I also wish to thank Professor J. B. Conway for pointing out the following mistake made in my basic work [8]. The assumption 2.8(a),(b) in [8] (appearing in the main results) should have been made earlier and replace its weaker form (2.3). The gap in the proof of Proposition 2.5 resulted from the possibility of the situation where there are more than one nontrivial Gleason parts of $R(\bar{\Omega})$. This gap is then filled as follows. If U is the connected component of $\operatorname{int}(\bar{\Omega})$ containing Ω, then points of ∂U either belong to the same part of $R(\bar{\Omega})$ as U, or are peak points for $R(\bar{\Omega})$ (cf. Exercise 8 in Chapter VI of T. Gamelin's *Uniform Algebras*). Thus, any other nontrivial part of $R(\bar{\Omega})$ must lie in $\partial\Omega\setminus\partial U$, a subset of the inner boundary of Ω, which by (2.8)(a) is small. S decomposes as a direct sum of operators with spectra contained in the closures of nontrivial parts of $R(\bar{\Omega})$. All summands except one are normal, since $\partial\Omega\setminus\partial_f\Omega$, being a peak (hence closed) set of small Hausdorff dimension, has zero area, but the purity assumption implies that no nontrivial normal summands exist.

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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
Św. Tomasza 30, 31-027 Kraków, Poland

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Inequalities for exponentials in Banach algebras

by

A. J. PRYDE (Clayton, Vic.)

Abstract. For commuting elements x, y of a unital Banach algebra $\mathscr{B}$ it is clear that $\|e^{x+y}\| \leqslant \|e^x\| \|e^y\|$. On the other hand, M. Taylor has shown that this inequality remains valid for a self-adjoint operator x and a skew-adjoint operator y, without the assumption that they commute. In this paper we obtain similar inequalities under conditions that lie between these extremes. The inequalities are used to deduce growth estimates of the form $\|e^{i\langle x,\xi\rangle}\| \leqslant c(1+|\xi|)^s$ for all $\xi \in \mathbf{R}^m$, where $x = (x_1, \ldots, x_m) \in \mathscr{B}^m$ and c, s are constants.

1. Introduction. Let $\mathscr{B}$ be a unital Banach algebra. Then $e^{x+y} = e^x e^y$ for all commuting pairs $(x, y) \in \mathscr{B}^2$ and hence

$$\|e^{x+y}\| \leqslant \|e^x\| \|e^y\|. \tag{1}$$

In this paper we consider the validity of this inequality, and modifications of it, in the case that x, y do not commute. To begin with, let $\mathscr{B} = \mathscr{M}_3$, the space of 3 by 3 complex matrices, together with any norm. The following example shows that there is no constant $c > 0$ such that for all $x, y \in \mathscr{M}_3$,

$$\|e^{x+y}\| \leqslant c\|e^x\| \|e^y\|. \tag{2}$$

EXAMPLE 1.1. Let $x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $y = \begin{bmatrix} 0 & t & -t^2/2 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix}$ where $t \in \mathbf{R}$. Then

$$e^x = \begin{bmatrix} e & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad e^y = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}, \quad e^{x+y} = \begin{bmatrix} e & t(e-1) & t^2(e-3)/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

So $\|e^x\| = O(1)$, $\|e^y\| = O(t)$, $\|e^{x+y}\| = O(t^2)$ as $t \to \infty$.

Nevertheless, inequality (1) remains valid for certain classes of not necessarily commuting pairs x, y. For example, Taylor [7] obtained (1) for self-adjoint operators x and skew-adjoint y. In Section 3 we show that (1) holds whenever x, y are normal elements of a C^*-algebra.

In Section 4 we obtain a weaker estimate for triangular matrices. Indeed, let $\mathscr{T}_n$ be the space of n by n (upper) triangular matrices with any suitable norm.

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We show that there is a constant $c > 0$ such that for all diagonal matrices $D \in \mathcal{T}_n$ and all nilpotent matrices $N \in \mathcal{T}_n$,

$$\|e^{D+N}\| \leqslant c\,\|e^D\|\,\|e^{\operatorname{mod} N}\|. \tag{3}$$

Here, $\operatorname{mod} N \in \mathcal{T}_n$ is defined by $(\operatorname{mod} N)_{ij} = |N_{ij}|$.

We also describe a class of norms for which the constant in (3) is $c = 1$. This enables us in Section 5 to obtain inequalities for exponentials of triangular operators on separable Hilbert space.

Our motivation for studying these inequalities is to deduce Paley–Wiener type growth estimates for m-tuples $(x_1, \ldots, x_m)$ of elements of $\mathcal{B}$. This is described in Section 2 and developed in subsequent sections.

2. Paley–Wiener inequalities. As elsewhere, let $\mathcal{B}$ denote a unital Banach algebra. Let $x = (x_1, \ldots, x_m) \in \mathcal{B}^m$. We shall say that x is of *type s* if $s \geqslant 0$ and there exists $c > 0$ such that for all $\xi \in \mathbf{R}^m$,

$$\|e^{i\langle x, \xi\rangle}\| \leqslant c(1+|\xi|)^s, \tag{4}$$

where, for $\zeta \in \mathbf{C}^m$, $\langle x, \zeta\rangle = \sum_{j=1}^m x_j\zeta_j$.

Moreover, we shall say that $x = (x_1, \ldots, x_m)$ is of *Paley–Wiener type* (s, r) if $s \geqslant 0$, $r \geqslant 0$ and there exists $c > 0$ such that

$$\|e^{i\langle x, \zeta\rangle}\| \leqslant c(1+|\zeta|)^s e^{r|\operatorname{Im}\zeta|} \quad \text{for all } \zeta \in \mathbf{C}^m. \tag{5}$$

The following proposition follows immediately from the fact that inequality (1) holds for commuting elements (namely, $\|e^{x+y}\| \leqslant \|e^x\|\,\|e^y\|$).

Proposition 2.1. *Let $x = (x_1, \ldots, x_m)$ be a commuting m-tuple in $\mathcal{B}^m$. If x_j is of type s_j for $1 \leqslant j \leqslant m$ then x is of type $s = s_1 + \ldots + s_m$. If x_j is of Paley–Wiener type (s_j, r_j) for $1 \leqslant j \leqslant m$ then x is of Paley–Wiener type (s, r) where $s = s_1 + \ldots + s_m$ and $r = (r_1^2 + \ldots + r_m^2)^{1/2}$.*

To provide some initial examples of vectors having a Paley–Wiener type we consider the algebra $\mathcal{M}_n$ of n by n complex matrices provided with any suitable norm. For $A \in \mathcal{M}_n$, or more generally $A \in \mathcal{B}$, let $\sigma(A)$ denote the *spectrum* of A and $r(A)$ the *spectral radius*.

It is proved in McIntosh, Pryde and Ricker [2], for example, that each $A \in \mathcal{M}_n$ with real spectrum is of type $J(A)-1$, where $J(A)$ is the size of the largest block in the Jordan decomposition of A. The following proposition improves that result.

Proposition 2.2. *If $A \in \mathcal{M}_n$ has real spectrum then A is of Paley–Wiener type (s, r) where $s = J(A)-1$ and $r = r(A)$.*

Proof. From the Jordan decomposition theorem we know there exists an invertible matrix Q, a diagonal matrix D, and a nilpotent matrix N commuting with D such that $A = Q(D+N)Q^{-1}$. For $\zeta \in \mathbf{C}$,

$$e^{iA\zeta} = Qe^{iD\zeta}e^{iN\zeta}Q^{-1} = Qe^{iD\zeta}\sum_{k=0}^{s}\frac{1}{k!}(iN\zeta)^k Q^{-1},$$

from which the result follows.

From Propositions 2.1 and 2.2 we obtain the following result (which will be improved in Section 4).

Corollary 2.3. *Let $A = (A_1, \ldots, A_m)$ be an m-tuple of commuting matrices with real spectra. Then A is of Paley–Wiener type (s, r) where $s = J(A_1) + \ldots + J(A_m) - m$ and $r = (r(A_1)^2 + \ldots + r(A_m)^2)^{1/2}$.*

Other examples of m-tuples of vectors with a Paley–Wiener type will appear in subsequent sections. Our motivation for discussing them is in the construction of joint functional calculi. See for example [1]–[4].

Indeed, for a Banach space X, let $L(\mathcal{S}(\mathbf{R}^m), X)$ denote the space of *X-valued tempered distributions*. This is the space of continuous linear functions W from the Schwartz space $\mathcal{S}(\mathbf{R}^m)$ of rapidly decreasing functions into X.

An m-tuple $x \in \mathcal{B}^m$ of type s determines a $\mathcal{B}$-valued tempered distribution W_x by the formula

$$W_x(f) = (2\pi)^{-m}\int_{\mathbf{R}^m} e^{i\langle x, \xi\rangle}\hat{f}(\xi)\,d\xi$$

where $\hat{f}(\xi) = \int_{\mathbf{R}^m} e^{-i\langle\lambda, \xi\rangle} f(\lambda)\,d\lambda$ is the Fourier transform of the function $f \in \mathcal{S}(\mathbf{R}^m)$ and the first integral is a Bochner integral. So W_x is the Fourier transform of the entire function $\zeta \mapsto e^{i\langle x, \zeta\rangle}\colon \mathbf{C}^m \to \mathcal{B}$.

The type (s, r) conditions arise in the Paley–Wiener theorem below (whose proof follows readily from the corresponding theorem for scalar-valued distributions: for the latter, see for example Reed and Simon [6]).

Paley–Wiener Theorem 2.4. *Let $W \in L(\mathcal{S}(\mathbf{R}^m), X)$ where X is a Banach space. Then W has compact support if and only if W is the Fourier transform of an entire function $e\colon \mathbf{C}^m \to X$ satisfying $\|e(\zeta)\| \leqslant c(1+|\zeta|)^s e^{r|\operatorname{Im}\zeta|}$ for all $\zeta \in \mathbf{C}^m$ and some $c > 0$, $s \geqslant 0$, $r \geqslant 0$. In that case, W has support in $\{\lambda \in \mathbf{R}^m\colon |\lambda| \leqslant r\}$.*

Corollary 2.5. *Let $x \in \mathcal{B}^m$ be of Paley–Wiener type (s, r) for some $s \geqslant 0$, $r \geqslant 0$. Then the tempered distribution W_x has support in $\{\lambda \in \mathbf{R}^m\colon |\lambda| \leqslant r\}$.*

3. Normal elements of a C^*-algebra. There is a classical theorem of Lie for matrices that extends to unital Banach algebras:

Lie Product Formula 3.1. *Let $x, y \in \mathcal{B}$, a unital Banach algebra. Then*

$$e^{x+y} = \lim_{n\to\infty}(e^{x/n}e^{y/n})^n.$$

Proof. Reed and Simon [6, Theorem VIII.29] supply a proof for matrices which also holds in this setting.

THEOREM 3.2. *Let x, y be normal elements of a C^*-algebra $\mathscr{B}$. Then inequality (1) holds, namely*

$$\|e^{x+y}\| \leqslant \|e^x\|\,\|e^y\|.$$

Proof. When x is normal, so is e^x and $\|e^x\| = r(e^x)$, the spectral radius of e^x. Hence,

$$\|e^{x+y}\| = \|\lim_{n\to\infty}(e^{x/n}e^{y/n})^n\| \leqslant \lim_{n\to\infty}\|e^{x/n}\|^n\|e^{y/n}\|^n$$

$$= \lim_{n\to\infty}[r(e^{x/n})r(e^{y/n})]^n = \lim_{n\to\infty} r(e^x)r(e^y) = \|e^x\|\,\|e^y\|.$$

This result was obtained, with a similar proof, by Taylor [7], for the case of a self-adjoint operator x and a skew-adjoint operator y.

Thompson [8] obtained a stronger result for Hermitian matrices x, y and any unitarily invariant norm:

$$\|e^{x+y}\| \leqslant \|e^x e^y\|. \tag{6}$$

For unbounded operators on a Hilbert space, weaker estimates are valid. For example, Segal's lemma [6, Theorem X.57] asserts that for semibounded self-adjoint operators x, y for which the closure of $x+y$ is self-adjoint,

$$\|e^{-x-y}\| \leqslant \|e^{-x/2}e^{-y}e^{-x/2}\|. \tag{7}$$

COROLLARY 3.3. *If $x_1, \ldots, x_m$ are self-adjoint elements of a C^*-algebra, then $x = (x_1, \ldots, x_m)$ is of Paley–Wiener type $(0, \|x\|)$ where $\|x\| = (\|x_1\|^2 + \ldots + \|x_m\|^2)^{1/2}$.*

Proof. For $\zeta \in \mathbf{C}^m$ set $\zeta = \xi + i\eta$ where $\xi, \eta \in \mathbf{R}^m$. Then $\|e^{i\langle x,\zeta\rangle}\| \leqslant \|e^{i\langle x,\xi\rangle}\|\,\|e^{-\langle x,\eta\rangle}\| \leqslant e^{\|x\|\,|\eta|}$.

4. Triangular matrices. In this section we prove an inequality for matrices that allows us to generalize Corollary 2.3.

Let $\mathscr{T}_n$, $\mathscr{T}_n^0$ and $\mathscr{D}_n$ denote the subalgebras of $\mathscr{M}_n$ consisting of upper triangular, nilpotent upper triangular and diagonal matrices respectively. For each $A = (A_{ij})$ in $\mathscr{M}_{m,n}$, the space of m by n matrices, define a matrix $\operatorname{mod} A \in \mathscr{M}_{m,n}$ by $(\operatorname{mod} A)_{ij} = |A_{ij}|$.

Example 1.1 shows that there is no constant c such that for all $D \in \mathscr{D}_3$ and $N \in \mathscr{T}_3^0$,

$$\|e^{D+N}\| \leqslant c\|e^D\|\,\|e^N\|. \tag{8}$$

However, a weaker version of this inequality, with N replaced by $\operatorname{mod} N$ on the right side, is valid. Firstly we require

LEMMA 4.1. *Let $T \in \mathscr{M}_n$ be a matrix of the form $T = \left[\begin{smallmatrix} A & B \\ 0 & C\end{smallmatrix}\right]$ where $A \in \mathscr{M}_k$, $B \in \mathscr{M}_{k,n-k}$ and $C \in \mathscr{M}_{n-k}$. Then for each $\xi \in \mathbf{R}$,*

$$e^{T\xi} = \begin{bmatrix} e^{A\xi} & Q(\xi) \\ 0 & e^{C\xi}\end{bmatrix} \quad \text{where} \quad Q(\xi) = \int_0^\xi e^{At}Be^{C(\xi-t)}\,dt.$$

Proof. Let $f(\xi) = \begin{bmatrix} e^{A\xi} & Q(\xi) \\ 0 & e^{C\xi}\end{bmatrix}$. Then $f'(\xi) = \begin{bmatrix} e^{A\xi}A & e^{A\xi}B + Q(\xi)C \\ 0 & e^{C\xi}C\end{bmatrix} = f(\xi)T$. Since $f(0) = I$, the identity matrix, we conclude that $f(\xi) = e^{T\xi}$.

LEMMA 4.2. *Let $D \in \mathscr{D}_n$, $N \in \mathscr{T}_n^0$ and $T = D+N$. Then*

$$|(e^T)_{ij}| \leqslant r(e^D)(e^{\operatorname{mod} N})_{ij} \quad \text{for } 1 \leqslant i, j \leqslant n. \tag{9}$$

Proof. Since the result is trivial for $n = 1$ we use induction on n. Write $T = \left[\begin{smallmatrix} A & B \\ 0 & \alpha\end{smallmatrix}\right]$ where $A \in \mathscr{M}_{n-1}$, $B \in \mathscr{M}_{n-1,1}$ and $\alpha \in \sigma(D)$, the spectrum of D. So $A = D_1 + N_1$ where

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & \alpha\end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} N_1 & B \\ 0 & 0\end{bmatrix}.$$

Defining $g(A, \alpha) = \int_0^1 e^{At+\alpha(1-t)I}\,dt$ and using Lemma 4.1, we find

$$e^T = \begin{bmatrix} e^A & g(A,\alpha)B \\ 0 & e^\alpha\end{bmatrix}, \quad e^D = \begin{bmatrix} e^{D_1} & 0 \\ 0 & e^\alpha\end{bmatrix},$$

$$e^{\operatorname{mod} N} = \begin{bmatrix} e^{\operatorname{mod} N_1} & g(\operatorname{mod} N_1, 0)\operatorname{mod} B \\ 0 & 1\end{bmatrix}.$$

For $1 \leqslant i, j \leqslant n-1$ it follows from the induction hypothesis that $|(e^T)_{ij}| \leqslant r(e^D)(e^{\operatorname{mod} N})_{ij}$. For $i = n$ and $1 \leqslant j \leqslant n$ the result is trivial. Finally, suppose $1 \leqslant i \leqslant n-1$ and $j = n$. Then

$$|(e^T)_{in}| = |(g(A,\alpha)B)_i| \leqslant \int_0^1 \sum_{j=1}^{n-1} |(e^{At+\alpha(1-t)I})_{ij}|\,|B_j|\,dt$$

$$\leqslant \int_0^1 \sum_{j=1}^{n-1} r(e^{D_1t+\alpha(1-t)I})(e^{\operatorname{mod} N_1t})_{ij}(\operatorname{mod} B)_j\,dt$$

(by the induction hypothesis again)

$$\leqslant r(e^D)\int_0^1 (e^{\operatorname{mod} N_1t}\operatorname{mod} B)_i\,dt = r(e^D)(g(\operatorname{mod} N, 0)\operatorname{mod} B)_i$$

$$= r(e^D)(e^{\operatorname{mod} N})_{in}, \quad \text{as required.}$$

From Lemma 4.2 there follows immediately

THEOREM 4.3. *For any norm on $\mathcal{T}_n$ there is a constant c such that for all $D \in \mathcal{D}_n$ and all $N \in \mathcal{T}_n^0$ inequality* (3) *holds, namely* $\|e^{D+N}\| \leqslant c\|e^D\|\,\|e^{\operatorname{mod} N}\|$.

We will define a class of norms for which the constant in this last inequality is $c = 1$.

For this, let $\mathcal{M}$ be a subspace of $\mathcal{M}_n$. We shall call a norm $\|\ \|$ *monotone* on $\mathcal{M}$ if $\|A\| \leqslant \|B\|$ whenever $A, B \in \mathcal{M}$ and $|A_{ij}| \leqslant |B_{ij}|$ for all i, j.

For example, let $\|\ \|_p$ denote the operator norm on $\mathcal{M}_n$ induced by the l_p norm on $\mathbf{C}^n$ for $1 \leqslant p \leqslant \infty$. Then $\|A\|_1 = \max_j \sum_i |A_{ij}|$ and $\|A\|_\infty = \max_i \sum_j |A_{ij}|$ are monotone operator norms on $\mathcal{M}_n$. The Frobenius norm $\|A\|_F = (\sum_{i,j} |A_{ij}|^2)^{1/2}$ is also monotone on $\mathcal{M}_n$. It is easy to see that the only monotone operator norm on $\mathcal{D}_n$ is $\|D\| = r(D)$.

From Lemma 4.2 we obtain

PROPOSITION 4.4. *For any monotone norm on $\mathcal{T}_n$ and all $D \in \mathcal{D}_n$ and $N \in \mathcal{T}_n^0$*

$$\|e^{D+N}\| \leqslant r(e^D)\|e^{\operatorname{mod} N}\|. \tag{10}$$

In particular, for any monotone operator norm on $\mathcal{T}_n$, inequality (3) *holds with* $c = 1$.

We come now to an application of inequalities (3) and (10). We will use the *spectral set*

$$\gamma(A) = \{\lambda \in \mathbf{R}^m:\ 0 \in \sigma(\sum_{j=1}^{m} (A_j - \lambda_j)^2)\}$$

and the corresponding *joint spectral radius* $r(A) = \sup\{|\lambda|:\ \lambda \in \gamma(A)\}$ defined for m-tuples $A = (A_1, \ldots, A_m) \in \mathcal{M}_n^m$. This set was also used in [1]–[4].

THEOREM 4.5. *Let $A = (A_1, \ldots, A_m)$ where the A_j are simultaneously triangularizable matrices in $\mathcal{M}_n$ with real spectra. Then A is of Paley–Wiener type (s, r) where $s = n-1$ and $r = r(A)$.*

Proof. Let Q be a matrix such that for each j, $Q^{-1}A_jQ = T_j = D_j + N_j$ where $D_j \in \mathcal{D}_n$ and $N_j \in \mathcal{T}_n^0$. Set $T = (T_1, \ldots, T_m)$, $D = (D_1, \ldots, D_m)$ and $N = (N_1, \ldots, N_m)$. For $\xi, \eta \in \mathbf{R}^m$ and $\zeta = \xi + i\eta$ we have $e^{i\langle A,\zeta\rangle} = Qe^{i\langle T,\zeta\rangle}Q^{-1}$.

Take $\|\ \| = \|\ \|_1$ for example. Then

$$\begin{aligned}\|e^{i\langle A,\zeta\rangle}\| &\leqslant \|Q\|\,\|Q^{-1}\|\,\|e^{i\langle T,\zeta\rangle}\| \leqslant \|Q\|\,\|Q^{-1}\|\,r(e^{i\langle D,\zeta\rangle})\,\|e^{\operatorname{mod}\langle N,\zeta\rangle}\| \\ &\leqslant \|Q\|\,\|Q^{-1}\|\,r(e^{-\langle D,\eta\rangle})\left\|\sum_{k=0}^{n-1}\frac{1}{k!}(\operatorname{mod}\langle N,\zeta\rangle)^k\right\| \\ &\leqslant \|Q\|\,\|Q^{-1}\|\,e^{r(D)|\eta|}\sum_{k=0}^{n-1}\frac{1}{k!}\|N\|^k|\zeta|^k \leqslant c(1+|\zeta|)^{n-1}e^{r(A)|\eta|}\end{aligned}$$

where

$$\|N\| = (\sum_{j=1}^{m}\|N_j\|^2)^{1/2} \quad\text{and}\quad c = \|Q\|\,\|Q^{-1}\|\max_{0\leqslant k\leqslant n-1}\frac{(n-k-1)!}{(n-1)!}\|N\|^k.$$

In order to prove analogous results for operators on infinite-dimensional Hilbert space we obtain a better value for the constant c in the proof of Theorem 4.5. Recall that the spectral norm $\|\ \|_2$ on $\mathcal{M}_n$ is the operator norm induced by the l_2 norm on $\mathbf{C}^n$.

PROPOSITION 4.6. *Let $A_1, \ldots, A_m$ be simultaneously triangularizable matrices in $\mathcal{M}_n$ with real spectra and set $\delta = 2(\sum_{j=1}^m \|A_j\|_2^2)^{1/2}$. Then for all $\zeta \in \mathbf{C}^m$*

$$\|e^{i\langle A,\zeta\rangle}\|_2 \leqslant (1+\delta|\zeta|)^{n-1}e^{r(A)|\operatorname{Im}\zeta|}. \tag{11}$$

Proof. We use the notation of the proof of Theorem 4.5. The triangularization of the matrices A_j can be effected by a unitary similarity matrix Q. (Perhaps the simplest proof of this fact is by induction on n, using the fact that the A_j have a common unit eigenvector and the result of Radjavi [5] that a semigroup $\mathcal{S}$ of matrices is triangularizable if and only if $\operatorname{trace}(ABC - BAC) = 0$ for all $A, B, C \in \mathcal{S}$.) Now recall that the spectral and Frobenius norms are related by $\|A\|_2 \leqslant \|A\|_F \leqslant (\operatorname{rank} A)^{1/2}\|A\|_2$ for all $A \in \mathcal{M}_n$. Moreover, $N_j = T_j - D_j$, so $\|N_j\|_2 \leqslant \|T_j\|_2 + \|D_j\|_2 = \|T_j\|_2 + r(D_j) \leqslant 2\|T_j\|_2 = 2\|A_j\|_2$. Hence, $\|N_j\|_F \leqslant 2\sqrt{n-1}\,\|A_j\|_2$ and $\|N\|_F \leqslant \delta\sqrt{n-1}$.

By inequality (10),

$$\begin{aligned}\|e^{i\langle A,\zeta\rangle}\|_2 = \|e^{i\langle T,\zeta\rangle}\|_2 &\leqslant \|e^{i\langle T,\zeta\rangle}\|_F \\ &\leqslant r(e^{i\langle D,\zeta\rangle})\|e^{\operatorname{mod}\langle N,\zeta\rangle}\|_F \\ &\leqslant e^{r(A)|\eta|}\sum_{k=0}^{n-1}\frac{1}{k!}\|N\|_F^k|\zeta|^k \leqslant (1+\delta|\zeta|)^{n-1}e^{r(A)|\eta|}\end{aligned}$$

since

$$\max_{0\leqslant k\leqslant n-1}\frac{(n-k-1)!}{(n-1)!}(n-1)^{k/2} = 1.$$

5. Triangular operators on Hilbert space. Let $\mathcal{B} = \mathcal{B}(H)$, the space of bounded linear operators on the separable Hilbert space H. We exhibit noncommuting but triangularizable m-tuples in $\mathcal{B}$ of Paley–Wiener type.

For $x = (x_1, \ldots, x_m) \in \mathcal{B}^m$ we again set $\gamma(x) = \{\lambda \in \mathbf{R}^m:\ 0 \in \sigma(\sum_{j=1}^m (x_j - \lambda_j)^2)\}$ and $r(x) = \sup\{|\lambda|:\ \lambda \in \gamma(x)\}$.

We shall impose the following condition on $x_1, \ldots, x_m$.

CONDITION 5.1. *There is an orthogonal decomposition $H = \bigoplus_{k=1}^\infty H_k$ where for each k and each j, the subspace H_k is an x_j-invariant subspace of H. Moreover, H_k is of finite dimension $n_k \leqslant n$ and the restriction of $x = (x_1, \ldots, x_m)$ to H_k is triangularizable.*

PROPOSITION 5.2. *Let $x_1, \ldots, x_m$ be operators in $\mathscr{B}(H)$ with real spectra and satisfying condition* 5.1. *Let $\delta = 2(\sum_{j=1}^m \|x_j\|^2)^{1/2}$. Then for all $\zeta \in \mathbf{C}^m$*

$$\|e^{i\langle x,\zeta\rangle}\| \leqslant (1+\delta|\zeta|)^{n-1} e^{r(x)|\mathrm{Im}\,\zeta|}. \tag{12}$$

In particular, $(x_1, \ldots, x_m)$ is of Paley–Wiener type $(n-1, r(x))$.

Proof. Relative to the decomposition $H = \bigoplus_{k=1}^\infty H_k$ we have $x_j = \bigoplus_{k=1}^\infty x_{jk}$ where $x_{jk} \in \mathscr{B}(H_k)$ and $\sigma(x_{jk}) \subseteq \mathbf{R}$. For each k the m-tuple $x_{\cdot k} = (x_{1k}, \ldots, x_{mk})$ is triangularizable in $\mathscr{B}(H_k)$. By Proposition 4.6

$$\|e^{i\langle x_{\cdot k},\zeta\rangle}\| \leqslant (1+\delta_k|\zeta|)^{n_k-1} e^{r(x_{\cdot k})|\eta|}$$

for all $\zeta \in \mathbf{C}^m$, where $\delta_k = 2(\sum_{j=1}^m \|x_{jk}\|^2)^{1/2}$ and $\eta = \mathrm{Im}\,\zeta$.

Now $\|x_j\| = \sup_k \|x_{jk}\|$, so $\delta_k \leqslant \delta$. Also $\gamma(x) = \bigcup_{k=1}^\infty \gamma(x_{\cdot k})$, so $r(x) = \sup_k r(x_{\cdot k})$. Finally, $x = \bigoplus_{k=1}^\infty x_{\cdot k}$, so $e^{i\langle x,\zeta\rangle} = \bigoplus_{k=1}^\infty e^{i\langle x_{\cdot k},\zeta\rangle}$. Hence

$$\|e^{i\langle x,\zeta\rangle}\| = \sup_k \|e^{i\langle x_{\cdot k},\zeta\rangle}\| \leqslant (1+\delta|\zeta|)^{n-1} e^{r(x)|\eta|}$$

as claimed.

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DEPARTMENT OF MATHEMATICS
MONASH UNIVERSITY
Clayton, Victoria 3168, Australia

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