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Weighted-BMO and the Hilbert transform

by

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Abstract. In 1967, E. M. Stein proved that the Hilbert transform is bounded from L^∞ to BMO. In 1976, Muckenhoupt and Wheeden gave an analogue of Stein's result. They gave a necessary and sufficient condition for the boundedness of the Hilbert transform from L_w^∞ to BMO_w. We improve the results of Muckenhoupt and Wheeden's and give a necessary and sufficient condition for the boundedness of the Hilbert transform from BMO_w to BMO_w.

Introduction. Let $f(x)$ and $w(x)$ be locally integrable in $\mathbf{R}^n$ and $w(x)\geqslant 0$. Then we say that $f\in \mathrm{BMO}_w(\mathbf{R}^n)$ if there is a constant C such that

$$\frac{1}{w(I)}\int_I |f(x)-f_I|\,dx\leqslant C$$

for all n-dimensional cubes I whose edges are parallel to the coordinate axes. Here $f_I=(1/|I|)\int_I f\,dx$, $w(I)=\int_I w\,dx$. The norm in $\mathrm{BMO}_w(\mathbf{R}^n)$ is defined as

$$\|f\|_{*,w}=\sup_I \frac{1}{w(I)}\int_I |f(x)-f_I|\,dx.$$

The case $w=1$ corresponds to that of John and Nirenberg.

A function f is said to belong to $L_w^\infty(\mathbf{R}^n)$ if $fw^{-1}\in L^\infty(\mathbf{R}^n)$. The norm in $L_w^\infty(\mathbf{R}^n)$ is defined as

$$\|f\|_{\infty,w}=\|fw^{-1}\|_\infty.$$

Finally, if there is a constant C such that

$$\int_{I^c}\frac{w(t)}{|x_I-t|^{2n}}\,dt\leqslant C\frac{1}{|I|^2}\int_I w(t)\,dt$$

for all cubes I, then we say $w\in B_2$. Here x_I is the center of I. From [1] we know $w\in A_2$ implies $w\in B_2$.

Only the case $n=1$ is considered in the following.

In [2] Muckenhoupt and Wheeden considered the modified version of the Hilbert transform: let

$$Hf(x)=\lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon}\left[\frac{1}{x-y}+\frac{\eta(y)}{y}\right]f(y)\,dy$$

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where $\eta(y)$ is the characteristic function of $\{|y| \geq 1\}$. The following results were given in [2]:

THEOREM A. *Let w be nonnegative and locally integrable. Then a necessary and sufficient condition that there exists a constant C such that*

$$\frac{1}{w(I)}\int_I |Hf-(Hf)_I|\,dx \leq C\|f\|_{\infty,w}$$

for all intervals I and all $f \in L_w^\infty(\mathbf{R})$ is that $w \in A_\infty \cap B_2$.

THEOREM B. *Let w be as above. Then a necessary and sufficient condition that there exists a constant C such that*

$$\left(\operatorname*{ess\,sup}_I \frac{1}{w}\right)\frac{1}{|I|}\int_I |Hf-(Hf)_I|\,dx \leq C\|f\|_{\infty,w}$$

for all intervals I and all $f \in L_w^\infty(\mathbf{R})$ is that $w \in A_1$.

THEOREM C. *Let w be as in Theorem A. Then a necessary and sufficient condition that there exists a constant C such that*

$$\left(\frac{1}{|I|}\int_I w^{-1/(p-1)}\,dx\right)^{p-1}\frac{1}{|I|}\int_I |Hf-(Hf)_I|\,dx \leq C\|f\|_{\infty,w}$$

for all intervals I and all $f \in L_w^\infty(\mathbf{R})$ is that $w \in A_p \cap B_2$.

Obviously, $L_w^\infty(\mathbf{R}) \subset \mathrm{BMO}_w(\mathbf{R})$ and $\|f\|_{*,w} \leq 2\|f\|_{\infty,w}$ if $f \in L_w^\infty(\mathbf{R})$. It is natural to ask whether H is a bounded operator from $\mathrm{BMO}_w(\mathbf{R})$ to $\mathrm{BMO}_w(\mathbf{R})$ under the condition in Theorem A and whether there are similar results with Theorems B and C. The answers are positive. Here are our results:

THEOREM 1. *Let w be as in Theorem A. Then a necessary and sufficient condition that there exists a constant C such that*

$$\frac{1}{w(I)}\int_I |Hf-(Hf)_I|\,dx \leq C\|f\|_{*,w}$$

for all intervals I and all $f \in \mathrm{BMO}_w(\mathbf{R})$ is that $w \in A_\infty \cap B_2$.

THEOREM 2. *A necessary and sufficient condition for*

$$\left(\operatorname*{ess\,sup}_I \frac{1}{w}\right)\frac{1}{|I|}\int_I |Hf-(Hf)_I|\,dx \leq C\|f\|_{*,w}$$

is that $w \in A_1$.

THEOREM 3. *A necessary and sufficient condition for*

$$\left(\frac{1}{|I|}\int_I w^{-1/(p-1)}\,dx\right)^{p-1}\frac{1}{|I|}\int_I |Hf-(Hf)_I|\,dx \leq C\|f\|_{*,w}$$

is that $w \in A_p \cap B_2$.

In Section 1 we establish two lemmas. In Section 2 we give the proofs of Theorems 1–3.

1. Two lemmas

LEMMA 1. *If $w \in B_2$, then there is a constant C such that*

$$\sum_{l=1}^{\infty}\frac{1}{2^{2l}}w(2^l I) \leq Cw(I)$$

for all intervals I. Here $2^l I$ is the interval having the same center as I and $|2^l I| = 2^l|I|$.

Proof. $w(x)dx$ is a doubling measure because $w \in B_2$. Thus there is a constant C such that $w(I) \leq Cw(E)$ for all I, where E is any subinterval of I with $|E| = \frac{1}{2}|I|$. Let x_I be the center of I. We have

$$\begin{aligned}\int_{x \notin I}\frac{w(x)}{(x-x_I)^2}\,dx &= \sum_{l=1}^{\infty}\int_{2^l I - 2^{l-1}I}\frac{w(x)}{(x-x_I)^2}\,dx \\ &\geq C\sum_{l=1}^{\infty}\int_{2^l I - 2^{l-1}I}\frac{w(x)}{2^{2l}|I|^2}\,dx \geq C\sum_{l=1}^{\infty}\frac{1}{2^{2l}|I|^2}\int_{2^l I} w(x)\,dx \\ &= \frac{C}{|I|^2}\sum_{l=1}^{\infty}\frac{1}{2^{2l}}w(2^l I).\end{aligned}$$

Using B_2 we finish the proof of Lemma 1. Here the second inequality is true since $w(x)dx$ is a doubling measure.

LEMMA 2. *Suppose $f \in \mathrm{BMO}_w(\mathbf{R})$, $w \in B_2$, x_I is the center of the interval I. Then*

$$(*)\qquad \int_{(2I)^c}\frac{1}{(y-x_I)^2}|f(y)-f_I|\,dy \leq \frac{C}{|I|^2}w(I)\|f\|_{*,w}.$$

Proof. In fact, the left side of $(*)$ is not greater than

$$\begin{aligned}C\sum_{l=1}^{\infty}\int_{2^{l+1}I-2^l I}\frac{1}{2^{2l}|I|^2}|f(y)-f_I|\,dy &\leq \frac{C}{|I|^2}\sum_{l=1}^{\infty}\int_{2^{l+1}I}\frac{1}{2^{2l}}|f(y)-f_I|\,dy \\ &\leq \frac{C}{|I|^2}\sum_{l=1}^{\infty}\frac{1}{2^{2l}}\Big[\int_{2^{l+1}I}|f-f_{2^{l+1}I}|\,dy+\int_{2^{l+1}I}|f_{2^{l+1}I}-f_I|\,dy\Big] \\ &\leq \frac{C}{|I|^2}\sum_{l=1}^{\infty}\frac{1}{2^{2l}}\big[\|f\|_{*,w}w(2^{l+1}I)+|2^{l+1}I|\,|f_{2^{l+1}I}-f_I|\big].\end{aligned}$$

From Lemma 1 we see that

$$\sum_{l=1}^{\infty}\frac{1}{2^{2l}}\|f\|_{*,w}w(2^{l+1}I) \leq C\|f\|_{*,w}w(I),$$

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{1}{2^{2i}}|2^{i+1}I|\,|f_{2^{i+1}I}-f_I| &\leqslant \sum_{i=1}^{\infty}\frac{1}{2^{i-1}}|I|\sum_{j=0}^{i}|f_{2^{j+1}I}-f_{2^jI}|\\ &\leqslant \sum_{i=1}^{\infty}\frac{|I|}{2^{i-1}}\sum_{j=0}^{i}\frac{1}{2^j|I|}\int_{2^{j+1}I}|f-f_{2^{j+1}I}|\,dx\\ &\leqslant \sum_{i=0}^{\infty}\frac{2}{2^i}\sum_{j=0}^{i}\frac{1}{2^j}w(2^{j+1}I)\|f\|_{*,w}\\ &= 2\sum_{j=0}^{\infty}\frac{1}{2^j}\sum_{i=j}^{\infty}\frac{1}{2^i}w(2^{j+1}I)\|f\|_{*,w}\\ &= 4\sum_{j=0}^{\infty}\frac{1}{2^{2j}}w(2^{j+1}I)\|f\|_{*,w}\\ &\leqslant Cw(I)\|f\|_{*,w}.\end{aligned}$$

2. Proof of Theorems 1–3

Sufficiency in Theorem 1. Suppose I is an interval, $f\in \mathrm{BMO}_w(\mathbf{R})$. Write f as

$$\begin{aligned}f(x) &= [f(x)-f_{2I}]\chi_{2I}(x)+[f_{2I}-f_I]\chi_{2I}(x)+[f-f_I]\chi_{(2I)^c}(x)+f_I\\ &= f_1(x)+f_2(x)+f_3(x)+f_4(x).\end{aligned}$$

First, we prove that there exists $r\in(1,\infty)$ such that $f_1\in L^r(dx)$. Since $w\in A_\infty$, we can find $r\in(1,\infty)$ such that the reverse Hölder inequality holds for w, that is, there exists a constant C such that

$$\left(\frac{1}{|I|}\int_I w(x)^r\,dx\right)^{1/r}\leqslant \frac{C}{|I|}\int_I w(x)\,dx$$

for all intervals.

Now we consider the local maximal function and the local $\#$-function. We define

$$f_I^*(x)=\sup_{\substack{J\ni x\\ J\subset 2I}}\frac{1}{|J|}\int_J|f(x)|\,dx,\qquad f_I^{\#}(x)=\sup_{\substack{J\ni x\\ J\subset 2I}}\frac{1}{|J|}\int_J|f-f_J|\,dx.$$

As $f\in\mathrm{BMO}_w(\mathbf{R})$, for any interval $S\subset\mathbf{R}$ we have

$$\frac{1}{|S|}\int_S|f-f_S|\,dx\leqslant\frac{1}{|S|}w(S)\|f\|_{*,w},$$

from which we see that $f_I^{\#}(x)\leqslant w_I^*(x)\|f\|_{*,w}$.

According to [3], p. 272, and the definition of f_1, there exists a constant C (which has nothing to do with $2I$) such that

$$\begin{aligned}\int_{\mathbf{R}}|f_1(x)|^r\,dx &= \int_{2I}|f-f_{2I}|^r\,dx\leqslant C\int_{2I}f_I^{\#}(x)^r\,dx\leqslant C\int_{2I}w_I^*(x)^r\,dx\cdot\|f\|_{*,w}^r\\ &\leqslant C\int_{2I}w(x)^r\,dx\cdot\|f\|_{*,w}^r\leqslant\frac{C}{|I|^{r-1}}\Big(\int_{2I}w\,dx\Big)^r\|f\|_{*,w}^r.\end{aligned}$$

It follows that $f_1\in L^r(\mathbf{R})$.

We denote by $\tilde f$ the Hilbert transform of f. It is easy to see that Hf_1 is locally integrable, so

$$\begin{aligned}\int_I|Hf_1-(Hf_1)_I|\,dx &\leqslant 2\int_I|\tilde f_1|\,dx\leqslant 2\Big(\int_{\mathbf{R}}|\tilde f_1|^r\,dx\Big)^{1/r}|I|^{1/r'}\qquad (1/r+1/r'=1)\\ &\leqslant C\Big(\int_{\mathbf{R}}|f_1|^r\,dx\Big)^{1/r}|I|^{1/r'}\leqslant C\|f\|_{*,w}\int_{2I}w(x)\,dx\cdot\frac{1}{|I|^{1-1/r}}|I|^{1/r'}\\ &= C\|f\|_{*,w}w(2I)\leqslant C\|f\|_{*,w}w(I).\end{aligned}$$

For f_2 we also easily get

$$\int_I|Hf_2-(Hf_2)_I|\,dx\leqslant C\|f\|_{*,w}w(I).$$

We now consider f_3. Let x_I be the center of I, and

$$a_I=Hf_3(x_I)=\lim_{\varepsilon\to0^+}\int_{|x_I-y|>\varepsilon}\left[\frac{1}{x_I-y}+\frac{\eta(y)}{y}\right]f_3(y)\,dy.$$

From Lemma 2 it is easy to see that $|a_I|<\infty$. Now,

$$\begin{aligned}\int_I|Hf_3-a_I|\,dx &= \int_I\left|\int_{\mathbf{R}}(f(t)-f_I)\chi_{(2I)^c}(t)\left(\frac{1}{x-t}-\frac{1}{x_I-t}\right)dt\right|dx\\ &\leqslant\int_{(2I)^c}|f(t)-f_I|\int_I\left|\frac{1}{x-t}-\frac{1}{x_I-t}\right|dx\,dt\\ &\leqslant C|I|^2\int_{(2I)^c}|f(t)-f_I|\frac{1}{(t-x_I)^2}\,dt\leqslant Cw(I)\|f\|_{*,w},\end{aligned}$$

the last inequality following from Lemma 2. We have thus finished the proof of sufficiency in Theorem 1.

It is clear that the necessity part of Theorem 1 is contained in Theorem A. As in the proofs of Theorems B–C, we can now easily obtain Theorems 2–3.

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A model for some analytic Toeplitz operators

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Abstract. We present a change of variable method and use it to prove the equivalence to bundle shifts for certain analytic Toeplitz operators on the Banach spaces $H^p(\mathscr{G})$ $(1 \leqslant p < \infty)$. In Section 2 we see this approach applied in the analysis of essential spectra. Some partial results were obtained in [9] in the Hilbert space case.

1. Functional model. Given a bounded plane domain $\mathscr{G}$ and a nonconstant function $\varphi \in H^\infty(\mathscr{G})$, we consider the multiplication by φ operator $\mathrm{T} = T_\varphi$ acting on the Hardy space $H^p(\mathscr{G})$, where $1 \leqslant p < \infty$. Let $\Omega = \varphi(\mathscr{G})$. Since for complicated symbols φ the handling of this operator presents many difficulties, it is quite useful to know whether T is isometrically equivalent to a "shift" T_E of the Hardy class $H^p[E]$ related to some analytic vector bundle E over Ω. Indeed, T_E is the multiplication by a perfectly simple function: the complex coordinate, i.e. $(T_E f)(\lambda) = \lambda f(\lambda)$ for $f \in H^p[E]$. (The basic notation can be found in [1] and [8]. By an *analytic vector bundle* over Ω we mean here a complex manifold E together with a holomorphic projection π: $E \to \Omega$ whose fibres $E_\lambda = \pi^{-1}\{\lambda\}$ are Banach spaces linked in a regular manner: for any point λ we can find its open neighbourhood U, a Banach space K and an analytic isomorphism $\pi^{-1}U \to K \times U$ whose restriction to E_λ is a linear isometry onto $K \times \{\lambda\}$. The space $H^p[E]$ consists of those analytic mappings f: $\Omega \to E$ which are cross-sections of E (i.e. satisfy $\pi(f(\lambda)) = \lambda$, $\lambda \in \Omega$) and for which the function $\lambda \to \|f(\lambda)\|^p$ has a harmonic majorant on Ω.) Our method depends on the properties of the set Ω rather than on the domain $\mathscr{G}$, which can even be replaced (under suitable conditions) by a Riemann surface. Since the shift T_E shows some "inner-like" behaviour, the natural requirement on φ is that it "maps the boundary $(\partial\mathscr{G})$ of $\mathscr{G}$ into $\partial\Omega$" in the sense described by our "boundary condition" (b) given below. The latter has clear motivation in the case $p = 2$, when analytic Toeplitz operators are subnormal. A characteristic feature of the subnormals S equivalent to bundle shifts over Ω is that the minimal normal extension N of S satisfies $\sigma(N) \subseteq \partial\Omega$, while $\sigma(S) \subseteq \bar{\Omega}$.

Let us fix some notation. We can always take analytic universal covers τ: $\mathbf{D} \to \mathscr{G}$ of $\mathscr{G}$ and t: $\mathbf{D} \to \Omega$ of Ω with $\varphi(\tau(0)) = t(0)$. Let μ be the normalized

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