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On separation theorems for subadditive and superadditive functionals

by

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Abstract. We generalize the well known separation theorems for subadditive and superadditive functionals to some classes of not necessarily Abelian semigroups. We also consider the problem of supporting subadditive functionals by additive ones in the not necessarily commutative case. Our results are motivated by similar extensions of the Hyers stability theorem for the Cauchy functional equation. In this context the so-called weakly commutative and amenable semigroups appear naturally. The relations between these two classes of semigroups are discussed at the end of the paper.

1. Introduction. In this paper we are concerned with the problem of separation of subadditive and superadditive functionals defined on not necessarily commutative semigroups. Results of this type, for Abelian semigroups, were first obtained by R. Kaufman [8] and P. Kranz [10]. They can also be derived from the celebrated separation theorem of G. Rodé [12] (cf. also H. König [9]) which represents a far-reaching generalization of the classical Hahn–Banach theorem. In spite of its highly abstract setting, Rodé's theorem does not yield any extensions of Kaufman's and Kranz's results beyond the class of Abelian semigroups (some special noncommutative versions of Rodé's theorem have recently been discussed by A. Chaljub-Simon and P. Volkmann [1]). The main purpose of the present work is to replace the commutativity assumption in separation theorems of Kranz's type by some essentially weaker conditions of algebraic or analytic nature. In this regard, we follow the lines along which the Hyers stability theorem for the Cauchy functional equation (see D. H. Hyers [7]) was generalized to certain classes of not necessarily commutative semigroups.

In what follows $\mathbf{R}$ and $\mathbf{N}$ denote the sets of all reals and positive integers, respectively, whereas $(S, \cdot)$ stands for a semigroup or, occasionally, a group. To emphasize the fact that the binary operation in S does not have to be commutative we use for it the multiplicative notation.

We recall that a functional $f\colon S\to\mathbf{R}$ is said to be *subadditive* iff

$$f(xy) \leqslant f(x)+f(y), \quad x, y\in S. \tag{1}$$

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A functional $g\colon S\to\mathbf{R}$ is called *superadditive* iff $f:=-g$ is subadditive or, equivalently, iff g satisfies

(2) $$g(xy)\geqslant g(x)+g(y),\quad x,y\in S.$$

Finally, $a\colon S\to\mathbf{R}$ is *additive* iff it is simultaneously subadditive and superadditive; in other words, it satisfies the *Cauchy functional equation*

(3) $$a(xy)=a(x)+a(y),\quad x,y\in S.$$

In the sequel we shall consider the question whether for any pair of functionals $f,g\colon S\to\mathbf{R}$ satisfying conditions (1), (2) respectively and such that

(4) $$g(x)\leqslant f(x),\quad x\in S,$$

one can find an additive functional $a\colon S\to\mathbf{R}$ which *separates* f and g, i.e.,

(5) $$g(x)\leqslant a(x)\leqslant f(x),\quad x\in S.$$

Notice that, without additional hypotheses on the semigroup S, the answer to this question may be negative. This becomes apparent if we call to mind the example given by G. L. Forti in [3] (see also [4]). Taking for S the free group with two generators Forti constructed a function $\varphi\colon S\to\mathbf{R}$ with the following properties:

(6) $$\varphi(xy)-\varphi(x)-\varphi(y)\in\{-1,0,1\},\quad x,y\in S,$$

(7) there exists no additive functional $a\colon S\to\mathbf{R}$ for which $a-\varphi$ is uniformly bounded on S.

If we now put $f:=\varphi+1$ and $g:=\varphi-1$, then (6) implies that f is subadditive, g is superadditive and obviously $g\leqslant f$. On the other hand, by (7), there is no additive functional separating f and g.

Forti's example was originally intended to show that the Hyers theorem on stability of the Cauchy functional equation does not hold on the free group generated by two elements. It is known, however, that there are two kinds of assumptions on the semigroup S, either of which is much weaker than commutativity, yet ensures that the Hyers theorem remains valid on S (see [11], [14] and [13], [4]). We recall here these assumptions, since they will play the crucial role in our further discussion of separation theorems.

The assumption of the first kind may be formulated in purely algebraic terms:

(8) $$\underset{x,y\in S}{\forall}\ \underset{n\in\mathbf{N}}{\exists}\ (xy)^{2^n}=x^{2^n}y^{2^n}.$$

This implies that for any $x,y\in S$ there exists a sequence of positive integers n_k (depending on x and y) such that $n_k\to\infty$ as $k\to\infty$ and

$$(xy)^{2^{n_k}}=x^{2^{n_k}}y^{2^{n_k}},\quad k\in\mathbf{N}.$$

A semigroup S satisfying (8) will be called *weakly commutative* (cf. e.g. [14]). It is clear that every Abelian semigroup is weakly commutative, but there exist non-Abelian semigroups and even groups which satisfy (8). A simple example is provided by the multiplicative group consisting of the quaternions $1,-1,i,-i,j,-j,k,-k$ (i,j and k being the quaternion imaginary units).

The assumption of the second type involves analytic properties of the space conjugate to the Banach space $\mathscr{B}(S,\mathbf{R})$ of all bounded real-valued functions defined on S (with the supremum norm). A linear functional $M\in\mathscr{B}(S,\mathbf{R})^*$ is called a *right* (resp. *left*) *invariant mean* iff

(9) $$\inf_{x\in S}\varphi(x)\leqslant M(\varphi)\leqslant\sup_{x\in S}\varphi(x)\quad\text{for all }\varphi\in\mathscr{B}(S,\mathbf{R});$$

(10) $$M(\varphi_a)=M(\varphi)\quad(\text{resp. } M({}_a\varphi)=M(\varphi))\quad\text{for all }\varphi\in\mathscr{B}(S,\mathbf{R})\text{ and }a\in S,$$

where φ_a and ${}_a\varphi$ are the right and left translates of φ defined by

$$\varphi_a(x):=\varphi(xa),\quad {}_a\varphi(x):=\varphi(ax),\quad x\in S.$$

We say that the semigroup S is *amenable* iff $\mathscr{B}(S,\mathbf{R})^*$ contains at least one right or left invariant mean (cf. e.g. [2]). One can prove that every Abelian semigroup is amenable, but again, there are lots of non-Abelian amenable semigroups and even groups. For instance, it is well known that every solvable group admits an invariant mean (see [2]).

In this paper we confine our considerations to subadditive and superadditive functionals which assume only finite real values, although in the respective results of Kaufman and Kranz the value $-\infty$ is admissible. This restriction is forced by the methods we apply and it enters into the price we pay for the substantial weakening of the commutativity assumption.

2. Preliminary results. Let $(S,\cdot)$ be an arbitrary semigroup and let x_0 be a fixed element of S. A functional $f\colon S\to\mathbf{R}$ satisfying

(11) $$f(x_0^k)=kf(x_0)$$

for some $k\in\mathbf{N}$ is said to be *k-homogeneous at x_0*. If (11) holds for every $k\in\mathbf{N}$, then f is **N**-*homogeneous at x_0*. Moreover, if f is k- (resp. **N**-) homogeneous at every point of S, then we simply say that it is k- (resp. **N**-) *homogeneous.*

LEMMA 1. *Let $(S,\cdot)$ be a semigroup and let $f\colon S\to\mathbf{R}$ be subadditive or superadditive. If f is 2-homogeneous, then it is* **N**-*homogeneous.*

Proof. First we consider the case where f is subadditive. If f is 2-homogeneous, then a simple induction shows that it is 2^m-homogeneous for every $m\in\mathbf{N}$.

Now, choose an arbitrary $x_0\in S$ and a positive integer k which is different from 2^m for any $m\in\mathbf{N}$. Then either $k=1$ and the k-homogeneity of f is trivial, or $k=2^m+r$ with some $m\in\mathbf{N}$ and $r\in\{1,\ldots,2^m-1\}$. Consequently,

$$2^{m+1}f(x_0)=f(x_0^{2^{m+1}})=f(x_0^kx_0^{2^m-r})\leqslant f(x_0^k)+(2^m-r)f(x_0).$$

Hence $kf(x_0) \leqslant f(x_0^k)$, whereas the converse inequality follows from the fact that f is subadditive.

If f is superadditive, just consider $g := -f$.

LEMMA 2. *Let $(S, \cdot)$ be a semigroup and let f: $S \to \mathbf{R}$ be subadditive or superadditive. If f is 2-homogeneous, then*

(12) $$f(xy) = f(yx), \quad x, y \in S.$$

Proof. We only consider the subadditive case. By the previous lemma, the 2-homogeneity of f implies its $\mathbf{N}$-homogeneity. Therefore, for every integer $k \geqslant 2$ and for any $x, y \in S$ we have

$$f(xy) = \frac{f((xy)^k)}{k} = \frac{f(x(yx)^{k-1}y)}{k} \leqslant \frac{f(x)+f(y)}{k} + \frac{k-1}{k} f(yx).$$

Letting $k \to \infty$ we get $f(xy) \leqslant f(xy)$, which yields (12) since x and y are interchangeable.

The next lemma is of the key importance for the proofs of separation theorems on weakly commutative groups or semigroups. Roughly speaking, it allows us to replace the subadditive and superadditive functionals we want to separate by new ones which display some additional suitable properties.

LEMMA 3. *Let $(S, \cdot)$ be a weakly commutative semigroup and assume that f, g: $S \to \mathbf{R}$ satisfy (1), (2) and (4). Then there exist f^*, g^*: $S \to \mathbf{R}$ with the following properties:*

(i) $g(x) \leqslant g^*(x) \leqslant f^*(x) \leqslant f(x)$, $x \in S$;
(ii) *f^* is subadditive and g^* is superadditive;*
(iii) *f^* and g^* are $\mathbf{N}$-homogeneous;*
(iv) *$f^*(xy) = f^*(yx)$ and $g^*(xy) = g^*(yx)$, $x, y \in S$.*

Moreover, if f (resp. g) is $\mathbf{N}$-homogeneous at a point $x_0 \in S$, then

(v) *$f^*(x_0) = f(x_0)$ (resp. $g^*(x_0) = g(x_0)$).*

Proof. For $x \in S$ and $n \in \mathbf{N}$ we put

$$f_n(x) := f(x^{2^n})/2^n, \quad g_n(x) := g(x^{2^n})/2^n.$$

By a simple recurrence based on the subadditivity of f and the superadditivity of g, one can check that

(13) $$g(x) \leqslant g_n(x) \leqslant g_{n+1}(x) \leqslant f_{n+1}(x) \leqslant f_n(x) \leqslant f(x), \quad n \in \mathbf{N}.$$

For each $x \in S$, the sequences $\{f_n(x)\}_{n\in\mathbf{N}}$ and $\{g_n(x)\}_{n\in\mathbf{N}}$, being monotone and bounded, are convergent in $\mathbf{R}$. Therefore we may define f^*, g^*: $S \to \mathbf{R}$ by

$$f^*(x) := \lim_{n\to\infty} f_n(x), \quad g^*(x) := \lim_{n\to\infty} g_n(x), \quad x \in S.$$

From (13) it follows immediately that (i) holds.

Now fix $x, y \in S$ and, using the weak commutativity of S, find a sequence $\{n_k\}_{k\in\mathbf{N}}$ of positive integers such that $n_k \to \infty$ as $k \to \infty$ and

$$(xy)^{2^{n_k}} = x^{2^{n_k}} y^{2^{n_k}}, \quad k \in \mathbf{N}.$$

Then we get

$$f^*(xy) = \lim_{k\to\infty} f((xy)^{2^{n_k}})/2^{n_k} = \lim_{k\to\infty} f(x^{2^{n_k}} y^{2^{n_k}})/2^{n_k}$$
$$\leqslant \lim_{k\to\infty} f(x^{2^{n_k}})/2^{n_k} + \lim_{k\to\infty} f(y^{2^{n_k}})/2^{n_k} = f^*(x) + f^*(y),$$

which means that f^* is subadditive. A similar argument ensures the superadditivity of g^*.

Further, observe that for each $x \in S$, one has

$$f^*(x^2) = \lim_{n\to\infty} f(x^{2^{n+1}})/2^n = 2\lim_{n\to\infty} f(x^{2^{n+1}})/2^{n+1} = 2f^*(x)$$

and analogously,

$$g^*(x^2) = 2g^*(x).$$

On account of Lemma 1, the last two identities guarantee the $\mathbf{N}$-homogeneity of f^* and g^*, whereas Lemma 2 implies (iv).

Finally, if f is $\mathbf{N}$-homogeneous at $x_0 \in S$, then $f_n(x_0) = f(x_0)$ for all $n \in \mathbf{N}$. Consequently, $f^*(x_0) = \lim_{n\to\infty} f_n(x_0) = f(x_0)$. The same argument works for g^*.

3. Separation theorems. We start this section with two results which rely upon the weak commutativity of S.

THEOREM 1. *Suppose that $(S, \cdot)$ is a weakly commutative semigroup and let f, g: $S \to \mathbf{R}$ fulfil (1), (2) and (4). Moreover, assume that*

(14) $$\sup\{f(x) - g(x): x \in S\} < \infty.$$

Then there exists exactly one additive functional a: $S \to \mathbf{R}$ which satisfies (5).

Proof. Let f^*, g^*: $S \to \mathbf{R}$ be the functionals associated with f and g according to Lemma 3. Using assertions (iii) and (i) of that lemma, we obtain

$$k(f^*(x) - g^*(x)) = f^*(x^k) - g^*(x^k) \leqslant f(x^k) - g(x^k)$$

for every $x \in S$ and $k \in \mathbf{N}$, which combined with (14) implies that $f^* = g^*$. Evidently, $a := f^*$ is then additive and satisfies (5).

If b: $S \to \mathbf{R}$ is another additive functional separating f and g, then

$$k|a(x) - b(x)| = |a(x^k) - b(x^k)| \leqslant |f(x^k) - g(x^k)|$$

for each $x \in S$ and $k \in \mathbf{N}$, whence $b = a$ in virtue of (14). This finishes the proof.

Hypothesis (14) is fairly restrictive. It can be omitted if we assume that S is a weakly commutative group (and not just a semigroup).

THEOREM 2. *Let $(S, \cdot)$ be a weakly commutative group. If $f, g\colon S\to\mathbf{R}$ satisfy* (1), (2) *and* (4), *then there exists an additive functional $a\colon S\to\mathbf{R}$ such that* (5) *holds true.*

Proof. Consider the family $\mathscr{F}$ of all pairs (φ, ψ) where $\varphi\colon S\to\mathbf{R}$ is subadditive, $\psi\colon S\to\mathbf{R}$ superadditive and $\psi(x)\leqslant\varphi(x)$, $x\in S$. In $\mathscr{F}$ we introduce a partial order $\preccurlyeq$ by putting $(\varphi_1, \psi_1)\preccurlyeq(\varphi_2, \psi_2)$ iff $\psi_1(x)\leqslant\psi_2(x)\leqslant\varphi_2(x)\leqslant\varphi_1(x)$, $x\in S$. It is easily seen that every linearly ordered subfamily of $\mathscr{F}$ has an upper bound in $\mathscr{F}$. Since $(f, g)\in\mathscr{F}$, the Kuratowski–Zorn lemma implies that there exists in $\mathscr{F}$ a maximal pair, say (F, G), which succeeds (f, g) in the sense of the order $\preccurlyeq$.

Now, let F^* and G^* be the functionals linked with F and G by Lemma 3. Owing to the maximality of (F, G) we get $F^* = F$ and $G^* = G$. In particular,

$$F(xy) = F(yx),\qquad G(xy) = G(yx),\qquad x, y\in S. \tag{15}$$

Let e stand for the identity element of S. Since the value of any subadditive (resp. superadditive) functional at e is nonnegative (resp. nonpositive), it is clear, by the maximality of (F, G), that $F(e) = G(e) = 0$ (otherwise we could modify the value of F or G at e so as to obtain a pair $(\tilde{F}, \tilde{G})\in\mathscr{F}$ exceeding (F, G)). We would be through with the proof if we knew that F and G coincide on the whole of S. Then $a := F = G$ would be the required additive functional separating f and g. For the indirect proof suppose that there exist $c\in S$ and a real number r such that

$$G(c) < r < F(c). \tag{16}$$

We shall show that then either

A) $mr+F(s)\geqslant G(c^m s)$ for every $m\in\mathbf{N}_0$ and $s\in S$, or

B) $F(c^n t)\geqslant nr+G(t)$ for every $n\in\mathbf{N}_0$ and $t\in S$,

where $\mathbf{N}_0 := \mathbf{N}\cup\{0\}$ and we adopt the convention that the 0th power of any element of S is equal to e. Supposing that neither A) nor B) holds, we would be able to find $m, n\in\mathbf{N}_0$ and $s, t\in S$ such that

$$mr+F(s) < G(c^m s)\quad\text{and}\quad F(c^n t) < nr+G(t),$$

whence,

$$nF(s)+mF(c^n t) < mG(t)+nG(c^m s).$$

Consequently, referring several times to the fact that F is subadditive, G is superadditive and they fulfil (15), we would derive

$$\begin{aligned}F(s^n t^m c^{nm}) &\leqslant nF(s)+F(t^m c^{nm}) = nF(s)+F(t^{m-1}c^{n(m-1)}c^n t)\\ &\leqslant nF(s)+F(t^{m-1}c^{n(m-1)})+F(c^n t)\leqslant\ldots\\ &\leqslant nF(s)+mF(c^n t) < mG(t)+nG(c^m s)\\ &\leqslant G(t^m)+(n-2)G(c^m s)+G(c^m s)+G(sc^m)\\ &\leqslant G(t^m)+(n-2)G(c^m s)+G(c^m s^2 c^m)\\ &= G(t^m)+(n-2)G(c^m s)+G(c^{2m}s^2)\leqslant\ldots\\ &\leqslant G(t^m)+G(c^{nm}s^n)\leqslant G(c^{nm}s^n t^m) = G(s^n t^m c^{nm}),\end{aligned}$$

which is impossible since G is dominated by F.

Further we assume A) and, for each $x\in S$, we put

$$F_0(x) := \inf\{mr+F(s)\colon\ c^m s = x,\ m\in\mathbf{N}_0,\ s\in S\}.$$

Directly from A) and (16) it follows that

$$G(x)\leqslant F_0(x)\leqslant F(x),\qquad x\in S, \tag{17}$$

$$F_0(c)\leqslant r < F(c). \tag{18}$$

Next we choose arbitrary $m, n\in\mathbf{N}_0$ and $x, y\in S$. The subadditivity of F jointly with (15) yields

$$\begin{aligned}mr+F(c^{-m}x)+nr+F(c^{-n}y) &= (m+n)r+F(yc^{-n})+F(c^{-m}x)\\ &\geqslant (m+n)r+F(yc^{-(m+n)}x) = (m+n)r+F(c^{-(m+n)}xy).\end{aligned}$$

Hence, by the definition of F_0, we get

$$F_0(x)+F_0(y)\geqslant F_0(xy),$$

which means that F_0 is subadditive. By (17), the pair (F_0, G) belongs to $\mathscr{F}$, and $(F, G)\preccurlyeq(F_0, G)$. This, combined with (18), contradicts the maximality of (F, G) and completes the proof in case A).

Assuming B), we put

$$G_0(x) := \sup\{nr+G(t)\colon\ c^n t = x,\ n\in\mathbf{N}_0,\ t\in S\}$$

for every $x\in S$. Then, similarly to case A), one can show that

$$G(x)\leqslant G_0(x)\leqslant F(x),\qquad x\in S, \tag{19}$$

$$G(c) < r < G_0(c) \tag{20}$$

and G_0 is superadditive. By (19) and (20), we have $(F, G)\preccurlyeq(F, G_0)$ and $(F, G_0)\neq(F, G)$, contrary to the maximality of the latter pair. Thus, the whole proof is finished.

Now we present a series of results in which the amenability assumption on S intervenes. We start with the following

THEOREM 3. *Suppose* $(S, \cdot)$ *to be an amenable semigroup and let* $f, g\colon S \to \mathbf{R}$ *satisfy* (1), (2) *and* (4). *Moreover, assume that*

$$(21)\qquad \sup\{|f(xy)-f(yx)|\colon x, y \in S\} < \infty.$$

Then there exists an additive functional $a\colon S \to \mathbf{R}$ *fulfilling* (5).

Proof. To fix ideas, assume that there exists a left invariant mean M in $\mathscr{B}(S, \mathbf{R})^*$. Notice that, by (4) and (2), we have

$$f(xy)-g(y) \geqslant g(xy)-g(y) \geqslant g(x), \qquad x, y \in S.$$

Therefore, a functional $h\colon S \to \mathbf{R}$ is well defined by

$$h(x) := \inf\{f(xy)-g(y)\colon y \in S\}, \qquad x \in S.$$

Making use of (1), we easily derive

$$(22)\qquad h(xy) = \inf\{f(xyz)-g(z)\colon z \in S\} \leqslant \inf\{f(x)+f(yz)-g(z)\colon z \in S\}$$
$$= f(x)+\inf\{f(yz)-g(z)\colon z \in S\} = f(x)+h(y), \qquad x, y \in S.$$

Hypothesis (21) implies that there exists a constant $K \geqslant 0$ such that $f(xy) \geqslant f(yx)-K$, $x, y \in S$. Hence and from (2) it follows that

$$(23)\qquad h(xy) = \inf\{f(xyz)-g(z)\colon z \in S\}$$
$$\geqslant \inf\{f(yzx)-K-g(zx)+g(x)\colon z \in S\}$$
$$= -K+g(x)+\inf\{f(yzx)-g(zx)\colon z \in S\}$$
$$\geqslant -K+g(x)+\inf\{f(yw)-g(w)\colon w \in S\}$$
$$= -K+g(x)+h(y), \qquad x, y \in S.$$

Putting (22) and (23) together, we infer that for any $x \in S$, the function $S \ni y \to h(xy)-h(y) \in \mathbf{R}$ is bounded from above by $f(x)$ and from below by $g(x)-K$.

Now, we define $a\colon S \to \mathbf{R}$ as follows:

$$a(x) := M_y\big(h(xy)-h(y)\big), \qquad x \in S,$$

where the subscript y indicates that the mean M is applied to a function of the variable y. The following simple calculation based on the left invariance and linearity of M shows that a is additive:

$$a(u)+a(v) = M_y\big(h(uy)-h(y)\big)+M_y\big(h(vy)-h(y)\big)$$
$$= M_y\big(h(uvy)-h(vy)\big)+M_y\big(h(vy)-h(y)\big)$$
$$= M_y\big(h(uvy)-h(y)\big) = a(uv), \qquad u, v \in S.$$

From (9) and the definition of a we deduce that

$$(24)\qquad g(x)-K \leqslant a(x) \leqslant f(x), \qquad x \in S.$$

It remains to eliminate the constant K. Take a positive integer n. Then, by the additivity of a and the superadditivity of g combined with (24), we obtain

$$na(x) = a(x^n) \geqslant g(x^n)-K \geqslant ng(x)-K, \qquad x \in S,$$

which, jointly with (24), implies that

$$f(x) \geqslant a(x) \geqslant g(x)-K/n, \qquad x \in S,\ n \in \mathbf{N}.$$

Passing to the limit as $n \to \infty$, we arrive at (5).

For a right invariant mean, the proof is analogous, with h replaced by

$$\tilde{h}(x) := \inf\{f(yx)-g(y)\colon y \in S\}, \qquad x \in S.$$

Remark 1. Theorem 3 remains valid if we replace (21) by

$$(21')\qquad \sup\{|g(xy)-g(yx)|\colon x, y \in S\} < \infty.$$

Indeed, if we assume (21′), then Theorem 3 (in its original form) allows us to separate $-g$ and $-f$ by an additive functional $\tilde{a}\colon S \to \mathbf{R}$. It is evident that $a := -\tilde{a}$ is also additive and fulfils (5), as required.

We do not know whether amenability of S alone (without (21) and (21′)) is enough to ensure that Theorem 3 holds true. On the other hand, it is worth noticing that without the amenability assumption Theorem 3 is no longer true, even if (21) and (21′) are both satisfied. To see this, we refer once more to the example of G. L. Forti mentioned in the introduction. It is well known that the free group generated by two elements is not amenable (see e.g. [6], Theorem 17.16) and, as one can easily verify, Forti's function φ satisfies

$$\varphi(xy)-\varphi(yx) \in \{-1, 0, 1\}$$

for all x and y. Consequently, the functionals $f := \varphi+1$ and $g := \varphi-1$ satisfy (21) and (21′), respectively. It was already noticed, however, that f and g do not admit separation by an additive functional.

Our next result is an analogue of Theorem 1 for amenable semigroups. We obtain it as a corollary from Theorem 3 (cf. also [5], where a direct proof of a similar result was given).

THEOREM 4. *Assume that* $(S, \cdot)$ *is an amenable semigroup. If* $f, g\colon S \to \mathbf{R}$ *satisfy* (1), (2), (4) *and* (14), *then there exists a unique additive functional* $a\colon S \to \mathbf{R}$ *which fulfils* (5).

Proof. Using (1), (2) and (4), we obtain

$$f(xy)-f(yx) \leqslant f(xy)-g(yx) \leqslant f(x)+f(y)-g(y)-g(x)$$
$$= [f(x)-g(x)]+[f(y)-g(y)], \qquad x, y \in S.$$

Hence, on account of (14) and the symmetry between x and y, we infer that f satisfies (21). Now, the existence assertion is a direct consequence of Theorem 3. To show the uniqueness, we argue in exactly the same way as in the proof of Theorem 1.

Remark 2. Prof. A. Smajdor observed that our Theorems 1 and 4 may also be deduced from the stability of the Cauchy functional equation (which is valid on both weakly commutative and amenable semigroups). In fact this uniform argument works for any semigroup on which the Hyers stability theorem holds true.

THEOREM 5. *Let $(S, \cdot)$ be an amenable semigroup and let $f, g\colon S\to \mathbf{R}$ satisfy* (1), (2) *and* (4). *Moreover, assume that f or g is 2-homogeneous. Then there exists an additive functional $a\colon S\to\mathbf{R}$ fulfilling* (5).

Proof. In virtue of Lemma 2, either $f(xy) = f(yx)$, $x, y\in S$, or $g(xy) = g(yx)$, $x, y\in S$. Accordingly, we have either (21) or (21') trivially satisfied. To accomplish the proof it is enough to apply Theorem 3.

Theorems 3, 4 and 5 contain various supplementary hypotheses concerning f or g (apart from the minimal system of assumptions consisting of (1), (2) and (4)). If the semigroup S is amenable and, at the same time, weakly commutative, then these additional hypotheses become redundant.

THEOREM 6. *Suppose that $(S, \cdot)$ is a semigroup which is both amenable and weakly commutative. Then for any functionals $f, g\colon S\to\mathbf{R}$ satisfying* (1), (2) *and* (4) *there exists an additive functional $a\colon S\to\mathbf{R}$ fulfilling* (5).

Proof. We associate with f and g the functionals f^* and g^* given by Lemma 3. In particular, assertion (iv) of that lemma implies that f^* and g^* satisfy (21) and (21'), respectively. Thus, Theorem 3 together with assertions (i) and (ii) of Lemma 3 ensure that f^* and g^* can be separated by an additive functional $a\colon S\to\mathbf{R}$ which, automatically, separates f and g too.

4. Supporting functionals. In this section $(S, \cdot)$ is assumed to be a group (not necessarily Abelian). We consider the question whether for a given subadditive functional $f\colon S\to\mathbf{R}$ and a point $x_0\in S$, one can find an additive functional $a\colon S\to\mathbf{R}$ with

$$a(x)\leqslant f(x),\qquad x\in S, \tag{22}$$

$$a(x_0) = f(x_0). \tag{23}$$

If (22) and (23) are fulfilled, then we say that *a supports f at x_0*. From the separation theorem of P. Kranz [10] one can easily deduce that, provided S is commutative, the subadditive functional f is supported by at least one additive functional at any point of S at which f is $\mathbf{N}$-homogeneous. In what follows we present similar results for weakly commutative and amenable groups.

First notice that regardless of whether S is Abelian or not, the $\mathbf{N}$-homogeneity of f at $x_0\in S$ is necessary for the existence of an additive functional supporting f at x_0. Indeed, if f is subadditive and an additive functional $a\colon S\to\mathbf{R}$ supports f at x_0, then for any $n\in\mathbf{N}$ we have

$$a(x_0^n)\leqslant f(x_0^n)\leqslant nf(x_0) = na(x_0) = a(x_0^n),$$

whence $f(x_0^n) = nf(x_0)$.

THEOREM 7. *Let $(S, \cdot)$ be a weakly commutative group and let $f\colon S\to\mathbf{R}$ be subadditive. If f is $\mathbf{N}$-homogeneous at $x_0\in S$, then there exists an additive functional $a\colon S\to\mathbf{R}$ satisfying* (22) *and* (23).

Proof. One can readily check that $g\colon S\to\mathbf{R}$ defined by

$$g(x) := -f(x^{-1}),\quad x\in S,$$

is superadditive and lies below f. Consider f^*, g^* given by Lemma 3. Since f is $\mathbf{N}$-homogeneous at x_0, Lemma 3(v) states that $f^*(x_0) = f(x_0)$.

By the subadditivity of f^*, we have

$$f^*(x_0^n)-f^*(x^{-1}x_0^n)\leqslant f^*(x),\qquad x\in S,\ n\in\mathbf{N}.$$

We also know that f^* is $\mathbf{N}$-homogeneous (in fact, we need here its $\mathbf{N}$-homogeneity at x_0 only). Hence

$$f^*(x_0)+f^*(x_0^n) = f^*(x_0^{n+1}),\qquad n\in\mathbf{N}.$$

Referring again to the subadditivity of f^*, we obtain

$$f^*(x^{-1}x_0^{n+1})\leqslant f^*(x^{-1}x_0^n)+f^*(x_0),\qquad x\in S,\ n\in\mathbf{N},$$

which, in conjunction with the previous identity, yields

$$f^*(x_0^n)-f^*(x^{-1}x_0^n)\leqslant f^*(x_0^{n+1})-f^*(x^{-1}x_0^{n+1}),\qquad x\in S,\ n\in\mathbf{N}.$$

As a result, for each fixed $x\in S$, the sequence $\{f^*(x_0^n)-f^*(x^{-1}x_0^n)\}_{n\in\mathbf{N}}$ is nondecreasing and bounded above by $f^*(x)$. We put

$$h(x) := \lim_{n\to\infty}[f^*(x_0^n)-f^*(x^{-1}x_0^n)],\qquad x\in S.$$

Then

$$h(x)\leqslant f^*(x)\leqslant f(x),\ x\in S,$$

$$h(x_0) = f^*(x_0) = f(x_0).$$

We now verify that h is superadditive. By Lemma 3,

$$\begin{aligned} h(x)+h(y) &= \lim_{n\to\infty}[f^*(x_0^n)-f^*(x^{-1}x_0^n)]+\lim_{n\to\infty}[f^*(x_0^n)-f^*(y^{-1}x_0^n)] \\ &= \lim_{n\to\infty}[f^*(x_0^{2n})-f^*(x^{-1}x_0^n)-f^*(x_0^n y^{-1})] \\ &\leqslant \lim_{n\to\infty}[f^*(x_0^{2n})-f^*(x^{-1}x_0^{2n}y^{-1})] \\ &= \lim_{n\to\infty}[f^*(x_0^{2n})-f^*(y^{-1}x^{-1}x_0^{2n})] = h(xy). \end{aligned}$$

Finally, with the aid of Theorem 2, we separate h and f^* by an additive functional a: $S \to \mathbf{R}$. Since h, f^* and f assume the same value at x_0, it is clear that a supports f at this point and the proof is finished.

COROLLARY 1. *If $(S, \cdot)$ is a weakly commutative group and f: $S \to \mathbf{R}$ is a subadditive and 2-homogeneous functional, then f is supported by an additive functional at any point of S.*

THEOREM 8. *Assume that $(S, \cdot)$ is an amenable group and let f: $S \to \mathbf{R}$ be a subadditive functional satisfying* (21). *If f is* N*-homogeneous at $x_0 \in S$, then there exists an additive functional a: $S \to \mathbf{R}$ which fulfils* (22) *and* (23).

Proof. By the same reasoning as in the proof of the preceding theorem (with f^* replaced by f) we can show that, for each fixed $x \in S$, the sequence $\{f(x_0^n) - f(x^{-1}x_0^n)\}_{n \in \mathbf{N}}$ is nondecreasing and bounded above by $f(x)$. Thus

$$h(x) := \lim_{n \to \infty} [f(x_0^n) - f(x^{-1}x_0^n)], \quad x \in S,$$

satisfies

$$h(x) \leqslant f(x), \quad x \in S, \quad h(x_0) = f(x_0).$$

Hypothesis (21) guarantees the existence of a constant $K \geqslant 0$ such that

$$f(xy) \geqslant f(yx) - K, \quad x, y \in S.$$

If we combine this relation with the N-homogeneity of f at x_0 and its subadditivity, then as in the proof of Theorem 7 we get the estimate

$$h(x) + h(y) \leqslant h(xy) + 2K, \quad x, y \in S.$$

Consequently, the functional $\tilde{h}(x) := h(x) - 2K$, $x \in S$, is superadditive and, obviously, it is dominated by f. By Theorem 3, f and $\tilde{h}$ can be separated by an additive functional a, i.e.

$$h(x) - 2K \leqslant a(x) \leqslant f(x), \quad x \in S.$$

Applying an argument similar to that completing the proof of Theorem 3, we may remove the constant $2K$ from the last inequality. Then clearly a supports f at x_0, which was to be shown.

COROLLARY 2. *If $(S, \cdot)$ is an amenable group and f: $S \to \mathbf{R}$ is subadditive and 2-homogeneous, then f is supported by an additive functional at every point of S.*

This clearly results from Theorem 8 and Lemma 2.

5. Concluding remarks and examples. In fact, in Theorems 7 and 8 it is enough to assume that f is 2^n-homogeneous at x_0 for every $n \in \mathbf{N}$. This assumption implies that f is N-homogeneous at x_0, which may be deduced from the second paragraph of the proof of Lemma 1. We should also mention

that the mere 2-homogeneity of f at x_0 does not necessarily force its N-homogeneity at this point and is too weak for the assertions of Theorems 7 and 8 to hold true (even in the Abelian case). We owe this observation to the referee, whom we wish to thank most cordially at this place.

In this paper we have dealt with two classes of semigroups: weakly commutative and amenable. The question arises of relation between these two classes. It turns out that they intersect (both contain the class of commutative semigroups) but neither of them is included in the other. The latter fact is illustrated by the following two examples:

EXAMPLE 1. Let $(S, \cdot)$ be the group of all nonconstant affine mappings of the real line onto itself (with superposition as the binary operation). Then S can be identified with the set $(\mathbf{R} \setminus \{0\}) \times \mathbf{R}$ in which the group operation is defined as follows:

$$(a, b)(c, d) := (ac, ad + b), \quad a, c \in \mathbf{R} \setminus \{0\}, \ b, d \in \mathbf{R}.$$

The group S is solvable and hence amenable. To see the solvability of S, one can check that the sequence $\{(1, 0)\} \subset \{(1, b)\colon b \in \mathbf{R}\} \subset S$ has Abelian factors. On the other hand, if $a > 1$, then there is no integer $k \geqslant 2$ for which

$$((1, a) \cdot (a, 0))^k = (1, a)^k \cdot (a, 0)^k. \tag{24}$$

Indeed, it is easy to verify that

$$((1, a) \cdot (a, 0))^k = (a^k, a + a^2 + \ldots + a^k),$$

whereas

$$(1, a)^k \cdot (a, 0)^k = (a^k, ka).$$

If (24) were satisfied, we would have $a + a^2 + \ldots + a^k = ka$, which is false whenever $a > 1$ and $k \geqslant 2$. In particular, S is not weakly commutative.

EXAMPLE 2. Let $S := X \times Y$, where X and Y are two sets with card $X \geqslant 2$ and card $Y \geqslant 2$. If we introduce a binary operation in S by setting

$$(x_1, y_1) \cdot (x_2, y_2) := (x_2, y_1), \quad (x_i, y_i) \in S, \ i = 1, 2,$$

then S becomes a semigroup. It is weakly commutative, since

$$((x_1, y_1) \cdot (x_2, y_2))^2 = (x_2, y_1) = (x_1, y_1)^2 \cdot (x_2, y_2)^2$$

for all $(x_i, y_i) \in S$, $i = 1, 2$. On the other hand, there exists neither right nor left invariant mean in $\mathscr{B}(S, \mathbf{R})^*$ (i.e., S is not amenable). Supposing, for instance, that $M \in \mathscr{B}(S, \mathbf{R})^*$ is a right invariant mean, we take a nonconstant bounded function φ: $X \to \mathbf{R}$ and we put

$$\psi(x, y) := \varphi(x), \quad (x, y) \in S.$$

Then $\psi \in \mathscr{B}(S, \mathbf{R})$ and $\psi_{(a,b)} = \varphi(a) =$ const. for each fixed $(a, b) \in S$. Consequently,

$$M(\psi) = M(\psi_{(a,b)}) = \varphi(a), \quad a \in S,$$

which is impossible, since we have chosen a nonconstant φ. The nonexistence of a left invariant mean in $\mathscr{B}(S, \mathbf{R})^*$ may be shown in a similar way.

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A multiplier theorem for *H*-type groups

by

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Abstract. We prove an L^p-boundedness result for a convolution operator with rough kernel supported on a hyperplane of a group of Heisenberg type.

In recent years, several results have been proved in which the Calderón–Zygmund theory of singular integrals has been extended to the more general setting of nilpotent Lie groups (see e.g. [8], [15], [19]). In particular, F. Ricci ([15]) showed that the classical theory of Calderón–Zygmund kernels on $\mathbf{R}^n$ has very natural extensions to kernels on nilpotent Lie groups.

As a further generalization, more singular convolution operators, for instance convolution with distributions which are extensions of Calderón–Zygmund kernels supported on submanifolds, have been considered.

In this context, we would like to mention some results concerning Hilbert transforms along homogeneous curves in Stein and Wainger ([19]) and Christ ([2]). Subsequently, Geller and Stein ([7]) studied smooth homogeneous kernels supported by a hyperplane of the $(2n+1)$-dimensional Heisenberg group; such operators arise in the study of the $\bar{\partial}$-Neumann problem on the Siegel upper half-space. Müller ([13], [14]) showed that Theorem 1.1 in [7] has rather general extensions for more general homogeneous Lie groups and for an even larger class of submanifolds.

Recently, Ricci and Stein unified and extended in a series of papers ([16]–[18]) some results of these previous works; particularly, in ([17]), they considered singular integral operators on homogeneous Lie groups defined by smooth kernels supported on lower-dimensional analytic submanifolds and having the critical degree of homogeneity. To prove the L^p-boundedness, they make a strong use of the smoothness assumption.

In this paper we work on groups of Heisenberg type; we study convolution operators with rough kernels carried by the subspace complementary to the center. We improve Geller and Stein's result, by proving the L^p-boundedness of the convolution operator under some minimal assumptions on the regularity of

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