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On the principle of local reflexivity

by

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Abstract. We prove a version of the local reflexivity theorem which is, in a sense, the most general one: our main theorem characterizes the conditions which can be imposed additionally on the usual local reflexivity map provided that these conditions are of a certain general type. It is then shown how known and new local reflexivity theorems can be derived. In particular, the compatibility of the local reflexivity map with subspaces and operators is investigated.

1. Introduction. The by now classical version of the local reflexivity theorem reads as follows:

1.1. THEOREM [11, 13]. *Let X be a Banach space, $E \subset X''$ and $F \subset X'$ finite-dimensional subspaces, and $\varepsilon > 0$. Then there is an isomorphism $T : E \to X$ such that*

(i) $\|T\|, \|T^{-1}\| \le 1 + \varepsilon$,
(ii) $x'(Tx'') = x''(x')$ *for* $x'' \in E$ *and* $x' \in F$,
(iii) $Tx'' = x''$ *for* $x'' \in E \cap X$.

New proofs have been given in [6] and [14], variants where it is shown that T may be assumed to satisfy certain additional conditions are studied in [2, 4, 5, 7, 8, 9, 12]. The applications of the local reflexivity theorem are abundant, and it is undoubtedly one of the most fundamental theorems in Banach space theory.

The aim of this paper is to state and prove a local reflexivity theorem which is in a sense the most far-reaching one (this will be made precise shortly).

At least formally all known local reflexivity results are covered by our main theorem, and we will indicate how some of them can be derived easily as corollaries. A systematic investigation of how to apply the new local reflexivity technics to situations where variants of the classical theorem have

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already been used successfully is not intended here. We preferred to provide some new results and to indicate some new applications.

In order to find generalizations of Theorem 1.1 one surely has to ask first: What are the properties the local reflexivity operator should always have and what kind of additional conditions can reasonably be expected? We will regard the properties 1.1(i) and (ii) as fundamental; 1.1(iii) will play a different role, and the reader is warned to check carefully the results to follow (is 1.1(iii) satisfied or not?) before they are applied to concrete situations.

As to the further conditions, we start with the observation that there are two types which cover all generalizations of Theorem 1.1 considered up to now, namely:

1. *Exact* conditions (T has to satisfy certain equations, e.g. 1.1(iii)).
2. Conditions which have to be satisfied *approximately* up to a prescribed but fixed accuracy.

Further, these generalizations show that:

— The conditions occurring in "1" are usually linear: they can be expressed by certain operator conditions as will be made precise in Definition 1.2 below. The only exception known to the author is Bernau's local reflexivity theorem for Banach lattices. Such situations seem to demand a different approach.

— Conditions of type "2" can be rephrased by demanding that T or a suitable operator image of T can be approximated arbitrarily well by elements of a certain convex set.

These observations give rise to the

1.2. Definition. Let X be a Banach space and $E \subset X''$ a finite-dimensional subspace.

(i) Let $F \subset X'$ be finite-dimensional and $\varepsilon > 0$. An isomorphism $T : E \to X$ satisfying 1.1(i) and (ii) will be called an *ε-isomorphism along F*.

(ii) Let $A_i : L(E,X) \to Y_i$ and $B_j : L(E,X) \to Z_j$ be linear and continuous operators for $i = 1,\ldots,n$ and $j = 1,\ldots,m$. Further, let $y_i \in Y_i$ and $C_j \subset Z_j$ be given such that the C_j are convex. E is said to satisfy

— the *exact conditions* (A_i, y_i), $i = 1,\ldots,n$, and
— the *approximate conditions* (B_j, C_j), $j = 1,\ldots,m$,

if for finite-dimensional $F \subset X'$ and $\varepsilon > 0$ there always exists an ε-isomorphism $T : E \to X$ along F such that $A_i(T) = y_i$ and $B_j(T) \in (C_j)_\varepsilon$ $(:= \{z \mid \operatorname{dist}(z, C_j) \le \varepsilon\})$ for all i, j.

Example. As an illustration consider Theorem 1.1. Its assertion can be rephrased by saying that, for arbitrary finite-dimensional $E \subset X''$, E satisfies the exact condition (A, y_0) with $A : L(E,X) \to L(E \cap X, X)$ the restriction operator and $y_0 =$ the identity from $E \cap X$ to X.

Conditions which can be exactly or approximately fulfilled will be characterized in the following section. This characterization will then be applied to derive known and new local reflexivity theorems concerned with certain subspace and operator compatibility conditions (see Sections 3 and 5).

It should be noted that the methods which lead to our general results can also be used to give a new elementary proof of the "classical" Theorem 1.1. This is sketched in Remark 2 following Theorem 2.3 (see also [3]).

2. The characterization theorem.

First we will concentrate on the simplest case, namely to characterize the property introduced in 1.2(ii) when E is one-dimensional. We will see that the case of arbitrary finite-dimensional E can be reduced to this situation by using a theorem of Dean's.

Also, to simplify notation, we will assume that there is only one exact and only one approximate condition under consideration. The case of arbitrary n and m in 1.2(ii) can always be rewritten in this way by introducing new operators $T \mapsto (A_i(T))$ and $T \mapsto (B_j(T))$ from $L(E,X)$ to $\prod Y_i$ and $\prod Z_j$, respectively.

Then 1.2(ii) is equivalent to (i) of the following theorem, which thus provides the desired characterization:

2.1. Theorem. *Let X, Y, Z be Banach spaces, $A : X \to Y$, $B : X \to Z$ linear and continuous operators, $x_0'' \in X''$ with $\|x_0''\| = 1$, $y_0 \in Y$, $C \subset Z$ convex. Then the following assertions are equivalent:*

(i) *For every finite-dimensional subspace $F \subset X'$ and every $\varepsilon > 0$ there is an $x_0 \in X$ such that*
— $(1+\varepsilon)^{-1} \le \|x_0\| \le 1+\varepsilon$,
— $x_0''(x') = x'(x_0)$ *for all* $x' \in F$,
— $Ax_0 = y_0$,
— $Bx_0 \in (C)_\varepsilon$.

(ii) *x_0'' is weak*-continuous on the weak*-closure of* range A', $A''x_0'' = y_0$, *and $B''x_0''$ lies in the weak*-closure of C (where C is considered as a subset of Z'').*

Proof. (i)⇒(ii). Let $y' \in Y'$ be arbitrary. An application of (i) with $F := \operatorname{lin}\{A'y'\}$ leads to

$$y'(y_0) = y'(Ax_0) = x_0''(y' \circ A) = (A''x_0'')(y'),$$

i.e. $y_0 = A''x_0''$. Similarly it is shown that $B''x_0''$ cannot be separated from the weak*-closure of C.

In order to prove that x_0'' is weak*-continuous on $(\operatorname{range} A')^{-\sigma^*}$ choose first any x_0 with $Ax_0 = y_0$ $(= A''x_0'')$. We claim that x_0'' and x_0 coincide on the weak*-closure of range A'. Let x' in this closure be given. If we apply (i) with $F = \text{lin}\{x'\}$ we obtain an $\widetilde{x}_0$ such that $A\widetilde{x}_0 = y_0$ and $x_0''(x') = x'(\widetilde{x}_0)$. x_0 and $\widetilde{x}_0$ coincide on range A' and thus at x', and we conclude that $x_0''(x') = x'(\widetilde{x}_0) = x'(x_0)$.

(ii)⇒(i). We will make use of the following elementary fact: If $F_0 \subset X'$ is a weak*-closed subspace and $x_0'' \in X''$ is weak*-continuous on F_0, then the same holds with F_0 replaced by $F_0 + F$ for any finite-dimensional $F \subset X'$.

First we will show: For $\varepsilon > 0$ and $\widetilde{F} \subset X'$ finite-dimensional there is an $x_0 \in X$ with $\|x_0\| \le 1 + \varepsilon$ such that $Ax_0 = y_0$ and $x'(x_0) = x_0''(x')$ for $x' \in \widetilde{F}$.

Let such $\widetilde{F}, \varepsilon$ be given. We write the weak*-closed space $(\text{range}\, A')^{-\sigma^*} + \widetilde{F}$ as the polar G° of a closed subspace G of X. Then, with ω_G = the canonical map from X to X/G, the weak*-continuity of x_0'' on G° just means that $\omega_G''(x_0'')$ lies in X/G, considered as a subspace of $(X/G)''$. Choose any x_0 such that $\omega_G(x_0) = \omega_G''(x_0'')$ and $\|x_0\| \le 1 + \varepsilon$ (which is possible since $\|\omega_G''(x_0''\| \le 1)$.

Now let $\widetilde{\widetilde{F}}$ be a fixed finite-dimensional subspace of X' and $\varepsilon > 0$. We define $C_0 \subset Z$ to be the set $B(D(\widetilde{\widetilde{F}}, \varepsilon))$, where $D(\widetilde{\widetilde{F}}, \varepsilon) := \{x_0 \mid Ax_0 = A''x_0'',\ \|x_0\| \le 1 + \varepsilon, \text{ and } x'(x_0) = x_0''(x') \text{ for } x' \in \widetilde{\widetilde{F}}\}$. C_0 is obviously convex, and by the first part of the proof it is nonempty.

We claim that C_0 meets $(C)_\varepsilon$. If this were not the case we could find a $z' \in Z'$ and numbers a, b such that $\text{Re}\, z''|_C \ge a \ge b \ge \text{Re}\, z'|_{C_0}$. The first part of the proof, this time applied to $\widetilde{F} := \widetilde{\widetilde{F}} + \text{lin}\{B'z''\}$, guarantees the existence of an x_0 with $Bx_0 \in C_0$ and $(B'z')(x_0) = x_0''(B'z')$. But this means that z' would strictly separate $B''x_0''$ from C contrary to $B''x_0'' \in C^{-\sigma^*}$.

It is now simple to prove (i). For given F, ε select $x_\varepsilon' \in X'$ with $\|x_\varepsilon'\| = 1$ and $|x_0''(x_\varepsilon')| \ge (1+\varepsilon)^{-1}$. With $\widetilde{\widetilde{F}} := F + \text{lin}\{x_\varepsilon'\}$ we choose any x_0 in $D(\widetilde{\widetilde{F}}, \varepsilon)$ with $Bx_0 \in (C)_\varepsilon$, which obviously has the claimed properties.

Remarks. 1) It is important to have weak*-continuity of x_0'' on $(\text{range}\, A')^{-\sigma^*}$ and not only on range A'. Admittedly this might be difficult to check. However, if A has a closed range, then range A' is weak*-closed by the closed range theorem, and it only has to be shown that $A''x_0'' = y_0$ and $A''x_0'' \in \text{range}\, A$.

2) As noted above the case of exact conditions (A_i, y_i) and approximate conditions (B_j, C_j) for $i = 1, \ldots, n$, $j = 1, \ldots, m$ can be reduced to the case $n = m = 1$. The characterization theorem when applied to the operators describing this compound situation yields that x_0 with $(1+\varepsilon)^{-1} \le \|x_0\| \le 1 + \varepsilon$, $A_i x_0 = y_i$, $B_j x_0 \in (C_j)_\varepsilon$, $x'(x_0) = x_0''(x_0')$ for $x' \in F$ can be found for arbitrary F, ε iff x_0'' is weak*-continuous on the weak*-closure of $\sum \text{range}\, A_i'$, and $A_i''x_0'' = y_i$, $B_j''x_0'' \in C_j^{-\sigma^*}$ for all i, j.

This reveals a *remarkable difference* between the exact and approximate conditions. Approximate conditions can be put arbitrarily together: as soon as $B''x_0'' \in C^{-\sigma^*}$ is established one may demand $B(x_0) \in (C)_\varepsilon$ additionally in every local reflexivity theorem. Exact conditions, on the other hand, have to be investigated simultaneously: if x_0'' is weak*-continuous on $(\text{range}\, A_i')^{-\sigma^*}$ for all i one may in general not conclude that x_0'' is weak*-continuous on $(\sum \text{range}\, A_i')^{-\sigma^*}$. (As an example consider two subspaces G_1, G_2 of X for which the norm closure of $G_1^\circ + G_2^\circ$ is not weak*-closed and a normalized x_0'' in $G_1^{\circ\circ} \cap G_2^{\circ\circ}$ such that $x_0''(x') \ne 0$ for a suitable x' in $(G_1^\circ + G_2^\circ)^{-\sigma^*}$.)

We now turn to the *general case*. Crucial is the observation (due to Dean [6]) that arbitrary local reflexivity theorems can be derived from one-dimensional ones.

For the sake of a self-contained treatment we repeat (with some simplifications) Dean's argument. Let X and E be Banach spaces, E finite-dimensional. For $e \in E$ and $x' \in X'$ denote by $e \otimes x'$ the functional $T \mapsto x'(Te)$ on $L(E, X)$.

2.2. Theorem [6]. *Define* $\varphi : L(E,X)'' \to L(E, X'')$ *by* $y'' \mapsto (e \mapsto (x' \mapsto y''(e \otimes x')))$. *Then* φ *is an isometric isomorphism onto* $L(E, X'')$.

Proof. The assertion, which is a special case of well-known results on tensor product duality, can be obtained in an elemantary way as follows (the idea is similar to that of Jarchow's proof; see [10], p. 388):

— prove that the assertion is true if E is an l_n^1-space and that consequently φ is always a surjective isomorphism:

— for given $\varepsilon > 0$, choose $e_1, \ldots, e_m$ in E such that the convex hull of $\{e_1, \ldots, e_m\}$ lies between $B_E := \{e \mid \|e\| \le 1\}$ and $(1+\varepsilon)B_E$ and consider $\varphi_X : T \mapsto (Te_j)$ from $L(E, X)$ to X^m (the product of m copies of X, provided with the supremum norm); then φ_X'' satisfies $\|y''\| \le \|\varphi_X''(y'')\| \le (1+\varepsilon)\|y''\|$ for all y'' since the corresponding inequalities hold for φ_X, and using (with a similarly defined $\varphi_{X''}$) the identity $\varphi_{X''} \circ \varphi = (\varphi_X)''$ it is simple to show that $\|y''\| \le (1+\varepsilon)\|\varphi(y'')\|$; thus $\|y''\| \le \|\varphi(y'')\|$, and "≥" is trivially satisfied.

Of particular importance is the case when $E \subset X''$. We will denote by I that element of $L(E, X)''$ which corresponds under φ to the identity from E to X'', i.e. I is that functional for which $I(x'' \otimes x') = x''(x')$ (for all $x'' \in E$, $x' \in X'$).

A combination of 2.1 with 2.2 leads to the following description of possible local reflexivity theorems:

2.3. Theorem. *Let* $E \subset X''$ *be finite-dimensional*, $A_i : L(E, X) \to Y_i$, $B_j : L(E, X) \to Z_j$ *linear and continuous operators*, $y_i \in Y_i$, $C_j \subset Z_j$

convex ($i = 1, \dots, n$, $j = 1, \dots, m$). *Then the following are equivalent:*

(i) *E satisfies the exact conditions (A_i, y_i) and the approximate conditions (B_j, C_j) for all i, j.*

(ii) *I is weak*-continuous on the weak*-closure of $\sum \operatorname{range} A_i'$, and $A_i''(I) = y_i$, $B_j''(I) \in C_j^{-\sigma^*}$ for all i, j.*

Proof. We will make use of Theorem 2.1 with X and x_0'' replaced by $L(E, X)$ and I, respectively.

(i)⇒(ii). This follows from 2.1(i)⇒(ii) since every finite-dimensional subspace of $L(E,X)'$ is contained in a space of the form $\operatorname{lin}\{x'' \otimes x' \mid x'' \in E,\ x' \in F\}$ with finite-dimensional $F \subset X'$.

(ii)⇒(i). Let $F \subset X'$ be finite-dimensional and $\varepsilon > 0$. We choose a finite-dimensional subspace $F_\varepsilon \subset X'$ such that for $T \in L(E, X)$ the conditions $x'(Tx'') = x''(x')$ (for all $x'' \in E$, $x' \in F_\varepsilon$) imply that T is one-to-one with $\|T^{-1}\| \le 1 + \varepsilon$ (simply choose an F_ε which is "nearly norming" for the $x'' \in E$).

Now consider $\widetilde{F} := \operatorname{lin}\{x'' \otimes x' \mid x'' \in E,\ x' \in F + F_\varepsilon\} \subset L(E, X)'$. By 2.1(ii)⇒(i) we find a T with $A_i(T) = y_i$, $B_j(T) \in (C_j)_\varepsilon$, $(1+\varepsilon)^{-1} \le \|T\| \le 1 + \varepsilon$, $y'(T) = I(y')$ for $y' \in \widetilde{F}$. But this just means that $x'(Tx'') = x''(x')$ for $x'' \in E$ and $x' \in F + F_\varepsilon$ so that T has the properties which are asked for in (i).

Remarks. 1) If the operator $T \mapsto (A_i(T))$ from $L(E, X)$ to $\prod Y_i$ has a closed range then (ii) can be replaced by

There is a $T \in L(E, X)$ such that $A_i(T) = y_i = A_i''(I)$ for all i, and $B_j''(I) \in C_j^{-\sigma^*}$ for all j.

2) As an illustration how to apply the preceding theorem we indicate how some known results can easily be derived.

a) Theorem 1.1 follows immediately: one only has to note that the restriction operator $A : L(E, X) \to L(E \cap X, X)$ is onto and that $A''(I)$—being the identity from $E \cap X$ to X—lies in $L(E \cap X, X)$ and thus in range A; therefore E satisfies the exact condition (A, Id), which, as has already been noted, is just the assertion of Theorem 1.1 (cf. also [3]).

b) With the usual notation suppose that $C_1, \dots, C_n$ are convex subsets of X and that $x_{ik}'' \in E \cap C_k^{\circ\circ}$ for $k = 1, \dots, n$ and $i = 1, \dots, n_k$. Then, for $\varepsilon > 0$, the local reflexivity operator T can be chosen such that in addition to the usual properties 1.1(i)–(iii) also $Tx_{ik}'' \in (C_k)_\varepsilon$ for all i, k. (Apply Theorem 2.3 with the exact condition as in a) and the approximate conditions given by the $B_{ik} : L(E, X) \to X$, $T \mapsto Tx_{ik}''$, and the C_k.)

This is just—for the case of Banach spaces—the local reflexivity theorem of Geĭler and Chuchaev ([9], Th. 1.2).

c) Let K be a cone in X. Then one can find T in Theorem 1.1 such that $d(Tx'', K) \le d(x'', K^{\circ\circ}) + \varepsilon\|x''\|$ for all $x'' \in E$. The reader is invited to find suitable x_i'' in E and convex sets C_i in X such that this result (which is just Theorem 1 in [12]) follows from 2.3 using $B_i : L(E, X) \to X$, $T \mapsto Tx_i''$, and (A, y_0) as in a).

d) Let $G \subset X$ be a closed subspace. Consider $A : L(E, X) \to L(E \cap G^{\circ\circ}, X/G)$, $T \mapsto \omega_G \circ T|_{E\cap G^{\circ\circ}}$, $y_0 = 0$.

Then it is easy to see that 2.3(ii) is satisfied for (A, y_0), i.e. 2.3 provides a T with 1.1(i), (ii) and $A(T) = 0$, i.e. $T(E \cap G^{\circ\circ}) \subset G$.

This is a special case of Bellenot's main result ([4]); we will discuss this kind of problem in detail in Section 3.

3) Let $B_j : L(E, X) \to Z_j$ be any operators and C_j the closed ball in Z_j with radius $\|B_j''(I)\|$ ($j = 1, \dots, m$). Then surely $B_j''(I) \in C_j^{-\sigma^*}$ and Theorem 2.3 implies that for any E satisfying certain exact and approximate conditions one may additionally demand that the local reflexivity operator T satisfies $\|B_j(T)\| \le \|B_j''(I)\| + \varepsilon$. $\|B_j(T)\| \ge \|B_j''(I)\| - \varepsilon$ can also be achieved by prescribing T on suitable $z_j' \circ B_j$.

3. Best possible functional conditions and subspace restrictions.

Let $E \subset X''$ be finite-dimensional. Up to now we have investigated the problem whether there are ε-isomorphisms T from E to X along F (which possibly satisfy certain other conditions) for $\varepsilon > 0$ and *finite-dimensional* F. Is it possible to have results for larger spaces F? More precisely:

(∗) Let $F_0 \subset X'$ be a (not necessarily finite-dimensional) subspace of X' and $x_0'' \in E$. Does there exist, for $\varepsilon > 0$ and $F \subset X'$ finite-dimensional, an ε-isomorphism T along F such that, in addition, $x'(Tx_0'') = x_0''(x')$ for all $x' \in F_0$?

Suppose that x_0'' and a space F_0 have this property. Then it is easy to see (by a similar argument to the one in the proof of Theorem 2.1) that x_0'' is weak*-continuous on the weak*-closure of F_0 and that $x'(Tx_0'') = x_0''(x')$ for all $x' \in F_0^{-\sigma^*}$. Thus we may assume without loss of generality that F_0 is the polar G° of a subspace G of X.

It will be convenient to introduce a notation for the collection of all $x'' \in X''$ which are weak*-continuous on G°: this collection will be denoted by $G^\wedge$ here. We note in passing that $G^\wedge$ is just the space $G^{\circ\circ} + X$ and that $G^\wedge$ can be written as $\omega_G''^{-1}(X/G)$, which implies that it is norm-closed ($\omega_G : X \to X/G$ means as before the canonical map).

Using this notation we arrive at the following problem, which in view of the preceding discussion can be thought of as a natural generalization of (∗):

PROBLEM A. Let $\mathcal{G}$ be a finite family of closed subspaces of X. Does there exist, for $\varepsilon > 0$ and $F \subset X'$ finite-dimensional, an ε-isomorphism T from E to X along F such that, in addition, T satisfies the best possible functional relations with respect to the G°, i.e. $x'(Tx'') = x''(x')$ for $G \in \mathcal{G}$, $x' \in G^\circ$, $x'' \in E \cap G^\wedge$?

There seem to be no systematic investigations of this problem in the literature.

We now turn to *another problem* which looks to be completely different from Problem A. Let $G \subset X$ be a closed subspace and $x_0'' \in E \subset X''$. Can the local reflexivity map T be chosen such that $Tx_0'' \in G$? A moment's reflection shows that this is possible only if $x_0'' \in E \cap G^{\circ\circ}$, i.e. the problem is whether or not $T(E \cap G^{\circ\circ}) \subset G$ can be achieved. In the case of several subspaces this leads to

PROBLEM B. Let $\mathcal{G}$ be a finite family of closed subspaces of X. Does there exist for $\varepsilon > 0$ and $F \subset X'$ finite-dimensional an ε-isomorphism T from E to X along F such that $T(E \cap G^{\circ\circ}) \subset G$ for all $G \in \mathcal{G}$?

This has first been studied by Bellenot in [4] who not only provided some sufficient conditions but also pointed out that the property $T|_{E\cap X} = \mathrm{Id}$ (i.e. 1.1(iii)) of the local reflexivity map can be regarded as a special case of Problem B (if one puts $G = E \cap X$, then $T(E \cap G^{\circ\circ}) \subset G$ implies that $T|_{E\cap X} = \mathrm{Id}$ provided that one knows—which by a suitable enlarging can always be achieved—that F separates the points of $E \cap X$).

It is now a fundamental but completely elementary observation that a family $\mathcal{G}$ for which Problem A has a positive solution is also appropriate for Problem B (this follows from $G^{\circ\circ} \subset G^\wedge$ and the bipolar theorem). Therefore we may and will concentrate mainly on Problem A.

Let $\mathcal{G}$ be a fixed finite family of closed subspaces of X. Depending on E our problems will have positive solutions or not, and it is not hard to rephrase them as exact conditions and also to derive from Theorem 2.3 a characterization in terms of the weak*-continuity of I on a certain subspace of $L(E,X)'$. This characterization, however, is of little use in most concrete situations, and hence we will restrict ourselves to the investigation of certain sufficient conditions which can be easily checked in many cases.

3.1. DEFINITION. (i) Let $M = (a_{ji})$ be an $m \times n$-matrix with scalar entries a_{ji}. M induces an operator $(x_i) \mapsto (\sum_i a_{ji}x_i)$ from X^n to X^m which will be denoted by A_M (or A_M^X if X is not clear from the context).

(ii) A finite family $\mathcal{G}$ of closed subspaces of X will be called *matrix-closed* if $A_M(\prod G_i)$ is closed in X^m for every family $G_1, \ldots, G_n$ of (not necessarily distinct) members of $\mathcal{G}$ and every $m \times n$-matrix M.

Properties of matrix-closed families and *examples* will be discussed in

Section 4. Here we will use these results to prove our main theorem: For such families Problems A and B have an affirmative answer, and the local reflexivity map can be chosen such that it fixes the $x \in E \cap X$:

3.2. THEOREM. *Let $\mathcal{G}$ be a matrix-closed family* (for examples cf. Prop. 4.4 below). *Then for finite-dimensional subspaces $E \subset X''$ and $F \subset X'$ and $\varepsilon > 0$ there exists an ε-isomorphism $T : E \to X$ along F such that $T|_{E\cap X} = \mathrm{Id}$, $T(E \cap G^{\circ\circ}) \subset G$ for all G in $\mathcal{G}$, $(Tx'')(x') = x''(x')$ for $G \in \mathcal{G}$, $x'' \in E \cap G^\wedge$, $x' \in G^\circ$.*

Proof. The family $(\mathcal{G})_{\text{saturated}} := \{G_1 \cap \ldots \cap G_n \mid G_i \in \mathcal{G} \cup \{E \cap X\}\}$ is also matrix-closed by Prop. 4.4(ii), (iii), and therefore we may without loss of generality assume that $\mathcal{G} = (\mathcal{G})_{\text{saturated}}$. Thus—as the discussion following Problem B showed—it suffices to find an ε-isomorphism T along F such that

$$(*) \qquad (Tx'')(x') = x''(x') \quad \text{for } G \in \mathcal{G}, x'' \in E \cap G^\wedge, x' \in G^\circ.$$

Now fix a finite-dimensional subspace E of X''. As a preparation for the proof we rewrite $(*)$ as a suitable exact condition.

We noted already that $G^\wedge = \omega_G''^{-1}(X/G)$ so that ω_G'' maps $E \cap G^\wedge$ into X/G. Let $I_G \in L(E\cap G^\wedge, X/G)$ be the restriction of ω_G'' to $E \cap G^\wedge$. Further, define $A_G : L(E,X) \to L(E \cap G^\wedge, X/G)$ by $T \mapsto (\omega_G \circ T)|_{E\cap G^\wedge}$. It is then easy to check that $(*)$ is equivalent to

$$A_G(T) = I_G \quad (\text{for } G \in \mathcal{G}),$$

i.e. we will have to show that E satisfies the exact condition $(A_\mathcal{G}, I_\mathcal{G})$, where $A_\mathcal{G} : L(E,X) \to \prod_G L(E \cap G^\wedge, X/G)$ is defined by $T \mapsto (A_G(T))$, and $I_\mathcal{G} := (I_G)$.

By the remark following Theorem 2.3 it suffices to prove that

1° $(A_\mathcal{G})''(I) = I_\mathcal{G}$.

2° $I_\mathcal{G}$ lies in the range of $A_\mathcal{G}$.

3° $A_\mathcal{G}$ has a closed range.

The easy and canonical proof of 1° is omitted here. Before we are going to treat 2° and 3° we choose vectors $x_1'', \ldots, x_n'' \in E$ and an $m \times n$-matrix M such that the following holds:

a) $\{x_i'' \mid x_i'' \in E \cap G^\wedge\}$ spans $E \cap G^\wedge$ for all G,

b) $A_M^{X''}(x_1'', \ldots, x_n'') = 0$,

c) whenever $x_1, \ldots, x_n \in X$ are given such that $A_M(x_1, \ldots, x_n) = 0$ there is a $T \in L(E,X)$ with $Tx_i'' = x_i$ for $i = 1, \ldots, n$.

(Such x_i'' and M always exist: Choose arbitrary $x_1'', \ldots, x_n''$ in E which satisfy a) and assume that, without loss of generality, $x_1'', \ldots, x_r''$ is a basis for $\mathrm{lin}\{x_1'', \ldots, x_n''\}$. Write, for $j = r+1, \ldots, n$, $x_j'' = b_{j1}x_1'' + \ldots + b_{jr}x_r''$ and define the $(n-r)\times n$-matrix $M=(a_{ji})$ by $a_{ji}:=b_{j+r,i}$ if $i\le r$, $a_{j,r+j}=-1$, and all other $a_{ji}=0$.)

Proof of 2°. It has to be shown that there is a $T\in L(E,X)$ for which $(*)$ is satisfied.

Define, for $i=1,\ldots,n$,

$$G_i:=\bigcap\{G\mid x_i''\in G^\wedge\}.$$

Since $\mathcal{G}$ coincides with $(\mathcal{G})_{\text{saturated}}$ we have $G_i\in\mathcal{G}$, and 4.1(iii) implies that $x_i''\in G_i^\wedge$. Thus there are $x_i\in X$ with $x_i''(x')=x'(x_i)$ for $x'\in G_i^\circ$.

We claim that $A_M(x_1,\ldots,x_n)$ lies in the closure of $A_M(\prod G_i)$. Suppose that this were not the case. We could then find, by the separation theorem, functionals $x_1',\ldots,x_m'\in X'$ such that $(x_1',\ldots,x_m')$ annihilates $A_M(\prod G_i)$ but not $A_M(x_1,\ldots,x_n)$. It follows that $\sum_{j,i}a_{ji}x_j'(g_i)=0$ for arbitrary $g_i\in G_i$ so that $\sum_j a_{ji}x_j'$ lies in G_i° for $i=1,\ldots,n$.

Therefore $(\sum a_{ji}x_j')(x_i)=x_i''(\sum a_{ji}x_j')$ and hence $\sum a_{ji}x_j'(x_i)=\sum a_{ji}x_i''(x_j')$. The left hand side is different from zero (it is just the value of $(x_1',\ldots,x_m')$ at $A_M(x_1,\ldots,x_n)$), but the right hand side, which is just $A_M^{X''}(x_1'',\ldots,x_n'')(x_1',\ldots,x_m')$, vanishes. This contradiction proves our claim.

Since $\mathcal{G}$ is matrix-closed we find $g_1,\ldots,g_n$ in $\prod G_i$ with $A_M(x_1,\ldots,x_n)=A_M(g_1,\ldots,g_n)$. But then $A_M(x_1-g_1,\ldots,x_n-g_n)=0$, and by c) there is an operator $T\in L(E,X)$ with $Tx_i''=x_i-g_i$. We have to show that T satisfies $(*)$.

Let $G\in\mathcal{G}$ and $x'\in G^\circ$ be given and consider any i such that $x_i''\in G^\wedge$. Then $G_i\subset G$ by definition, and this yields $x'\in G_i^\circ$ and thus $x'(Tx_i'')=x_i''(x')$, Consequently $\{x''\mid x''\in E\cap G^\wedge,\ x''(x')=x'(Tx'')\text{ for all }x'\in G^\circ\}$ contains all x_i'' with $x_i''\in E\cap G^\wedge$. By a) this space is all of $E\cap G^\wedge$, which is just what has been claimed for T.

Proof of 3°. We will need the following

LEMMA. *Denote, for $G,\widetilde G\in\mathcal{G}$ with $G\subset\widetilde G$, by $\omega_{G\widetilde G}:X/G\to X/\widetilde G$ the canonical map (so that $\omega_{G\widetilde G}\circ\omega_G=\omega_{\widetilde G}$). Then for $(S_G)\in(\operatorname{range}A_{\mathcal{G}})^-$ and $G,\widetilde G\in\mathcal{G}$ with $G\subset\widetilde G$, one has*

$$\omega_{G\widetilde G}\circ S_G=S_{\widetilde G}|_{E\cap G^\wedge}.$$

Proof of the lemma. Let $x''\in E\cap G^\wedge$ and $\varepsilon>0$. By assumption there is a $T\in L(E,X)$ such that $\|\omega_GTx''-S_Gx''\|,\ \|\omega_{\widetilde G}Tx''-S_{\widetilde G}x''\|\le\varepsilon\|x''\|$. Since $\omega_{G\widetilde G}\omega_GT=\omega_{\widetilde G}T$ it follows that $\|\omega_{G\widetilde G}S_Gx''-S_{\widetilde G}x''\|\le2\varepsilon\|x''\|$. Hence $\omega_{G\widetilde G}S_Gx''=S_{\widetilde G}x''$.

Now let $(S_G)\in(\operatorname{range}A_{\mathcal{G}})^-$ be given. We will show that there is a $T\in L(E,X)$ such that $(S_G)=A_{\mathcal{G}}(T)$.

With G_i as in the proof of 2° we choose $x_i\in X$ with $\omega_{G_i}(x_i)=S_{G_i}(x'')$. We claim that $A_M(x_1,\ldots,x_n)\in[A_M(\prod G_i)]^-$. Let $\varepsilon>0$ be given. By assumption there is a $T_\varepsilon\in L(E,X)$ such that $\|\omega_G\circ T_\varepsilon|_{E\cap G^\wedge}-S_G\|\le\varepsilon$ for all G, and in particular $\|\omega_{G_i}T_\varepsilon x_i''-\omega_{G_i}(x_i)\|\le K\varepsilon$ ($K:=\max\|x_i''\|$). Choose $g_i\in G_i$ such that $\|T_\varepsilon x_i''-x_i+g_i\|\le2K\varepsilon$. We have $A_M^{X''}(x_1'',\ldots,x_n'')=0$, hence $A_M(T_\varepsilon x_1'',\ldots,T_\varepsilon x_n'')=0$, and this implies that $\|A_M(x_1,\ldots,x_n)-A_M(g_1,\ldots,g_n)\|\le2K\|A_M\|\varepsilon$.

Since $\mathcal{G}$ is matrix-closed we may write $A_M(x_1,\ldots,x_n)$ as $A_M(g_1,\ldots,g_n)$ for suitable $g_i\in G_i$. By c) we get a $T\in L(E,X)$ with $Tx_i''=x_i-g_i$, and it remains to show that $A_G(T)=S_G$ for every G.

Let $G\in\mathcal{G}$ be arbitrary and x_i'' such that $x_i''\in G^\wedge$. Then $G_i\subset G$ by definition, and the lemma yields

$$S_Gx_i''=\omega_{G_iG}S_{G_i}x_i''=\omega_{G_iG}\omega_{G_i}Tx_i''=(\omega_G\circ T)(x_i'').$$

Hence S_G and ω_GT coincide on $\{x_i''\mid x_i''\in E\cap G^\wedge\}$ so that, by a), $S_G=\omega_G|_{E\cap G^\wedge}$. This proves that $A_{\mathcal{G}}(T)=(S_G)$.

Remarks. 1) The proof shows that in order to have

$$x'(Tx'')=x''(x')\quad(G\in\mathcal{G},x''\in E\cap G^\wedge,x'\in G^\circ)$$

for any not necessarily matrix-closed family $\mathcal{G}$ and any particular finite-dimensional $E\subset X''$ one only has to know that $\{E\cap G^\wedge\mid G\in\mathcal{G}\}$ is closed with respect to intersections and that there exist suitable $x_1'',\ldots,x_n''$ and a matrix M such that a)–c) are satisfied and $A_M(\prod G_i)$ is closed for this particular M. In some cases one can choose $M=0$ for arbitrary E; this is the idea which is behind Bellenot's "friendly collections" [4] (there, however, only the $G^{\circ\circ}$ and not the $G^\wedge$ are treated).

As a special case consider two subspaces G_1,G_2 such that

$$(*)^\wedge\qquad(G_1\cap G_2)^\wedge=G_1^\wedge\cap G_2^\wedge.$$

(This is true, e.g., if G_1+G_2 is closed.) Then, for arbitrary E, we may always find a basis $x_1'',\ldots,x_n''$ of E such that the x_i'' satisfy a) for $\mathcal{G}=\{G_1,G_2,G_1\cap G_2\}$. We may choose $M=0$ in the preceding proof, and we arrive at a local reflexivity theorem which asserts the existence of ε-isomorphisms $T:E\to X$ along F with $x'(Tx'')=x''(x')$ for $x''\in E\cap G_i^\wedge$, $x'\in G_i^\wedge$, $i=1,2$ (E, F finite-dimensional, $\varepsilon>0$). Similarly, if G_1,G_2 satisfy

$$(*)^{\circ\circ}\qquad(G_1\cap G_2)^{\circ\circ}=G_1^{\circ\circ}\cap G_2^{\circ\circ}$$

we can, by a similar argument, guarantee the existence of a local reflexivity map T (an ε-isomorphism along F) with $T(E\cap G_i^{\circ\circ})\subset G_i$ for $i=1,2$.

Note, however, that in both cases we do not claim that T can be chosen such that $T|_{E\cap X}=\mathrm{Id}$ (cf. the following discussion).

2) The preceding theorem provides conditions under which the $x''\in E\cap G^{\circ\circ}$ are mapped by T *exactly* into G. It should be noted that it is *always* possible to map these x'' *arbitrary close* to G. It suffices to define B_G from $L(E,X)$ to $L(E\cap G^{\circ\circ},X/G)$ by $T\mapsto\omega_GT|_{E\cap G^{\circ\circ}}$ and to observe

that $B''_G(I) = 0$. Hence, by the remark at the end of the preceding section we find the local reflexivity map T such that $\|B_G(T)\| \le \varepsilon$, which just means that the $x'' \in E \cap G^{\circ\circ}$ are mapped to points which are "close" to G; this can be achieved simultaneously for finitely many closed subspaces G and in addition to other admissible exact and approximate conditions. Note that this also follows from ([9], Th. 1.20) or ([12], Th. 1); cf. the notes following Theorem 2.3.

3) Let G be a closed subspace of X such that G° has a complementary subspace Y with $\|x' + y'\| = \|x'\| + \|y'\|$ for $x' \in G^\circ$, $y' \in Y$. Then, as can easily be verified, $G^{\circ\circ} + Y^\circ = X''$ with $\|x'' + y''\| = \max\{\|x''\|, \|y''\|\}$ ($x'' \in G^{\circ\circ}, y'' \in Y^\circ$).

Now let $x_1, \ldots, x_n \in X$ and $r_1, \ldots, r_n > 0$ be given such that there are $x \in X$, $g_j \in G$ with $\|x - x_j\|, \|g_j - x_j\| < r_j$ for all j. From the above norm condition it follows immediately that there is an $x'' \in G^{\circ\circ}$ with $\|x'' - x_j\| < r_j$, and Th. 3.2 provides a $g \in G$ with $\|g - x_j\| < r_j$.

This seems to be the by far shortest proof of one of the fundamental results in M-structure theory (see [1], Th. 2.17, a$\Rightarrow$c).

We are now going to discuss a little bit more carefully the role of the condition $T|_{E\cap X} = \mathrm{Id}$. This is an exact condition, and Theorem 2.3 indicates that one should not investigate it separately from other exact conditions. However, it is not clear up to now whether or not caution is really necessary for this particular exact condition. To state it otherwise: Can there be a general theorem asserting that E satisfies the exact conditions (A_i, y_i) together with $T|_{E\cap X} = \mathrm{Id}$ once it satisfies the (A_i, y_i)? We will show that the answer is in the negative, and this fact makes it clear why we preferred to regard the property $T|_{E\cap X} = \mathrm{Id}$ (i.e. 1.1(iii)) as less fundamental than 1.1(i) and (ii) for general local reflexivity theorems.

We will concentrate our discussion on exact conditions arising from Problem B which are easier to handle than those coming from Problem A. It is convenient to introduce

3.3. Definition. Let $\mathcal{G}$ be a finite family of closed subspaces of X. We will say that $\mathcal{G}$ is *admissible for the local reflexivity theorem* if for $E \subset X''$, $F \subset X'$ finite-dimensional and $\varepsilon > 0$ there is an ε-isomorphism $T : E \to X$ along F such that $T(E \cap G^{\circ\circ}) \subset G$ for all $G \in \mathcal{G}$.

$\mathcal{G}$ will be called *strongly admissible* if the operator T can be chosen such that in addition $T|_{E\cap X} = \mathrm{Id}$.

By the preceding results it is clear that $\mathcal{G}$ is strongly admissible iff $\mathcal{G} \cup \{E \cap X\}$ is admissible, Theorem 3.2 implies that matrix-closed families are strongly admissible and by the remark following this theorem families $\{G_1, G_2\}$ are admissible provided that $(G_1 \cap G_2)^{\circ\circ} = G_1^{\circ\circ} \cap G_2^{\circ\circ}$. This is a necessary condition as the following lemma shows:

3.4. Lemma. (i) *Let $\mathcal{G}$ be admissible. Then $(G_1 \cap \ldots \cap G_n)^{\circ\circ} = G_1^{\circ\circ} \cap \ldots \cap G_n^{\circ\circ}$ for $G_1, \ldots, G_n \in \mathcal{G}$.*

(ii) *Let $\mathcal{G}$ be strongly admissible and M any $m \times n$-matrix. Then $A_M^{X''}(\prod G_i^{\circ\circ}) \cap X^m \subset A_M(\prod G_i)$ for $G_1, \ldots, G_n \in \mathcal{G}$, i.e. the $G^{\circ\circ}$ "mimic" the $G \in \mathcal{G}$ when systems of linear equations with the right hand side in X are concerned.*

Proof. (i) Let $G_1, \ldots, G_n \in \mathcal{G}$ be given. Suppose that the norm closure of $G_1^\circ + \ldots + G_n^\circ$ were a proper subset of $(G_1 \cap \ldots \cap G_n)^\circ$. Then it is possible to choose $x' \in (G_1 \cap \ldots \cap G_n)^\circ$ and $x'' \in (G_1^\circ + \ldots + G_n^\circ)^\circ \subset G_1^{\circ\circ} \cap \ldots \cap G_n^{\circ\circ}$ with $x''(x') = 1$, and our assumption would provide an x $(= Tx'')$ with $x \in G_1 \cap \ldots \cap G_n$, $x'(x) = 1$. This contradicts $x' \in (G_1 \cap \ldots \cap G_n)^\circ$, and consequently we have shown that $(G_1^\circ + \ldots + G_n^\circ)^- = (G_1 \cap \ldots \cap G_n)^\circ$. From this one sees at once that $G_1^{\circ\circ} \cap \ldots \cap G_n^{\circ\circ} \subset (G_1 \cap \ldots \cap G_n)^{\circ\circ}$, and "$\subset$" is always valid.

(ii) Let $M = (a_{ji})$ be any $m \times n$-matrix, $G_1, \ldots, G_n \in \mathcal{G}$, and $x_i'' \in G_i^{\circ\circ}$ such that $x_j := \sum a_{ji} x_i'' \in X$ for all j. Define E to be the span of $x_1, \ldots, x_m, x_1'', \ldots, x_n''$ and choose T according to the assumption. Then $x_j = Tx_j = \sum a_{ji} T x_i''$, i.e. with $g_i := Tx_i''$ we have $(x_1, \ldots, x_m) = A_M(g_1, \ldots, g_n)$.

We are now going to present an example of a family $\mathcal{G}$ which is admissible but not strongly admissible. This example contains only two subspaces and thus represents the simplest case where counterexamples are possible (note that families $\mathcal{G} = \{G\}$ are matrix-closed and thus strongly admissible).

3.5. Proposition. *Let $X = c_0$, $G_1 = \{(x_i) \mid x_i = x_{i+1}$ for $i = 1, 3, 5, \ldots\}$, $G_2 := \{(x_i) \mid x_1 = 0,\ x_i = x_{i+1}$ for $i = 2, 4, 6, \ldots\}$, $\mathcal{G} = \{G_1, G_2\}$. Then $\mathcal{G}$ is admissible but not strongly admissible.*

Proof. We have $G_1 \cap G_2 = 0$ and $G_1^{\circ\circ} = \{(t_i) \mid (t_i) \in l^\infty,\ t_i = t_{i+1}$ for $i = 1, 3, 5, \ldots\}$, $G_2^{\circ\circ} = \{(t_i) \mid (t_i) \in l^\infty,\ t_1 = 0,\ t_i = t_{i+1}$ for $i = 2, 4, 6, \ldots\}$ so that $(G_1 \cap G_2)^{\circ\circ} = G_1^{\circ\circ} \cap G_2^{\circ\circ}$. Thus $\mathcal{G}$ is admissible by Remark 1 following Theorem 3.2. However, with $e_1 = (\delta_{1i}) \in c_0$ we have $e_1 \in G_1^{\circ\circ} + G_2^{\circ\circ}$, but $e_1 \notin G_1 + G_2$, and consequently our family cannot be strongly admissible by Lemma 3.4(iii).

4. Matrix-closed families.

These families have been introduced in Definition 3.1. In the present section we study their properties (some of which have been of importance in Section 3) and describe classes of examples in order to be able to apply Theorem 3.2.

Properties

4.1. Proposition. *Let $\mathcal{G}$ be a matrix-closed family and $G_1, \ldots, G_n \in \mathcal{G}$. Then*

(i) $(G_1 \cap \ldots \cap G_n)^\circ = G_1^\circ + \ldots + G_n^\circ$,
(ii) $(G_1 \cap \ldots \cap G_n)^{\circ\circ} = G^{\circ\circ} \cap \ldots \cap G_n^{\circ\circ}$,
(iii) $(G_1 \cap \ldots \cap G_n)^\wedge = G_1^\wedge \cap \ldots \cap G_n^\wedge$.

Proof. (i) Consider the operator $S : X \to \prod X/G_i$, $x \mapsto (\omega_{G_i}(x))$. We will show that S has a closed range; then, by the closed range theorem, range $S' = G_1^\circ + \ldots + G_n^\circ$ will be the annihilator of $\ker S = G_1 \cap \ldots \cap G_n$.

Let $(\omega_{G_i}(x_i)) \in (\text{range}\, S)^-$ be given. For $\varepsilon > 0$ we choose $x \in X$ and $g_i \in G_i$ with $\|x - x_i + g_i\| \le \varepsilon$. Then $\|(x_1 - x_i) - (g_1 - g_i)\| \le 2\varepsilon$ for $i = 2, \ldots, n$. We thus have shown that, with $M = (a_{ji})$ defined by

$$a_{j1} = 1, \quad a_{j,j+1} = -1, \quad \text{all other } a_{ji} = 0 \qquad (j = 1, \ldots, n-1,\ i = 1, \ldots, n),$$

$A_M(x_1, \ldots, x_n)$ lies in the closure of $A_M(\prod G_i)$. Since $\mathcal{G}$ is matrix-closed there are $g_1, \ldots, g_n$ in $G_1, \ldots, G_n$, respectively, with $A_M(x_1, \ldots, x_n) = A_M(g_1, \ldots, g_n)$. It follows that $x := x_1 - g_1 = x_2 - g_2 = \ldots = x_n - g_n$ so that $(\omega_{G_i}(x_i)) = Sx$ lies in range S.

(ii) This is an immediate consequence of (i).

(iii) Let $x'' \in G_1^\wedge \cap \ldots \cap G_n^\wedge$. By assumption there are $x_1, \ldots, x_n \in X$ such that x'' coincides with x_i on G_i° for $i = 1, \ldots, n$. We claim that $A_M(x_1, \ldots, x_n)$ lies in the closure of $A_M(\prod G_i)$ (where M is as in the proof of (i)). Suppose that this were not the case. Then we find $x_1', \ldots, x_{n-1}'$ in X such that $(x_1', \ldots, x_{n-1}')(A_M(x_1, \ldots, x_n)) = 1$, but $(x_1', \ldots, x_{n-1}')(A_M(g_1, \ldots, g_n)) = 0$ whenever $g_i \in G_i$. This just means that

$$x_1'(x_1 - x_2) + \ldots + x_{n-1}'(x_1 - x_n) = 1,$$
$$x_1'(g_1 - g_2) + \ldots + x_{n-1}'(g_1 - g_n) = 0 \quad \text{for } g_i \in G_i.$$

Hence $x_1' + \ldots + x_{n-1}' \in G_1^\circ$, $x_j' \in G_{j+1}^\circ$ for $j = 1, \ldots, n-1$, and we conclude that $(x_1' + \ldots + x_{n-1}')(x_1) = x''(x_1' + \ldots + x_{n-1}')$, $x_j'(x_{j+1}) = x''(x_j')$ for $j = 1, \ldots, n-1$. But this leads to $x_1'(x_1 - x_2) + \ldots + x_{n-1}'(x_1 - x_n) = 0$, a contradiction.

Since our family is matrix-closed we may choose $g_1, \ldots, g_n$ with $A_M(x_1, \ldots, x_n) = A_M(g_1, \ldots, g_n)$. Then $x := x_1 - g_1 = x_2 - g_2 = \ldots = x_n - g_n$ satisfies $x''(x') = x'(x)$ for $x' \in G_1^\circ + \ldots + G_n^\circ$ $(= (G_1 \cap \ldots \cap G_n)^\circ$ by (i)), and this proves that $x'' \in (G_1 \cap \ldots \cap G_n)^\wedge$.

The inclusion "$\subset$" is trivially satisfied.

Examples. Let $G_1, \ldots, G_n$ be closed subspaces of X and M an $m \times n$-matrix. We have to investigate whether $A_M(\prod G_i)$ is closed in X^m. The following elementary observation will be useful:

4.2. Lemma. *Let $A_1 : X^n \to X^n$ and $A_2 : X^m \to X^m$ be surjective isomorphisms such that A_1 and A_1^{-1} leave $\prod G_i$ invariant. Then $A_M(\prod G_i)$ is closed provided that $(A_2 A_M A_1)(\prod G_i)$ is closed.*

4.3. Lemma. *Let $G_1, \ldots, G_{n-1}$ be such that $A_M(\prod_{i=1}^{n-1} G_i)$ is closed for every $m \times (n-1)$-matrix M. Suppose that G_n is another closed subspace such that $G_i \subset G_n$ for $i = 1, \ldots, n-1$. Then $A_M(\prod G_i)$ is closed for all $m \times n$-matrices M.*

Proof. Let $M = (a_{ji})$ be an $m \times n$-matrix. If all a_{jn} vanish nothing has to be shown. So suppose that, without loss of generality, $a_{mn} = 1$. Consider

— the $m \times n$-matrix $M_1 = (b_{ji})$, where $b_{ii} = 1$ for $i = 1, \ldots, n$, $b_{ni} = -a_{mi}$ for $i = 1, \ldots, n-1$, all other $b_{ji} = 0$;
— the $m \times n$-matrix $M_2 = (d_{ji})$, where $d_{jj} = 1$ for $j = 1, \ldots, m$, $d_{jm} = -c_{jn}$ for $j = 1, \ldots, m-1$, all other $d_{ji} = 0$ (here c_{ji} are the entries of MM_1).

Then $M_2 M M_1 = (e_{ij})$ is an $m \times n$-matrix for which $e_{mn} = 1$ and $e_{mi} = e_{jn} = 0$ for $i = 1, \ldots, n-1$, $j = 1, \ldots, m-1$. Hence $A_{M_2 M M_1}(\prod G_i)$ is closed as the direct product of $A_{\widetilde{M}}(\prod_{i=1}^{n-1} G_i)$ with G_n, where $\widetilde{M} = (e_{ji})_{j=1,\ldots,m-1, i=1,\ldots,n-1}$. M_1 and M_1^{-1} are matrices which differ from the identity matrix only (possibly) in the mth row. Hence, since G_n contains all G_i, the associated operators A_{M_1} and $(A_{M_1})^{-1}$ leave $\prod G_i$ invariant, and by 4.2 it follows that $A_M(\prod G_i)$ is closed.

4.4. Proposition. (i) *Let $H_1, \ldots, H_r$ be closed subspaces such that $H_1 \subset \ldots \subset H_r$. Then $\{H_1, \ldots, H_r\}$ is matrix-closed.*

(ii) *Let $\mathcal{G}$ be matrix-closed and G a finite-dimensional subspace of X. Then $\mathcal{G} \cup \{G\}$ is also matrix-closed.*

(iii) *If $\mathcal{G}$ is matrix-closed then so is*

$$\{G_1 \cap \ldots \cap G_n \mid n \in \mathbb{N},\ G_1, \ldots, G_n \in \mathcal{G}\}.$$

(iv) *Let $H_1, \ldots, H_r$ be closed subspaces such that the sum $H_1 + \ldots + H_r$ is a direct sum. Further, let $\mathcal{G}_\varrho$ $(\varrho = 1, \ldots, r)$ be matrix-closed families with $G_\varrho \subset H_\varrho$ for $G_\varrho \in \mathcal{G}_\varrho$. Then $\bigcup \mathcal{G}_\varrho$ is matrix-closed.*

(v) *With $\mathcal{G}$ also $\{G_1 + \ldots + G_n \mid n \in \mathbb{N},\ G_1, \ldots, G_n \in \mathcal{G}\}$ is matrix-closed.*

(vi) *Let $H_1, \ldots, H_r$ be as in* (iv). *Then $\{\sum_{i \in \Delta} H_i \mid \Delta \subset \{1, \ldots, r\}\}$ is matrix-closed.*

(vii) *A family $\mathcal{G}$ is matrix-closed iff $\{G^{\circ\circ} \mid G \in \mathcal{G}\}$ is matrix-closed.*

(viii) *Every finite family of M-ideals is matrix-closed* (for definitions see [1]; here we only note that two-sided closed ideals in C^*-algebras are M-ideals).

Proof. (i) This follows at once from the preceding lemma.

(ii) Let $G_1, \ldots, G_n$ be in $\mathcal{G} \cup \{G\}$, where $G_1 = G_2 = \ldots = G_r = G$ and $G_{r+1}, \ldots, G_n \in \mathcal{G}$. We write $\prod G_i$ as the direct sum of $H_1 := \prod_{i=1}^r G_i$ and $H_2 := \prod_{i=r+1}^n G_i$. Then, for an $m \times n$-matrix M, $A_M(\prod G_i)$ has the form $A_{M_1}(H_1) + A_{M_2}(H_2)$ for suitable $m \times r$- and $m \times (n-r)$-matrices M_1 and M_2.

But $A_{M_2}(H_2)$ is closed by assumption, and $A_{M_1}(H_1)$ is finite-dimensional. This proves that $A_M(\prod G_i)$ is closed.

(iii) We indicate the idea by showing that $A_M(\prod G_i)$ is always closed for $G_1,\ldots,G_{n-1}\in\mathcal{G}$, $G_n=G_1\cap G_2$. Let any $m\times n$-matrix $M=(a_{ji})$ be given. We define the spaces $H_1,\ldots,H_{n+1}$ to be just the $G_1,\ldots,G_{n-1},G_1,G_2$ and an $(m+1)\times(n+1)$-matrix $M=(b_{ji})$ by

$$b_{ji}=a_{ji}\quad\text{for } j=1,\ldots,m,\ i=1,\ldots,n,$$
$$b_{m+1,n}=1,\qquad b_{m+1,n+1}=-1,$$
$$b_{m+1,i}=b_{j,n+1}=0\quad\text{for } i=1,\ldots,n-1,\ j=1,\ldots,m.$$

Then $A_M(\prod H_i)$ is closed in X^{m+1} by assumption. Hence $A_M(\prod H_i)\cap(X^m\times\{0\})$ is also closed, and this space can be identified with $A_M(\prod G_i)$.

(iv) and (v) are easily verified, and (vi) is an immediate consequence of these assertions.

(vii) follows from the closed range theorem and the fact that the bidual of $\prod G_i$ can be identified with $\prod G_i^{\circ\circ}$.

(viii) For M-ideals $J_1,\ldots,J_n$ in X the bipolars $J_1^{\circ\circ},\ldots,J_n^{\circ\circ}$ are M-summands in X''. Thus, since the M-summands form a Boolean algebra, we find M-summands $H_1,\ldots,H_r$ in X'' such that $\{J_i^{\circ\circ}\mid i=1,\ldots,n\}\subset\{\sum_{i\in\Delta}H_i\mid\Delta\subset\{1,\ldots,r\}\}$. Hence it suffices to combine (vi) and (vii) to finish the proof.

A counterexample. In view of the importance of matrix-closed families in the present context it would be desirable to have a simple criterion at hand. Surely, if $\mathcal{G}$ is matrix-closed then $G_1+\ldots+G_n$ is closed for $G_1,\ldots,G_n\in\mathcal{G}$, and one might ask whether the converse is also true. The following counterexample shows that this is not the case.

4.5. PROPOSITION. *There are a Banach space X and closed subspaces $H_1,\ldots,H_4$ such that*

(i) *$H_i+H_j=X$, $H_i\cap H_j=\{0\}$ for $i\neq j$; thus $G_1+\ldots+G_n$ is closed for $G_1,\ldots,G_n\in\mathcal{G}:=\{H_1,\ldots,H_4\}$,*

(ii) *there is a 2×4-matrix M such that $A_M(\prod H_i)$ is not closed.*

Proof. Let $X_0:=\mathbf{K}^2$, provided with any norm, $U_1^k:=\text{lin}\{(1,1)\}$, $U_2^k:=\text{lin}\{(-1,1+1/k)\}$, $U_3^k:=\text{lin}\{(1,0)\}$, $U_4^k:=\text{lin}\{(0,1)\}$ for $K\in\mathbf{N}$. We define $X:=c_0(X_0)$ $(=\{(x_k)\mid x_k\in X_0,\ x_k\to0\})$, $H_i:=\{(x_k)\mid x_k\in U_i^k \text{ for all } k\}$ for $i=1,\ldots,4$. Then obviously (i) holds.

For the proof of (ii) let M be the matrix

$$\begin{pmatrix}-1&0&1&1\\0&-1&-1&1\end{pmatrix}.$$

We claim that $A_M(\prod H_i)$ is not closed.

Let $x=((\alpha_k,\beta_k))_k\in X$ be arbitrary and suppose that, for suitable $h_i\in H_i$, $A_M(h_1,h_2,h_3,h_4)=(0,x)$, i.e.

$$h_1=h_3+h_4,\qquad h_2+x=-h_3+h_4.$$

Writing $h_1=((a_k,a_k))_k$, $h_2=((-b_k,b_k(1+1/k))_k$, $h_3=((c_k,0))_k$, $h_4=((0,d_k))_k$ we conclude that $a_k=c_k=d_k$ and $(-b_k+\alpha_k,b_k(1+1/k)+\beta_k)=(-a_k,a_k)$. Therefore $-a_k=k\beta_k+(k+1)\alpha_k$, and consequently $(0,x)\in A_M(\prod H_i)$ is equivalent to $k\beta_k+(k+1)\alpha_k\to0$; in particular, there are $x\in X$ with $(0,x)\notin A_M(\prod H_i)$. But *all* $(0,x)$ lie in $[A_M(\prod H_i)]^-$: since for arbitrary $(\alpha,\beta)\in X_0$ and $k\in\mathbf{N}$ there are $u_i^k\in U_i^k$ with $A_M(u_1^k,u_2^k,u_3^k,u_k^4)=(0,(\alpha,\beta))$ it follows that $A_M(\prod H_i)$ contains all $(0,x)$ with $x\in X$ a finite sequence, and these $(0,x)$ are dense in $\{0\}\times X$.

5. Operator restrictions in the local reflexivity theorem. Let $R:X\to X$ be an operator. Can the local reflexivity map T be chosen such that it respects R? There are several approaches which make this demand precise and make it possible to prove a corresponding local reflexivity theorem ([2], [4], [7]).

We will present a result which contains the previous ones.

5.1. LEMMA. *Let $R_1,\ldots,R_r:X\to Y$ be operators, $E\subset X''$ finite-dimensional, and $1\geq\varepsilon>0$. There are operators $B_i:L(E,X)\to Y$, convex sets $C_i\subset Y$ $(i=1,\ldots,n)$ and a finite-dimensional space $F_\varepsilon\subset X'$ such that*

(i) *$B''(I)\in C_i^{-\sigma^*}$, i.e. E satisfies the approximate conditions (B_i,C_i) for all i,*

(ii) *whenever $T\in L(E,X)$ is an ε-isomorphism along F_ε such that $B_i(T)\in(C_i)_{\varepsilon/2}$ for all i, then*

$$(*)\qquad \Big\|\sum_{\varrho=1}^r R''_\varrho x''_\varrho\Big\|(1+\varepsilon)^{-1}\leq\Big\|\sum_{\varrho=1}^r R_\varrho\circ Tx''_\varrho\Big\|\leq\Big\|\sum_{\varrho=1}^r R''_\varrho x''_\varrho\Big\|+\varepsilon$$

for all $x''_1,\ldots,x''_r\in E$ with $\|x''_\varrho\|\leq1$.

Proof. We will make use of the approximate conditions described in Remark 3 at the end of Section 2. For fixed $e=(x''_1,\ldots,x''_r)\in E^r$ we consider $B_e:L(E,X)\to Y$ defined by $T\mapsto\sum R_\varrho\circ Tx''_\varrho$ and $C_e:=$ the closed ball in Y with radius $\|\sum R''_\varrho x''_\varrho\|$. Since, as can easily be checked, $B''_e(I)=\sum R''_\varrho x''_\varrho$, we have $B''_e(I)\in(C_e)^{-\sigma^*}$.

Now let $\delta>0$ be arbitrary. We choose a δ-net $(e_i)_{i=1,\ldots,n}$ in $\{(x''_1,\ldots,x''_r)\mid x''_\varrho\in E,\ \|x''_\varrho\|\leq1\}\subset E^r$ and define $B_i:=B_{e_i}$, $C_i:=C_{e_i}$.

Choose a finite-dimensional $F\subset Y'$ such that $\|y''\|(1+\varepsilon)^{-1}\leq\sup\{|y''(y')|\mid y'\in F,\|y'\|=1\}$ for all y'' in $\{\sum R''_\varrho x''_\varrho\mid x''_\varrho\in E,\|x''_\varrho\|\leq1\}$. It is then clear that every ε-isomorphism T along $F_\varepsilon:=\text{lin}\{R'_\varrho y'\mid y'\in F,\ \varrho=1,\ldots,r\}$ satisfies the first inequality in $(*)$.

If $B_i(T) \in (C_i)_{\varepsilon/2}$ for all i, then $\|\sum R_\varrho T x''_\varrho\| \le \|\sum R''_\varrho x''_\varrho\| + \varepsilon/2$ for all $(x''_1,\ldots,x''_r)$ in $\{e_i \mid i = 1,\ldots,n\}$, and consequently

$$\Big\|\sum R_\varrho T x''_\varrho\Big\| \le \Big\|\sum R''_\varrho x''_\varrho\Big\| + \varepsilon/2 + r\delta \max\|R_\delta\| \quad \text{if } \|x''_\varrho\| \le 1.$$

Hence $(*)$ will be satisfied provided that δ has been chosen sufficiently small.

Note. There are operators R which are one-to-one but for which there exist x'' with $\|x''\| = 1$, $R''x'' = 0$. Consequently, it is not to be expected that the right side of $(*)$ can be replaced by $\|\sum R''_\varrho x''_\varrho\|(1+\varepsilon)$.

5.2. THEOREM. *Let $R_{\varrho\sigma} : X \to Y_\sigma$ be operators ($\sigma = 1,\ldots,s$, $\varrho = 1,\ldots,r_\sigma$), $E \subset X''$ finite-dimensional. Then for every finite-dimensional $F \subset X'$ and every $\varepsilon > 0$ there exists an ε-isomorphism T along F such that*

(i) $T|_{E\cap X} = \mathrm{Id}$,

(ii) $\|\sum_\varrho R''_{\varrho\sigma} x''_\varrho\|(1+\varepsilon)^{-1} \le \|\sum_\varrho R_{\varrho\sigma} T x''_\varrho\| \le \|\sum_\varrho R''_{\varrho\sigma} x''_\varrho\| + \varepsilon$ *for all σ and all $x''_1,\ldots,x''_{r_\sigma} \in E$ with $\|x''_\varrho\| \le 1$.*

More generally, (i) *can be replaced by any exact and approximate conditions satisfied by E, e.g. by $T(E \cap G^{\circ\circ}) \subset G$ for G in a matrix-closed family.*

Proof. By Lemma 5.1(ii) the assertion in (ii) can be deduced from certain approximate conditions, and by 5.1(i), E satisfies these conditions. Thus the result follows from Theorem 2.3.

Note. The case when all $r_\sigma = 1$ together with subspace restrictions for a "friendly collection" has been treated in [4].

We are now going to state some consequences of the theorem as corollaries for the sake of easy reference:

5.3. COROLLARY [2]. *Let $R_1,\ldots,R_r : X \to X$ be operators, $E \subset X''$ and $F \subset X'$ finite-dimensional, and $\varepsilon > 0$.*

(i) *Suppose that*

$$(*) \qquad \dim \widetilde{E} = (r+1)\dim E, \quad \text{where} \quad \widetilde{E} := E + \sum R''_\varrho E.$$

Then there is an ε-isomorphism $\widetilde{T} : \widetilde{E} \to X$ along F such that

(1) $\widetilde{T}|_{E\cap X} = \mathrm{Id}$,

(2) $\widetilde{T} \circ R''_\varrho|_E = R_\varrho \circ \widetilde{T}|_E$ *for* $\varrho = 1,\ldots,r$.

(ii) *Without the assumption $(*)$ the conclusion of* (i) *is true with* (2) *replaced by*

(2′) $\|(\widetilde{T} \circ R''_\varrho - R_\varrho \circ \widetilde{T})|_E\| \le \varepsilon$ *for* $\varrho = 1,\ldots,r$.

Proof. Fix any $\delta > 0$. By Theorem 5.2 (applied to $\mathrm{Id}, R_1,\ldots,R_r$) there is a δ-isomorphism $T : E \to X$ along $F + R'_1 + \ldots + R'_r F$ with $T|_{E\cap X} = \mathrm{Id}$ and

$$\big|\|x''_0 + R''_1 x''_1 + \ldots + R''_r x''_r\| - \|Tx''_0 + R_1 T x''_1 + \ldots + R_r T x''_r\|\big| \le \delta$$

for $x''_1,\ldots,x''_r \in E$, $\|x''\| \le 1$.

Define $\widetilde{T} : \widetilde{E} \to X$ by $x''_0 + R''_1 x''_1 + \ldots + R''_r x''_r \mapsto Tx''_0 + R_1 T x''_1 + \ldots + R_r T x''_r$. $\widetilde{T}$ is well defined by $(*)$, $\widetilde{T}$ maps along F, and (1) and (2) are obviously satisfied. In order to show that $\widetilde{T}$ is an ε-isomorphism we first note that there is a constant η such that $\|x''_\varrho\| \le \eta$ for all ϱ provided that $\|x''_0 + R''_1 x''_1 + \ldots + R''_r x''_r\| \le 1$ (since the map $(x''_0,\ldots,x''_r) \mapsto x''_0 + R''_1 x''_1 + \ldots + R''_r x''_r$ is an isomorphism from E^{r+1} to $\widetilde{E}$ by $(*)$). But this implies that

$$1 - \delta\eta \le \|\widetilde{T}(x''_0 + R''_1 x''_1 + \ldots + R''_r x''_r)\| \le 1 + \delta\eta$$

for normalized $x''_0 + \ldots + R''_r x''_r$, and consequently $\widetilde{T}$ is an ε-isomorphism for small δ.

(ii) follows from (i) by an easy perturbation argument. One only has to apply (i) to operators $\widetilde{R}_\varrho$ which are close to R_ϱ and satisfy $(*)$ as well as $(\widetilde{R}''_\varrho x'')(x') = (R''_\varrho x'')(x')$ for all $x'' \in E$, $x' \in F$, and then (i)(2) applied for the $\widetilde{R}_\varrho$ yields (2′) for the R_ϱ. For details see [2].

5.4. COROLLARY: *Let $R_\varrho : X \to Y_\varrho$ ($\varrho = 1,\ldots,r$) be operators, $E \subset X''$ and $F \subset X'$, $F_\varrho \subset Y'_\varrho$ finite-dimensional, and $\varepsilon > 0$. Suppose that*

(a) $\ker R''_\varrho = (\ker R_\varrho)^{\circ\circ}$ *for all* ϱ,

(b) *the family* $\{\ker R_\varrho \mid \varrho = 1,\ldots,r\}$ *is matrix-closed.*

(c) $R_\varrho(E \cap X) = R''_\varrho(E) \cap Y_\varrho$ *for all* ϱ.

Then there are ε-isomorphisms $T : E \to X$ along F, $T_\varrho : R''_\varrho E \to Y_\varrho$ along F_ϱ ($\varrho = 1,\ldots,r$) such that

(1) $T|_{E\cap X} = \mathrm{Id}$, $T_\varrho|_{R''_\varrho(E)\cap Y_\varrho} = \mathrm{Id}$,

(2) $R_\varrho \circ T = T_\varrho \circ R''_\varrho|_E$ *for* $\varrho = 1,\ldots,r$.

Proof. Theorem 3.2 yields, for arbitrary $\delta > 0$, the existence of a δ-isomorphism $T : E \to X$ along $F + R'_1 F_1 + \ldots + R'_r F_r$ such that $T|_{E\cap X} = \mathrm{Id}$, $T(E \cap \ker R''_\varrho) \subset R_\varrho$. Define $T_\varrho : R''_\varrho E \to Y_\varrho$ by $T_\varrho R''_\varrho x'' := R_\varrho T x''$. The T_ϱ are well defined, and (1) and (2) are obviously valid. That the T_ϱ are ε-isomorphisms provided that δ is small follows as in the proof of the preceding corollary.

Note. Domański has treated in [7] the case $R_\varrho = S_\varrho \circ \ldots \circ S_1$ for a family of operators $S_\varrho : X_{\varrho-1} \to X_\varrho$. His result follow from 5.4 since in this case the $\ker R_\varrho$ are increasing, which implies that (b) is satisfied by 4.4(i).

In [8] he has presented a more elaborate version of this idea (the T_ϱ can be defined not only on $R''_\varrho E$ but on larger finite-dimensional subspaces).

There seems to be no simple way to obtain this as a direct consequence of Theorem 5.2.

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Almost everywhere summability of Laguerre series

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Abstract. We apply a construction of generalized twisted convolution to investigate almost everywhere summability of expansions with respect to the orthonormal system of functions $\ell_n^a(x) = (n!/\Gamma(n+a+1))^{1/2} e^{-x/2} L_n^a(x)$, $n = 0,1,2,\ldots$, in $L^2(\mathbf{R}_+, x^a\,dx)$, $a \geq 0$. We prove that the Cesàro means of order $\delta > a+2/3$ of any function $f \in L^p(x^a\,dx)$, $1 \leq p \leq \infty$, converge to f a.e. The main tool we use is a Hardy–Littlewood type maximal operator associated with a generalized Euclidean convolution.

1. Introduction. The problem of mean convergence of Laguerre expansions has attracted considerable attention in the last thirty years or so. The articles by Askey and Wainger [2] and Muckenhoupt [12, 13] are fundamental in the subject but also papers by Freud and Knapowski [6], Poiani [15] and Długosz [5] brought interesting results. A new impulse was given to the field in the '80s by Görlich and Markett in a series of papers [7–11]. Their method of investigation of the mean convergence problem was based on a convolution structure for Laguerre polynomials defined first by McCully and extended by Askey. An underlying device there is Watson's product formula for Laguerre polynomials.

In contrast with mean convergence surprisingly little is known for almost everywhere convergence of Laguerre series. The first result in this direction was obtained by Muckenhoupt for expansions with respect to the Laguerre polynomials.

Let

$$L_n^a(x) = (n!)^{-1} e^x x^{-a} (d/dx)^n (e^{-x} x^{n+a})$$

denote the nth Laguerre polynomial of order $a > -1$. Then the normalized polynomials

$$(1.1)\qquad \widetilde{L}_n^a(x) = (n!/\Gamma(n+a+1))^{1/2} L_n^a(x), \quad n = 0,1,2,\ldots,$$

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