidean*) *generalized translation operators*

$$T_E^x f(y) = \int_0^\pi f((x,y)_\theta)\,d\nu(\theta)\,. \tag{3.2}$$

Here as before $d\mu(y) = y^{2\alpha-1}\,dy$, $d\nu$ and $(x,y)_\theta$ are given by (2.4) and (2.7) and the range of the parameter α can now be even enlarged to $\alpha > 1/2$. As in the case of T^x an equivalent description of T_E^x is

$$T_E^x f(y) = \int_{|x-y|}^{x+y} f(t)\,dW_{x,y}(t) \tag{3.3}$$

with $dW_{x,y}(t)$ given by (2.9). It is well known that $L^1(\mu)$ with (3.1) as multiplication is a commutative Banach algebra. Moreover, the second order differential operator

$$L_E = -\left(\frac{d^2}{dx^2} + \frac{2\alpha-1}{x}\frac{d}{dx}\right) \tag{3.4}$$

has $\mathcal{J}_{\alpha-1}(\lambda x)$, $\lambda \ge 0$, as its eigenfunctions:

$$L_E\mathcal{J}_{\alpha-1}(\lambda x) = \lambda^2\mathcal{J}_{\alpha-1}(\lambda x)\,.$$

These functions are also kernels of the *Fourier–Bessel* (also called *Hankel*) *transform*

$$f \mapsto \widehat{f}(\lambda) = \int_0^\infty f(x)\mathcal{J}_{\alpha-1}(\lambda x)\,d\mu(x)\,, \quad \lambda \ge 0\,,$$

which satisfies $(f * g)^\wedge(\lambda) = \widehat{f}(\lambda)\widehat{g}(\lambda)$ and $(L_E f)^\wedge(\lambda) = \lambda^2\widehat{f}(\lambda)$ for any $f,g \in L^1(\mu)$.

From our point of view the most important connection between the twisted and Euclidean convolution is expressed by the inequality

$$|f \times g(x)| \le |f| * |g|(x)\,,$$

immediately implied by $|T^x f(y)| \le T_E^x(|f|)(y)$, which in turn is a consequence of (2.3). We will also make use of the heat kernel

$$W_t(x) = \frac{2}{\Gamma(\alpha)}(4t)^{-\alpha}\exp(-x^2/4t), \quad t>0, \tag{3.5}$$

for the operator L_E.

With the convolution structure (3.1) we also associate a maximal operator

$$f^*(x) = \sup_{\varepsilon>0}\frac{1}{\mu(0,\varepsilon)}\int_0^\varepsilon T_E^s(|f|)(x)\,d\mu(s) \tag{3.6}$$

defined for any reasonable function f on $[0,\infty)$. We also define the dilations f_ε, $\varepsilon>0$, by setting

$$f_\varepsilon(x) = \varepsilon^{-2\alpha}f(x/\varepsilon). \tag{3.7}$$

The following proposition justifies the consideration of this operator.

PROPOSITION 3.1. *Suppose ω is a positive nonincreasing function on $[0,\infty)$ with $\|\omega\|_1<\infty$. Then*

$$\sup_{\varepsilon>0}|\omega_\varepsilon * f(x)| \le \|\omega\|_1 f^*(x). \tag{3.8}$$

Proof. It suffices to prove (3.8) when ω is a simple function $\omega = \sum_{j=1}^m c_j\chi_{[0,a_j]}$, a_j and c_j being positive reals, and this may be obtained by straightforward calculations. The general case then follows by a simple limiting argument.

By a *space of homogeneous type* we mean a topological space X equipped with a continuous pseudometric ϱ and a positive measure m satisfying the doubling condition

$$m(B_{2\varepsilon}(x)) \le Cm(B_\varepsilon(x)) \tag{3.9}$$

with a constant C independent of $x\in X$ and $\varepsilon>0$. Here $B_\varepsilon(x)=\{y\in X: \varrho(x,y)<\varepsilon\}$. Let (X,ϱ,m) be a space of homogeneous type. For any locally integrable function f on X define

$$Mf(x) = \sup_{\varepsilon>0} m(B_\varepsilon(x))^{-1}\int_{B_\varepsilon(x)}|f(y)|\,dm(y). \tag{3.10}$$

It is well known (cf. [4]) that the maximal operator $f\mapsto Mf$ is of weak type $(1,1)$, i.e.

$$m(\{x: Mf(x)>s\}) \le \frac{C}{s}\int_X |f(x)|\,dm(x),$$

with $C>0$ independent of $s>0$ and $f\in L^1(X,dm)$, and, in virtue of Marcinkiewicz' interpolation theorem, is of strong type (p,p), $1<p<\infty$, i.e.

$$\int_X Mf(x)^p\,dm(x) \le C_p\int_X |f(x)|^p\,dm(x),$$

with $C_p>0$ independent of $f\in L^p(X,dm)$.

The following may be proved by a straightforward calculation.

PROPOSITION 3.2. *Consider $X=[0,\infty)$ equipped with the Euclidean metric and the measure $d\mu(x)=x^{2\alpha-1}\,dx$, $\alpha>1/2$. Then $d\mu$ satisfies the doubling condition* (3.9).

We are going to show that the maximal function f^* is majorized by Mf. We start with the following

PROPOSITION 3.3. *There exists a constant $C>0$ independent of $0<t<x$ such that*

$$\|T_E^x\chi_{[0,t]}\|_\infty \le C(t/x)^{2\alpha-1}. \tag{3.11}$$

Proof. For any function f on $[0,\infty)$ set $\tau_\varepsilon f(x)=f(\varepsilon x)$, $\varepsilon>0$. In virtue of the easily verified identity

$$T_E^x\chi_{[0,t]} = \tau_{1/t}(T_E^{x/t}\chi_{[0,1]})$$

it suffices to show (3.11) for $t=1$ and $x>1$. Clearly we can assume that x is large enough, say $x>10$. The function $T_E^x\chi_{[0,1]}$ is supported in $[x-1,x+1]$. Thus using (3.3) we write for $y\in[x-1,x+1]$

$$T_E^x\chi_{[0,1]}(y) = c(\alpha)\frac{1}{x^{2\alpha-1}}\frac{x}{y}\int_{|x-y|}^1\left(\frac{\Delta(x,y,t)}{y}\right)^{2\alpha-3} t\,dt. \tag{3.12}$$

Since $x/y<2$ we need only check that the integrals on the right side of (3.12) are uniformly bounded on $x,y>0$, $|x-y|<1$. Clearly this is the case when $\alpha\ge 3/2$ since then $\Delta(x,y,t)\le\frac{1}{2}yt$ and $2\alpha-3\ge 0$. Suppose now $1/2<\alpha<3/2$. Then for $|x-y|<t<1$

$$\Delta(x,y,t) = \tfrac14\left[((x+y)^2-t^2)(t^2-(x-y)^2)\right]^{1/2} \ge \tfrac12 y(t^2-(x-y)^2)^{1/2}.$$

Hence, with $b=|x-y|$, the integral in (3.12) is, up to a constant, bounded by

$$\int_b^1 (t^2-b^2)^{\alpha-3/2}\,t\,dt = \frac{1}{2\alpha-1}(1-b^2)^{\alpha-1/2} \le \frac{1}{2\alpha-1}.$$

This finishes the proof of Proposition 3.3.

We now come to the main result of this section.

THEOREM 3.4. *There exists a constant C such that*

$$f^*(x) \le CMf(x). \tag{3.13}$$

In this way the maximal operator $f \mapsto f^*$ is of weak type $(1,1)$ and strong type (p,p), $1 < p \le \infty$.

Proof. We can assume that f is a nonnegative function. Thus let $f \ge 0$ and denote by f_1^*, f_2^* two auxiliary maximal functions

$$f_1^*(x) = \sup_{\varepsilon < x} \frac{1}{\mu(0,\varepsilon)} \int_0^\varepsilon T_E^s f(x)\, d\mu(s),$$

$$f_2^*(x) = \sup_{\varepsilon \ge x} \frac{1}{\mu(0,\varepsilon)} \int_0^\varepsilon T_E^s f(x)\, d\mu(s).$$

Clearly $f^*(x) \le f_1^*(x) + f_2^*(x)$, so we will show that $f_i^*(x) \le CMf(x)$, $i = 1,2$. Since for $\varepsilon \ge x$ we have $\varepsilon^{-2\alpha} \le 2^{2\alpha}(\varepsilon + x)^{-2\alpha}$ and $T_E^x \chi_{[0,\varepsilon]} \le \chi_{[0,\varepsilon+x]}$ we write

$$\frac{1}{\mu(0,\varepsilon)} \int_0^\varepsilon T_E^x f(s)\, d\mu(s) = \frac{2\alpha}{\varepsilon^{2\alpha}} \langle T_E^x f, \chi_{[0,\varepsilon]}\rangle = \frac{2\alpha}{\varepsilon^{2\alpha}} \langle f, T_E^x \chi_{[0,\varepsilon]}\rangle$$

$$\le \frac{\alpha 2^{2\alpha+1}}{(\varepsilon + x)^{2\alpha}} \int_0^{\varepsilon+x} f(s)\, d\mu(s).$$

Thus $f_2^*(x) \le CMf(x)$. To complete the proof we use Proposition 3.3 writing for $\varepsilon < x$

$$\langle T_E^x f, \chi_{[0,\varepsilon]}\rangle = \langle f, T_E^x \chi_{[0,\varepsilon]}\rangle \le C(\varepsilon/x)^{2\alpha-1} \langle f, \chi_{[x-\varepsilon,x+\varepsilon]}\rangle, \tag{3.14}$$

since, as one can note, $T_E^x \chi_{[0,\varepsilon]}$ is supported in $[x-\varepsilon, x+\varepsilon]$. Furthermore, $x^{2\alpha-1}\varepsilon \ge C\mu(x-\varepsilon, x+\varepsilon)$ for $\varepsilon < x$ and thus

$$\frac{1}{\mu(0,\varepsilon)} \int_0^\varepsilon T_E^x f(s)\, d\mu(s) \le \frac{C}{x^{2\alpha-1}\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(s)\, d\mu(s)$$

$$\le C \frac{1}{\mu(x-\varepsilon, x+\varepsilon)} \int_{x-\varepsilon}^{x+\varepsilon} f(s)\, d\mu(s).$$

This gives $f_1^*(x) \le CMf(x)$ and concludes the proof.

4. Heisenberg generalized convolution.

In this section we are going to explain a natural connection between the twisted convolution we investigated and a convolution structure considered by the author in [17, 18]. Let $\mathbf{H}_n = \mathbf{C}^n \times \mathbf{R}$, $n = 1,2,\ldots$, denote the $(2n+1)$-dimensional Heisenberg group with the product

$$(z,t)(z',t') = (z + z', t + t' + \operatorname{Im}\langle z, z'\rangle)$$

where $\langle z, z'\rangle = \sum_{j=1}^n z_j \overline{z_j'}$. Similarly denote by $\widetilde{\mathbf{H}}_n$ the quotient $\mathbf{H}_n/\mathbf{K}$, $\mathbf{K} = \{(0,0,2k\pi) : k \in \mathbf{Z}\}$, isomorphic to $\mathbf{C}^n \times \mathbf{T}$ with multiplication

$$(z, e^{it})(z', e^{it'}) = (z + z', e^{i(t+t'+\operatorname{Im}\langle z,z'\rangle)}).$$

For any function F on $\mathbf{C}^n$ define $F^\natural(z, e^{i\theta}) = F(z)e^{-i\theta}$. It is easy to check that

$$F^\natural * G^\natural = (F \times G)^\natural \tag{4.1}$$

where $*$ stands for the convolution in $\widetilde{\mathbf{H}}_n$ and $F \times G$ is the twisted convolution given by (2.1). A function F on $\widetilde{\mathbf{H}}_n$ is called *radial* provided $F(z,t) = F(|z|,t)$. Clearly $F^\natural$ is radial on $\widetilde{\mathbf{H}}_n$ if and only if F is radial on $\mathbf{C}^n$. It is well known that the algebra of radial integrable functions on $\widetilde{\mathbf{H}}_n$ is commutative. Thus (4.1) shows that the algebra of radial integrable functions on $\mathbf{C}^n$ with $\times$ as product is commutative, too.

We now recall a construction of a generalized convolution considered in [17]. Let $\alpha > 1$ be a fixed parameter, the same as in the previous two sections (since in the limiting case $\alpha = 1$ definitions are slightly different than those for $\alpha > 1$ we decided, for simplicity, to consider only the case $\alpha > 1$). Keeping the notation from Section 2 we endow the space $X = [0,\infty) \times \mathbf{R}$ with the measure $d\mu_X(x,t) = d\mu(x)\,dt$ and denote by $L^p(X)$ the corresponding Lebesgue spaces. For $\xi = (x,t)$, $\eta = (y,u) \in X$ and $\theta, \varphi \in \mathbf{R}$ we define the (θ,φ)*-product*

$$(\xi, \eta)_{\theta,\varphi} = ((x,y)_\theta, t - u + xy\cos\varphi\sin\theta)$$

and the probability measure ν_X on $[0,\pi) \times [0,\pi]$ by

$$d\nu_X(\theta,\varphi) = \frac{\alpha}{\pi}(\sin\varphi)^{2\alpha-3}(\sin\theta)^{2\alpha-2}\, d\varphi\, d\theta.$$

The generalized translation operators T^η, $\eta \in X$, and the convolution $f * g$, are then defined by

$$T^\eta f(\xi) = \int_0^\pi \int_0^\pi f((\xi,\eta)_{\theta,\varphi})\, d\nu_X(\theta,\varphi), \qquad f \in L^p(X), \tag{4.2}$$

$$f * g(\xi) = \int_X T^\eta f(\xi) g(\eta)\, d\mu_X(\eta), \tag{4.3}$$

where, for instance, $f \in L^p(X)$, $1 \le p \le \infty$, and $g \in L^1(X)$.

We proved in [17] that $L^1(X)$ endowed with (4.3) is a commutative Banach algebra. Moreover, the generalized translations T^η, $\eta \in X$, are submarkovian contractions on every $L^p(X)$, $1 \le p \le \infty$. Furthermore, an underlying differential operator is

$$L_{\mathbf{H}} = -\left(\frac{\partial^2}{\partial x^2} + \frac{2\alpha - 1}{x}\frac{\partial}{\partial x} + x^2 \frac{\partial^2}{\partial t^2}\right).$$

We mention that for $\alpha = n$, (4.3) describes the Heisenberg convolution of two radial functions and $L_{\mathbf{H}}$ is the radial part of the Heisenberg sublaplacian.

Next consider a periodic analogy of the situation described above. By $\widetilde{X}$ we denote the measure space $[0,\infty) \times \mathbf{T}$ equipped with $d\widetilde{\mu}_X(x) = d\mu(x)\,dt$, dt being now the normalized Lebesgue measure on $\mathbf{T}$. Analogously we define the generalized translations $\widetilde{T}^\eta$, $\eta \in \widetilde{X}$, and the convolution $f * g$ of two functions $f, g \in L^1(\widetilde{X})$. As before, for a function f on $[0,\infty)$ write $f^\natural(x, e^{i\theta}) = f(x)e^{-i\theta}$. Then, as one can check, for $\xi = (x, e^{it})$, $\eta = (y, e^{iu}) \in \widetilde{X}$,

$$\widetilde{T}^\eta f^\natural(\xi) = e^{-i(t-u)} T^x f(y)\,,$$

which immediately implies

$$f^\natural * g^\natural = (f \times g)^\natural\,.$$

Furthermore, $\widetilde{L}_{\mathbf{H}} f^\natural = (Lf)^\natural$.

5. Almost everywhere convergence results. We are now in a position to prove our main results. It is well known that the convergence of the Cesàro (or, equivalently, Riesz) means of any nonnegative order implies the convergence of the Abel–Poisson means (both, in L^p-norm and almost everywhere). Nevertheless, to illustrate the simplicity of the method used we include here the results concerning the Abel–Poisson means.

We start with finding the heat kernel associated to the operator

$$L = -\left(\frac{d^2}{dx^2} + \frac{2\alpha - 1}{x}\frac{d}{dx} - x^2\right).$$

Using the same character L to denote its selfadjoint extension we write $L = \int_0^\infty \lambda\, dE_\lambda$ for the spectral decomposition. Then we define

$$P_t f = \int_0^\infty e^{-t\lambda}\, dE_\lambda\,.$$

We are going to show that $\{P_t\}_{t>0}$ constitutes a twisted convolution semigroup with positive integrable kernels. Normalizing the eigenfunctions $\varphi_n(x)$ we write

$$\varphi_n^*(x) = (2n!/\Gamma(n+\alpha))^{1/2} e^{-x^2/2} L_n^{\alpha-1}(x^2)\,.$$

Then $\varphi_n^* = \frac{1}{\Gamma(\alpha)}(2\Gamma(n+\alpha)/n!)^{1/2}\varphi_n$ and φ_n^*, $n = 0, 1, 2, \ldots$, is an orthonormal sequence in $L^2(\mu)$. Next

$$P_t f(x) = \sum_{n=0}^\infty e^{-4(n+\alpha/2)t}\langle f, \varphi_n^*\rangle \varphi_n^*(x) = \int_0^\infty K_t(x,y) f(y)\, d\mu(y)$$

with

$$K_t(x,y) = \frac{2e^{-2t\alpha}}{\Gamma(\alpha)}\sum_{n=0}^\infty e^{-4tn}\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)n!}\varphi_n(x)\varphi_n(y)$$
$$= \frac{2e^{-2t\alpha}}{\Gamma(\alpha)} T^x\left(\sum_{n=0}^\infty e^{-4tn}\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)n!}\varphi_n\right)(y)\,.$$

Substituting $r = \exp(-4t)$ and using a formula for the generating function for Laguerre polynomials we get

$$\sum_{n=0}^\infty r^n \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)n!}\varphi_n(y) = e^{-y^2/2}\sum_{n=0}^\infty r^n L_n^{\alpha-1}(y^2)$$
$$= (1-r)^{-\alpha}\exp\left(-\frac{1}{2}\frac{1+r}{1-r}y^2\right).$$

Finally, $P_t f = p_t \times f$ with

$$p_t(y) = \frac{1}{\Gamma(\alpha)2^{\alpha-1}}(\sinh 2t)^{-\alpha}\exp\left(-\frac{y^2}{2}\coth 2t\right).$$

An easy evaluation also shows that $\|p_t\|_1 = (\cosh 2t)^{-\alpha}$. This gives rise to the following mean convergence result.

THEOREM 5.1. *Let $f \in L^p(x^a\,dx)$, $a \ge 0$, $1 \le p < \infty$. Then the Abel–Poisson means*

$$\sum_{n=0}^\infty r^n \langle f, \ell_n^a\rangle_{L^2(x^a\,dx)}\ell_n^a$$

converge in L^p-norm to f as $r \to 1^-$.

Proof. We work with $\alpha \ge 1$ and prove Theorem 5.1 with a replaced with $\alpha - 1$. The mean convergence above is clearly implied by

$$\lim_{r\to 1^-}\Big\|\sum_{n=0}^\infty r^n\langle f, \varphi_n^*\rangle\varphi_n^* - f\Big\|_p = 0\,,$$

$f \in L^p(x^{2\alpha-1}\,dx)$, $1 \le p < \infty$, which in turn is a consequence of the uniform boundedness of $\|p_t\|_1$, $t > 0$.

As far as almost everywhere convergence is concerned, note that by $\sinh 2t > 2t$, $\coth 2t > 1/2t$, for $t > 0$, we estimate

$$p_t(y) \le Ct^{-\alpha}e^{-y^2/4t} = C_1 W_t(y)\,,$$

where W_t is the heat kernel associated to L_E (cf. (3.5)). Therefore, due to Proposition 3.1,

$$\sup_{t>0}|p_t \times f(x)| \le C_1 \sup_{t>0} W_t * |f|(x) \le C_2 f^*(x)\,,$$

which obviously implies

THEOREM 5.2. *For every $f \in L^p(x^a\,dx)$, $1 \le p \le \infty$, $a \ge 0$, the Abel–Poisson means*

$$\sum_{n=0}^{\infty} r^n \langle f, \ell_n^a \rangle_{L^2(x^a\,dx)} \ell_n^a$$

converge almost everywhere to $f(x)$ as $r \to 1^-$.

We now pass to the Cesàro means. We write

$$C_n^\delta f(x) = (A_n^\delta)^{-1} \sum_{k=0}^{n} A_{n-k}^\delta \langle f, \varphi_k^* \rangle \varphi_k^*(x)$$
$$= (A_n^\delta)^{-1} \int_0^\infty K_n(x,y) f(y)\, y^{2\alpha-1}\,dy$$

for the nth Cesàro mean of a function $f \in L^p(\mu)$. Here

$$K_n(x,y) = \frac{2}{\Gamma(\alpha)^2} \sum_{k=0}^{n} A_{n-k}^\delta \frac{\Gamma(k+\alpha)}{k!} \varphi_k(x)\varphi_k(y)$$
$$= T^x\Big(\frac{2}{\Gamma(\alpha)^2} \sum_{k=0}^{n} A_{n-k}^\delta \frac{\Gamma(k+\alpha)}{k!} \varphi_k\Big)(y)\,.$$

Using the formula

$$\sum_{k=0}^{n} L_{n-k}^\delta(s) L_k^\gamma(t) = L_n^{\delta+\gamma+1}(s+t)$$

and the identity $A_{n-k}^\delta = L_{n-k}^\delta(0)$, we finally obtain

$$\frac{2}{\Gamma(\alpha)^2} \sum_{k=0}^{n} A_{n-k}^\delta \frac{\Gamma(k+\alpha)}{k!} \varphi_k(y) = \frac{2}{\Gamma(\alpha)} e^{-y^2/2} \sum_{k=0}^{n} A_{n-k}^\delta L_k^{\alpha-1}(y^2)$$
$$= \frac{2}{\Gamma(\alpha)} e^{-y^2/2} L_n^{\delta+\alpha}(y^2)\,.$$

Consequently, $C_n^\delta f = \Phi_{n,\delta} \times f$ with

$$\Phi_{n,\delta}(y) = (A_n^\delta)^{-1} \frac{2}{\Gamma(\alpha)} e^{-y^2/2} L_n^{\delta+\alpha}(y^2).$$

We now want to know for which values of δ

(5.1) $$\sup_n \|\Phi_{n,\delta}\|_1 < \infty\,,$$

which is equivalent to

$$\sup_n n^{-\delta} \int_0^\infty |L_n^{\alpha+\delta}(x)| e^{-x/2} x^{\alpha-1}\,dx < \infty$$

(we used $A_n^\delta \sim n^\delta$ where $a_n \sim b_n$ stands for $a_n = O(b_n)$ and $b_n = O(a_n)$ as $n \to \infty$). Applying the estimate (cf. [10], Lemma 1)

$$\int_0^\infty |L_n^{a+b}(x)| e^{-x/2} x^{a/2}\,dx \sim \begin{cases} n^{(a+1)/2}, & b < 3/2,\\ n^{(a+1)/2}\ln n, & b = 3/2,\\ n^{a/2+b-1}, & b > 3/2,\end{cases}$$

valid for $a+b > -1$, $a > -2$, we easily verify that (5.1) holds if and only if $\delta > \alpha - 1/2$.

We thus come to the following mean convergence result.

THEOREM 5.3. *Let $1 \le p < \infty$, $\alpha \ge 1$ and $\delta > \alpha - 1/2$. Then $\lim_{n\to\infty} \|C_n^\delta f - f\|_p = 0$ for any $f \in L^p(x^{2\alpha-1}\,dx)$.*

Proof. Straightforward consequence of (5.1).

Next, working with $a = \alpha - 1$, $\alpha \ge 1$, we immediately obtain Theorem 1.3. Our next goal is to obtain an almost everywhere convergence result. We start with

LEMMA 5.4. *Let $\alpha \ge 1$, $\delta > \alpha - 1/3$ and $\varepsilon > 0$ be such that $\delta > \alpha - 1/3 + 2\varepsilon$. Define $\omega(x) = (1+x^2)^{-(\alpha+\varepsilon)}$. Then*

$$|\Phi_{n,\delta}(x)| \le C\omega_{\varepsilon(n)}(x)$$

with $\varepsilon(n) = n^{-1/2}$ and C independent of n where ω_ε is defined by (3.7).

The lemma above is sufficient to prove

THEOREM 5.5. *Let $1 \le p \le \infty$, $\alpha \ge 1$ and $\delta > \alpha - 1/3$. Then for any $f \in L^p(x^{2\alpha-1}\,dx)$*

$$C_n^\delta f(x) \to f(x)$$

almost everywhere as $n \to \infty$.

Proof. Given $\delta > \alpha - 1/3 + 2\varepsilon$, $\varepsilon > 0$, $f \in L^p(x^{2\alpha-1}\,dx)$, $1 \le p \le \infty$, we estimate

$$\sup_n |C_n^\delta f(x)| \le \sup_n |\Phi_{n,\delta}| * |f|(x) \le C \sup_n \omega_{\varepsilon(n)} * |f|(x) \le C f^*(x)\,,$$

which clearly gives almost everywhere convergence for the Cesàro means. Consequently, Theorem 5.5 implies Theorem 1.1.

Proof of Lemma 5.4. Since $A_n^\delta \sim n^\delta$ we write

$$|\Phi_{n,\delta}(x)| \le C\, n^{-\delta} e^{-x^2/2} |L_n^{\alpha+\delta}(x^2)|$$

and thus the inequality we are going to prove is

$$n^{-\delta}e^{-y/2}|L_n^{\alpha+\delta}(y)| \le C\, n^{\alpha}(1+ny)^{-(\alpha+\varepsilon)}. \tag{5.2}$$

But

$$n^{-\delta}e^{-y/2}|L_n^{\alpha+\delta}(y)| \le C\, n^{(\alpha-\delta)/2}y^{-(\alpha+\delta)/2}|\mathcal{L}_n^{\alpha+\delta}(y)|,$$

therefore, to prove (5.2) it is sufficient to obtain

$$|\mathcal{L}_n^{\alpha+\delta}(y)| \le C\,(ny)^{(\alpha+\delta)/2}/(1+ny)^{\alpha+\varepsilon}. \tag{5.3}$$

We prove (5.3) using the fundamental estimate for Laguerre functions (cf. [13], (2.5))

$$|\mathcal{L}_n^{\alpha+\delta}(y)| \le C \begin{cases} (y\nu)^{(\alpha+\delta)/2} & \text{if } 0 \le y \le 1/\nu, \\ (y\nu)^{-1/4} & \text{if } 1/\nu < y \le \nu/2, \\ (\nu(\nu^{1/3}+|y-\nu|))^{-1/4} & \text{if } \nu/2 < y \le 3\nu/2, \\ e^{-\gamma y} & \text{if } 3\nu/2 < y, \end{cases}$$

where $\nu = 4n + 2(\alpha+\delta) + 2$ and γ is a positive constant. We verify (5.3) step by step:

a) $0 \le y \le 1/\nu$; then (5.3) is obvious.

b) $1/\nu < y \le \nu/2$; then $\nu y > 1$ and $C \le (\nu y)^{(\alpha+\delta)/2+1/4}/(1+\nu y)^{\alpha+\varepsilon}$ since $\delta > \alpha - 1/2 + 2\varepsilon$ by the assumption.

c) $\nu/2 < y \le 3\nu/2$; then $\nu y > \nu^2/2$. What we now need is the estimate

$$(\nu(\nu^{1/3}+|y-\nu|))^{-1/4} \le C(ny)^{(\alpha+\delta)/2}/(1+ny)^{\alpha+\varepsilon}$$

or, equivalently,

$$y^{1/4}(\nu^{1/3}+|y-\nu|)^{-1/4} \le C(\nu y)^{(\alpha+\delta)/2+1/4}/(1+\nu y)^{\alpha+\varepsilon}. \tag{5.4}$$

Clearly, on $(\nu/2, 3\nu/2]$ the left-hand side of (5.4) is bounded from above by $C\nu^{1/6}$, while the right-hand side of (5.4) is bounded from below by

$$C(\nu^2)^{(\alpha+\delta)/2+1/4-(\alpha+\varepsilon)}.$$

By the assumption $\delta > \alpha - 1/3 + 2\varepsilon$ so $\nu^{1/6} \le C\nu^{\delta-\alpha+1/2-2\varepsilon}$, which proves (5.4).

d) $3\nu/2 < y$; what we need is

$$C \le e^{\gamma y}(ny)^{(\alpha+\delta)/2}/(1+ny)^{\alpha+\varepsilon},$$

which is clearly implied by

$$e^{\gamma y} \ge C(ny)^{(\alpha-\delta)/2+\varepsilon}.$$

Since $(\alpha-\delta)/2 + \varepsilon < 1/6$ we are done if $e^{\gamma y} \ge C(ny)^{1/6}$. But this is valid when $y > 3\nu/2$. This concludes the proof of Lemma 5.4.

A final remark. It would be interesting to know if we could enlarge the range of the parameter a in Theorems 1.1 and 1.3 to all $a > -1$ or lower the index δ in Theorem 1.1 to $\delta > a + 1/2$.

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