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A new convexity property that implies a fixed point property for L_1

by

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Abstract. In this paper we prove a new convexity property for L_1 that resembles uniform convexity. We then develop a general theory that leads from the convexity property through normal structure to a fixed point property, via a theorem of Kirk. Applying this theory to L_1, we get the following type of normal structure: any convex subset of L_1 of positive diameter that is compact for the topology of convergence locally in measure, must have a radius that is smaller than its diameter. Indeed, a stronger result holds. The Chebyshev centre of any norm bounded, convergence locally in measure compact subset of L_1 must be norm compact. Immediately from normal structure, we get a new proof of a fixed point theorem for L_1 due to Lami Dozo and Turpin.

Introduction. We prove a new convexity property for $L_1(\mu)$ that resembles and has similar consequences to uniform convexity: the uniform Kadec–Klee property for the topology of convergence locally in measure.

Browder [Br] showed that in a uniformly convex Banach space $(X, \|\cdot\|)$, for every closed, bounded, convex set $C \subseteq X$, every nonexpansive mapping $T : C \to C$ has a fixed point. Kirk [Ki$_1$] extended this result to show that if X is a Banach space with weak normal structure (i.e. if $C \subseteq X$ is convex, weakly compact and has positive diameter, then the radius of C must be less than its diameter) then X has the weak fixed point property for nonexpansive mapping (FPP(weak)). Indeed, every uniformly convex space has (weak) normal structure.

Van Dulst and Sims [D-S] (see also Istrăţescu and Partington [I-P]) showed, using the work of Brodskiĭ and Mil'man [B-M], that if a Banach space X has the uniform Kadec–Klee property for the weak topology (a property of a wide class of spaces that strictly includes the uniformly convex ones), then X has weak normal structure. Consequently, X has the

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FPP(weak). Also in [D-S] analogous implications for the weak* topology on a dual Banach space are established.

Moreover, Kirk [Ki_2], building on work of Penot [Pe], developed a more abstract setting in which normal structure properties imply fixed point properties for nonexpansive mappings.

In this paper we extend to a more abstract setting the van Dulst and Sims [D-S] scheme wherein uniform Kadec–Klee properties imply normal structure properties. A consequence of this is the new result that $L_1(\mu)$ has normal structure for the topology of convergence locally in measure. This result has been independently proven by Besbes [Be]. Moreover, $L_1(\mu)$ has the stronger property that the Chebyshev centres of norm bounded, convergence locally in measure compact subsets are norm compact. Using Kirk [Ki_2] it follows from normal structure that $L_1(\mu)$ has the FPP for nonexpansive mappings w.r.t. the topology of convergence locally in measure. This is a new proof of a result of Lami Dozo and Turpin [L-T]. We remark that the theorem of [L-T] is more general, in the setting of generalized Orlicz spaces; and also convex domains of nonexpansive mappings are replaced by star-shaped ones. For related results see Khamsi and Turpin [K-T].

We begin with Kadec–Klee properties. By a Banach space $(X, \|\cdot\|)$ having "the Kadec–Klee property" we mean that whenever a sequence converges with respect to the weak topology to a point of the space, and the norms of that sequence converge to the norm of the point, it follows that the sequence converges in norm (and necessarily to the weak limit). This is sometimes called "the Radon–Riesz property" or "property H". Many Banach spaces have this property, e.g. ℓ_p, $1 \le p < \infty$. On the other hand, $L_1[0,1]$, c_0 and ℓ_∞ fail the Kadec–Klee property.

Nevertheless, it is well known that if $\{f_n\}_{n=1}^{\infty}$ is a sequence in $L_1[0,1]$ that converges almost everywhere to $f \in L_1[0,1]$ and $\|f_n\|_1 \xrightarrow[n]{} \|f\|_1$ then $\|f_n - f\|_1 \xrightarrow[n]{} 0$. Convergence in measure in $L_1[0,1]$ is the topological notion behind almost everywhere convergence. Indeed, the above Kadec–Klee result still holds if we replace almost everywhere convergence by convergence in measure. We are led by this positive result to study $L_1[0,1]$ endowed with the topology of convergence in measure.

The result in Section 1 is about uniform Kadec–Klee (UKK) properties. Huff [H] introduced this idea for the weak topology of a Banach space. In this paper we call Huff's original property "the uniform Kadec–Klee–Huff property" (UKKH), and the reformulation of van Dulst and Sims [D-S], "the uniform Kadec–Klee property" (UKK). As we shall see, these ideas coincide for the case of the weak topology and Proposition 1.2 gives a reasonable sufficient condition for them to coincide under more general circumstances. We remark that the ε-UKKH property, discussed in Section 1 and used in Section 3, is strictly weaker than the UKKH property in general (see [D-S] for an example).

In Section 2, we establish the key result that $L_1(\mu)$, for μ a σ-finite measure, has the UKK property for the topology of convergence locally in measure.

In Section 3 we generalize the implication ε-UKK $\Rightarrow$ normal structure to a setting involving an abstract weak topology τ on a Banach space X, that includes the topology of convergence locally in measure on an $L_1(\mu)$-space. We also observe that the UKK property for the τ topology implies that the Chebyshev centres of norm bounded, τ-compact sets in X must be norm compact. We then conclude that $L_1(\mu)$ has normal structure w.r.t. the topology of convergence locally in measure.

A result of Kirk [Ki_2] gives us, in Section 4, that normal structure implies the FPP in our setting. We are then able to state two criteria for using UKK properties to recognize fixed point properties for nonexpansive mappings. We are led immediately to a new proof that $L_1(\mu)$, for μ a σ-finite measure, has the fixed point property for nonexpansive mappings in convex sets that are compact for the topology of convergence locally in measure.

Section 5 contains an application of the L_1 fixed point theorem.

We remark that in the paper of Khamsi [Kh] it is shown (generalizing a result of Khamsi and Turpin [K-T]) that the norm boundedness hypothesis used throughout this paper on our τ-compact, convex sets in X is redundant.

The proof that L_1 has the above-described uniform Kadec–Klee property extends to $L_p(\mu)$ for $0 < p < 1$. The link to some kind of normal structure property appears to be open; although by [L-T] it is known that these spaces have the fixed point property for nonexpansive mappings on star-shaped, norm bounded sets that are compact for the topology of convergence locally in measure.

Other papers on uniform Kadec–Klee properties are Carothers, Dilworth, Lennard and Trautman [C-D-L-T], van Dulst and de Valk [D-V], Lau and Mah [L-M], Partington [Pa] and Lennard [L_2].

Sections 1 and 2 form part of the author's Ph.D. dissertation [L_1].

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0. Preliminaries. $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of real numbers, respectively. When we want to talk about the real or complex numbers, but do not care which one, we will use the expression "the scalars".

For a Banach space X, $\mathbf{B}_X$ and $B(x,\delta)$ will always denote the closed unit ball and the closed ball with centre x and radius δ, respectively.

Let (Ω,Σ,μ) be a positive σ-finite measure space. μ will always be assumed to be countably additive. $L_1(\mu)$ is the Banach space of all (equivalence classes of) measurable functions $f:\Omega\to$ the scalars for which $\|f\|_1<\infty$, where

$$\|f\|_1 := \int_\Omega |f(\omega)|\,d\mu(\omega).$$

The equivalence relation mentioned above is the usual one where we identify functions that differ only on a measurable subset of Ω of μ-measure zero.

Throughout this paper let $(X,\|\cdot\|)$ be a Banach space and let τ be a topological vector space topology on X that is weaker than the norm topology.

A standard example of such a pair (X,τ) is where X is a Banach space and τ is the weak topology on X. Another example is where X is a dual Banach space, with a predual Y say, and τ is the weak*-topology (with respect to Y). Yet another example is when X is an $L_1(\mu)$-space and τ is the topology **clm** of convergence locally in measure.

The topological vector space topology **clm** of convergence locally in measure on the set of all scalar-valued Σ-measurable functions on Ω, $L_0(\mu)$, is generated by the following translation-invariant metric. Let $(E_n)_{n=1}^\infty$ be a $\widetilde{\Sigma}$-partition of Ω, where

$$\widetilde{\Sigma} := \{E\in\Sigma : \mu(E)\in(0,\infty)\}.$$

Define d_0 by

$$d_0(f,g) := \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{1}{\mu(E_n)}\int_{E_n}\frac{|f-g|}{1+|f-g|}\,d\mu,\quad \text{for all } f,g\in L_0.$$

If $\mu(\Omega)<\infty$ then the simpler metric

$$d_0(f,g) := \int_\Omega \frac{|f-g|}{1+|f-g|}\,d\mu,\quad \text{for all } f,g\in L_0,$$

generates the **clm** topology. In this case we simply refer to **clm** as the topology of convergence in measure, denoted by **cm**. $L_0(\mu)$ is complete with respect to the above metric. Indeed, $(L_0(\mu),\mathbf{clm})$ is an F-space.

For sequences, **clm**-convergence reduces, in a sense, to almost everywhere convergence. Indeed, any sequence in L_0 that converges almost everywhere to $f\in L_0$ must converge to f locally in measure. On the other hand, every **clm**-convergent sequence of scalar-valued measurable functions

has a subsequence that converges almost everywhere to the same limit function.

Note that when we discuss L_1, **clm** or **cm** will denote the topologies introduced above, restricted to L_1.

For $p=0$ or 1, $L_p[0,1]$ denotes the space $L_p(\lambda_1)$, where λ_1 is Lebesgue measure on $[0,1]$.

1. Uniform Kadec–Klee properties. The measure of compactness of a nonempty subset S of X, $\gamma(S)$, is defined by

$$\gamma(S) := \sup\{\operatorname{sep}\{x_n\}_{n=1}^\infty : \{x_n\}_{n=1}^\infty \text{ is a sequence in } S\},$$

where

$$\operatorname{sep}\{x_n\}_{n=1}^\infty := \inf_{n\neq m}\|x_n-x_m\|,$$

for all sequences $\{x_n\}_{n=1}^\infty$ in X.

Recall that a function $f:(X,\tau)\to\mathbf{R}$ is called *lower semicontinuous* (lsc or τ-lsc) if whenever a net $\{x_\alpha\}_{\alpha\in A}$ converging to $x\in X$ with respect to τ is such that $f(x_\alpha)\le f(x)$ for all α then

$$f(x_\alpha)\xrightarrow[\alpha]{} f(x).$$

f is called *sequentially lower semicontinuous* (sequentially lsc or τ-sequentially lsc) if it has the property that we get from the one in the previous paragraph by replacing nets everywhere by sequences.

1.1. DEFINITION. (a) For some $\varepsilon>0$, X is said to have the *ε-uniform Kadec–Klee property* with respect to τ, denoted by ε-UKK(τ), if there exists $\delta\in(0,1)$ so that whenever S is a τ-compact subset of $\mathbf{B}_X$ with $\gamma(S)>\varepsilon$, it follows that

$$S\cap B(0,1-\delta)\neq\emptyset.$$

(b) X is said to have the *uniform Kadec–Klee property* with respect to τ, denoted by UKK(τ), if for every $\varepsilon>0$, X has ε-UKK(τ).

(c) For some $\varepsilon>0$, X has the *ε-uniform Kadec–Klee–Huff property* with respect to τ, denoted by ε-UKKH(τ), if there exists $\delta\in(0,1)$ so that whenever $\{x_n\}_{n=1}^\infty$ is a sequence in $\mathbf{B}_X$ such that $x_n\xrightarrow[n]{} x\in X$ with respect to τ and $\inf_{n\neq m}\|x_n-x_m\|>\varepsilon$, it follows that

$$\|x\|\le 1-\delta.$$

(d) X has the *uniform Kadec–Klee–Huff property* with respect to the topology τ, abbreviated to UKKH(τ), if for every $\varepsilon>0$, X has ε-UKKH(τ).

We remark that if X is a Banach space then $\|\cdot\|_X$ is weak-lsc, while if X is a dual Banach space then $\|\cdot\|_X$ is weak*-lsc. Both these facts follow from the Hahn–Banach separation theorem.

Further, the $L_1(\mu)$-norm is **clm**-lsc. This follows from Fatou's lemma and the fact that **clm** is a metric topology.

We now describe the relationship between the UKK and UKKH properties. For the proof of the following result see [L₁].

1.2. PROPOSITION. *Suppose that for τ, every compact subset of $\mathbf{B}_X$ is sequentially compact. Then the following statements are true.*

(1) *Suppose that $\|\cdot\|_X$ is τ-sequentially lsc. For all $\varepsilon > 0$,*

$$X \text{ is } \varepsilon\text{-UKKH}(\tau) \Leftrightarrow X \text{ is } \varepsilon\text{-UKK}(\tau).$$

(2) *X is UKKH(τ) $\Leftrightarrow$ X is UKK(τ) and $\|\cdot\|_X$ is τ-sequentially lsc.*

Note that the above proposition applies in the case where X is a Banach space and τ is the weak topology on X, by the Eberlein–Shmul'yan theorem. The observation of van Dulst and Sims [D-S] that for a dual Banach space X, UKKH(weak*) and UKK(weak*) coincide when $\mathbf{B}_X$ is weak*-sequentially compact, is another special case of the above result. Proposition 1.2 also applies in any case where τ is the topology of convergence locally in measure (for a σ-finite measure), because τ is a metric topology. Indeed, this result holds any time τ, when restricted to $\mathbf{B}_X$, is a metric topology.

2. A new convexity property of L_1.

The aim of this section is to show (Theorem 2.3) that $L_1(\mu)$, for μ a σ-finite measure, has the UKKH(**clm**) property; which, by Proposition 1.2, is equivalent to the UKK(**clm**) property.

2.1. LEMMA. *Let (Ω,Σ,μ) be a σ-finite measure space. If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $L_1(\mu)$, $f\in L_1(\mu)$ and $f_n \xrightarrow[n]{} f$ almost everywhere, then given $\varepsilon>0$ there exists $T\in\Sigma$ with $\mu(T)<\infty$ and with the following additional properties:*

(1) $$\|f\chi_{\Omega\setminus T}\|_1 < \varepsilon\,,$$

(2) $$\|(f_n-f)\chi_T\|_1 \xrightarrow[n]{} 0\,.$$

Proof. Fix $\varepsilon>0$. The equations

$$\nu(E)=\int_E |f|\,d\mu,\qquad E\in\Sigma\,,$$

define a finite, positive measure ν on Σ that is absolutely continuous with respect to μ. Choose $\delta>0$ such that $\nu(S)<\varepsilon/2$ for all $S\in\Sigma$ with $\mu(S)<\delta$. Now, choose $F\in\Sigma$ with $\mu(F)<\infty$ such that $\nu(\Omega\setminus F)<\varepsilon/2$; this is possible as (Ω,Σ,μ) is σ-finite.

Next, apply Egorov's theorem to produce $T\in\Sigma$ with $T\subseteq F$ such that $\mu(F\setminus T)<\delta$ and $\|(f_n-f)\chi_T\|_\infty \xrightarrow[n]{} 0$. Clearly then $\nu(F\setminus T)<\varepsilon/2$, so that

$$\|f\chi_{\Omega\setminus T}\|_1=\nu(\Omega\setminus T)=\nu(\Omega\setminus F)+\nu(F\setminus T)<\varepsilon/2+\varepsilon/2=\varepsilon\,.$$

Further, since $\mu(T)<\infty$, it follows that

$$\|(f_n-f)\chi_T\|_1\le\mu(T)\|(f_n-f)\chi_T\|_\infty\to 0\quad\text{ as } n\to\infty\,. \blacksquare$$

2.2. THEOREM. *Let (Ω,Σ,μ) be a σ-finite measure space. Then $L_1(\mu)$ has the following property: For every $\varepsilon>0$ there exists $\delta\in(0,1)$ such that whenever $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathbf{B}_{L_1}$, $f\in L_1$, $\{f_n\}_n$ converges to f almost everywhere and $\inf_{n\ne m}\|f_n-f_m\|_1>\varepsilon$, it follows that $\|f\|_1\le 1-\delta$.*

Proof. Fix $\delta\in(0,1)$ and then fix a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbf{B}_{L_1}$ such that $f_n\xrightarrow[n]{} f\in L_1$ almost everywhere and $\|f\|_1>1-\delta$. For each $g\in L_1$ and for each set $S\in\widetilde{\Sigma}$, define

$$\alpha_S(g):=\|g\chi_{\Omega\setminus S}\|_1\,.$$

It is clear that for all $g\in\mathbf{B}_{L_1}$ and $S\in\widetilde{\Sigma}$,

(1) $$\alpha_S(g)\le 1-\|g\chi_S\|_1\,.$$

As $f_n\xrightarrow[n]{} f$ almost everywhere there exists, by Lemma 2.1, $T\in\widetilde{\Sigma}$ such that

(2) $$\alpha_T(f)<\|f\|_1-(1-\delta)\,,$$

(3) $$\|(f_n-f)\chi_T\|_1\xrightarrow[n]{} 0\,.$$

So, using (2), we see that

$$\begin{aligned}\|f_n\chi_T\|_1&\ge\|f\chi_T\|_1-\|(f_n-f)\chi_T\|_1\\&=(\|f\|_1-\alpha_T(f))-\|(f_n-f)\chi_T\|_1>1-\delta-\|(f_n-f)\chi_T\|_1\,.\end{aligned}$$

It follows from (1) that $\alpha_T(f_n)\le\delta+\|(f_n-f)\chi_T\|_1$ for all $n\in\mathbb{N}$, and consequently for each n and m in $\mathbb{N}$ we have

$$\begin{aligned}\|f_n-f_m\|_1&\le\|(f_n-f_m)\chi_T\|_1+\alpha_T(f_n)+\alpha_T(f_m)\\&\le 2(\delta+\|(f_n-f)\chi_T\|_1+\|(f_m-f)\chi_T\|_1)\,.\end{aligned}$$

Thus, from (3), $\inf_{n\ne m}\|f_n-f_m\|_1\le 2\delta$. Finally, if $\varepsilon>0$ is given, with $\varepsilon<2$, set $\delta=\varepsilon/2$. Then for all sequences $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbf{B}_{L_1}$ such that $f_n\xrightarrow[n]{} f\in L_1$ almost everywhere we have

$$\|f\|_1>1-\delta\ \Rightarrow\ \inf_{n\ne m}\|f_n-f_m\|_1\le\varepsilon\,. \blacksquare$$

2.3. THEOREM. *Let (Ω,Σ,μ) be a σ-finite measure space. Then $L_1(\mu)$ has the uniform Kadec–Klee–Huff property with respect to the topology of convergence locally in measure.*

Proof. Fix $\varepsilon > 0$. Choose $\delta \in (0,1)$ satisfying the conclusions of Theorem 2.2. Now fix a sequence $\{f_n\}_{n\in\mathbf{N}}$ in $\mathbf{B}_{L_1}$ such that $\{f_n\}_n$ converges locally in measure to some $f \in L_1$ and $\inf_{n\neq m}\|f_n - f_m\|_1 > \varepsilon$.

As (Ω, Σ, μ) is σ-finite, there exists a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ such that $f_{n_k} \xrightarrow[k]{} f$ almost everywhere. Moreover, $\inf_{k\neq l}\|f_{n_k} - f_{n_l}\|_1 \geq \inf_{n\neq m}\|f_n - f_m\|_1 > \varepsilon$. Consequently, invoking the above theorem, it follows that $\|f\|_1 \leq 1 - \delta$. ■

3. Uniform Kadec–Klee properties imply normal structure. In this section $(X, \|\cdot\|)$ is a Banach space and τ is a topological vector space topology that is weaker than the norm topology.

X is said to have *normal structure* w.r.t. τ (NS(τ)) if for all norm bounded, τ-compact, convex subsets C of X with two or more points, we have

$$\mathrm{rad}(C) < \mathrm{diam}(C).$$

Here $\mathrm{rad}(C)$, the *radius* of C, is defined by

$$\mathrm{rad}(C) := \inf_{y\in C}\sup_{x\in C}\|x - y\|;$$

while $\mathrm{diam}(C)$, the *diameter* of C, is given by

$$\mathrm{diam}(C) := \sup_{y\in C}\sup_{x\in C}\|x - y\|.$$

A subset K of X is called *diametral* if $\mathrm{rad}(K) = \mathrm{diam}(K)$. For each $y \in C$, the *radius of C w.r.t. y*, denoted by $\mathrm{rad}(y; C)$, is defined by

$$\mathrm{rad}(y; C) := \sup_{x\in C}\|x - y\|.$$

The *Chebyshev centre* of C is the set of all $y \in C$ such that $\mathrm{rad}(y; C) = \mathrm{rad}(C)$.

3.1. THEOREM. *Fix $\varepsilon \in (0,1)$. Suppose $(X, \|\cdot\|)$ has the ε-UKKH(τ) property. Suppose also that every τ-compact subset of $\mathbf{B}_X$ is τ-sequentially compact and norm separable. Then X has normal structure with respect to τ.*

The construction used in the proof below is a modification of one of Brodskiĭ and Mil'man [B-M]. The key new idea exploits the norm separability of the convex set involved.

Proof. Suppose, to get a contradiction, that X fails NS(τ). Then there exists a subset C of X such that C is τ-compact, norm bounded, convex and diametral with $\mathrm{diam}(C) > 0$. We may translate and scale C, preserving its other properties and gaining: $\mathrm{diam}(C) = 1$ and $\theta \in C$, where θ is the zero element in X.

Fix $\{u_k\}_{k\in\mathbf{N}}$, a norm dense sequence in C. Let $\delta > 0$ correspond to our given $\varepsilon \in (0,1)$ as in the definition of the property ε-UKKH(τ). Define $a \in (0,1)$ by

$$a := \tfrac{1}{2}\min\{1 - \varepsilon, \delta/4\}.$$

Choose $x_1 := \theta$ and $C_1 := \mathrm{co}\{x_1, u_1\}$. Our initial aim is to construct a sequence $\{x_k\}_{k\in\mathbf{N}}$ in C such that, for all $n \in \mathbf{N}$,

$$(\clubsuit)\qquad \mathrm{dist}(x_{n+1}, \overline{\mathrm{co}}\{x_1, \ldots, x_n, u_1, \ldots, u_n\}) \geq 1 - a.$$

So fix $n \in \mathbf{N}$. Suppose that $x_1, \ldots, x_n \in C$ have been chosen such that

$$\mathrm{dist}(x_k, \overline{\mathrm{co}}\{x_1, \ldots, x_{k-1}, u_1, \ldots, u_{k-1}\}) \geq 1 - a,$$

for all $k = 2, \ldots, n$. We wish to construct $x_{n+1} \in C$ with property ($\clubsuit$). Define

$$C_n := \mathrm{co}\{x_1, \ldots, x_n, u_1, \ldots, u_n\}.$$

C_n is a subset of C. Also, let b be the barycentre of C_n:

$$b := \frac{1}{2n}\Big(\sum_{j=1}^{n} x_j + \sum_{j=1}^{n} u_j\Big).$$

Fix $\eta \in (0,1)$. Since C is diametral, there exists $z_\eta \in C$ such that $\|z_\eta - b\| > 1 - \eta$. Next, fix $x \in C_n$. It is straightforward to verify that

$$y := \frac{1}{1-\lambda}(b - \lambda x), \quad \text{where} \quad \lambda := \frac{1}{2n},$$

is a member of C_n. Thus, $\lambda x + (1-\lambda)y = b$. So

$$x = \frac{1}{\lambda}(b - (1-\lambda)y) = 2nb - (2n-1)y.$$

Consequently, $x - z_\eta = 2n(b - z_\eta) - (2n-1)(y - z_\eta)$. It follows that

$$\begin{aligned}\|x - z_\eta\| &\geq 2n\|b - z_\eta\| - (2n-1)\|y - z_\eta\| \\ &\geq 2n(1-\eta) - (2n-1)(1) = 1 - 2n\eta.\end{aligned}$$

Choose $\eta := a/(2n)$ and $x_{n+1} := z_\eta$. Then $\|x - x_{n+1}\| \geq 1 - a$, for all $x \in C_n$. The construction of $\{x_k\}_{k\in\mathbf{N}}$ is complete.

We begin the final part of the proof by recalling that C is τ-compact; and therefore τ-sequentially compact, by hypothesis. Consequently, we can find a subsequence $\{x_{n_k}\}_{k\in\mathbf{N}}$ of $\{x_n\}_n$ and some $w \in C$ so that $x_{n_k} \xrightarrow[k]{} w$ w.r.t. τ. Now, since $\{u_n\}_{n\in\mathbf{N}}$ is norm dense in C, there is some $N \in \mathbf{N}$ for which $\|w - u_N\| < \delta/4$. Define, for every $k \in \mathbf{N}$, $z_k := x_{n_k} - x_{N+1}$. Then each $\|z_k\| \leq \mathrm{diam}\, C = 1$, and $z_k \xrightarrow[k]{} w - x_{N+1}$ w.r.t. τ.

Fix $k, l \in \mathbf{N}$ with $k < l$. Since $n_k < n_l$, we have from inequality ($\clubsuit$)

$$\|z_k - z_l\| = \|x_{n_k} - x_{n_l}\| \geq 1 - a > 1 - (1 - \varepsilon) = \varepsilon.$$

So, $\operatorname{sep}\{z_k\}_{k\in\mathbb{N}} \geq 1 - a > \varepsilon$. But X has the ε-UKKH(τ) property, and therefore by our choice of δ,

$$(\dagger) \qquad \|w - x_{N+1}\| \leq 1 - \delta .$$

However, again using (♣), we see that

$$\begin{aligned}\|x_{N+1} - w\| &\geq \|x_{N+1} - u_N\| - \|u_N - w\| \\ &\geq (1-a) - \delta/4 \geq 1 - \delta/4 - \delta/4 = 1 - \delta/2 ,\end{aligned}$$

which contradicts (†). ∎

We remark that if we strengthen our assumption in Theorem 3.1 to the UKKH(τ) property on X, then by the proof of [D-S, Theorem 2], or the proof of [I-P, Theorem 1], we get the following theorem.

3.2. THEOREM. *Suppose $(X, \|\cdot\|)$ has the* UKKH(τ) *property. Let $C \subseteq X$, where C is nonempty, τ-compact and norm bounded. Also suppose that in $\mathbf{B}_X$, τ-compact sets are τ-sequentially compact. Then the Chebyshev centre of C is norm compact and nonempty.*

So, directly from [B-M] (without the adaptation of their construction that we used to prove Theorem 3.1), we get the corollary below.

3.3. COROLLARY. *Suppose that $(X, \|\cdot\|)$ has the* UKKH(τ) *property. Also suppose that τ-compact subsets of $\mathbf{B}_X$ are τ-sequentially compact. Then X has normal structure w.r.t. τ.*

We remark that strengthening the UKK hypothesis on X in the above result has enabled us to drop the assumption of norm separability of τ-compact sets that we used in Theorem 3.1. Note also that, by Proposition 1.2, if X has the UKKH(τ) property then $\|\cdot\|$ is τ-sequentially lsc—a fact that one needs when verifying Theorem 3.2.

From Theorem 2.3 and Corollary 3.3 we have the following new result concerning L_1; it has been proven independently by Besbes [Be].

3.4. THEOREM. *Let (Ω, Σ, μ) be a σ-finite measure space. Then $L_1(\mu)$ has normal structure w.r.t. the topology of convergence locally in measure.*

We remark that one can show that in every $L_1(\mu)$, μ σ-finite, **clm**-compact sets must be norm separable. So the previous theorem also follows from Theorems 2.3 and 3.1.

4. Normal structure implies a fixed point property. In this section, as usual, $(X, \|\cdot\|_X)$ is a Banach space and τ is a topological vector space topology on X that is weaker than the norm topology.

X is said to have the *fixed point property* w.r.t. τ (FPP(τ)) if the following holds. For each nonempty, norm bounded, τ-compact, convex subset C of X, every nonexpansive mapping $T : C \to C$ has a fixed point. Here T is a *nonexpansive mapping* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \text{for all } x, y \in C .$$

4.1. THEOREM. *Suppose that $\|\cdot\|$ is a τ-lsc function. If X has normal structure w.r.t. τ, then X has the fixed point property w.r.t. τ.*

Proof. By our hypothesis on $\|\cdot\|$, the norm closed balls of X are also τ-closed. By [Ki$_2$, Theorem 2], the result immediately follows. ∎

It is also true that if $\|\cdot\|$ is simply τ-sequentially lsc in Theorem 4.1 and the τ-compact subsets of $\mathbf{B}_X$ are τ-sequentially compact, then NS(τ) implies FPP(τ). Combining Theorem 3.1 and Corollary 3.3 with the above remark, we have the following result.

4.2. THEOREM. *Let $(X, \|\cdot\|)$ be such that τ-compact subsets of $\mathbf{B}_X$ are τ-sequentially compact.*

(a) *If X has the ε-UKKH(τ) property for some $\varepsilon \in (0,1)$, $\|\cdot\|$ is τ-sequentially lsc and τ-compact subsets of $\mathbf{B}_X$ are norm separable, then X has the* FPP(τ).

(b) *If X has the* UKKH(τ) *property, then X has the* FPP(τ).

Our work above gives a new proof, via normal structure, of a fixed point theorem for L_1 due to Lami Dozo and Turpin [L-T].

4.3. THEOREM. *Let (Ω, Σ, μ) be a σ-finite measure space. Then $L_1(\mu)$ has the fixed point property w.r.t. the topology of convergence locally in measure on L_1.*

5. An application of the L_1 fixed point theorem. In this section our Banach space X is $L_1[0,1]$ and the weaker t.v.s. topology τ on X is **cm**, the topology of convergence in measure restricted to $L_1[0,1]$.

We begin with an example of a norm bounded, **cm**-compact, convex subset S of $L_1[0,1]$ that is not norm compact. Then we define a nonexpansive mapping $T : C \to C$ and apply the fixed point theorem to gain information about certain subsets of S.

A Lebesgue-measurable function $f : [0,1] \to$ the scalars is called *nonincreasing almost everywhere* (a.e.) if there exists a nonincreasing function $g : [0,1] \to$ the scalars such that $f(x) = g(x)$ almost everywhere.

Define the subset S of $L_1[0,1]$ by

$$S := \{f \in L_1[0,1] : f \geq 0 \text{ a.e., } f \text{ is nonincreasing a.e. and } \|f\|_1 \leq 1\} .$$

It is easy to see that S is norm bounded, convex and **cm**-closed.

S is not $\|\cdot\|_1$-compact. Indeed, $\{2^n \chi_{[0,1/2^n]} : n \in \mathbb{N}\}$ is a subset of S that is 1-separated in L_1-norm.

We can see that S is **cm**-compact in the following way. Note that for each $\varepsilon > 0$, the set $Q_\varepsilon := \{f\chi_{[\varepsilon,1]} : f \in S\}$ is of essentially-uniformly bounded variation, and is also essentially-uniformly bounded. Applying Helly's Selection Principle (see, for example, Kolmogorov and Fomin [K-F]) iteratively to the sets $Q_{1/n}$ ($n \in \mathbb{N}$), and using a diagonal argument, we can show that every sequence in S has a subsequence that converges almost everywhere.

Consider $T : L_0[0,1] \to L_0[0,1]$, where Tf is defined for all $f \in L_0$ by

$$(Tf)(x) := \tfrac{1}{2}f(x/2), \quad \text{for all } x \in [0,1].$$

T maps $L_1[0,1]$ into $L_1[0,1]$. Indeed, for all $f \in L_1$,

$$\|Tf\|_1 = \int_0^1 \tfrac{1}{2}|f(x/2)|\,dx = \int_0^{1/2} |f(y)|\,dy \le \|f\|_1 .$$

Similarly, $\|Tf - Tg\|_1 \le \|f - g\|_1$ for all $f, g \in L_1$. It is now clear that T maps S into S and T is nonexpansive on S.

Note that $T\theta = \theta$, where θ is the zero function. Moreover, the following result, which will enable us to apply the fixed point theorem (Theorem 4.3), is true.

5.1. LEMMA. *θ is the unique fixed point of T in $L_1[0,1]$.*

Proof. Suppose $f \in L_1[0,1]$ and $Tf = f$. Then

$$\tfrac{1}{2}f(x/2) = f(x), \quad \text{for almost all } x \in [0,1]. \tag{1}$$

By changing the values of f on a set of measure zero, we may assume that f is nonincreasing on $[0,1]$. Of course, equation (1) still holds. Moreover, since f is also in L_1,

$$\lim_{t \to 0+} tf(t) = 0 . \tag{2}$$

From (1), there exists a λ_1-null set $N_1 \in \Lambda_1$ such that $\frac{1}{2}f(x/2) = f(x)$, for all $x \in \Omega_1 := [0,1] \setminus N_1$. For each $j \in \mathbb{N}$ with $j > 1$ define

$$N_j := \{y \in [0,1] : y = 2^{j-1}z, \text{ for some } z \in N_1\}.$$

Further define $N := \bigcup_{j=1}^{\infty} N_j$ and $\Omega := [0,1] \setminus N$. Clearly $\lambda_1(N) = 0$, and so $\lambda_1(\Omega) = 1$.

Fix $x \in \Omega$. Hence $x \in [0,1] \setminus N_j$ for all $j \in \mathbb{N}$. Then for each $j \in \mathbb{N}$, $x/2^{j-1} \in [0,1] \setminus N_1 = \Omega_1$, or equivalently $\frac{1}{2}f(x/2^j) = f(x/2^{j-1})$. Consequently, for all $k \in \mathbb{N}$,

$$f(x) = \frac{1}{2^k} f\left(\frac{x}{2^k}\right),$$

and thus, by (2),

$$xf(x) = \frac{x}{2^k} f\left(\frac{x}{2^k}\right) \xrightarrow[k]{} 0 .$$

So, for all $x \in \Omega$ with $x \ne 0$, we have $f(x) = 0$. We conclude, therefore, that $f(x) = 0$ almost everywhere. ∎

Note that T does have nonzero fixed points in $L_0 \setminus L_1$. For example, $Tf = f$ for the function $1/\mathbf{x}$, where $\mathbf{x}$ is defined by $\mathbf{x}(x) := x$ for all $x \in [0,1]$.

5.2. PROPOSITION. *Let C be a **cm**-compact, convex subset of S, not containing the zero function θ. Then C fails to be invariant under the mapping T.*

Proof. If T did map C into C, then by hypothesis and Theorem 4.3, T would have a nonzero fixed point $f \in C$. But C is a subset of $L_1[0,1]$, and so from Lemma 5.1 we have a contradiction. ∎

There are many examples of subsets C of S that satisfy the hypotheses of the above proposition and are not norm compact. So that for these sets, Theorem 4.3 cannot be replaced by the Schauder–Tikhonov fixed point theorem in the above proof. For instance, fix $g \in S$ with $0 < \|g\|_1 < 1$. Define

$$I_g := \{f \in L_1[0,1] : f \ge g \text{ a.e., } f \text{ is nonincreasing a.e. and } \|f\|_1 \le 1\}.$$

I_g is a **cm**-closed subset of S. Consequently, I_g is **cm**-compact. It is also norm bounded and convex. Moreover, θ is not in I_g. Finally, using a similar argument to that for S, it is simple to show that I_g is not norm totally bounded.

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On the principle of local reflexivity

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Abstract. We prove a version of the local reflexivity theorem which is, in a sense, the most general one: our main theorem characterizes the conditions which can be imposed additionally on the usual local reflexivity map provided that these conditions are of a certain general type. It is then shown how known and new local reflexivity theorems can be derived. In particular, the compatibility of the local reflexivity map with subspaces and operators is investigated.

1. Introduction.

The by now classical version of the local reflexivity theorem reads as follows:

1.1. THEOREM [11, 13]. *Let X be a Banach space, $E \subset X''$ and $F \subset X'$ finite-dimensional subspaces, and $\varepsilon > 0$. Then there is an isomorphism $T : E \to X$ such that*

(i) $\|T\|, \|T^{-1}\| \le 1 + \varepsilon$,
(ii) $x'(Tx'') = x''(x')$ *for* $x'' \in E$ *and* $x' \in F$,
(iii) $Tx'' = x''$ *for* $x'' \in E \cap X$.

New proofs have been given in [6] and [14], variants where it is shown that T may be assumed to satisfy certain additional conditions are studied in [2, 4, 5, 7, 8, 9, 12]. The applications of the local reflexivity theorem are abundant, and it is undoubtedly one of the most fundamental theorems in Banach space theory.

The aim of this paper is to state and prove a local reflexivity theorem which is in a sense the most far-reaching one (this will be made precise shortly).

At least formally all known local reflexivity results are covered by our main theorem, and we will indicate how some of them can be derived easily as corollaries. A systematic investigation of how to apply the new local reflexivity technics to situations where variants of the classical theorem have

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