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Spaces of sequences, sampling theorem, and functions of exponential type

by

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Abstract. We introduce certain spaces of sequences which can be used to characterize spaces of functions of exponential type. We present a generalized version of the sampling theorem and a "nonorthogonal wavelet decomposition" for the elements of these spaces of sequences. In particular, we obtain a discrete version of the so-called φ-transform studied in [6]–[8]. We also show how these new spaces and the corresponding decompositions can be used to study multiplier operators on Besov spaces.

§ 0. Introduction. It is a well known fact that if a function is in $L^p(\mathbf{R})$ and its Fourier transform has compact support on, say, the interval $(-\pi, \pi)$, then its $L^p(\mathbf{R})$-norm is comparable to the $L^p(\mathbf{Z})$-norm of its samples on the integer numbers (see e.g. [2], p. 101). It is then natural to ask whether it is possible to tell, by just looking at the samples, if the function is also in some other space of distributions. We show in this paper that the answer to this question is, in general, yes.

Which spaces of functions other than $L^p(\mathbf{R})$ should we consider? Suppose that we want to deduce smoothness properties of a function from its samples. Then, from an heuristic point of view, we may think that we should have to measure how "smooth" the collection of samples is when considered as a function over the integers. This is apparently in conflict with the lack of a nontrivial differentiable structure on the set of integers. On the other hand, band limited functions, i.e. functions whose Fourier transforms are compactly supported, are automatically C^∞ (and in fact, analytic). Thus, it is only meaningful to consider smoothness properties of functions in connection with some other properties such as "size", "growth", and "oscillation". We are then led to look at Besov-type spaces of functions on $\mathbf{R}$ and their possible counterparts on the integers. We still have to address the problem of finding a device to intrinsically define smooth functions on a nonsmooth and discrete object. It is not surprising that this device is given by a version of the so-called Littlewood–Paley theory.

Once these spaces of distributions on the integers are defined, it is then natural to study their properties such as duality, interpolation, boundedness

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of operators, etc. As is well known, for spaces of functions in $\mathbf{R}$ (or $\mathbf{R}^n$), the φ-transform theory (see [6], [8] and [23]) is a very useful tool to study that kind of properties. The basic idea is to decompose the elements of the spaces under study into some "building blocks" which greatly facilitate many computations. We develop a discrete version of the φ-transform which serves for the same purpose but in the context of distribution spaces on the integers.

Although we will not carry out the details here, we also want to mention that it is possible to develop a "dual" theory, and study spaces of distributions on $\mathbf{R}/2\pi\mathbf{Z}$, using a periodic version of the φ-transform (details will appear in [10]). A similar approach has been taken in [12] and [15], where some distribution spaces on $\mathbf{R}/2\pi\mathbf{Z}$ are studied from the "orthonormal-wavelet" point of view. Of course, as the reader familiar with the subject knows, in terms of decomposition of function spaces in $\mathbf{R}^n$, the φ-transform and wavelet theories mainly differ in the greater flexibility but lack of orthogonality in the decompositions obtained with the first one compared to the decompositions obtained with the second (for details and references about the wavelet theory, see the book by Meyer [16]). These decompositions can be viewed as discrete versions of the so-called Calderón reproducing formula, or more generally, as an example of the more recently developed Feichtinger–Gröchenig theory of decompositions of Banach spaces through group representations [5] (see also [11] for a self-contained description of the relationship between these theories). Nevertheless, the decomposition of the sequence spaces introduced in this paper does not fit into the general framework of [5]. What we want to emphasize is the "sampling theorem" point of view. We believe that the φ-transform decompositions on $\mathbf{R}$, $\mathbf{Z}$, and $\mathbf{R}/2\pi\mathbf{Z}$ could be three applications of a possibly more general sampling theorem in an abstract group-theoretic setting. Perhaps, it should be worthy to pursue this idea further in order to achieve a more elegant and unified theory. Some related work in this direction has recently been done in [17].

This paper is organized as follows. In the first section, we recall certain facts about periodic distributions on $\mathbf{R}$ and we give a generalized version of the sampling theorem which allows us to sample arbitrary distributions with compactly supported Fourier transforms. In § 2, we define the (homogeneous) Besov spaces $\dot{B}_p^{\alpha,q}(\mathbf{Z})$ by mimicking the Littlewood–Paley definition of the Besov spaces $\dot{B}_p^{\alpha,q}(\mathbf{R})$ as is presented, for example, in [18]. Using the Plancherel–Pólya inequality and our version of the sampling theorem, we show that for $p > 1$ the space $\dot{B}_p^{\alpha,q}(\mathbf{Z})$ can be identified with the space of distributions in $\dot{B}_p^{\alpha,q}(\mathbf{R})$ whose Fourier transforms are supported on $[-\pi, \pi]$. In contrast, for $0 < p \leqslant 1$, the latter space is strictly "smaller" than the former. Finally, in § 3 we present the φ-transform decomposition of the spaces $\dot{B}_p^{\alpha,q}(\mathbf{Z})$. For $L^2(\mathbf{Z})$ a similar decomposition, but more related to orthonormal bases and quadrature mirror filters, have already been studied in [3] and [14]. In this section, we also give an application of our decomposition to the study of Fourier multipliers, and we discuss further extensions and generalizations of these results.

The decomposition results of § 3 depend heavily on the work of M. Frazier and B. Jawerth about decomposition of function spaces. In particular, some of the arguments used are borrowed from [6]. In addition, as was pointed out in that work, the importance and the usefulness of the Plancherel–Pólya inequality in connection with the study of Besov spaces is already described in the book of J. Peetre [18]. The author gladly acknowledges that the discussion at the beginning of § 11 of that book and a lecture given by M. Frazier motivated some of the results presented here. He wants to thank B. Jawerth, R. Rochberg, and G. Welland for some conversations and comments. He also wants to thank H. Feichtinger for pointing out a reference.

§ 1. Periodic distributions and sampling theorem. We start by introducing some notation and by recalling some facts about periodic distributions on $\mathbf{R}$. The reader is referred to [20], § 2.11, for further details.

As usual, let $\mathscr{D}(\mathbf{R})$ and $\mathscr{S}(\mathbf{R})$ be, respectively, the subspaces of $C^\infty(\mathbf{R})$ of compactly supported functions and of Schwartz rapidly decreasing functions. Their topological duals are the spaces of distributions $\mathscr{D}'(\mathbf{R})$ and $\mathscr{S}'(\mathbf{R})$ (tempered distributions), while $\mathscr{E}'(\mathbf{R})$ (compactly supported distributions) is the dual of $C^\infty(\mathbf{R})$. The pairing between distributions and test functions is denoted by $\langle\cdot,\cdot\rangle$, which is assumed to be linear in both entries. For a function g, its translate by $h \in \mathbf{R}$ is the function $\tau_h g(x) = g(x-h)$, and $\tilde{g}(x) = g(-x)$. These operations extend to distributions in the usual manner. The Fourier transform of a tempered distribution f is denoted by $\hat{f}$, which is defined, in case f is an integrable function, by $\hat{f}(\xi) = \int f(x) e^{-ix\xi}\, dx$. The inverse Fourier transform of f is denoted by $\check{f}$. For $A \subseteq \mathbf{R}$, χ_A is the characteristic function of the set A.

A distribution $f \in \mathscr{S}'(\mathbf{R})$ is said to be 2π-*periodic* if $\tau_{2\pi n} f = f$ for all $n \in \mathbf{Z}$. Let $f \in \mathscr{S}'(\mathbf{R})$ and assume that $\hat{f}$ is 2π-periodic. Then it can be shown that there exists a sequence of numbers, $\{a_n\}_{n\in\mathbf{Z}}$, such that

$$f = \sum_{n\in\mathbf{Z}} a_n \tau_{-n} \delta,$$

where δ is the Dirac mass at the origin, and where the convergence of the sum is weak* in $\mathscr{S}'(\mathbf{R})$. It follows that

$$\hat{f} = \sum_{n\in\mathbf{Z}} a_n e^{-in\xi}.$$

Moreover, given a sequence of numbers $\{a_n\}_{n\in\mathbf{Z}}$, the series $\sum_{n\in\mathbf{Z}} a_n \tau_{-n}\delta$ converges weak* to a distribution $f \in \mathscr{S}'(\mathbf{R})$ if and only if there exist two nonnegative constants C and N such that

$$|a_n| \leqslant C(1+|n|)^N.$$

In such a case $\hat{f}$ is a 2π-periodic distribution. In addition, if $\hat{f}$ is a locally integrable function, then

$$a_n = (2\pi)^{-1} \int \hat{f}(\xi) \chi_{[-\pi,\pi]}(\xi) e^{in\xi}\, d\xi.$$

It is well known that the space of 2π-periodic distributions on $\mathbf{R}$ can be identified with the space of distributions on the unit circle, or, more precisely, distributions on the group $\mathbf{R}/2\pi\mathbf{Z}$ (see [13], p. 43). In fact, this identification is usually done by letting a 2π-periodic distribution

$$g = \sum_{n\in\mathbf{Z}} a_n e^{-in\xi}$$

(originally defined on $\mathscr{S}(\mathbf{R})$) act on 2π-periodic C^∞-functions by the formula

$$\langle g, \varphi\rangle \equiv \sum_{n\in\mathbf{Z}} a_n \int \varphi(\xi)\chi_{[-\pi,\pi]}(\xi) e^{-in\xi}\, d\xi = \sum_{n\in\mathbf{Z}} a_n \langle \chi_{[-\pi,\pi]} e^{-in\xi}, \varphi\rangle.$$

The convergence of the above sum is guaranteed by the (at most) polynomial growth of the sequence $\{a_n\}$ and the rapid decay of the "Fourier coefficients" of $\chi_{[-\pi,\pi]}\varphi$. Notice that there is no ambiguity with this extension of g because a function in $\mathscr{S}(\mathbf{R})$ is 2π-periodic if and only if it is identically zero. In addition, we obviously have $a_n = (2\pi)^{-1}\langle g, e^{in\xi}\rangle$. In the particular case that g is a locally integrable function, it is not hard to see that

$$a_n = (2\pi)^{-1} \int g(\xi)\chi_{[-\pi,\pi]}(\xi) e^{in\xi}\, d\xi.$$

We want now to relate periodic distributions with compactly supported distributions. The first thing to recall is that, by the Paley–Wiener theorem, a distribution with compactly supported Fourier transform is actually the restriction to the reals of an analytic function of exponential type. It does make sense then to talk about its value at a point. Moreover, if $\operatorname{supp}\hat{f} \subseteq [-t, t]$, then there exist two positive constants, C and N, such that

$$|f(z)| \leqslant C(1+|z|)^N e^{t|y|}$$

for all $z = x+iy \in \mathbf{C}$.

For $t > 0$, let $E_t(\mathbf{R})$ denote the space of *distributions of exponential type t*, i.e., the subspace of $\mathscr{S}'(\mathbf{R})$ consisting of all tempered distributions whose Fourier transforms are supported on the interval $[-t, t]$. Suppose that $f \in L^2(\mathbf{R}) \cap E_\pi(\mathbf{R})$. Then, of course, its *periodic extension*

$$\hat{F}(\xi) \equiv \sum_{n\in\mathbf{Z}} (2\pi)^{-1} \int \hat{f}(x) e^{inx}\, dx\, e^{-in\xi} = \sum_{n\in\mathbf{Z}} f(n) e^{-in\xi}$$

is a 2π-periodic distribution, which is still a locally integrable function, and whose restriction to $[-\pi, \pi]$ is $\hat{f}$. Notice also that $\chi_{(-\pi,\pi)}\hat{f}$ or $\chi_{[-\pi,\pi)}\hat{f}$ give rise, in the above way, to the same periodic distribution. In addition, by Plancherel's theorem,

$$\sum_{n\in\mathbf{Z}} |f(n)|^2 = \|f\|^2_{L^2(\mathbf{R})}.$$

Conversely, suppose the sequence $\{a_n\}_{n\in\mathbf{Z}} \in L^2(\mathbf{Z})$ is given. Then

$$\hat{F}(\xi) \equiv \sum_{n\in\mathbf{Z}} a_n e^{-in\xi}$$

is a periodic distribution which is a locally integrable function, and such that $f = (\chi_{[-\pi,\pi]}\hat{F})^\vee$ is in $L^2(\mathbf{R})$ with

$$\|f\|^2_{L^2(\mathbf{R})} = \sum_{n\in\mathbf{Z}} |a_n|^2.$$

The above correspondence is one-to-one. If we now consider an arbitrary distribution $f \in E_\pi(\mathbf{R})$, then the situation is a bit different. Not only $\hat{f}$, $\chi_{(-\pi,\pi)}\hat{f}$, and $\chi_{[-\pi,\pi)}\hat{f}$ (whenever it makes sense) could be different objects, but also, two different distributions could give rise to the same periodic extension. In fact, we have the following simple result.

LEMMA 1. *Let $f \in E_\pi(\mathbf{R})$. Then the periodic extension of $\hat{f}$,*

$$\hat{F} = \sum_{n\in\mathbf{Z}} \tau_{2\pi n}\hat{f} = \sum_{n\in\mathbf{Z}} f(n) e^{-in\xi} \tag{1.1}$$

(weak convergence in $\mathscr{S}'(\mathbf{R})$), is a 2π-periodic distribution. Moreover, given $f, g \in E_\pi(\mathbf{R})$, then $\hat{F}$ and $\hat{G}$, the periodic extensions of $\hat{f}$ and $\hat{g}$, agree as distributions if and only if for some $M \geqslant 0$ and complex numbers c_k, $k = 0, \ldots, M$,*

$$g = f + \sum_{k=0}^{M} c_k x^k \sin \pi x. \tag{1.2}$$

Proof. Let $\varphi \in \mathscr{S}(\mathbf{R})$ and let $L > 0$. Then

$$\Big\langle \sum_{|n|\leqslant L} \tau_{2\pi n}\hat{f}, \varphi\Big\rangle = \Big\langle \hat{f}, \sum_{|n|\leqslant L} \tau_{-2\pi n}\varphi\Big\rangle.$$

Since $\hat{f} \in \mathscr{E}'(\mathbf{R})$, and for any $\varphi \in \mathscr{S}(\mathbf{R})$, $\sum_{n\in\mathbf{Z}} \tau_{-2\pi n}\varphi$ converges in $C^\infty(\mathbf{R})$,

$$\lim_{L\to\infty} \Big\langle \sum_{|n|\leqslant L} \tau_{2\pi n}\hat{f}, \varphi\Big\rangle = \Big\langle \hat{f}, \sum_{n\in\mathbf{Z}} \tau_{-2\pi n}\varphi\Big\rangle.$$

The rest of (1.2) follows from the Poisson summation formula for a function in $\mathscr{S}$.

Assume now that $f, g \in \mathscr{S}'(\mathbf{R})$ have Fourier transforms supported on the interval $[-\pi, \pi]$. If $\hat{F} = \hat{G}$, we must have $f(n) = g(n)$ for all $n \in \mathbf{Z}$. Then, for any $\varphi \in \mathscr{D}(\mathbf{R})$ with $\operatorname{supp}\varphi \subseteq (-\pi, \pi)$,

$$\langle \hat{f}, \varphi\rangle = \Big\langle \hat{f}, \sum_{n\in\mathbf{Z}} \tau_{-2\pi n}\varphi\Big\rangle = \langle \hat{F}, \varphi\rangle = \langle \hat{G}, \varphi\rangle = \Big\langle \hat{g}, \sum_{n\in\mathbf{Z}} \tau_{-2\pi n}\varphi\Big\rangle = \langle \hat{g}, \varphi\rangle.$$

Thus, $\operatorname{supp}(\hat{f}-\hat{g}) \subseteq \{-\pi, \pi\}$. That is, for some $M \geqslant 0$ and complex numbers a_k, b_k, $k = 0, \ldots, M$,

$$\hat{f} - \hat{g} = \sum_{k=0}^{M} (a_k \partial^k \delta_{-\pi} + b_k \partial^k \delta_\pi).$$

The condition $f(n) = g(n)$ easily implies that $a_k = -b_k$. Hence,

$$f - g = \sum_{k=0}^{M} a_k (\partial^k \delta_{-\pi} - \partial^k \delta_\pi)^\vee = (2\pi)^{-1} \sum_{k=0}^{M} a_k (ix)^k (e^{-ix\pi} - e^{ix\pi}),$$

and (1.2) follows. Conversely, if f and g satisfy (1.2), the same computations show that $f(n) = g(n)$ for all n and $\hat{F} = \hat{G}$. ∎

We now want to associate to a sequence of numbers with at most polynomial growth a tempered distribution in $E_\pi(\mathbf{R})$. First, we need to introduce some additional notation. Given a sequence of complex numbers $s = \{s_n\}_{n\in\mathbf{Z}}$, we define the *order* of s to be the (extended) real number

$$\operatorname{ord}(s) = \inf\{t \in \mathbf{R}\colon\ |s_n| \leqslant C_t(1+|n|)^t\},$$

with the convention that $\operatorname{ord}(s) = -\infty$ if $|s_n| \leqslant C_t(1+|n|)^t$ for every negative t, and $\operatorname{ord}(s) = \infty$ if the set $\{t\in\mathbf{R}\colon |s_n| \leqslant C_t(1+|n|)^t\}$ is empty.

Let $\operatorname{sinc} x$ denote the function $(\pi x)^{-1}\sin \pi x$. We have the following version of the sampling theorem (cf. [2], p. 107).

THEOREM 1 (Generalized sampling theorem). *Suppose that $s = \{s_n\}_{n\in\mathbf{Z}}$ is a sequence of complex numbers with $\operatorname{ord}(s) < \infty$. Let $N = \min\{M\in\mathbf{Z},\ M \geqslant 0\colon \{(1+|n|)^{-M}s_n\} \in L^p(\mathbf{Z})$ for some $1 < p < \infty\}$. Then*

$$E(s) = x^N\Big(\sum_{n\neq 0} n^{-N} s_n \operatorname{sinc}(x-n)\Big) + s_0 \operatorname{sinc} x \tag{1.3}$$

converges weak in $\mathscr{S}'(\mathbf{R})$. Moreover, the distribution $E(s)$ so defined satisfies* $\operatorname{supp} E(s)^\wedge \subseteq [-\pi, \pi]$ *and* $E(s)(n) = s_n$ *for all* $n \in \dot{\mathbf{Z}}$.

Proof. For any $\varphi \in \mathscr{S}(\mathbf{R})$ and $L > 0$,

$$\Big\langle \Big(x^N \sum_{|n|=1}^{L} n^{-N} s_n \operatorname{sinc}(x-n)\Big)^\wedge, \varphi\Big\rangle = \Big\langle \partial^N \sum_{|n|=1}^{L} (-in)^{-N} s_n \chi_{[-\pi,\pi]} e^{-in\xi}, \varphi\Big\rangle$$
$$= \Big\langle \sum_{|n|=1}^{L} (in)^{-N} s_n \chi_{[-\pi,\pi]} e^{-in\xi}, \partial^N\varphi\Big\rangle$$
$$= \sum_{|n|=1}^{L} (in)^{-N} s_n \int_{-\pi}^{\pi} e^{-in\xi}\partial^N\varphi(\xi)\,d\xi.$$

Now, since $\partial^N\varphi$ is a function of bounded variation on $[-\pi, \pi]$, we have

$$\Big|\int_{-\pi}^{\pi} e^{-in\xi}\partial^N\varphi(\xi)\,d\xi\Big| \leqslant C|n|^{-1}.$$

Then

$$\Big|\Big\langle \Big(x^n \sum_{|n|=1}^{L} n^{-N} s_n \operatorname{sinc}(x-n)\Big)^\wedge, \varphi\Big\rangle\Big| \leqslant C \sum_{|n|=1}^{L} |n|^{-N-1}|s_n|,$$

and using Hölder's inequality we obtain

$$\Big|\Big\langle \Big(x^n \sum_{|n|=1}^{L} n^{-N} s_n \operatorname{sinc}(x-n)\Big)^\wedge, \varphi\Big\rangle\Big| \leqslant C\,\|\{(1+|n|)^{-N}s_n\}\|_{L^p(\mathbf{Z})}.$$

It follows that the expression in (1.3) defines a tempered distribution. Since the Fourier transform of each partial sum in (1.3) is supported in $[-\pi, \pi]$, so is $E(s)^\wedge$. In particular, both $E(s)^\wedge$ and the Fourier transform of each partial sum are distributions in $\mathscr{E}'(\mathbf{R})$, and we can compute

$$E(s)(k) = (2\pi)^{-1}\langle E(s)^\wedge, e^{ik\xi}\rangle$$
$$= (2\pi)^{-1}\Big(\sum_{n\neq 0} (in)^{-N} s_n \langle \chi_{[-\pi,\pi]} e^{-in\xi}, \partial^N e^{ik\xi}\rangle + s_0\langle \chi_{[-\pi,\pi]}, e^{ik\xi}\rangle\Big) = s_k. \ ∎$$

We want to remark for later use that, if $\varphi \in \mathscr{S}(\mathbf{R})$ and $\partial^k\varphi(-\pi) = \partial^k\varphi(\pi)$ for all $k \leqslant N$, then a simple integration by parts shows that the distribution defined in the above theorem satisfies

$$\langle E(s)^\wedge, \varphi\rangle = \sum_{n\in\mathbf{Z}} s_n \langle \chi_{[-\pi,\pi]} e^{-in\xi}, \varphi\rangle.$$

Notice also that if $f \in E_\pi(\mathbf{R})$, then, in general,

$$f = x^N\Big(\sum_{n\neq 0} n^{-N} f(n)\operatorname{sinc}(x-n)\Big) + f(0)\operatorname{sinc} x + \sum_{k=0}^{M} c_k x^k \sin \pi x,$$

where N is the number defined in the statement of Theorem 1. Thus, it is not always possible to reconstruct a distribution in $E_\pi(\mathbf{R})$ from its samples on the integers. This is a well known fact. Nevertheless, if $\operatorname{supp}\hat{f} \subseteq (-\pi, \pi)$, then, for any $\varphi \in \mathscr{D}(\mathbf{R})$ satisfying $\operatorname{supp}\varphi \subseteq (-\pi, \pi)$, and $\varphi \equiv 1$ on $\operatorname{supp}\hat{f}$,

$$f = f * \varphi$$
$$= \Big\{x^N\Big(\sum_{n\neq 0} n^{-N} f(n)\operatorname{sinc}(x-n)\Big) + f(0)\operatorname{sinc} x + \sum_{k=0}^{M} c_k x^k \sin \pi x\Big\} * \varphi$$
$$= \Big\{x^N\Big(\sum_{n\neq 0} n^{-N} f(n)\operatorname{sinc}(x-n)\Big) + f(0)\operatorname{sinc} x\Big\} * \varphi,$$

and we can recover f from its samples. In this particular case there is another simpler way to reconstruct f from its samples, which essentially consists in replacing the functions $\operatorname{sinc}(x-n)$ by functions with a "faster" decay at infinity (cf. Lemma 7 below). The advantage of formula (1.3) is that we can use it even for distributions with the support of their Fourier transforms equal to the interval $[-\pi, \pi]$. Given $f \in E_\pi(\mathbf{R})$, we will call the distribution

$$S(f) = x^N\Big(\sum_{n\neq 0} n^{-N} f(n)\operatorname{sinc}(x-n)\Big) + f(0)\operatorname{sinc} x$$

(for lack of a better name) the *sampled part* of f, and, when f agrees with its sampled part, we will say that f is in *sampled form*. Notice that, in general, we always have

$$S(f)^\wedge - \hat{f} = \sum_{k=0}^{M} a_k(\partial^k\delta_{-\pi} - \partial^k\delta_\pi).$$

As we have already mentioned in the introduction, if we assume that f is in $E_\pi(\mathbf{R})$, then, for any $0 < p < \infty$,

$$\sum_{n\in\mathbf{Z}} |f(n)|^p \leqslant C_p \|f\|_{L^p(\mathbf{R})}^p. \tag{1.4}$$

On the other hand, if $f \in E_{\pi-\varepsilon}(\mathbf{R})$ for some $\varepsilon > 0$, then

$$\|f\|_{L^p(\mathbf{R})}^p \leqslant C_{p,\varepsilon} \sum_{n\in\mathbf{Z}} |f(n)|^p \tag{1.5}$$

(see [2], p. 197). Notice that for $\varepsilon = 0$, (1.5) may fail to be true. Obvious examples are provided by the function $\operatorname{sinc} x$ if $0 < p \leqslant 1$, or the function $\operatorname{sinc} x + \sin \pi x$ if $1 < p < \infty$. The next lemma says that, at least for $1 < p < \infty$, (1.5) holds with $\varepsilon = 0$ for the sampled part of f. More precisely, we have

LEMMA 2. *Let $1 < p < \infty$, and assume that $f \in E_\pi(\mathbf{R})$. Then*

$$\|S(f)\|_{L^p(\mathbf{R})}^p \approx \sum_{n\in\mathbf{Z}} |f(n)|^p, \tag{1.6}$$

and if $f \in L^p(\mathbf{R}) \cap E_\pi(\mathbf{R})$, then $f = S(f)$.

Proof. Notice that since $S(f)(n) = f(n)$ for all $n \in \mathbf{Z}$, we only need to check that for $\sum_{n\in\mathbf{Z}} |f(n)|^p < \infty$,

$$\|S(f)\|_{L^p(\mathbf{R})}^p \leqslant C_p \sum_{n\in\mathbf{Z}} |f(n)|^p.$$

Moreover, it is enough to check that for every $\varphi \in \mathscr{S}(\mathbf{R})$,

$$|\langle S(f), \varphi\rangle| \leqslant C\Big(\sum_{n\in\mathbf{Z}} |f(n)|^p\Big)^{1/p} \|\varphi\|_{L^{p'}},$$

where $1/p' + 1/p = 1$. But

$$\begin{aligned} |\langle S(f), \varphi\rangle| &= C|\langle S(f)^\wedge, \hat\varphi\rangle| = C\Big|\sum_{n\in\mathbf{Z}} f(n)\langle \chi_{[-\pi,\pi]} e^{-in\xi}, \hat\varphi\rangle\Big| \\ &\leqslant C\Big(\sum_{n\in\mathbf{Z}} |f(n)|^p\Big)^{1/p} \Big(\sum_{n\in\mathbf{Z}} |(\chi_{[-\pi,\pi]}\hat\varphi)^\vee(n)|^{p'}\Big)^{1/p'}. \end{aligned}$$

If we recall that $\chi_{[-\pi,\pi]}$ is a Fourier multiplier for every $L^p(\mathbf{R})$, $1 < p < \infty$ (see [22], p. 100), then

$$\|(\chi_{-\pi,\pi]}\hat\varphi)^\vee\|_{L^{p'}(\mathbf{R})} \leqslant C_p \|\varphi\|_{L^{p'}(\mathbf{R})}.$$

In particular, $(\chi_{[-\pi,\pi]}\hat\varphi)^\vee \in L^{p'}(\mathbf{R})$, and by (1.4) we finally obtain

$$|\langle S(f), \varphi\rangle| \leqslant C_p\Big(\sum_{n\in\mathbf{Z}} |f(n)|^p\Big)^{1/p} \|(\chi_{[-\pi,\pi]}\hat\varphi)^\vee\|_{L^{p'}(\mathbf{R})} \leqslant C_p\Big(\sum_{n\in\mathbf{Z}} |f(n)|^p\Big)^{1/p} \|\varphi\|_{L^{p'}(\mathbf{R})}. \ \blacksquare$$

Clearly, if $f \in L^p(\mathbf{R}) \cap E_\pi(\mathbf{R})$, $0 < p < \infty$, then by another well known fact about functions of exponential type, $f \in L^q(\mathbf{R})$ for every $q \geqslant p$. It follows then that if $f \in L^p(\mathbf{R}) \cap E_\pi(\mathbf{R})$, $0 < p \leqslant 1$, we must also have $f = S(f)$. That is, $S(f)$ is the only function of exponential type giving rise to the same periodic distribution as f that can be in $L^p(\mathbf{R})$ for some $0 < p < \infty$. Notice also that, by (1.5), if f is in $E_{\pi-\varepsilon}(\mathbf{R})$ and $S(f)$ is in $L^p(\mathbf{R})$, then again $S(f) = f$.

§ 2. Function spaces on the integers. We need to recall the definition of Besov spaces on $\mathbf{R}$. Among all the equivalent different characterizations of these spaces, we find it most convenient to employ the following one. Let $\varphi \in \mathscr{S}(\mathbf{R})$ satisfy $\operatorname{supp}\hat\varphi \subsetneq \{\xi\colon \pi/4 < |\xi| < \pi\}$, and for some $C, \varepsilon > 0$, $|\hat\varphi(\xi)| > C$ on $\{\xi\colon \pi/4 + \varepsilon < |\xi| < \pi - \varepsilon\}$. For $\nu \in \mathbf{Z}$, let $\varphi_\nu(x) = 2^\nu \varphi(2^\nu x)$. For $\alpha \in \mathbf{R}$ and $0 < p, q < \infty$, the (homogeneous) Besov space $\dot B_p^{\alpha,q}(\mathbf{R})$ is the collection of all $f \in \mathscr{S}'/\mathscr{P}(\mathbf{R})$ (tempered distributions modulo polynomials on $\mathbf{R}$) such that

$$\|f\|_{\dot B_p^{\alpha,q}(\mathbf{R})} = \Big(\sum_{\nu\in\mathbf{Z}} (2^{\nu\alpha}\|f * \varphi_\nu\|_{L^p(\mathbf{R})})^q\Big)^{1/q} < \infty.$$

These spaces are Banach spaces if $1 \leqslant p, q < \infty$, and quasi-Banach spaces otherwise; their definition is independent of the choice of φ. For appropriate choices of the parameters α, p and q, these spaces can be identified with other more classical spaces. For example, $\dot B_2^{\alpha,2}(\mathbf{R}) = \dot L_\alpha^2(\mathbf{R})$ (the homogeneous Sobolev spaces of order α). We refer to [18] and [24] for details and further properties.

We will now define the analogs of these spaces on the integers. Let $\mathscr{S}'(\mathbf{Z})$ be the collection of all sequences of finite order and let $\mathscr{S}(\mathbf{Z})$ be the collection of all sequences of order $-\infty$. As we have already mentioned, $\mathscr{S}'(\mathbf{Z})$ can be identified with the space of distributions on $\mathbf{R}/2\pi\mathbf{Z}$. On the other hand, we can also identify $\mathscr{S}(\mathbf{Z})$ with the space of C^∞-functions on $\mathbf{R}/2\pi\mathbf{Z}$, namely, the sequence $r = \{r_n\}_{n\in\mathbf{Z}}$ is identified with the periodic function $\sum_{n\in\mathbf{Z}} r_n e^{-in\xi}$. With these identifications, we can view $\mathscr{S}'(\mathbf{Z})$ as the dual of $\mathscr{S}(\mathbf{Z})$. Alternatively (see [12]), the duality between $\mathscr{S}(\mathbf{Z})$ and $\mathscr{S}'(\mathbf{Z})$ can be obtained directly by considering in $\mathscr{S}(\mathbf{Z})$ the topology of a Fréchet space induced by the family of seminorms

$$\|r\|_N = \sum_{0\leqslant j\leqslant N} \sup_{n\in\mathbf{Z}} |n^j r_n|, \qquad N = 0, 1, \ldots$$

For $s \in \mathscr{S}'(\mathbf{Z})$ and $r \in \mathscr{S}(\mathbf{Z})$, it is convenient for us to define the action of s on r by

$$\langle s, r\rangle = \sum_{n\in\mathbf{Z}} s_n r_n = \Big\langle \sum_{n\in\mathbf{Z}} s_n e^{-in\xi}, \sum_{n\in\mathbf{Z}} r_{-n} e^{-in\xi}\Big\rangle_{\mathbf{R}/2\pi\mathbf{Z}},$$

where, in order to be consistent with our notation, we assume that the pairing $\langle\cdot,\cdot\rangle$ in $\mathbf{R}/2\pi\mathbf{Z}$ is linear in both entries, and, in the case that the distribution is a locally integrable function, its action on a function is given by the integral of their product with respect to Haar measure. More precisely, if $P(s) \in \mathscr{S}'(\mathbf{R})$ is the distribution

$$P(s) = \sum_{n\in\mathbf{Z}} s_n \tau_{-n}\delta,$$

$r = \{r_n\}$ is the sequence given by

$$r_n = (2\pi)^{-1} \int \chi_{[-\pi,\pi]} \psi(\xi) e^{in\xi}\, d\xi$$

for some smooth 2π-periodic function ψ, and we extend the action of $P(s)^\wedge$ to smooth 2π-periodic functions in the way described in the previous section, then

$$\langle s, r\rangle = \sum_{n\in\mathbf{Z}} s_n r_n = (2\pi)^{-1} \sum_{n\in\mathbf{Z}} s_n \langle e^{in\xi}\chi_{[-\pi,\pi]}, \psi\rangle = (2\pi)^{-1} \sum_{n\in\mathbf{Z}} s_n \langle e^{in\xi}\chi_{[-\pi,\pi]}, \tilde{\psi}\rangle$$
$$= (2\pi)^{-1} \Big\langle \sum_{n\in\mathbf{Z}} s_n e^{in\xi}, \tilde{\psi}\Big\rangle = (2\pi)^{-1}\langle P(s)^\wedge, \tilde{\psi}\rangle.$$

If for a continuous function f on $\mathbf{R}$, we let $R(f)$ be the sequence obtained by the restriction of f to the integers, then for $\varphi \in \mathscr{S}(\mathbf{R})$, $R(\varphi) \in \mathscr{S}(\mathbf{Z})$. We also have

$$\langle s, R(\varphi)\rangle = \sum_{n\in\mathbf{Z}} s_n \varphi(n) = (2\pi)^{-1} \sum_{n\in\mathbf{Z}} s_n \langle e^{in\xi}, \hat{\varphi}\rangle = (2\pi)^{-1} \sum_{n\in\mathbf{Z}} s_n \langle e^{-in\xi}, (\hat{\varphi})^\sim\rangle$$
$$= (2\pi)^{-1} \Big\langle \sum_{n\in\mathbf{Z}} s_n e^{-in\xi}, \hat{\varphi}\Big\rangle = (2\pi)^{-1}\langle P(s)^\wedge, (\hat{\varphi})^\sim\rangle.$$

Finally, if $f \in E_\pi(\mathbf{R})$, $\varphi \in \mathscr{S}(\mathbf{R})$, and $\chi_{[-\pi,\pi]}\hat{\varphi} = \chi_{[-\pi,\pi]}\psi$, where ψ is a smooth 2π-periodic function, then

$$\langle f, \varphi\rangle = (2\pi)^{-1}\langle \hat{f}, (\hat{\varphi})^\sim\rangle = (2\pi)^{-1}\langle S(f)^\wedge, \tilde{\varphi}\rangle = (2\pi)^{-1}\langle S(f)^\wedge, \tilde{\psi}\rangle$$
$$= (2\pi)^{-1}\big\langle P\big(R(S(f))\big)^\wedge, \tilde{\psi}\big\rangle = (2\pi)^{-1}\langle P(R(f))^\wedge, \tilde{\psi}\rangle = \langle R(f), r\rangle,$$

where $r = R\big((\chi_{[-\pi,\pi]}\psi)^\vee\big) = R\big((\chi_{[-\pi,\pi]}\hat{\varphi})^\vee\big)$. In particular, this shows that the linear operator $R\colon E_\pi(\mathbf{R}) \to \mathscr{S}'(\mathbf{Z})$ is continuous with respect to the weak* topologies.

For $s \in \mathscr{S}'(\mathbf{Z})$, its (integer) translates, $\tau_k s$, and the distribution $\tilde{s}$ are defined in the obvious way. The convolution of $s \in \mathscr{S}'(\mathbf{Z})$ with $r \in \mathscr{S}(\mathbf{Z})$ is given by

$$(s * r)_n = \langle s, (\tau_n r)^\sim\rangle = \sum_{k\in\mathbf{Z}} s_k r_{n-k}.$$

Clearly $s * r \in \mathscr{S}'(\mathbf{Z})$ and $P(s*r)^\wedge = P(r)^\wedge P(s)^\wedge$. It follows from the above remarks that, if $f \in E_\pi(\mathbf{R})$ and $\varphi \in \mathscr{S}(\mathbf{R})$ with $\partial^\gamma\hat{\varphi}(-\pi) = \partial^\gamma\hat{\varphi}(\pi)$ for all γ, then

$$f * \varphi(k) = \langle f, (\tau_k\varphi)^\sim\rangle = (2\pi)^{-1}\langle \hat{f}, (\tau_k\varphi)^\wedge\rangle = (2\pi)^{-1}\langle \hat{f}, e^{-ik\xi}\hat{\varphi}\rangle$$
$$= (2\pi)^{-1}\big\langle \hat{f}, \big((e^{-ik\xi}\hat{\varphi})^\sim\big)^\sim\big\rangle = \big\langle R(f), R\big(\big((\chi_{[-\pi,\pi]}e^{-ik\xi}\hat{\varphi})^\sim\big)^\vee\big)\big\rangle$$
$$= \big\langle R(f), R\big(\big((\chi_{[-\pi,\pi]}e^{-ik\xi}\hat{\varphi})^\vee\big)^\sim\big)\big\rangle = \big\langle R(f), R\big(\big(\tau_k((\chi_{[-\pi,\pi]}\hat{\varphi})^\vee)\big)^\sim\big)\big\rangle$$
$$= \big\langle R(f), \big(\tau_k R\big((\chi_{[-\pi,\pi]}\hat{\varphi})^\vee\big)\big)^\sim\big\rangle = \big(R(f) * R\big((\chi_{[-\pi,\pi]}\hat{\varphi})^\vee\big)\big)_k.$$

We have thus proved

LEMMA 3. *If $f \in E_\pi(\mathbf{R})$, and $\varphi \in \mathscr{S}(\mathbf{R})$, with $\partial^\gamma\hat{\varphi}(-\pi) = \partial^\gamma\hat{\varphi}(\pi)$ for all γ, then*

$$R(f * \varphi) = R(f) * R\big((\chi_{[-\pi,\pi]}\hat{\varphi})^\vee\big). \tag{2.1}$$

After these remarks we can now define the spaces $\dot{B}_p^{\alpha,q}(\mathbf{Z})$. Let $\varphi \in \mathscr{S}(\mathbf{R})$ be as in the definition of the spaces $\dot{B}_p^{\alpha,q}(\mathbf{R})$, and assume further that $\hat{\varphi} \equiv 1$ in a small neighborhood of $\{-\pi/2, \pi/2\}$ and that $\hat{\varphi} = (\hat{\varphi})^\sim$ (this can always be done). For $\nu \leq 1$, let $\varphi_\nu^d \in \mathscr{S}(\mathbf{Z})$ be the sequence $R\big((\chi_{[-\pi,\pi]}\hat{\varphi}_\nu)^\vee\big) = R\big((\chi_{[-\pi,\pi]}\hat{\varphi}(2^{-\nu}\xi))^\vee\big)$. Also, let $\mathscr{S}'/\mathscr{P}(\mathbf{Z})$ be the space of distributions in $\mathscr{S}'(\mathbf{Z})$ modulo $\mathscr{P}(\mathbf{Z}) = R(\mathscr{P}(\mathbf{R})) = \{R(f)\colon f \in \mathscr{P}(\mathbf{R})\}$. For $\alpha \in \mathbf{R}$ and $0 < p, q < \infty$, the (homogeneous) Besov space $\dot{B}_p^{\alpha,q}(\mathbf{Z})$ is the collection of all $s \in \mathscr{S}'/\mathscr{P}(\mathbf{Z})$ such that

$$\|s\|_{\dot{B}_p^{\alpha,q}(\mathbf{Z})} = \Big(\sum_{\nu\leq 1} (2^{\nu\alpha}\|s * \varphi_\nu^d\|_{L^p(\mathbf{Z})})^q\Big)^{1/q} < \infty.$$

That this definition is independent of the choice of φ will follow from the decomposition results of the next section. For $1 < p < \infty$, this also follows from our next result, which shows the relation between $\dot{B}_p^{\alpha,q}(\mathbf{Z})$ and $\dot{B}_p^{\alpha,q}(\mathbf{R})$.

THEOREM 2. *Let $\alpha \in \mathbf{R}$, $0 < q < \infty$.*

(a) *If $f \in \dot{B}_p^{\alpha,q}(\mathbf{R}) \cap E_\pi(\mathbf{R})$ then $R(f) \in \dot{B}_p^{\alpha,q}(\mathbf{Z})$, $0 < p < \infty$.*

(b) *If $f \in E_{\pi-\varepsilon}(\mathbf{R})$ for some $\varepsilon > 0$, then $f \in \dot{B}_p^{\alpha,q}(\mathbf{R})$ if and only if $R(f) \in \dot{B}_p^{\alpha,q}(\mathbf{Z})$, $0 < p < \infty$.*

(c) *If $s \in \dot{B}_p^{\alpha,q}(\mathbf{Z})$, $1 < p < \infty$, and $f = E(s) \in E_\pi(\mathbf{R})$ is the distribution defined by the generalized sampling theorem, then there is a unique function g of the form*

$$g = \sum_{k=0}^{M} c_k x^k \sin \pi x$$

such that $f + g \in \dot{B}_p^{\alpha,q}(\mathbf{R})$.

Proof. The proof of (a) is immediate. In fact, since $f \in \dot{B}_p^{\alpha,q}(\mathbf{R})$ we have, in particular, $f * \varphi_\nu = f * (\chi_{[-\pi,\pi]}\hat{\varphi}_\nu)^\vee \in L^p(\mathbf{R})$, for all $\nu \leq 1$. By Lemma 3 and (1.4),

$$\|R(f) * \varphi_\nu^d\|_{L^p(\mathbf{Z})} \leq C\|f * \varphi_\nu\|_{L^p(\mathbf{R})},$$

and, hence,

$$\|R(f)\|_{\dot{B}_p^{\alpha,q}(\mathbf{Z})} \leq C\|f\|_{\dot{B}_p^{\alpha,q}(\mathbf{R})}.$$

Similarly, (b) follows from (a) and (1.5). To prove (c), we first observe that if $s \in \dot{B}_p^{\alpha,q}(\mathbf{Z})$, f is the distribution defined by the generalized sampling theorem, and g is any function of the form

$$g = \sum_{k=0}^{M} c_k x^k \sin \pi x,$$

then for $\nu \leq 0$,

$$f * \varphi_\nu = (f+g) * \varphi_\nu.$$

On the other hand,

$$(f+g) * \varphi_1 = (f * \varphi_1) + g.$$

Now, since $s = R(f) \in \dot{B}_p^{\alpha,q}(\mathbf{Z})$, $R(f)*\varphi_\nu^d \in L^p(\mathbf{Z})$ for all $\nu \leqslant 1$. In addition, for $\nu \leqslant 0$, $\operatorname{supp} f*\varphi_\nu \subseteq (-\pi, \pi)$, and therefore

$$\|(f+g)*\varphi_\nu\|_{L^p(\mathbf{R})} = \|f*\varphi_\nu\|_{L^p(\mathbf{R})} \leqslant C\|R(f*\varphi_\nu)\|_{L^p(\mathbf{Z})} = C\|R(f)*\varphi_\nu^d\|_{L^p(\mathbf{Z})}.$$

Thus, to finish the proof of the theorem we only need to show that for some g of the above form, $(f+g)*\varphi_1 \in L^p(\mathbf{R})$. But, since $1 < p < \infty$, we can use Lemma 2 to obtain

$$\|S(f*\varphi_1)\|_{L^p(\mathbf{R})} \leqslant C\|R(f*\varphi_1)\|_{L^p(\mathbf{Z})} = C\|s*\varphi_1^d\|_{L^p(\mathbf{Z})}.$$

If we now choose g such that $S(f*\varphi_1) = (f*\varphi_1)+g = (f+g)*\varphi_1$, we finally get

$$\|(f+g)*\varphi_1\|_{L^p(\mathbf{R})} \leqslant C\|s*\varphi_1^d\|_{L^p(\mathbf{Z})}.$$

The remarks in the previous section show that g is unique. ∎

Part (c) of the above theorem is in general not true if $0 < p \leqslant 1$. To show this, we need to recall the definition and some properties of the inhomogeneous Besov spaces. These spaces are essentially defined by replacing, in the definition of the homogeneous Besov spaces, all the terms corresponding to the "low frequencies" of f by a single one. More precisely, let $\Phi \in \mathscr{S}(\mathbf{R})$ satisfy $\operatorname{supp}\hat{\Phi} \subseteq (-\pi, \pi)$ and $|\hat{\Phi}(\xi)| > C > 0$ on $(-\pi+\varepsilon, \pi-\varepsilon)$ for some $C, \varepsilon > 0$. For $\alpha \in \mathbf{R}$ and $0 < p, q < \infty$, the inhomogeneous Besov space $B_p^{\alpha,q}(\mathbf{R})$ is the collection of all $f \in \mathscr{S}'(\mathbf{R})$ such that

$$\|f\|_{B_p^{\alpha,q}(\mathbf{R})} = \|f*\Phi\|_{L^p(\mathbf{R})} + \Big(\sum_{\nu \geqslant 1} (2^{\nu\alpha}\|f*\varphi_\nu\|_{L^p(\mathbf{R})})^q\Big)^{1/q} < \infty.$$

There is nothing particular about where we "cut the lower frequencies of f", and, in fact, if we let $\Phi_k(x) = 2^k\Phi(2^k x)$, $k \in \mathbf{Z}$, then it can be shown that

$$\|f*\Phi_k\|_{L^p(\mathbf{R})} + \Big(\sum_{\nu \geqslant k+1} (2^{\nu\alpha}\|f*\varphi_\nu\|_{L^p(\mathbf{R})})^q\Big)^{1/q} < \infty$$

defines an equivalent norm (see [8]). From this it easily follows that

$$B_p^{\alpha,q}(\mathbf{R}) \cap E_\pi(\mathbf{R}) = L^p(\mathbf{R}) \cap E_\pi(\mathbf{R}). \tag{2.2}$$

It is equally easy to see that if $\hat{f}$ vanishes in a neighborhood of zero, then $f \in B_p^{\alpha,q}(\mathbf{R})$ if and only if $f \in \dot{B}_p^{\alpha,q}(\mathbf{R})$. We can now prove:

THEOREM 3. *For each $0 < p \leqslant 1$ there exists $s \in \dot{B}_p^{\alpha,q}(\mathbf{Z})$ such that no $f \in E_\pi(\mathbf{R})$, with $R(f) = s$, belongs to $\dot{B}_p^{\alpha,q}(\mathbf{R})$.*

Proof. Let $h \in \mathscr{D}(\mathbf{R})$ satisfy $\operatorname{supp} h \subseteq (-\pi, \pi)$ and $h \equiv 1$ on a neighborhood of $[-\pi/2, \pi/2]$. Let $\hat{f} = \chi_{[-\pi,\pi]} - h$ and let $s = R(f)$. We have

$$\|R(f)\|_{\dot{B}_p^{\alpha,q}(\mathbf{Z})} \leqslant C(\|R(f)*\varphi_0^d\|_{L^p(\mathbf{Z})} + \|R(f)*\varphi_1^d\|_{L^p(\mathbf{Z})}) < \infty.$$

Suppose now that there exists a function g of the form

$$g = \sum_{k=0}^{M} c_k x^k \sin \pi x$$

such that $f+g \in \dot{B}_p^{\alpha,q}(\mathbf{R})$. We shall see that this leads to a contradiction. In fact, since $\check{h} \in \mathscr{S}(\mathbf{R})$ and $\operatorname{supp} h \subseteq (-\pi, \pi)$,

$$\|\check{h}\|_{L^p(\mathbf{R})} \leqslant C\|R(\check{h})\|_{L^p(\mathbf{Z})} < \infty.$$

On the other hand, since $(f+g)^\wedge$ vanishes in a neighborhood of zero, we would have

$$f+g \in B_p^{\alpha,q}(\mathbf{R}) \cap E_\pi(\mathbf{R}) = L^p(\mathbf{R}) \cap E_\pi(\mathbf{R}).$$

But $f+g = \operatorname{sinc} x + \check{h} + g$, and hence we would have $\operatorname{sinc} x + g \in L^p(\mathbf{R})$, which is not possible because $0 < p \leqslant 1$. ∎

We observe that it is not really meaningful to consider the inhomogeneous version of the Besov spaces over the integers. In fact, let $\Phi_k^d = R(\Phi_k)$, $k \leqslant 0$, and define the Besov space $B_p^{\alpha,q}(\mathbf{Z})$ to be the collection of all $s \in \mathscr{S}'(\mathbf{Z})$ such that

$$\|s\|_{B_p^{\alpha,q}(\mathbf{Z})} = \|s*\Phi_k^d\|_{L^p(\mathbf{Z})} + \Big(\sum_{k+1 \leqslant \nu \leqslant 1} (2^{\nu\alpha}\|s*\varphi_\nu^d\|_{L^p(\mathbf{Z})})^q\Big)^{1/q} < \infty,$$

or, equivalently,

$$\|s\|_{B_p^{\alpha,q}(\mathbf{Z})} = \|s*\Phi_k^d\|_{L^p(\mathbf{Z})} + \sum_{k+1 \leqslant \nu \leqslant 1} \|s*\varphi_\nu^d\|_{L^p(\mathbf{Z})} < \infty. \tag{2.3}$$

We have the following result (cf. [24], p. 17).

THEOREM 4. *For all $\alpha \in \mathbf{R}$ and $0 < p, q < \infty$, $B_p^{\alpha,q}(\mathbf{Z}) = L^p(\mathbf{Z})$ (which is consistent with $B_p^{\alpha,q}(\mathbf{R}) \cap E_\pi(\mathbf{R}) = L^p(\mathbf{R}) \cap E_\pi(\mathbf{R})$).*

Proof. Assume first that $s \in L^p(\mathbf{Z})$. If $0 < p \leqslant 1$,

$$\|s*\Phi_k^d\|_{L^p(\mathbf{Z})}^p = \sum_{n\in\mathbf{Z}} \Big|\sum_{j\in\mathbf{Z}} s_j(\tau_n\tilde{\Phi}_k^d)_j\Big|^p \leqslant \sum_{n\in\mathbf{Z}}\sum_{j\in\mathbf{Z}} |s_j|^p |(\tau_n\tilde{\Phi}_k^d)_j|^p$$

$$\leqslant \sum_{j\in\mathbf{Z}} |s_j|^p \sum_{n\in\mathbf{Z}} |(\tau_n\tilde{\Phi}_k^d)_j|^p \leqslant C\sum_{j\in\mathbf{Z}} |s_j|^p < \infty,$$

since $\Phi_k^d \in \mathscr{S}(\mathbf{Z})$. Similar estimates hold for the other terms in (2.3). Thus,

$$\|s\|_{B_p^{\alpha,q}(\mathbf{Z})} \leqslant C\|s\|_{L^p(\mathbf{Z})}. \tag{2.4}$$

If $1 < p < \infty$, (2.4) follows by applying Young's inequality to each term of (2.3). To prove the converse, we may require that Φ and φ satisfy the additional condition

$$\hat{\Phi}(\xi) + \sum_{\nu \leqslant 1} \hat{\varphi}(2^{-\nu}\xi) \equiv 1.$$

Then the conclusion

$$C\|s\|_{L^p(\mathbf{Z})} \leqslant \|s\|_{B_p^{\alpha,q}(\mathbf{Z})}$$

follows from

$$s = s*\Big(\Phi_k^d + \sum_{k+1 \leqslant \nu \leqslant 1} \varphi_\nu^d\Big),$$

together with the triangle (or p-triangle) inequality. ∎

Remarks(¹). Assume $1 \leqslant p, q < \infty$, $0 < \alpha < 1$. A more classical definition of the Besov(–Lipschitz) spaces on $\mathbf{R}$ is given by

$$\dot{B}_p^{\alpha,q}(\mathbf{R}) = \Big\{f \in \mathscr{S}'/\mathscr{P}\colon \Big(\int_{\mathbf{R}} \big(|t|^{-\alpha} \|f(\cdot+t)-f(\cdot)\|_{L^p(\mathbf{R})}\big)^q\, dt/|t|\Big)^{1/q} < \infty\Big\},$$

$$B_p^{\alpha,q}(\mathbf{R}) = \dot{B}_p^{\alpha,q}(\mathbf{R}) \cap L^p(\mathbf{R}).$$

For $\alpha \geqslant 1$, the expression $f(\cdot+t)-f(\cdot)$ in the above definitions has to be replaced by higher order differences. See [22], § 5. Using these definitions and following the comments in [21], p. 190, we can reinterpret (2.2) and Theorem 4 in the following way. Observe that if $\|f\|_{L^p(\mathbf{R})} < \infty$, then the integral appearing in the above definition of $\dot{B}_p^{\alpha,q}(\mathbf{R})$ is clearly convergent for t large. Thus, for $f \in L^p(\mathbf{R})$, it is only the behavior of f at "small scales" ($t \to 0$) what determines whether the function is in $B_p^{\alpha,q}(\mathbf{R})$ or not. If in addition $f \in E_\pi$, then its behavior at small scales is a priori controlled by the fact that the function is analytic, and we obtain (2.2). On the other hand, in the discrete case there is no scale smaller than one, and the condition $s \in L^p(\mathbf{Z})$ becomes equivalent to the condition $s \in B_p^{\alpha,q}(\mathbf{Z})$ (Theorem 4).

The situation in the homogeneous case is different. When no a priori L^p-estimate on the function of exponential type is assumed, then its homogeneous Besov norm is not only determined by the growth at infinity, but also by the oscillations at "large scales" ($t \to \infty$). By Theorem 2, the same situation must hold in the discrete case. We want to illustrate this point with some explicit examples. For simplicity in the computations that follow we will consider the case $p = q = 2$, and we will invoke some classical results about trigonometric series. Additional examples for other specific values of p and q can be constructed by adapting to the discrete case some of the results discussed in [22], pp. 159–164.

Let $0 < \gamma < 1$, and let $s_\gamma = \{s_\gamma(n)\}_{n \in \mathbf{Z}}$ be the sequence given by $s_\gamma(n) = n^{-\gamma}$ if $n > 0$, and $s_\gamma(n) = 0$ if $n \leqslant 0$. For $0 < \beta < 1$, let $s_{\gamma,\beta} = \{s_\gamma(n) e^{in^\beta}\}_{n \in \mathbf{Z}}$, and for $0 < \delta < 2\pi$, let $s_{\gamma,\beta,\delta} = \{s_{\gamma,\beta}(n) e^{in\delta}\}_{n \in \mathbf{Z}}$. Clearly $|s_\gamma(n)| = |s_{\gamma,\beta}(n)| = |s_{\gamma,\beta,\delta}(n)|$ for all n. Now, if f_γ, $f_{\gamma,\beta}$, and $f_{\gamma,\beta,\delta}$ are the distributions obtained from s_γ, $s_{\gamma,\beta}$, and $s_{\gamma,\beta,\delta}$ by the generalized sampling theorem, then

$$\hat{f}_\gamma(\xi) = \sum_{n \geqslant 1} n^{-\gamma} e^{-in\xi} \chi_{[-\pi,\pi]}(\xi),$$

$$\hat{f}_{\gamma,\beta}(\xi) = \sum_{n \geqslant 1} n^{-\gamma} e^{in^\beta} e^{-in\xi} \chi_{[-\pi,\pi]}(\xi),$$

$$\hat{f}_{\gamma,\beta,\delta}(\xi) = \sum_{n \geqslant 1} n^{-\gamma} e^{in^\beta} e^{in\delta} e^{-in\xi} \chi_{[-\pi,\pi]}(\xi).$$

Assume $0 < \gamma \leqslant 1/2$, so that s_γ, $s_{\gamma,\beta}$, and $s_{\gamma,\beta,\delta}$ are not in $L^2(\mathbf{Z})$. It can be shown (see [24], Vol. I, pp. 68–70 and Vol. II, pp. 133–136) that

$$\hat{f}_\gamma = \big(c_\gamma |\xi|^{\gamma-1} e^{i\pi\gamma/2 \operatorname{sig} \xi} + r_\gamma(\xi)\big)\chi_{[-\pi,\pi]}(\xi),$$

where c_γ is a constant and r_γ is a bounded function in $[-\pi, \pi]$.

(¹) The author thanks the referee for suggesting the inclusion of some of these remarks.

Recall that $\dot{B}_2^{\alpha,2}(\mathbf{R})$ coincides with the (homogeneous) Sobolev space of order α. Then $f_\gamma \in \dot{B}_2^{\alpha,2}(\mathbf{R})$ if and only if $|\xi|^\alpha \hat{f}_\gamma \in L^2(\mathbf{R})$. It follows from Theorem 2 that $s_\gamma \in \dot{B}_2^{\alpha,2}(\mathbf{Z})$ if and only if $\alpha > 1/2-\gamma$.

Assume now that $\gamma+\beta < 1$. Then the presence of the "mildly oscillatory" factor e^{in^β} in $s_{\gamma,\beta}$ produces a higher order singularity at the origin in the Fourier transform side. In fact, it is possible to prove that $\hat{f}_{\gamma,\beta}(\xi)$ behaves at 0 like $|\xi|^{(\gamma-1+\beta/2)/(1-\beta)}$, and is continuous outside the origin. See [25], Vol. I, pp. 200–202. It follows that $s_{\gamma,\beta} \in \dot{B}_2^{\alpha,2}(\mathbf{Z})$ only for $\alpha > (1/2-\gamma)/(1-\beta)$. (Notice that $\dot{B}_p^{\alpha,q}(\mathbf{Z}) \subseteq \dot{B}_p^{\alpha',q}(\mathbf{Z})$ if $\alpha' \geqslant \alpha$.)

Finally, a simple computation shows that $\hat{f}_{\gamma,\beta,\delta}$ is the restriction to the interval $[-\pi, \pi]$ of the 2π-periodic extension of the function $\hat{f}_{\gamma,\beta,\delta}(\xi-\delta)$. We then see that the "highly oscillatory" factor $e^{in\delta}$ prevents $s_{\gamma,\beta,\delta}$ from being in any of the spaces $\dot{B}_2^{\alpha,2}(\mathbf{Z})$, since for every $\alpha \in \mathbf{R}$ the function $|\xi|^\alpha \hat{f}_{\gamma,\beta,\delta}$ has a non-square-integrable singularity at the point $\delta \neq 0$.

§ 3. The φ-transform for spaces of sequences. We will now describe an "almost orthogonal" decomposition of the spaces $\dot{B}_p^{\alpha,q}(\mathbf{Z})$. We will closely follow the pattern in [6]. Consider again a function $\varphi \in \mathscr{S}(\mathbf{R})$ satisfying $\operatorname{supp} \hat{\varphi} \subseteq \{\xi\colon \pi/4 < |\xi| < \pi\}$, $\hat{\varphi} \equiv 1$ in a neighborhood of the set $\{-\pi/2, \pi/2\}$, and so that for some $C, \varepsilon > 0$, $|\hat{\varphi}| > C$ on $\{\xi\colon \pi/4+\varepsilon < |\xi| < \pi-\varepsilon\}$. Let ψ be another function in $\mathscr{S}(\mathbf{R})$ satisfying the same properties as φ and selected so that

$$\sum_{\nu \in \mathbf{Z}} \hat{\bar{\varphi}}(2^{-\nu}\xi)\hat{\psi}(2^{-\nu}\xi) = 1 \quad \text{for all } \xi \neq 0.$$

For $\nu, k \in \mathbf{Z}$, let $\varphi_{\nu k}(x) = 2^{-\nu/2}\varphi_\nu(x-2^{-\nu}k) = 2^{\nu/2}\varphi(2^\nu x-k)$. Define $\psi_{\nu k}$ in the same way. In order to study discrete decompositions of the spaces $\dot{B}_p^{\alpha,q}(\mathbf{R})$, Frazier and Jawerth introduced the following associated spaces of sequences. For $\alpha \in \mathbf{R}$, and $0 < p, q < \infty$, let $\dot{b}_p^{\alpha,q}$ be the collection of all doubly indexed sequences $t = \{t_{\nu k}\}_{\nu,k}$ such that

$$\|t\|_{\dot{b}_p^{\alpha,q}(\mathbf{R})} = \Big(\sum_{\nu \in \mathbf{Z}} \Big(\sum_{k \in \mathbf{Z}} (2^{\nu(\alpha+1/2-1/p)}|t_{\nu k}|)^p\Big)^{q/p}\Big)^{1/q}.$$

Although $\dot{b}_p^{\alpha,q}$ is a space of sequences, we will denote it by $\dot{b}_p^{\alpha,q}(\mathbf{R})$ to differentiate it from the space $\dot{b}_p^{\alpha,q}(\mathbf{Z})$ that we will be considering later. The φ-transform decomposition for the spaces $\dot{B}_p^{\alpha,q}(\mathbf{R})$ is given by the following result.

THEOREM 5 (Frazier and Jawerth [6]). *Every $f \in \dot{B}_p^{\alpha,q}(\mathbf{R})$ can be written in the form*

$$f = \sum_{\nu \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} \langle f, \varphi_{\nu k}\rangle \psi_{\nu k},$$

where the sequence of φ-transform coefficients $S_\varphi f = \{\langle f, \varphi_{\nu k}\rangle\}_{\nu,k}$ is in $\dot{b}_p^{\alpha,q}(\mathbf{R})$. Conversely, for every sequence $t = \{t_{\nu k}\}_{\nu,k} \in \dot{b}_p^{\alpha,q}(\mathbf{R})$, its inverse φ-transform

$$T_\psi t = \sum_{\nu \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} t_{\nu k}\psi_{\nu k}$$

is an element of $\dot{B}_p^{\alpha,q}(\mathbf{R})$. *Moreover, the operators* $S_\varphi\colon \dot{B}_p^{\alpha,q}(\mathbf{R}) \to \dot{b}_p^{\alpha,q}(\mathbf{R})$ *and* $T_\psi\colon \dot{b}_p^{\alpha,q}(\mathbf{R}) \to \dot{B}_p^{\alpha,q}(\mathbf{R})$ *so defined are bounded and* $T_\psi \circ S_\varphi$ *is the identity on* $\dot{B}_p^{\alpha,q}(\mathbf{R})$. *In particular,*

$$\|f\|_{\dot{B}_p^{\alpha,q}(\mathbf{R})} \approx \|S_\varphi f\|_{\dot{b}_p^{\alpha,q}(\mathbf{R})}.$$

We will prove an analogous result for the spaces $\dot{B}_p^{\alpha,q}(\mathbf{Z})$. The starting point for the decomposition of the spaces $\dot{B}_p^{\alpha,q}(\mathbf{R})$ is the following simple identity: for every distribution $f \in \mathscr{S}'(\mathbf{R})$, and with convergence in $\mathscr{S}'/\mathscr{P}(\mathbf{R})$,

$$f = \sum_{\nu \in \mathbf{Z}} f * \tilde{\varphi}_\nu * \psi_\nu. \tag{3.1}$$

A proof of this identity can be found in [18], pp. 52–54. A more detailed proof is also given in [9]. The correct interpretation of (3.1) is that, given $f \in \mathscr{S}'(\mathbf{R})$ there exist a number $M \geqslant 0$, and a sequence of polynomials $\{P_k\}_{k \geqslant 0}$ of degree at most M, such that

$$f = \lim_{k \to \infty} \Big(\sum_{\nu=-k}^{\infty} f * \tilde{\varphi}_\nu * \psi_\nu + P_k \Big) + P_0 \quad \text{in } \mathscr{S}'(\mathbf{R}). \tag{3.2}$$

For $\nu \leqslant 1$, let us define, as in the previous section, $\varphi_\nu^d = R\big((\chi_{[-\pi,\pi]}\hat{\varphi}_\nu)^\vee\big)$, and $\psi_\nu^d = R\big((\chi_{[-\pi,\pi]}\hat{\psi}_\nu)^\vee\big)$. Then we have

$$\sum_{\nu \in \mathbf{Z}} \big(P(\tilde{\varphi}_\nu^d)\big)^\wedge(\xi)\big(P(\psi_\nu^d)\big)^\wedge(\xi) = 1 \quad \text{for all } \xi \notin 2\pi\mathbf{Z}.$$

From this it is straightforward to check that the proof of (3.1) given in [9] can be adapted to our discrete situation, so that for every $s \in \mathscr{S}'(\mathbf{Z})$,

$$s = \sum_{\nu \leqslant 1} s * \tilde{\varphi}_\nu^d * \psi_\nu^d, \tag{3.3}$$

where the convergence is now in $\mathscr{S}'/\mathscr{P}(\mathbf{Z})$. We can also prove (3.2) by using Lemma 3. In fact, if for $s \in \mathscr{S}'(\mathbf{Z})$, $E(s)$ is the distribution defined by the generalized sampling theorem, then by (3.2),

$$E(s) = \lim_{k \to \infty} \Big(\sum_{-k \leqslant \nu \leqslant 1} E(s) * \tilde{\varphi}_\nu * \psi_\nu + P_k \Big) + P_0.$$

But, by Lemma 3 and the continuity of the operator R,

$$s = R\big(E(s)\big) = \lim_{k \to \infty} \Big(\sum_{-k \leqslant \nu \leqslant 1} s * \tilde{\varphi}_\nu^d * \psi_\nu^d + R(P_k) \Big) + R(P_0),$$

with convergence in $\mathscr{S}'(\mathbf{Z})$.

For the next step in our program, we need to use the following version of the sampling theorem whose proof is contained in the proof of Theorem 4, and which turns out to be of crucial importance (see also [9]). For convenience, in the rest of this section we will denote the nth term of a sequence $s \in \mathscr{S}'(\mathbf{Z})$ by $s(n)$.

LEMMA 4. *Suppose* $f \in \mathscr{S}'(\mathbf{R})$, $h \in \mathscr{S}(\mathbf{R})$, *and* $\operatorname{supp} \hat{f}$, $\operatorname{supp} \hat{h} \subseteq \{\xi\colon |\xi| < 2^\nu \pi\}$ *for some* $\nu \in \mathbf{Z}$. *Then*

$$f * h(x) = \sum_{k \in \mathbf{Z}} 2^{-\nu} f(2^{-\nu}k) h(x - 2^{-\nu}k).$$

Let $A_\nu = \bigcup_{n \in \mathbf{Z}} \{\xi\colon |\xi - 2n\pi| < 2^\nu \pi\}$. From Lemma 4 we obtain the following discrete version.

LEMMA 5. *Suppose* $s \in \mathscr{S}'(\mathbf{Z})$, $r \in \mathscr{S}(\mathbf{Z})$, *and* $\operatorname{supp} P(s)^\wedge, \operatorname{supp} P(r)^\wedge \subseteq A_\nu$ *for some* $\nu \leqslant 0$. *Then*

$$s * r(n) = \sum_{k \in \mathbf{Z}} 2^{-\nu} s(2^{-\nu}k)(n - 2^{-\nu}k).$$

Proof. Notice that if $\nu = 0$, there is nothing to prove and, moreover, in this case we may allow $P(s)^\wedge$ and $P(r)^\wedge$ to be supported in all $\mathbf{R}$. For $\nu < 0$, let $E(s)$ and $E(r)$ be the distributions defined by the generalized sampling theorem. We have

$$s * r = R\big(E(s)\big) * R\big(E(r)\big).$$

On the other hand, because of the condition on the supports of $P(s)^\wedge$ and $P(r)^\wedge$, if $\phi \in C^\infty(\mathbf{R})$, $\operatorname{supp}\phi \subseteq (-\pi, \pi)$, and $\phi \equiv 1$ on $(\operatorname{supp} P(s)^\wedge \cup \operatorname{supp} P(r)^\wedge) \cap (-\pi, \pi)$, then

$$R\big(E(s)\big) * R\big(E(r)\big) = R\Big(\big(\phi E(s)^\wedge\big)^\vee\Big) * R\Big(\big(\phi E(r)^\wedge\big)^\vee\Big).$$

It follows from Lemmas 3 and 4 that

$$\begin{aligned} s * r &= R\Big(\big(\phi E(s)^\wedge\big)^\vee * \big(\phi E(r)^\wedge\big)^\vee\Big) \\ &= R\Big(\sum_{k \in \mathbf{Z}} 2^{-\nu} \big(\phi E(s)^\wedge\big)^\vee (2^{-\nu}k) \big(\phi E(r)^\wedge\big)^\vee (x - 2^{-\nu}k)\Big). \end{aligned}$$

Since the convergence in Lemma 4 is pointwise, we finally get

$$\begin{aligned} s * r(n) &= \sum_{k \in \mathbf{Z}} 2^{-\nu} \big(\phi E(s)^\wedge\big)^\vee (2^{-\nu}k) \big(\phi E(r)^\wedge\big)^\vee (n - 2^{-\nu}k) \\ &= \sum_{k \in \mathbf{Z}} 2^{-\nu} E(s)(2^{-\nu}k) E(r)(n - 2^{-\nu}k) = \sum_{k \in \mathbf{Z}} 2^{-\nu} s(2^{-\nu}k)(n - 2^{-\nu}k). \quad \blacksquare \end{aligned}$$

The other tool used in [6] to prove Theorem 5 is the Plancherel–Pólya inequality.

LEMMA 6. *Let* $0 < p < \infty$ *and* $\nu \in \mathbf{Z}$. *Suppose that* $f \in \mathscr{S}'(\mathbf{R})$ *and* $\operatorname{supp} \hat{f} \subseteq \{\xi\colon |\xi| < 2^\nu \pi\}$. *Then*

$$\sum_{k \in \mathbf{Z}} \sup_{z \in [2^{-\nu}k, 2^{-\nu}(k+1))} |f(z)|^p \leqslant C_p 2^\nu \|f\|_{L^p(\mathbf{R})}^p.$$

For a proof see e.g. [2], p. 101. The discrete analog of this lemma is easy to obtain.

LEMMA 7. *Let* $0 < p < \infty$ *and* $\nu \leqslant 0$. *Suppose that* $s \in \mathcal{S}'(\mathbf{Z})$ *and* $\operatorname{supp} P(s)^\wedge \subseteq A_\nu$. *Then*

$$\sum_{k\in\mathbf{Z}} \sup_{n\in[2^{-\nu}k,\,2^{-\nu}(k+1))} |s_n|^p \leqslant C_p 2^\nu \|s\|^p_{L^p(\mathbf{Z})}.$$

Proof. As in Lemma 4, for $\nu = 0$, there is nothing to prove and the condition on $\operatorname{supp} P(s)^\wedge$ is unnecessary. For $\nu < 0$,

$$s = R(E(s)) = R\big((\phi E(s)^\wedge)^\vee\big),$$

where $\phi \in C^\infty(\mathbf{R})$, $\operatorname{supp}\phi \subseteq (-\pi, \pi)$, and $\phi \equiv 1$ on $\operatorname{supp} P(s)^\wedge \cap (-\pi, \pi)$. But then, by Lemma 6,

$$\sum_{k\in\mathbf{Z}} \sup_{n\in[2^{-\nu}k,\,2^{-\nu}(k+1))} |s_n|^p = \sum_{k\in\mathbf{Z}} \sup_{n\in[2^{-\nu}k,\,2^{-\nu}(k+1))} \big|(\phi E(s)^\wedge)^\vee(n)\big|^p$$

$$\leqslant \sum_{k\in\mathbf{Z}} \sup_{z\in[2^{-\nu}k,\,2^{-\nu}(k+1))} \big|(\phi E(s)^\wedge)^\vee(z)\big|^p \leqslant C_p 2^\nu \big\|(\phi E(s)^\wedge)^\vee\big\|^p_{L^p(\mathbf{R})}$$

$$\leqslant C_p 2^\nu \big\|R\big((\phi E(s)^\wedge)^\vee\big)\big\|^p_{L^p(\mathbf{Z})} = C_p 2^\nu \|s\|^p_{L^p(\mathbf{Z})}. \ \blacksquare$$

For $\nu \leqslant 0$ and $k \in \mathbf{Z}$, we define $\varphi^d_{\nu k} = R(\varphi_{\nu k})$ and $\psi^d_{\nu k} = R(\psi_{\nu k})$, while for $\nu = 1$ and $k \in \mathbf{Z}$, $\varphi^d_{1k} = \tau_k \varphi^d_1$ and $\psi^d_{1k} = \tau_k \psi^d_1$. The different definition for $\nu = 1$ is necessary, as we will see, because $\chi_{[-\pi,\pi]}\hat{\varphi}_{\nu k}$ and $\chi_{[-\pi,\pi]}\hat{\psi}_{\nu k}$ do not have smooth 2π-periodic extensions if $k \neq 0$. We also define for $\alpha \in \mathbf{R}$, and $0 < p, q < \infty$, $\dot{b}^{\alpha,q}_p(\mathbf{Z})$ to be the collection of all doubly indexed sequences of complex numbers, $t = \{t_{\nu k}\}_{\nu \leqslant 1, k\in\mathbf{Z}}$, such that

$$\|t\|_{\dot{b}^{\alpha,q}_p(\mathbf{Z})} = \Big(\sum_{\nu\leqslant 1}\Big(\sum_{k\in\mathbf{Z}} (2^{\nu(\alpha+1/2-1/p)}|t_{\nu k}|)^p\Big)^{q/p}\Big)^{1/q} < \infty.$$

We can now prove the first part of the φ-transform decomposition of the spaces $\dot{B}^{\alpha,q}_p(\mathbf{Z})$.

THEOREM 6. *Every* $s \in \dot{B}^{\alpha,q}_p(\mathbf{Z})$ *can be written in the form*

$$\sum_{\nu\leqslant 1}\sum_{k\in\mathbf{Z}} \langle s, \varphi^d_{\nu k}\rangle \psi^d_{\nu k},$$

where the sequence of φ*-transform coefficients* $S_\varphi s = \{\langle s, \varphi^d_{\nu k}\rangle\}_{\nu,k}$ *is in* $\dot{b}^{\alpha,q}_p(\mathbf{Z})$ *and satisfies*

$$\|S_\varphi s\|_{\dot{b}^{\alpha,q}_p(\mathbf{Z})} \leqslant C\|s\|_{\dot{B}^{\alpha,q}_p(\mathbf{Z})}.$$

Proof. Let $s \in \dot{B}^{\alpha,q}_p(\mathbf{Z})$. By Lemma 5 and (3.3), we have

$$s(\cdot) = \sum_{\nu\leqslant 0}\sum_{k\in\mathbf{Z}} 2^{-\nu} s * \tilde{\varphi}^d_\nu(2^{-\nu}k)\psi^d_\nu(\cdot - 2^{-\nu}k) + \sum_{k\in\mathbf{Z}} 2^{-1} s * \tilde{\varphi}^d_1(k)\psi^d_1(\cdot - k)$$

$$= \sum_{\nu\leqslant 1}\sum_{k\in\mathbf{Z}} \langle s, \varphi^d_{\nu k}\rangle \psi^d_{\nu k}.$$

Thus, we only need to check that $S_\varphi s = \{\langle s, \varphi^d_{\nu k}\rangle\}$ is in $\dot{b}^{\alpha,q}_p(\mathbf{Z})$. For $\nu \leqslant 0$, we have from Lemma 7,

$$\Big(\sum_{k\in\mathbf{Z}} (2^{\nu(\alpha+1/2-1/p)}|\langle s, \varphi^d_{\nu k}\rangle|)^p = \sum_{k\in\mathbf{Z}} (2^{\nu(\alpha+1/2-1/p)}|2^{-\nu/2} s * \tilde{\varphi}^d_\nu(2^{-\nu}k)|)^p$$

$$\leqslant C 2^{\nu\alpha p}\|s * \tilde{\varphi}^d_\nu\|^p_{L^p(\mathbf{Z})},$$

and, for $\nu = 1$,

$$\sum_{k\in\mathbf{Z}} (2^{\alpha+1/2-1/p}|\langle s, \varphi^d_{1k}\rangle|)^p = \sum_{k\in\mathbf{Z}} \big(2^{\alpha+1/2-1/p}|s * \tilde{\varphi}^d_1(k)|\big)^p \leqslant C 2^{\alpha p}\|s * \tilde{\varphi}^d_1\|^p_{L^p(\mathbf{Z})}.$$

Thus,

$$\|S_\varphi s\|_{\dot{b}^{\alpha,q}_p(\mathbf{Z})} = \Big(\sum_{\nu\leqslant 1}\Big(\sum_{k\in\mathbf{Z}} (2^{\nu(\alpha+1/2-1/p)}|\langle s, \varphi^d_{\nu k}\rangle|)^p\Big)^{q/p}\Big)^{1/q}$$

$$\leqslant C\Big(\sum_{\nu\leqslant 1} (2^{\nu\alpha}\|s * \tilde{\varphi}^d_\nu\|_{L^p(\mathbf{Z})})^q\Big)^{1/q} \leqslant C\|s\|_{\dot{B}^{\alpha,q}_p(\mathbf{Z})}. \ \blacksquare$$

To prove the converse of Theorem 6, we need to recall one more result from [6]. The following is a particular case of Lemma 3.3 in there (notice, however, our different normalization of the functions $\psi_{\nu k}$ with respect to the functions ψ_Q in [6]).

LEMMA 8. *For every* $M > N > 0$, *there exists a constant* $C > 0$ *such that*

$$|\varphi_\nu * \psi_{\mu k}(x)| \leqslant C 2^{\mu/2} 2^{(\nu-\mu)N}(1 + 2^\nu|x - 2^{-\nu}k|)^{-M}$$

if $\nu \leqslant \mu$,

$$\big|\varphi_\nu * \big(\tau_k(\chi_{[-\pi,\pi]}\hat{\psi}_1)^\vee\big)(x)\big| \leqslant C 2^{\nu N}(1 + 2^\nu|x - 2^{-\nu}k|)^{-M}$$

if $\nu < 1$, *and*

$$|\varphi_\nu * \psi_{\mu k}(x)| \leqslant C 2^{\mu/2} 2^{(\mu-\nu)N}(1 + 2^\mu|x - 2^{-\mu}k|)^{-M}$$

if $\mu \leqslant \nu$.

From this lemma, using the properties of the operator R, we immediately get

LEMMA 9. *For every* $M > N > 0$, *there exists a constant* $C > 0$ *such that*

(3.4) $$|\varphi^d_\nu * \psi^d_{\mu k}(n)| \leqslant C 2^{\mu/2} 2^{(\nu-\mu)N}(1 + 2^\nu|n - 2^{-\nu}k|)^{-M}$$

if $\nu \leqslant \mu \leqslant 0$,

(3.5) $$|\varphi^d_\nu * \psi^d_{1k}(n)| \leqslant C 2^{\nu N}(1 + 2^\nu|n - 2^{-\nu}k|)^{-M}$$

if $\nu < 1$, *and*

(3.6) $$|\varphi^d_\nu * \psi^d_{\mu k}(n)| \leqslant C 2^{\mu/2} 2^{(\mu-\nu)N}(1 + 2^\mu|n - 2^{-\mu}k|)^{-M}$$

if $\mu \leqslant \nu \leqslant 0$ *or* $\mu < \nu = 1$.

Notice also that we trivially have

$$|\varphi_1^d * \psi_{1k}^d(n)| = |\varphi_1^d * \tau_k \psi_1^d(n)| \leqslant C(1+|n-k|)^{-M}, \tag{3.7}$$

since φ_1^d and ψ_1^d are the sequences of "Fourier coefficients" of smooth periodic functions. We can now prove the converse of Theorem 6.

THEOREM 7. *For every sequence $t = \{t_{\nu k}\}_{\nu,k} \in \dot{b}_p^{\alpha,q}(\mathbf{Z})$, its inverse φ-transform*

$$T_\psi t = \sum_{\nu \leqslant 1} \sum_{k \in \mathbf{Z}} t_{\nu k} \psi_{\nu k}^d$$

is an element of $\dot{B}_p^{\alpha,q}(\mathbf{Z})$ and satisfies

$$\|T_\psi t\|_{\dot{B}_p^{\alpha,q}(\mathbf{Z})} \leqslant C \|t\|_{\dot{b}_p^{\alpha,q}(\mathbf{Z})}.$$

Proof. Assume first $0 < p \leqslant 1$. Then

$$\|T_\psi t\|^q_{\dot{B}_p^{\alpha,q}(\mathbf{Z})} = \sum_{\nu \leqslant 1} (2^{\nu\alpha} \|T_\psi t * \varphi_\nu^d\|_{L^p(\mathbf{Z})})^q \leqslant \sum_{\nu \leqslant 1} \Big(2^{\nu\alpha}\Big(\sum_{n \in \mathbf{Z}} \sum_{\mu \leqslant 1} \sum_{k \in \mathbf{Z}} |t_{\mu k}|^p |\psi_{\mu k}^d * \varphi_\nu^d(n)|^p\Big)^{1/p}\Big)^q$$

$$\leqslant \sum_{\nu \leqslant 1} \Big(2^{\nu\alpha}\Big(\sum_{\mu \leqslant \nu} \sum_{k \in \mathbf{Z}} |t_{\mu k}|^p \sum_{n \in \mathbf{Z}} |\psi_{\mu k}^d * \varphi_\nu^d(n)|^p\Big)^{1/p}\Big)^q$$

$$+ \sum_{\nu \leqslant 1} \Big(2^{\nu\alpha}\Big(\sum_{\nu < \mu \leqslant 1} \sum_{k \in \mathbf{Z}} |t_{\mu k}|^p \sum_{n \in \mathbf{Z}} |\psi_{\mu k}^d * \varphi_\nu^d(n)|^p\Big)^{1/p}\Big)^q.$$

If we now use (3.4–7) we get

$$\|T_\psi t\|^q_{\dot{B}_p^{\alpha,q}(\mathbf{Z})} \leqslant C \sum_{\nu \leqslant 1} \Big(2^{\nu\alpha}\Big(\sum_{\mu \leqslant \nu} 2^{-\mu\alpha p + (\mu-\nu)pN} \sum_{k \in \mathbf{Z}} 2^{\mu(\alpha - 1/p + 1/2)p} |t_{\mu k}|^p\Big)^{1/p}\Big)^q$$

$$+ C \sum_{\nu \leqslant 1} \Big(2^{\nu\alpha}\Big(\sum_{\nu < \mu \leqslant 1} 2^{\mu(1 - \alpha p - Np) + \nu(Np - 1)} \sum_{k \in \mathbf{Z}} 2^{\mu(\alpha - 1/p + 1/2)p} |t_{\mu k}|^p\Big)^{1/p}\Big)^q$$

$$\leqslant C \sum_{\nu \leqslant 1} \Big(\sum_{\mu \leqslant \nu} 2^{-(\nu-\mu)(N-\alpha)p} \sum_{k \in \mathbf{Z}} 2^{\mu(\alpha - 1/p + 1/2)p} |t_{\mu k}|^p\Big)^{q/p}$$

$$+ C \sum_{\nu \leqslant 1} \Big(\sum_{\nu < \mu \leqslant 1} 2^{-(\mu-\nu)(-1/p + N + \alpha)p} \sum_{k \in \mathbf{Z}} 2^{\mu(\alpha - 1/p + 1/2)p} |t_{\mu k}|^p\Big)^{q/p}.$$

If $q \leqslant p$,

$$\|T_\psi t\|^q_{\dot{B}_p^{\alpha,q}(\mathbf{Z})} \leqslant C \sum_{\nu \leqslant 1} \sum_{\mu \leqslant \nu} 2^{-(\nu-\mu)(N-\alpha)q} \Big(\sum_{k \in \mathbf{Z}} 2^{\mu(\alpha - 1/p + 1/2)p} |t_{\mu k}|^p\Big)^{q/p}$$

$$+ C \sum_{\nu \leqslant 1} \sum_{\nu < \mu \leqslant 1} 2^{-(\mu-\nu)(-1/p + N + \alpha)q} \Big(\sum_{k \in \mathbf{Z}} 2^{\mu(\alpha - 1/p + 1/2)p} |t_{\mu k}|^p\Big)^{q/p},$$

and, hence, taking N sufficiently large and noticing that the last two terms can be viewed as the $L^1(\mathbf{Z})$-norm of a convolution on $\mathbf{Z}$, we obtain

$$\|T_\psi t\|^q_{\dot{B}_p^{\alpha,q}(\mathbf{Z})} \leqslant C \sum_{\mu \leqslant 1} \Big(\sum_{k \in \mathbf{Z}} 2^{\mu(\alpha - 1/p + 1/2)p} |t_{\mu k}|^p\Big)^{q/p}. \tag{3.8}$$

On the other hand, if $q > p$, (3.8) follows from Young's inequality by taking again N sufficiently large.

Assume now that $1 < p < \infty$. Then

$$\|T_\psi t\|^q_{\dot{B}_p^{\alpha,q}(\mathbf{Z})} = \sum_{\nu \leqslant 1} (2^{\nu\alpha} \|T_\psi t * \varphi_\nu^d\|_{L^p(\mathbf{Z})})^q$$

$$\leqslant C \sum_{\nu \leqslant 1} \Big(2^{\nu\alpha} \sum_{\mu \leqslant \nu} \Big(\sum_{n \in \mathbf{Z}} \Big(\sum_{k \in \mathbf{Z}} |t_{\mu k}|\, |\psi_{\mu k}^d * \varphi_\nu^d(n)|\Big)^p\Big)^{1/p}\Big)^q$$

$$+ C \sum_{\nu \leqslant 1} \Big(2^{\nu\alpha} \sum_{\nu < \mu \leqslant 1} \Big(\sum_{n \in \mathbf{Z}} \Big(\sum_{k \in \mathbf{Z}} |t_{\mu k}|\, |\psi_{\mu k}^d * \varphi_\nu^d(n)|\Big)^p\Big)^{1/p}\Big)^q = I + II.$$

We will only estimate the first term. The estimates for the second one are entirely analogous. For more details see [6], p. 787. Using (3.4–7), we have

$$I \leqslant C \sum_{\nu < 1} \Big(2^{\nu\alpha} \sum_{\mu \leqslant \nu} \Big(\sum_{n \in \mathbf{Z}} \Big(\sum_{k \in \mathbf{Z}} |t_{\mu k}| 2^{\mu/2} 2^{(\mu-\nu)N} (1 + 2^\mu |n - 2^{-\mu} k|^{-M})\Big)^p\Big)^{1/p}\Big)^q$$

$$+ C \Big(\sum_{\mu \leqslant 1} \Big(\sum_{n \in \mathbf{Z}} \Big(\sum_{k \in \mathbf{Z}} |t_{\mu k}| 2^{\mu/2} 2^{(\mu-1)N} (1 + |n-k|^{-M})\Big)^p\Big)^{1/p}\Big)^q$$

$$\leqslant C \sum_{\nu \leqslant 1} \Big(2^{\nu\alpha} \sum_{\mu \leqslant \nu} \Big(2^{-\mu} \sum_{j \in \mathbf{Z}} \Big(\sum_{k \in \mathbf{Z}} |t_{\mu k}| 2^{\mu/2} 2^{(\mu-\nu)N} (1 + 2^\mu |2^{-\mu} j - 2^{-\mu} k|^{-M})\Big)^p\Big)^{1/p}\Big)^q$$

$$\leqslant C \sum_{\nu \leqslant 1} \Big(\sum_{\mu \leqslant \nu} \Big(2^{\mu(\alpha - 1/p + 1/2)p} 2^{(\mu-\nu)(N-\alpha)p} \sum_{j \in \mathbf{Z}} \Big(\sum_{k \in \mathbf{Z}} |t_{\mu k}| (1 + |j-k|^{-M})\Big)^p\Big)^{1/p}\Big)^q$$

$$\leqslant C \sum_{\nu \leqslant 1} \Big(\sum_{\mu \leqslant \nu} \Big(2^{\mu(\alpha - 1/p + 1/2)p} 2^{(\mu-\nu)(N-\alpha)p} \sum_{k \in \mathbf{Z}} |t_{\mu k}|^p\Big)^{1/p}\Big)^q.$$

Finally, by considering separately the case $q > 1$ and $q \leqslant 1$, similar arguments to the ones used in the case $0 < p \leqslant 1$ show that

$$I \leqslant C \sum_{\mu \leqslant 1} \Big(\sum_{k \in \mathbf{Z}} 2^{\mu(\alpha - 1/p + 1/2)p} |t_{\mu k}|^p\Big)^{q/p}. \ \blacksquare$$

As in the case of the spaces $\dot{B}_p^{\alpha,q}(\mathbf{R})$, the independence of the choice of φ in the definition of the spaces $\dot{B}_p^{\alpha,q}(\mathbf{Z})$ follows easily from Theorems 6 and 7 (see Remark 2.6 in [8]). We will now consider another application.

One of the important consequences of Theorem 5 is that it allows us to reduce the study of certain problems about distribution spaces to the study of equivalent ones on sequence spaces in a simplified way. Theorem 5 can be rephrased by saying that the following diagram is commutative:

$$\begin{array}{ccc} & \dot{b}_p^{\alpha,q}(\mathbf{R}) & \\ {\scriptstyle S_\varphi}\nearrow & & \searrow{\scriptstyle T_\psi} \\ \dot{B}_p^{\alpha,q}(\mathbf{R}) & \xrightarrow{\ \mathrm{Id}\ } & \dot{B}_p^{\alpha,q}(\mathbf{R}) \end{array}$$

It is also possible to extend this commutativity to the "operator level". In fact, let $\mathscr{L}(\dot{B}_p^{\alpha,q}(\mathbf{R}))$ and $\mathscr{L}(\dot{b}_p^{\alpha,q}(\mathbf{R}))$ denote, respectively, the spaces of bounded linear operators on $\dot{B}_p^{\alpha,q}(\mathbf{R})$ and $\dot{b}_p^{\alpha,q}(\mathbf{R})$, topologized with the operator norm. If we define the maps S_φ^*: $\mathscr{L}(\dot{B}_p^{\alpha,q}(\mathbf{R}))\to\mathscr{L}(\dot{b}_p^{\alpha,q}(\mathbf{R}))$ by

$$S_\varphi^* B = S_\varphi \circ B \circ T_\psi \quad \text{if } B\in\mathscr{L}(\dot{B}_p^{\alpha,q}(\mathbf{R})),$$

and T_ψ^*: $\mathscr{L}(\dot{b}_p^{\alpha,q}(\mathbf{R}))\to\mathscr{L}(\dot{B}_p^{\alpha,q}(\mathbf{R}))$ by

$$T_\psi^* A = T_\psi \circ A \circ S_\varphi \quad \text{if } A\in\mathscr{L}(\dot{b}_p^{\alpha,q}(\mathbf{R})),$$

then the following diagram is also commutative:

$$\begin{array}{ccccc} & & \mathscr{L}(\dot{b}_p^{\alpha,q}(\mathbf{R})) & & \\ & {\scriptstyle S_\varphi^*}\nearrow & & \searrow{\scriptstyle T_\psi^*} & \\ \mathscr{L}(\dot{B}_p^{\alpha,q}(\mathbf{R})) & & \xrightarrow{\text{Id}} & & \mathscr{L}(\dot{B}_p^{\alpha,q}(\mathbf{R})) \end{array}$$

For any $B\in\mathscr{L}(\dot{B}_p^{\alpha,q}(\mathbf{R}))$, we have

$$\|B\|_{\mathscr{L}(\dot{B}_p^{\alpha,q}(\mathbf{R}))} \approx \|S_\varphi^* B\|_{\mathscr{L}(\dot{b}_p^{\alpha,q}(\mathbf{R}))},$$

and the operator $S_\varphi^* B$ is given by the "matrix" with entries

$$a_{\nu k,\mu l} = \langle B\psi_{\mu l}, \varphi_{\nu k}\rangle;$$

i.e., for a sequence $t=\{t_{\nu k}\}\in\dot{b}_p^{\alpha,q}(\mathbf{R})$, $S_\varphi^* B(t)$ is the sequence given by

$$S_\varphi^* B(t)_{\nu k} = \sum_{\mu\in\mathbf{Z}}\sum_{l\in\mathbf{Z}} \langle B\psi_{\mu l}, \varphi_{\nu k}\rangle t_{\mu l}.$$

We refer the reader to [8] and [23] for more detailed explanations. Using Theorems 6 and 7, we can repeat all of the above for the spaces $\dot{B}_p^{\alpha,q}(\mathbf{Z})$ and $\dot{b}_p^{\alpha,q}(\mathbf{Z})$ and obtain retract diagrams

$$\begin{array}{ccccc} & & \dot{b}_p^{\alpha,q}(\mathbf{Z}) & & \\ & {\scriptstyle S_\varphi}\nearrow & & \searrow{\scriptstyle T_\psi} & \\ \dot{B}_p^{\alpha,q}(\mathbf{Z}) & & \xrightarrow{\text{Id}} & & \dot{B}_p^{\alpha,q}(\mathbf{Z}) \end{array} \qquad \begin{array}{ccccc} & & \mathscr{L}(\dot{b}_p^{\alpha,q}(\mathbf{Z})) & & \\ & {\scriptstyle S_\varphi^*}\nearrow & & \searrow{\scriptstyle T_\psi^*} & \\ \mathscr{L}(\dot{B}_p^{\alpha,q}(\mathbf{Z})) & & \xrightarrow{\text{Id}} & & \mathscr{L}(\dot{B}_p^{\alpha,q}(\mathbf{Z})) \end{array}$$

Notice that, even though $\dot{B}_p^{\alpha,q}(\mathbf{Z})$ is already a space of sequences, $\dot{b}_p^{\alpha,q}(\mathbf{Z})$ has a simpler structure since only "size" is involved in its definition and, in fact, it is a lattice.

Using the above techniques, we can describe a simple application to the study of Fourier multiplier operators. Boundedness properties of multipliers and related operators acting on $L^p(\mathbf{R})$ can be obtained from properties of some associated operators acting on $L^p(\mathbf{Z})$. This has been extensively studied. See for example [4] and, in particular, the more recent work [1]. Our techniques can be used to obtain similar results for $\dot{B}_p^{\alpha,q}(\mathbf{R})$ in a rather straightforward manner.

THEOREM 8. *Let $\alpha\in\mathbf{R}$ and $0<p,q<\infty$. Assume that $m\in\mathscr{S}'(\mathbf{R})$ satisfies* $\operatorname{supp} m \subseteq \{\xi\colon |\xi|\le h\}$. *Then the multiplier operator*

$$T_m f = (m\hat{f})^\vee,$$

initially defined on $\mathscr{S}(\mathbf{R})\cap\dot{B}_p^{\alpha,q}(\mathbf{R})$, extends to a bounded operator on $\dot{B}_p^{\alpha,q}(\mathbf{R})$ if and only if the operator

$$T_m^d s = \frac{\pi}{2h} R\left(\check{m}\left(\frac{\pi}{2h}\cdot\right)\right) * s,$$

initially defined on $\mathscr{S}(\mathbf{Z})\cap\dot{B}_p^{\alpha,q}(\mathbf{Z})$, extends to a bounded operator on $\dot{B}_p^{\alpha,q}(\mathbf{Z})$.

Proof. First, observe that for every $\lambda>0$,

$$\|f(\lambda\cdot)\|_{\dot{B}_p^{\alpha,q}(\mathbf{R})} \le C\lambda^{\alpha-1/p}\|f\|_{\dot{B}_p^{\alpha,q}(\mathbf{R})}$$

(see [23], p. 239) and, therefore, m defines a bounded Fourier multiplier operator on $\dot{B}_p^{\alpha,q}(\mathbf{R})$ if and only if $m((2h/\pi)\cdot)$ does. Next, as we have seen, $m((2h/\pi)\cdot)$ defines a bounded Fourier multiplier on $\dot{B}_p^{\alpha,q}(\mathbf{R})$ if and only if the matrix with entries

$$\left\langle \frac{\pi}{2h}\check{m}\left(\frac{\pi}{2h}\cdot\right) * \psi_{\mu l}, \varphi_{\nu k}\right\rangle, \qquad \nu,\mu,k,l\in\mathbf{Z},$$

defines a bounded operator on $\dot{b}_p^{\alpha,q}(\mathbf{R})$. Similarly, T_m^d is bounded in $\dot{B}_p^{\alpha,q}(\mathbf{Z})$ if and only if the matrix with entries

$$\left\langle \frac{\pi}{2h}R\left(\check{m}\left(\frac{\pi}{2h}\cdot\right)\right) * \psi_{\mu l}^d, \varphi_{\nu k}^d\right\rangle, \qquad \nu,\mu\le 1,\ k,l\in\mathbf{Z},$$

is bounded on $\dot{b}_p^{\alpha,q}(\mathbf{Z})$. But, since $\operatorname{supp} m((2h/\pi)\cdot)\subseteq[-\pi/2,\pi/2]$,

$$\left\langle \frac{\pi}{2h}\check{m}\left(\frac{\pi}{2h}\cdot\right) * \psi_{\mu l}, \varphi_{\nu k}\right\rangle = \left\langle \frac{\pi}{2h}R\left(\check{m}\left(\frac{\pi}{2h}\cdot\right)\right) * \psi_{\mu l}^d, \varphi_{\nu k}^d\right\rangle, \qquad \nu,\mu\le 1,\ k,l\in\mathbf{Z},$$

and

$$\left\langle \frac{\pi}{2h}\check{m}\left(\frac{\pi}{2h}\cdot\right) * \psi_{\mu l}, \varphi_{\nu k}\right\rangle = 0, \qquad \nu>1 \text{ or } \mu>1,\ k,l\in\mathbf{Z}.$$

The theorem now easily follows. ∎

We want to conclude this paper with a few remarks. First of all, although for simplicity in the exposition we have restricted ourselves to the one-dimensional case, all the results of this section easily extend to the case of $\mathbf{R}^n$ and $\mathbf{Z}^n$. It is possible to obtain more general "molecular decompositions" of the spaces $\dot{B}_p^{\alpha,q}(\mathbf{Z})$ and most of the Frazier–Jawerth theory about the spaces $\dot{B}_p^{\alpha,q}(\mathbf{R})$ passes through to the discrete case. As the reader may suspect, it should be also possible to consider discrete versions of the Triebel–Lizorkin spaces. We will not pursue this here any further.

Finally, we want to mention that a class of Besov spaces on the integers in the diagonal case, i.e., $\dot{B}_p^{1/p,p}(\mathbf{Z})$, have been considered by R. Rochberg in [21]. See also [19]. It would be of interest to relate those spaces to the ones defined in this paper. We think that the strategy of Theorem 8 may be carried over to the study of operators more complicated than multipliers, in particular, operators like the ones studied in [21] and [19].

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Weighted-BMO and the Hilbert transform

by

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Abstract. In 1967, E. M. Stein proved that the Hilbert transform is bounded from L^∞ to BMO. In 1976, Muckenhoupt and Wheeden gave an analogue of Stein's result. They gave a necessary and sufficient condition for the boundedness of the Hilbert transform from L_w^∞ to BMO_w. We improve the results of Muckenhoupt and Wheeden's and give a necessary and sufficient condition for the boundedness of the Hilbert transform from BMO_w to BMO_w.

Introduction. Let $f(x)$ and $w(x)$ be locally integrable in $\mathbf{R}^n$ and $w(x)\geqslant 0$. Then we say that $f\in \mathrm{BMO}_w(\mathbf{R}^n)$ if there is a constant C such that

$$\frac{1}{w(I)}\int_I |f(x)-f_I|\,dx\leqslant C$$

for all n-dimensional cubes I whose edges are parallel to the coordinate axes. Here $f_I=(1/|I|)\int_I f\,dx$, $w(I)=\int_I w\,dx$. The norm in $\mathrm{BMO}_w(\mathbf{R}^n)$ is defined as

$$\|f\|_{*,w}=\sup_I \frac{1}{w(I)}\int_I |f(x)-f_I|\,dx.$$

The case $w=1$ corresponds to that of John and Nirenberg.

A function f is said to belong to $L_w^\infty(\mathbf{R}^n)$ if $fw^{-1}\in L^\infty(\mathbf{R}^n)$. The norm in $L_w^\infty(\mathbf{R}^n)$ is defined as

$$\|f\|_{\infty,w}=\|fw^{-1}\|_\infty.$$

Finally, if there is a constant C such that

$$\int_{I^c}\frac{w(t)}{|x_I-t|^{2n}}\,dt\leqslant C\frac{1}{|I|^2}\int_I w(t)\,dt$$

for all cubes I, then we say $w\in B_2$. Here x_I is the center of I. From [1] we know $w\in A_2$ implies $w\in B_2$.

Only the case $n=1$ is considered in the following.

In [2] Muckenhoupt and Wheeden considered the modified version of the Hilbert transform: let

$$Hf(x)=\lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon}\left[\frac{1}{x-y}+\frac{\eta(y)}{y}\right]f(y)\,dy$$

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