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A model for some analytic Toeplitz operators

by

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Abstract. We present a change of variable method and use it to prove the equivalence to bundle shifts for certain analytic Toeplitz operators on the Banach spaces $H^p(\mathcal{G})$ $(1 \leqslant p < \infty)$. In Section 2 we see this approach applied in the analysis of essential spectra. Some partial results were obtained in [9] in the Hilbert space case.

1. Functional model. Given a bounded plane domain $\mathcal{G}$ and a nonconstant function $\varphi \in H^\infty(\mathcal{G})$, we consider the multiplication by φ operator $\mathrm{T} = T_\varphi$ acting on the Hardy space $H^p(\mathcal{G})$, where $1 \leqslant p < \infty$. Let $\Omega = \varphi(\mathcal{G})$. Since for complicated symbols φ the handling of this operator presents many difficulties, it is quite useful to know whether T is isometrically equivalent to a "shift" T_E of the Hardy class $H^p[E]$ related to some analytic vector bundle E over Ω. Indeed, T_E is the multiplication by a perfectly simple function: the complex coordinate, i.e. $(T_E f)(\lambda) = \lambda f(\lambda)$ for $f \in H^p[E]$. (The basic notation can be found in [1] and [8]. By an *analytic vector bundle* over Ω we mean here a complex manifold E together with a holomorphic projection π: $E \to \Omega$ whose fibres $E_\lambda = \pi^{-1}\{\lambda\}$ are Banach spaces linked in a regular manner: for any point λ we can find its open neighbourhood U, a Banach space K and an analytic isomorphism $\pi^{-1} U \to K \times U$ whose restriction to E_λ is a linear isometry onto $K \times \{\lambda\}$. The space $H^p[E]$ consists of those analytic mappings f: $\Omega \to E$ which are cross-sections of E (i.e. satisfy $\pi(f(\lambda)) = \lambda$, $\lambda \in \Omega$) and for which the function $\lambda \to \|f(\lambda)\|^p$ has a harmonic majorant on Ω.) Our method depends on the properties of the set Ω rather than on the domain $\mathcal{G}$, which can even be replaced (under suitable conditions) by a Riemann surface. Since the shift T_E shows some "inner-like" behaviour, the natural requirement on φ is that it "maps the boundary $(\partial\mathcal{G})$ of $\mathcal{G}$ into $\partial\Omega$" in the sense described by our "boundary condition" (b) given below. The latter has clear motivation in the case $p = 2$, when analytic Toeplitz operators are subnormal. A characteristic feature of the subnormals S equivalent to bundle shifts over Ω is that the minimal normal extension N of S satisfies $\sigma(N) \subseteq \partial\Omega$, while $\sigma(S) \subseteq \bar{\Omega}$.

Let us fix some notation. We can always take analytic universal covers τ: $\mathbf{D} \to \mathcal{G}$ of $\mathcal{G}$ and t: $\mathbf{D} \to \Omega$ of Ω with $\varphi(\tau(0)) = t(0)$. Let μ be the normalized

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Lebesgue measure on $\partial\mathbf{D}$. For $h \in H^p(\mathbf{D})$ denote by $\tilde{h} \in L^p(\mu)$ its boundary value (i.e. $h(rz) \to \tilde{h}(z)$ as $r \to 1^-$ for a.e. $z \in \partial\mathbf{D}$). Now we can formulate our first (necessary) condition:

(b) $(\varphi \circ \tau)^{\sim}(z) \in \partial\Omega$ *for almost every* $[\mu]$ *point* $z \in \partial\mathbf{D}$.

Let us define ϱ to be $(\varphi \circ \tau)^{\sim}$ treated as a measurable function on $\partial\mathbf{D}$. Now (b) simply states that $\mu(\varrho^{-1}\Omega) = 0$. Using the invariance of harmonic measures (statement (*) below), one can obtain an equivalent version of (b) in terms of cluster sets:

(b′) $\mathrm{cl}(\varphi, \lambda) \subset \partial\Omega$ *for a.e.* $\lambda \in \partial\mathscr{G}$.

Here a.e. means "almost everywhere w.r.t. harmonic measure for $\mathscr{G}$". If $\mathscr{G}$ is the unit disc $\mathbf{D}$, then the above condition holds when φ is either injective, a universal cover or an inner function. However, for $\mathscr{G} = \{z \in \mathbf{D}\colon 0 < \operatorname{Arg} z < 3\pi/2\}$, when $\varphi(z) = z^2$, it fails. The assumption (b) is, in a sense, justified by the following fact. (Here we assume for simplicity that $\partial\mathscr{G}$ consists of pairwise disjoint circles accumulating only on a countable set of points, cf. [10].)

1.1. Lemma. *If the condition* (b) *fails, then even for* $p = 2$ *our* T *cannot be equivalent to any bundle shift over* Ω.

Proof. First we show that $\operatorname{ess.rg}(\varrho) \cap \Omega \neq \emptyset$. If not, then to any point of Ω we can associate an open neighbourhood V with $\mu(\varrho^{-1}V) = 0$. By taking a countable subcover, we reach a contradiction: $\mu(\varrho^{-1}\Omega) = 0$. For $p = 2$, the minimal normal extension N of T can be identified (cf. [1], [8] and for the minimality [6], [10]) with multiplication by ϱ on the subspace of $L^2(\mu)$ composed of the functions automorphic w.r.t. the cover τ. Since the indicator (i.e. characteristic) function of the pre-image $\varrho^{-1}W$ of any set W is also automorphic, it is easy to verify that $\sigma(N) = \operatorname{ess.rg}(\varrho)$. ∎

Our remaining assumptions are of geometric nature and are satisfied by any finitely connected domain Ω. We summarize them as follows.

(g) $\varphi(\mathscr{G}) = \Omega$ *is conformally equivalent to a domain* Ω_1 *of Parreau–Widom type on which the Direct Cauchy Theorem holds and whose inner boundary has Hausdorff dimension less than one. Additionally, we assume that* $H^\infty(\mathscr{G})$ *is dense in* $H^p(\mathscr{G})$.

Although our assumptions are weaker (for $p = 2$) than the conditions of [8] necessary in the general setup, we have the following result.

1.2. Theorem. *If a nonconstant* $\varphi \in H^\infty(\mathscr{G})$ *satisfies* (b) *and* (g) *then* T_φ *is isometrically equivalent to a bundle shift over* Ω.

Proof. We begin by showing that the proof can be reduced to the case when (g) is satisfied with $\Omega = \Omega_1$. Indeed, in the general case denote by $\varkappa\colon \Omega_1 \to \Omega$ the conformal bijection and let $S = T_\psi$, where $\psi = \varkappa^{-1} \circ \varphi$. If φ satisfies (b), or (b′), then so does ψ. Suppose that we have shown the isometric equivalence of S to the shift T_F for a bundle F over Ω_1 (notation: $S \simeq T_F$). Now (g) implies (as in [8], p. 426) that $\varkappa$ is pointwise boundedly approximable on Ω_1 by a sequence of rational functions without poles in Ω_1^-. Consequently, in the sense of functional calculus, we have $T_\varphi = \varkappa(S) \simeq \varkappa(T_F)$, where $(\varkappa(T_F)h)(x) = \varkappa(z)h(z)$ for $h \in H^p[F]$, $z \in \Omega_1$. [That the action of a rational function, say r, of the given multiplication operator (by ξ) acts as multiplication (by $r \circ \xi$) is obvious and the mentioned approximability concludes the argument.] The conformal change of variable $w = \varkappa(z)$ induces the bundle E over Ω, corresponding to F, so that the isometry $f \to f \circ \varkappa^{-1}$ from $H^p[F]$ onto $H^p[E]$ carries $\varkappa(T_F)$ onto the bundle shift T_E. (The norming points, say $\mathcal{O}$, $\mathcal{O}_1$, should be chosen to satisfy $\mathcal{O} = \varkappa(\mathcal{O}_1)$.)

Thus, we have reduced the problem to the case $\Omega = \Omega_1$, with $S = T_\varphi$ satisfying (b) and (g). As before, put $\varrho = (\varphi \circ \tau)^{\sim}$. Define the measure ω on $\partial\Omega$ by $\omega = \varrho(\mu)$. In other words, $\int h\,d\omega = \int (h \circ \varrho)\,d\mu$ for any $h \in \mathscr{C}(\partial\Omega)$. From [8] or [9], we know that

(*) $\omega = \tilde{t}(\mu)$ *and* ω *is the harmonic measure for* Ω *evaluating at* $t(0)$.

Now, by the isometry $g \to g \circ \tau$ from $H^p(\mathscr{G})$ onto some subspace $\mathscr{N}$ of $H^p(\mathbf{D})$, we have $S \simeq T_{\varphi\circ\tau}|_{\mathscr{N}}$. Hence multiplication by $\varrho = (\varphi \circ \tau)^{\sim}$ on $L^p(\mu)$ is isometrically equivalent to an extension of S. The method of representing the operator of multiplication by ϱ as multiplication by the coordinate function on the direct L^p-integral $\mathscr{J}_p = \int^{\oplus} L^p(\nu_y)\,d\omega(y)$ of Banach spaces, presented in [2] for $p = 2$, applies also for $1 \leqslant p < \infty$. Here ν_y $(y \in \partial\Omega)$ are the measures on $\partial\mathbf{D}$ obtained by disintegration of μ with respect to ϱ and the space $\mathscr{J}_p$ consists of the functions $f(y, z)$, $y \in \partial\Omega$, $|z| = 1$, with $f(y, \cdot) \in L^p(\nu_y)$ for almost all $y \in \partial\Omega$ and such that the $L^p(\nu_y)$-norms $\|f(y, \cdot)\|_p$ define an $L^p(\omega)$-function in y. We also often write $f(y)$ for $f(y, \cdot)$, treating f as a vector-valued function. The norm in $\mathscr{J}_p$ is defined by

$$\|f\| = \left(\int \|f(y, \cdot)\|_p^p\,d\omega(y)\right)^{1/p}.$$

Given $F \in L^p(\mu)$, the mentioned correspondence defines $F(y, \cdot)$ as the element of $L^p(\nu_y)$ determined by F and it sends $\mathscr{N}$ onto a subspace $\mathscr{M}$ of $\mathscr{J}_p$. Thus, $S \simeq T_z|_{\mathscr{M}}$, where $(T_z f)(y, z) = zf(y, z)$. Now, our result will follow from analytic description of the subspace $\mathscr{M}$.

Clearly, $\mathscr{M}$ is pure invariant under multiplication by z:

(**) $z\mathscr{M} \subseteq \mathscr{M}$ and if $\mathscr{M}_1 \subseteq \mathscr{M}$, $\bar{z}\mathscr{M}_1 \subseteq \mathscr{M}_1 \supseteq z\mathscr{M}_1$, then $\mathscr{M}_1 = \{0\}$.

Indeed, if a subset $\mathscr{M}_1 \neq \{0\}$ of $\mathscr{M}$ satisfies the condition in (**), then so does its closed linear span and, by the Stone–Weierstrass theorem, $\mathscr{M}_1$ is invariant under multiplication by any continuous function on $\partial\Omega$—in particular, by z^{-k}. Hence (after returning to $\mathscr{N}$) we obtain a nonzero analytic function having a zero of arbitrarily high order.

Note that by (*), the mapping $f \to f \circ t$ is an isometry from $\mathcal{J}_p$ onto a subspace of $\int^{\oplus} L^p(v_{t(x)})\,d\mu(x)$ for any p. Here we write t instead of $\tilde{t}$ for notational convenience. Let $\mathcal{W}$ stand for the corresponding image of $\mathcal{M}$ ($\mathcal{W} = \mathcal{M} \circ t$). To our $\mathcal{W}$ there applies (a part of) the proof of Theorem 4.1 of [8]. But, unlike in [8], the elements of $\mathcal{W}$ are functions taking values in Banach (not Hilbert – unless $p = 2$) spaces and these spaces vary with x, $|x| = 1$. For $p = 2$ the latter difficulty is not essential: $\mathcal{J}_2$ is always a subspace of $L^2_X(\omega)$ for some Hilbert space X (cf. [2]). The version of Beurling's theorem ([5], Th. 9) for Hilbert-space-valued functions then applies, making all other steps of the analogous proof from [8] available. To handle the general case one needs the following version of Beurling's theorem for Banach-space-valued functions. It does not seem to have appeared in the literature.

Consider a measurable family of Borel measures $\varrho_x \geqslant 0$ on $\partial\mathbf{D}$, of mass 1. For a subspace $\mathcal{V}$ of $\int^{\oplus} L^p(\varrho_x)\,d\mu(x)$ put $\mathcal{V} \cap L^q = \{f \in \mathcal{V}\colon f(x) \in L^q(\varrho_x)$ for a.e. x, with $(x \to \|f(x)\|_q) \in L^q(\mu)\}$.

1.3. THEOREM. *Let $\mathcal{V}$ be a subspace of $\mathcal{J} = \int^{\oplus} L^p(\varrho_x)\,d\mu(x)$, invariant under multiplication by z and pure in the sense of* (**). *If $\mathcal{V} \cap L^2$ is dense in $\mathcal{V}$, then there exists a subspace $\mathcal{R}$ of $\mathcal{J}$ and a decomposable isometry $\Psi\colon L^p_{\mathcal{R}} \to \mathcal{J}$ such that $\mathcal{V} = \Psi H^p_{\mathcal{R}}$ and Ψ restricted to $L^p_{\mathcal{R}} \cap L^2$ is isometric into $\mathcal{J} \cap L^2$ in the respective Hilbert space (L^2) norms. The latter condition makes Ψ unique up to a constant unitary factor.*

Proof. Assume first that $1 \leqslant p \leqslant 2$. $\mathcal{V} \cap L^2$ is then a subspace of $\mathcal{J} \cap L^2$, closed in the L^2-norm and pure invariant (by (**)). Theorem 9 of [5] applies. The careful reading of its proof shows that the space $\mathcal{R}$ corresponding to $\mathcal{V} \cap L^2$, call it $\mathcal{R}_2$, can be chosen to be a subspace of $\mathcal{J}_2$. Moreover, the decomposable isometry Ψ takes then the canonical form (induced by inclusion) and it is a "universal isometry", i.e. it carries L^q-norms onto L^q-norms. As $p \leqslant 2$, the completions of $\mathcal{V} \cap L^2$ and of $H^2_{\mathcal{R}_2}$ in the L^p-norm are $\mathcal{V}$ and $H^p_{\mathcal{R}}$ respectively, while Ψ extends to a decomposable isometry between these two spaces. The case $2 \leqslant p < \infty$ follows by a duality argument. ■

Since $H^\infty(\mathcal{G})$ is dense in $H^p(\mathcal{G})$, the nature of all mappings defined so far yields the density of $\mathcal{M} \cap L^\infty$ in $\mathcal{M}$ and the appropriate spaces appearing in the proof of Theorem 4.1 in [8] satisfy the conditions of our Theorem 1.3. Now the rest of the proof of Theorem 1.2 proceeds as in [8]. ■

2. The essential spectra. In [3] the essential spectrum $\sigma_e(T_\varphi)$ is described as the union of the cluster sets $\mathrm{cl}(\varphi, y)$ over the set $\beta_e(\mathcal{G}, p)$ of all points $y \in \partial\mathcal{G}$ that are not "meromorphically removable" for $H^p(\mathcal{G})$. For domains with "small" inner boundary, $\beta_e(\mathcal{G}, p) = \partial\mathcal{G}$ and then $\sigma_e(T_\varphi)$ is the *global cluster set* of φ. By Theorem 1.2, there is another way of finding $\sigma_e(T_\varphi)$ if φ satisfies (b) and (g).

From now on, we fix $\mathrm{S} = T_E\colon H^p[E] \to H^p[E]$, the shift of an analytic bundle E over a bounded domain $\Omega \subseteq \mathbf{C}$. Since there exists a functional calculus in S, "based on $H^\infty(\Omega)$", $\sigma_e(\mathrm{S}) \subseteq \bar{\Omega}$ (cf. [7]).

Let k be the dimension (the same $\forall y \in \Omega$) of the fibre E_y of E over a point $y \in \Omega$. Then k is the codimension of $(\mathrm{S} - y)H^p[E] = \ker(f \to f(y))$, as one easily verifies. Also $\ker(\mathrm{S} - y) = \{0\}$. Hence,

(2.1) *$\sigma_e(\mathrm{S})$ is contained in $\partial\Omega$ if $k < \infty$, and is equal to $\bar{\Omega}$ otherwise.*

In special cases, this result can be strengthened:

(2.2) *If Ω is finitely connected and if $k < \infty$, then $\sigma_e(\mathrm{S}) = \partial\Omega$.*

Indeed, in this case S is similar to the shift $M \otimes \mathrm{I}$ of a trivial bundle $\Omega \times \mathbf{C}^k$ where I is the identity on $\mathbf{C}^k$ and M is the shift considered in [3] (Thm. 4.3), as follows from Theorem 2 of [1] applied to a smoothly bordered domain conformally equivalent to Ω.

Although the definition of quasisimilarity carries over verbatim to Banach space operators, the outstanding deep results on quasisimilar subnormal operators (like: $\sigma_e(\mathrm{T}) = \sigma_e(\mathrm{S})$) on Hilbert spaces lack any Banach space counterparts so far. The following application of Theorem 1.2 seems to be the first attempt in this direction.

2.3. COROLLARY. *If* T *and* S *are two quasisimilar Toeplitz operators on $H^p(\mathcal{G})$ with symbols $\varphi, \psi \in H^\infty(\mathcal{G})$ satisfying* (b) *and if $\Omega = \varphi(\mathcal{G}) = \psi(\mathcal{G})$ is finitely connected, then $\sigma_e(\mathrm{T}) = \sigma_e(\mathrm{S})$ and the operators are similar. If, moreover, φ is univalent and $\mathcal{G}$ is simply connected, then they are isometrically equivalent.*

Proof. To establish the second claim, check first that ψ is then injective (e.g. by noting that $\dim(\ker(\mathrm{T} - \lambda)^*) \leqslant 1$ for $\lambda \in \mathbf{C}$ implies the same for S). Now φ satisfies (b)&(g) with $\Omega = \varphi(\mathcal{G})$ and (2.2) yields $\sigma_e(\mathrm{T}) = \partial\Omega$. But the same is true of S, since $\varphi(\mathcal{G}) = \psi(\mathcal{G})$. To check the isometric equivalence, use Thm. B of [1]. As pointed out by the referee, one can also use a conformal change of variable in $\mathcal{G}$ that carries ψ to φ, and the corresponding isometry of L^p-spaces. In the case of $p = 2$, $\mathcal{G} = \mathbf{D}$, a part of the above corollary corresponds to a much stronger result of Cowen [4]. *Note also that if φ is injective, then* (b) *is satisfied automatically.*

Addendum. I wish to thank the referee for useful suggestions and for drawing my attention to a cycle of works related to [11], where an analytic functional model is given for a wide class of Toeplitz operators with rational, or even meromorphic symbols. The model is up to similarity, but the main difference is that in the case of a meromorphic function with no singularities (such as our φ) one obtains as its model the multiplication operator, whose symbol (the projection Π of the Riemann surface of φ) is not simpler than φ itself. The point in [11] is that while the symbol can have poles, the projection Π is regular. Multiplication by Π acts on spaces of functions on a bundle, unlike our T_E defined on spaces of cross-sections of E.

I also wish to thank Professor J. B. Conway for pointing out the following mistake made in my basic work [8]. The assumption 2.8(a),(b) in [8] (appearing in the main results) should have been made earlier and replace its weaker form (2.3). The gap in the proof of Proposition 2.5 resulted from the possibility of the situation where there are more than one nontrivial Gleason parts of $R(\bar{\Omega})$. This gap is then filled as follows. If U is the connected component of $\operatorname{int}(\bar{\Omega})$ containing Ω, then points of ∂U either belong to the same part of $R(\bar{\Omega})$ as U, or are peak points for $R(\bar{\Omega})$ (cf. Exercise 8 in Chapter VI of T. Gamelin's *Uniform Algebras*). Thus, any other nontrivial part of $R(\bar{\Omega})$ must lie in $\partial\Omega\setminus\partial U$, a subset of the inner boundary of Ω, which by (2.8)(a) is small. S decomposes as a direct sum of operators with spectra contained in the closures of nontrivial parts of $R(\bar{\Omega})$. All summands except one are normal, since $\partial\Omega\setminus\partial_f\Omega$, being a peak (hence closed) set of small Hausdorff dimension, has zero area, but the purity assumption implies that no nontrivial normal summands exist.

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Inequalities for exponentials in Banach algebras

by

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Abstract. For commuting elements x, y of a unital Banach algebra $\mathcal{B}$ it is clear that $\|e^{x+y}\| \leqslant \|e^x\|\,\|e^y\|$. On the other hand, M. Taylor has shown that this inequality remains valid for a self-adjoint operator x and a skew-adjoint operator y, without the assumption that they commute. In this paper we obtain similar inequalities under conditions that lie between these extremes. The inequalities are used to deduce growth estimates of the form $\|e^{i\langle x,\xi\rangle}\| \leqslant c(1+|\xi|)^s$ for all $\xi\in\mathbf{R}^m$, where $x=(x_1,\ldots,x_m)\in\mathcal{B}^m$ and c, s are constants.

1. Introduction. Let $\mathcal{B}$ be a unital Banach algebra. Then $e^{x+y}=e^xe^y$ for all commuting pairs $(x,y)\in\mathcal{B}^2$ and hence

$$\|e^{x+y}\| \leqslant \|e^x\|\,\|e^y\|. \tag{1}$$

In this paper we consider the validity of this inequality, and modifications of it, in the case that x, y do not commute. To begin with, let $\mathcal{B}=\mathcal{M}_3$, the space of 3 by 3 complex matrices, together with any norm. The following example shows that there is no constant $c>0$ such that for all $x, y\in\mathcal{M}_3$,

$$\|e^{x+y}\| \leqslant c\,\|e^x\|\,\|e^y\|. \tag{2}$$

EXAMPLE 1.1. Let $x=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$ and $y=\begin{bmatrix}0&t&-t^2/2\\0&0&t\\0&0&0\end{bmatrix}$ where $t\in\mathbf{R}$.

Then

$$e^x=\begin{bmatrix}e&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad e^y=\begin{bmatrix}1&t&0\\0&1&t\\0&0&1\end{bmatrix},\quad e^{x+y}=\begin{bmatrix}e&t(e-1)&t^2(e-3)/2\\0&1&t\\0&0&1\end{bmatrix}.$$

So $\|e^x\|=O(1)$, $\|e^y\|=O(t)$, $\|e^{x+y}\|=O(t^2)$ as $t\to\infty$.

Nevertheless, inequality (1) remains valid for certain classes of not necessarily commuting pairs x, y. For example, Taylor [7] obtained (1) for self-adjoint operators x and skew-adjoint y. In Section 3 we show that (1) holds whenever x, y are normal elements of a C^*-algebra.

In Section 4 we obtain a weaker estimate for triangular matrices. Indeed, let $\mathcal{T}_n$ be the space of n by n (upper) triangular matrices with any suitable norm.

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