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Automorphisms with finite exact uniform rank

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Abstract. The notion of *exact uniform rank*, EUR, of an automorphism of a probability Lebesgue space is defined. It is shown that each ergodic automorphism with finite EUR is a finite extension of some automorphism with rational discrete spectrum. Moreover, for automorphisms with finite EUR, the upper bounds of EUR of their factors and ergodic iterations are computed.

Introduction. Automorphisms with finite rank have been studied in [8], [5], [3], [4] and in other papers. They have zero entropy and enjoy the loosely Bernoulli property. If T is an ergodic automorphism of a Lebesgue space X, then the definition of the rank of T, $R(T)$, uses a sequence of Rokhlin's towers, converging to the whole σ-algebra of measurable subsets of X. Adding some extra conditions on the sequence of towers, some authors define the uniform rank of T, $UR(T)$, and the exact rank of T, $ER(T)$. Joining the two definitions we will obtain the notion of exact uniform rank of T, $EUR(T)$. One can prove that if $W \in \{R, UR, ER\}$, then the following statements hold:

(i) $W(S) \leqslant W(T)$ whenever S is a factor of T.
(ii) $W(T^n) \leqslant |n| \cdot W(T)$ whenever T^n is ergodic.

In this paper we will show that for $W = EUR$, the statement (i) is valid (Theorem 1), but (ii) fails to be true (Example): we will construct an automorphism T with $EUR(T) = 3$ and $EUR(T^2) = 9$. However, we will prove that $EUR(T^n) \leqslant EUR(T)^{|n|}$ whenever T^n is ergodic (Theorem 3). It can be proved that automorphisms with finite exact rank can not be mixing while there are mixing automorphisms with arbitrary finite uniform rank ([5]). We will show that each ergodic automorphism with finite exact uniform rank is a finite extension of some automorphism with rational discrete spectrum (Theorem 2). In particular, such an automorphism can not be even weakly mixing.

Definitions and notations. Let X be a Lebesgue space with a probability measure μ defined on a σ-algebra $\mathscr{B}$ of measurable subsets of X. In what follows all equations will be understood modulo null sets.

By a *finite semipartition* of X we mean any finite family $\mathbf{P} = (P_0, \ldots, P_{n-1})$ of measurable pairwise disjoint subsets of X. Note that in particular $P_i \neq P_j$ whenever $i \neq j$.

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By a *finite partition* of X we mean each finite semipartition $\mathbf{P}$ of X satisfying $\bigcup \mathbf{P} = X$.

If no confusion can arise we will shorten "finite semipartition" and "finite partition" to *semipartition* and *partition* respectively. The members of a semipartition are called *atoms*.

If $\mathbf{P}$, $\mathbf{Q}$ are semipartitions then $\mathbf{P}$ is called *finer* than $\mathbf{Q}$, $\mathbf{P} > \mathbf{Q}$, if each atom of $\mathbf{Q}$ is a union of some atoms of $\mathbf{P}$. If $\{\mathbf{P}^v\colon v \in V\}$ is a family of partitions of X then there exists a unique partition $\mathbf{P}$ (possibly infinite) of X satisfying the following properties (see [10]):

(a) For each $v \in V$, $\mathbf{P} > \mathbf{P}^v$.

(b) If $\mathbf{Q}$ is another partition satisfying (a) then $\mathbf{Q} > \mathbf{P}$.

The partition $\mathbf{P}$ will be denoted by $\bigvee_{v\in V} \mathbf{P}^v$. If the family $\{\mathbf{P}^v\colon v \in V\}$ is finite, $\{\mathbf{P}^v\colon v\in V\} = \{\mathbf{P}^1, \ldots, \mathbf{P}^n\}$, then we write $\bigvee_{v\in V}\mathbf{P}^v = \bigvee_{i=1}^n \mathbf{P}^i = \mathbf{P}^1 \vee \ldots \vee \mathbf{P}^n$.

Given a semipartition $\mathbf{P}$ we say that a measurable set B is *measurable with respect to* $\mathbf{P}$, or shortly $\mathbf{P}$*-measurable*, if B is a union of some atoms of $\mathbf{P}$. If $\mathbf{P}$ is a partition, then B is $\mathbf{P}$-measurable iff $\mathbf{P} > (B, X\setminus B)$. We say that a sequence $\mathbf{P}^n$, $n = 1, 2, \ldots$, of semipartitions *converges* to $\mathscr{B}$, $\mathbf{P}^n \to \mathscr{B}$, if for each measurable set A and for each $\varepsilon > 0$ there is a natural number n_0 such that for all $n > n_0$ we can find a $\mathbf{P}^n$-measurable set B_n satisfying $\mu(A \triangle B_n) < \varepsilon$ ([8]).

Now assume that T is a measure-theoretic automorphism of the Lebesgue space $(X, \mathscr{B}, \mu)$; this means that T is an invertible map and T and T^{-1} preserve the measure μ. A semipartition $\mathbf{P} = (P_0, \ldots, P_{n-1})$ is called a *stack for* T or just a *stack* if $P_{i+1} = TP_i$, $i = 0, \ldots, n-2$. The number n and the set P_0 are called the *height* and the *base* of $\mathbf{P}$ respectively. Atoms of a stack $\mathbf{P}$ are called the *levels* of $\mathbf{P}$.

After [8] we say that an automorphism T has *rank at most* r, $R(T) \leqslant r$, if there exists a sequence $\mathbf{P}^n$, $n \geqslant 1$, of semipartitions of X such that

R1 Each $\mathbf{P}^n$ consists of r pairwise disjoint stacks.

R2 $\mathbf{P}^n \to \mathscr{B}$.

An automorphism T has *rank* r, $R(T) = r$, if r is the smallest number satisfying $R(T) \leqslant r$. If there exists a number r such that $R(T) = r$ then T is called an automorphism with *finite rank*.

If the sequence $\mathbf{P}^n$, $n = 1, 2, \ldots$, satisfies R1, R2 and

U1 For each $n = 1, 2, \ldots$, $h_n^1 = \ldots = h_n^r$, where $h_n^1, \ldots, h_n^r$ denote the heights of the stacks in $\mathbf{P}^n$,

then we say that T has *uniform rank at most* r, $UR(T) \leqslant r$.

Now assume that the sequence $\mathbf{P}^n$, $n = 1, 2, \ldots$, satisfies R1, R2 and

E1 Each $\mathbf{P}^n$, $n = 1, 2, \ldots$, is a partition.

E2 There exists $\delta > 0$ such that $h_n^i \mu(F_n^i) \geqslant \delta$ for each n, i, where $\mathbf{P}^n = \{T^k F_n^i\colon i = 1, \ldots, r,\ k = 0, \ldots, h_n^i - 1\}$.

Then we say that T has *exact rank at most* r, $ER(T) \leqslant r$.

We say that T has *exact uniform rank at most* r, $EUR(T) \leqslant r$, if there exists a sequence $\mathbf{P}^n$, $n = 1, 2, \ldots$, of semipartitions of X satisfying R1, R2, U1 and E1. Obviously, $EUR(T) \leqslant r$ implies $UR(T) \leqslant r$. Observe that if $(F, TF, \ldots, T^{h-1}F)$ and $(G, TG, \ldots, T^{h-1}G)$ are disjoint stacks, then the family $((F \cup G), T(F\cup G), \ldots, T^{h-1}(F \cup G))$ is a stack as well. Therefore $EUR(T) \leqslant r$ implies $ER(T) \leqslant r$. Note that the relations $UR(T) = r$, $EUR(T) = r$ are defined analogously to $R(T) = r$.

From now on we assume that T is ergodic. For natural numbers h and r denote by $H_r^h(T)$ the family of all partitions of X consisting of r pairwise disjoint stacks each of height h:

$$\mathbf{P} \in H_r^h(T) \quad \text{iff} \quad \mathbf{P} = (F_1, \ldots, T^{h-1}F_1, F_2, \ldots, T^{h-1}F_2, \ldots, F_r, \ldots, T^{h-1}F_r) \text{ and } \bigcup \mathbf{P} = X.$$

For such a $\mathbf{P}$ we define $\hat{\mathbf{P}} \in H_1^h(T)$ setting $\hat{\mathbf{P}} = (\hat{F}, T\hat{F}, \ldots, T^{h-1}\hat{F})$ where $\hat{F} = F_1 \cup \ldots \cup F_r$. The partition $\hat{\mathbf{P}}$ is invariant with respect to T: $T\hat{\mathbf{P}} = \hat{\mathbf{P}}$.

Now we can rewrite the definition of exact uniform rank: $EUR(T) \leqslant r$ iff there exist a sequence of natural numbers $h(n)$, $n = 1, 2, \ldots$, and a sequence $\mathbf{P}^n \in H_r^{h(n)}(T)$, $n = 1, 2, \ldots$, such that $\mathbf{P}^n \to \mathscr{B}$.

Let $L^2(X, \mu)$ denote the Hilbert space of all complex square integrable functions on X. Denote by U_T the unitary operator on $L^2(X, \mu)$ defined by $U_T(f) = f \circ T$. We say that T has *discrete spectrum* if the set of all eigenfunctions of U_T is linearly dense in $L^2(X, \mu)$. T has *rational discrete spectrum* if T has discrete spectrum and each eigenvalue of U_T is of the form $\exp(2\pi ip/q)$, p, q integers. Automorphisms with discrete spectra are canonical (see [7]): if T has discrete spectrum and $T'\colon X' \to X'$ is isomorphic to T then for each ergodic $\tilde{T}\colon \tilde{X} \to \tilde{X}$ such that T and T' are factors of $\tilde{T}$ via maps f and f' respectively, $\{f^{-1}(x)\colon x \in X\} = \{f'^{-1}(x')\colon x' \in X'\}$ as measurable partitions (possibly infinite) of $\tilde{X}$. If T is a factor of an ergodic $\tilde{T}\colon \tilde{X} \to \tilde{X}$ via a map f and $c = \operatorname{card}\{f^{-1}(x)\} < \infty$ then we call $\tilde{T}$ a *finite extension* of T, or more precisely, a *c-extension* of T. In this case if T has discrete spectrum then by the canonicality property the number c does not depend on $\tilde{T}$ in the class of all automorphisms isomorphic to $\tilde{T}$. Observe that if T admits a T-invariant partition of cardinality h then $\exp(2\pi i/h)$ is an eigenvalue of U_T. Therefore if $\mathbf{P}^n \in H_r^{h(n)}(T)$, $n = 1, 2, \ldots$, then the group $\{\exp(2\pi it/h(n))\colon t = 1, 2, \ldots, n = 1, 2, \ldots\}$ is a subgroup of the group of all eigenvalues of U_T.

Results. Let $T\colon (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ be ergodic.

THEOREM 1. *If T has finite exact uniform rank and S is a factor of T then S has finite exact uniform rank and $EUR(S) \leqslant EUR(T)$.*

THEOREM 2. *If $EUR(T) < \infty$ then there exists a factor T_0 of T with rational discrete spectrum such that T is a c-extension of T_0 where $c \leqslant EUR(T)$.*

From the above two theorems we have

COROLLARY 1. *If $EUR(T) < \infty$ then each eigenvalue of U_T is rational.*

Indeed, if $\lambda = \exp(2\pi i\alpha)$, where α is an irrational number, is an eigenvalue of U_T, then the automorphism S: $S^1 = \{z: z \text{ complex}, |z| = 1\} \to S^1$ given by $S(z) = \lambda z$ is a factor of T. By Theorem 1, $EUR(S) \leqslant EUR(T) < \infty$. By Theorem 2, S has a factor S_0 with rational discrete spectrum such that S is a c-extension of S_0, where $c \leqslant EUR(S) < \infty$. But S has no rational eigenvalues different from 1. This is a contradiction.

COROLLARY 2. *$EUR(T) = 1$ if and only if T has rational discrete spectrum.*

THEOREM 3. *If $EUR(T) < \infty$ then for each nonzero integer n, $EUR(T^n) \leqslant EUR(T)^{|n|}$ whenever T^n is ergodic.*

Proofs. We start with some definitions to obtain a characterization of the finite exact uniform rank property in the language of the names of points (Lemma 1).

If t is a natural number then we will denote by N_t the set $\{0, 1, \ldots, t-1\}$. We will call *blocks* the elements of the set $N_t^* = \bigcup_{n\geqslant 1} N_t^n$. If B is a block, $B = b_0 b_1 \ldots b_{n-1}$, then for $0 \leqslant p \leqslant q \leqslant n-1$ we define $B[p, q] = b_p \ldots b_q$, $B[p] = B[p, p]$. If $B, C \in N_t^n$ then we define $d(B, C) = n^{-1}\operatorname{card}\{0 \leqslant k \leqslant n-1: B[k] \neq C[k]\}$. Then d is a metric on N_t^n (usually written $\bar{d}$).

Let r and h be natural numbers. Assume that $B_1, \ldots, B_r$ are blocks, $B_i \in N_t^h$, $i = 1, \ldots, r$. For natural n we define

$$A^n(B_1, \ldots, B_r) = \{B \in N_t^{nh}: B[kh, (k+1)h-1] \in \{B_1, \ldots, B_r\},\ k = 0, \ldots, n-1\}.$$

If T is an ergodic automorphism of a Lebesgue space $(X, \mathscr{B}, \mu)$ and $\mathbf{P} = (P_0, \ldots, P_{t-1})$ is a partition of X, then for $x \in X$ and for natural n we define a block $N_n^T(x, \mathbf{P}) \in N_t^n$ setting

$$N_n^T(x, \mathbf{P})[k] = j \quad \text{iff} \quad T^k x \in P_j,\ k = 0, 1, \ldots, n-1.$$

We call this block the $(\mathbf{P}, T)$-n *name* of x. For $x \in X$ we also define

$$d_{\mathbf{P}}(x, A^n(B_1, \ldots, B_r)) = \min\{d(N_{nh}^T(x, \mathbf{P}), B): B \in A^n(B_1, \ldots, B_r)\}.$$

If $\varepsilon > 0$ then let

$$E_n^\varepsilon((\mathbf{P}, T), (B_1, \ldots, B_r)) = \{x \in X: d_{\mathbf{P}}(x, A^n(B_1, \ldots, B_r)) < \varepsilon\},$$

$$E_\infty^\varepsilon((\mathbf{P}, T), (B_1, \ldots, B_r)) = \bigcup_{n=1}^\infty \bigcap_{k=n}^\infty E_k^\varepsilon((\mathbf{P}, T), (B_1, \ldots, B_r)).$$

If $\mathbf{P}$ and $\mathbf{Q}$ are partitions of X, $\mathbf{P} = (P_0, \ldots, P_{n-1})$, $\mathbf{Q} = (Q_0, \ldots, Q_{n-1})$, then we define the distance $\bar{d}$ between $\mathbf{P}$ and $\mathbf{Q}$ by

$$\bar{d}(\mathbf{P}, \mathbf{Q}) = \min\{\sum_{i=0}^{n-1} \mu(P_i \triangle Q_{\sigma(i)}): \sigma \in S_n\}$$

where S_n denotes the group of all permutations of N_n. If $\mathbf{R} = (R_0, \ldots, R_{m-1})$ with $n < m$ then we define

$$D(\mathbf{P}, \mathbf{R}) = \min\{\bar{d}(\mathbf{P}, \bar{\mathbf{R}}): \bar{\mathbf{R}} < \mathbf{R},\ \operatorname{card}(\bar{\mathbf{R}}) = n\}.$$

Now assume that T is an ergodic automorphism of a Lebesgue space $(X, \mathscr{B}, \mu)$.

LEMMA 1. *$EUR(T) \leqslant r$ if and only if for each $\varepsilon > 0$ and for any partition $\mathbf{Q} = (Q_0, \ldots, Q_{t-1})$ of X there are a natural number h, blocks $B_1, \ldots, B_r \in N_t^h$ and a partition $\mathbf{R} \in H_1^h(T)$, $\mathbf{R} = (R_0, \ldots, R_{h-1})$, such that $R_0 \subset E_\infty^\varepsilon((\mathbf{Q}, T), (B_1, \ldots, B_r))$.*

Proof. Assume that $EUR(T) \leqslant r$. Take $\varepsilon > 0$. Let $\mathbf{Q} = (Q_0, \ldots, Q_{t-1})$ be a partition of X. Put $\delta = \varepsilon/2$. Then for some h there is a partition $\mathbf{P} \in H_r^h(T)$ such that $D(\mathbf{Q}, \mathbf{P}) < \delta$. Let $\bar{\mathbf{P}} = (\bar{P}_0, \ldots, \bar{P}_{t-1})$ be a partition of X such that $\bar{\mathbf{P}} < \mathbf{P}$ and $\bar{d}(\mathbf{Q}, \bar{\mathbf{P}}) = D(\mathbf{Q}, \mathbf{P})$. Observe that if B is a base of some stack in the partition $\mathbf{P}$ then almost all points in B have the same $(\bar{\mathbf{P}}, T)$-h names. Let $B_i = N_h^T(x, \bar{\mathbf{P}})$ for x from the base of the ith stack of $\mathbf{P}$.

If $\bar{d}(\mathbf{Q}, \bar{\mathbf{P}}) = \sum_{i=0}^{t-1} \mu(Q_i \triangle \bar{P}_i)$ then $A = \sum_{i=0}^{t-1} Q_i \triangle \bar{P}_i$. Obviously $\mu(A) < \delta$. Let $\mathbf{R} = \hat{\mathbf{P}} = (\hat{P}_0, \ldots, \hat{P}_{h-1})$. By the Birkhoff Ergodic Theorem,

$$\frac{1}{m}\sum_{i=0}^{m-1} \chi_A \circ T^{-i}(x) < 2\delta$$

for a.e. $x \in \hat{P}_0$ and for m large enough ($m \geqslant N_x$). Therefore $d(N_m^T(x, \bar{\mathbf{P}}), N_m^T(x, \mathbf{Q})) < 2\delta$ for $m \geqslant N_x$. Observe that $N_{nh}^T(x, \bar{\mathbf{P}}) \in A^n(B_1, \ldots, B_r)$ for $x \in R_0 = \hat{P}_0$ and $n \geqslant 1$. Therefore $d_{\mathbf{Q}}(x, A^n(B_1, \ldots, B_r)) < 2\delta = \varepsilon$ for $x \in R_0$ whenever $nh \geqslant N_x$. This implies $x \in E_\infty^\varepsilon((\mathbf{Q}, T), (B_1, \ldots, B_r))$ and the "if" part of the proof is complete.

Assume now that $\varepsilon > 0$ and $\mathbf{Q} = (Q_0, \ldots, Q_{t-1})$ is a partition of X. We will construct a partition $\mathbf{P} \in H_r^h(T)$ satisfying $D(\mathbf{Q}, \mathbf{P}) < \varepsilon$, where h is given by the assumptions of our lemma for ε and $\mathbf{Q}$. Let $F_1, \ldots, F_r$ be disjoint sets given by

$$F_i = \{x \in R_0: d_{\mathbf{Q}}(x, A^1(B_1, \ldots, B_r)) = d(N_h^T(x, \mathbf{Q}), B_i) \text{ and } x \notin F_1 \cup \ldots \cup F_{i-1}\},\quad i = 1, \ldots, r.$$

Then $F_1 \cup \ldots \cup F_r = R_0$ and consequently $\bigcup_{i=1}^r \bigcup_{k=0}^{h-1} T^k F_i = X$. Obviously the sets $T^k F_i$, $k = 0, \ldots, h-1$, $i = 1, \ldots, r$, are pairwise disjoint. Thus we have defined a partition $\mathbf{P} = \{T^k F_i: k = 0, \ldots, h-1,\ i = 1, \ldots, r\} \in H_r^h(T)$. We intend to show that $D(\mathbf{P}, \mathbf{Q}) < \varepsilon$.

If $x \in F_i$ then for n large enough and for some sequence $i_2, \ldots, i_n$,

$$(*) \qquad d(N_{nh}^T(x, \mathbf{Q}), B_i B_{i_2} B_{i_3} \ldots B_{i_n}) < \varepsilon.$$

Indeed, since $F_i \subset R_0 \subset E_\infty^\varepsilon((\mathbf{Q}, T), (B_1, \ldots, B_r))$, $d(N_{nh}^T(x, \mathbf{Q}), B_{i_1} B_{i_2} \ldots B_{i_n}) < \varepsilon$ for n large enough and for some sequence $i_1, i_2, \ldots, i_n$. By the definition of F_i, $d(N_{nh}^T(x, \mathbf{Q}), B_i B_{i_2} \ldots B_{i_n}) \leqslant d(N_{nh}^T(x, \mathbf{Q}), B_{i_1} B_{i_2} \ldots B_{i_n})$. This forces $(*)$ to be true.

We define a partition $\bar{\mathbf{P}} < \mathbf{P}$, $\bar{\mathbf{P}} = (\bar{P}_0, \ldots, \bar{P}_{t-1})$, setting

$$T^k F_i \subset \bar{P}_j \quad \text{iff} \quad B_i[k] = j.$$

Let $x \in F_i$. Fix n. Let $U_n = N^T_{nh}(x, \mathbf{Q})$, $W_n = N^T_{nh}(x, \bar{\mathbf{P}})$. Then $W_n = B_i B_{i_2} \ldots B_{i_n}$ for some sequence $i_2, \ldots, i_n$. By the definition of the sets F_j, $j = 1, \ldots, r$, the inequality (*) is true for $i_2, \ldots, i_n$. By (*), $d(U_n, W_n) < \varepsilon$. Let $A = \bigcup_{i=0}^{t-1} (\bar{P}_i \triangle Q_i)$. Then

$$\mu(A) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \chi_A \circ T^{-k}(x) \leqslant \limsup_n d(U_n, W_n) \leqslant \varepsilon,$$

which completes the proof.

Assume that $EUR(T) \leqslant r$ and $B_1, \ldots, B_r, R_0, h$ are given by Lemma 1 for a fixed partition $\mathbf{Q}$ and $\varepsilon > 0$. For $x \in R_0$ and for n large enough define

$$M_n(x) = \{D \in A^n(B_1, \ldots, B_r):\ d(D, N^T_{nh}(x, \mathbf{Q})) < \varepsilon\}.$$

From Lemma 1 and from the Birkhoff Ergodic Theorem we have

LEMMA 2. *There are numbers $\alpha_1, \ldots, \alpha_r \in [0, 1]$ such that for almost all $x \in R_0$ and for n large enough ($n > n_x$) there exists a $D_n \in M_n(x)$ satisfying*

$$|n^{-1} \operatorname{card}\{0 \leqslant t \leqslant n-1:\ D_n[th, (t+1)h-1] = B_i\} - \alpha_i| < \varepsilon.$$

Proof. We define $F_1, \ldots, F_r$ in the same way as in the proof of Lemma 1. Let $\alpha_i = h\mu(F_i)$, $i = 1, \ldots, r$. Take $x \in R_0$. Let D be an infinite sequence of symbols from N_r given by

$$D[th, (t+1)h-1] = B_i \quad \text{iff} \quad T^{th} \in F_i, \ t = 0, 1, \ldots$$

Then for n large enough

$$d(N^T_{nh}(x, \mathbf{Q}), D[0, nh-1]) = d_{\mathbf{Q}}(x, A^n(B_1, \ldots, B_r)) < \varepsilon$$

since $x \in R_0 \subset E^\varepsilon_\infty((\mathbf{Q}, T), (B_1, \ldots, B_r))$.

From the Birkhoff Ergodic Theorem

$$|n^{-1} \operatorname{card}\{0 \leqslant t \leqslant n-1:\ T^{ts} x \in F_i\} - \alpha_i| < \varepsilon$$

for $n > n_x$. Therefore

$$\begin{aligned} &|n^{-1} \operatorname{card}\{0 \leqslant t \leqslant n-1:\ D[th, (t+1)h-1] = B_i\} - \alpha_i| \\ &\qquad \leqslant |n^{-1} \operatorname{card}\{0 \leqslant t \leqslant n-1:\ D[th, (t+1)h-1] = B_i\} \\ &\qquad\quad - n^{-1} \operatorname{card}\{0 \leqslant t \leqslant n-1:\ T^{th} x \in F_i\}| \\ &\qquad\quad + |n^{-1} \operatorname{card}\{0 \leqslant t \leqslant n-1:\ T^{th} x \in F_i\} - \alpha_i| < \varepsilon, \end{aligned}$$

which completes the proof.

Proof of Theorem 1. Let $S: (Y, \mathscr{A}, m) \to (Y, \mathscr{A}, m)$ be a factor of T. Let $f: X \to Y$ be a measurable map satisfying $\mu \circ f^{-1} = m$ and $f \circ T = S \circ f$. Assume that $EUR(T) = r < \infty$. Let $\mathbf{Q} = (Q_0, \ldots, Q_{t-1})$ be an arbitrary partition of Y. Set $\tilde{\mathbf{Q}} = f^{-1}(\mathbf{Q}) = (f^{-1}(Q_0), \ldots, f^{-1}(Q_{t-1})) = (\tilde{Q}_0, \ldots, \tilde{Q}_{t-1})$. Then $\tilde{\mathbf{Q}}$ is a partition of X and for each $x \in X$ and for each n the $(\tilde{\mathbf{Q}}, T)$-n name of x is equal to the $(\mathbf{Q}, S)$-n name of $f(x)$. Take $\varepsilon > 0$. By Lemma 1, there are blocks $B_1, \ldots, B_r$ of symbols from N_t, all of the same length, say h, and a partition $\mathbf{R} = (R_0, \ldots, R_{h-1}) \in H^h_1(T)$ such that $R_0 \subset E^{\varepsilon/2(r+1)}_\infty((\tilde{\mathbf{Q}}, T), (B_1, \ldots, B_r))$. Then $f(\mathbf{R}) \in H^{\bar{h}}_1(S)$ where $\bar{h}$ divides h. Let $h = s\bar{h}$. We divide each block B_i, $i = 1, \ldots, r$, into s segments of length $\bar{h}$. Let $C^0_1, \ldots, C^0_r, C^1_1, \ldots, C^{s-1}_r$ be the obtained blocks, this meaning that $B_i = C^0_i C^1_i \ldots C^{s-1}_i$, $i = 1, \ldots, r$. Then

$$f(R_0) \subset E^{\varepsilon/2(r+1)}_\infty((\mathbf{Q}, S), (C^0_1, \ldots, C^{s-1}_r)).$$

We will prove that there exists a t, $0 \leqslant t \leqslant s-1$, such that

(1) $$f(R_0) \subset E^\varepsilon_\infty((\mathbf{Q}, S), (C^t_1, \ldots, C^t_r)).$$

Assume that $\alpha_1, \ldots, \alpha_r$ are the numbers from Lemma 2 for $\tilde{\mathbf{Q}}$, $\varepsilon/2(r+1)$, h, R_0, $B_1, \ldots, B_r$. Let

$$d^k_i(t) = \min\{d(C^{k+t}_i, C^t_j):\ j = 1, \ldots, r\}, \qquad i = 1, \ldots, r,\ k, t = 0, \ldots, s-1,$$

where $k+t$ is taken modulo s.

Take $x_0 \in R_0$. Since $f(T^{-k\bar{h}} R_0) = f(R_0)$ for each $k = 0, \ldots, s-1$, there are $x_1, \ldots, x_{s-1} \in R_0$ such that $f(x_k) = f(T^{k\bar{h}} x_0)$, $k = 0, \ldots, s-1$. Obviously the $\tilde{\mathbf{Q}}$-names of x_k are equal to the $\tilde{\mathbf{Q}}$-names of $T^{k\bar{h}}(x_0)$, $k = 0, \ldots, s-1$. For n large enough and for each $k = 0, \ldots, s-1$ we can find a block $D^k \in A^n(B_1, \ldots, B_r)$ such that

(2) $$d(D^k, N^T_{nh}(x_k, \tilde{\mathbf{Q}})) < \varepsilon/2(r+1).$$

Denote by $\beta^k_i n$ the number of occurrences of the block B_i in D^k, $i = 1, \ldots, r$, $k = 0, \ldots, s-1$. By Lemma 2 we can assume that $|\beta^k_i - \alpha_i| < \varepsilon/2(r+1)$ (we can find a common n_x for all x from a set $A_\delta \subset R_0$ with $\mu(A_\delta) > 1/h - \delta$ for arbitrary $\delta > 0$). By (2), for $k = 0, \ldots, s-1$,

$$\frac{\sum_{t=0}^{s-1} \sum_{i=1}^{r} d^k_i(t) \beta^k_i n\bar{h}}{n\bar{h}s} < \varepsilon/2(r+1).$$

This implies

$$\frac{1}{s} \sum_{t=0}^{s-1} \Big(\sum_{k=0}^{s-1} \sum_{i=1}^{r} d^k_i(t) \beta^k_i \Big) < s\varepsilon/2(r+1).$$

Thus there exists a t such that

$$\sum_{k=0}^{s-1} \sum_{i=1}^{r} d^k_i(t) \beta^k_i < s\varepsilon/2(r+1)$$

or

$$\frac{1}{s}\sum_{k=0}^{s-1}\sum_{i=1}^{r} d_i^k(t)\beta_i^k < \varepsilon/2(r+1).$$

Since $|\beta_i^k-\alpha_i| < \varepsilon/2(r+1)$,

$$\left|\frac{1}{s}\sum_{k=0}^{s-1}\sum_{i=1}^{r} d_i^k(t)\beta_i^k - \frac{1}{s}\sum_{k=0}^{s-1}\sum_{i=1}^{r} d_i^k(t)\alpha_i\right| \leqslant \frac{1}{s}\sum_{k=0}^{s-1}\sum_{i=1}^{r} d_i^k(t)|\beta_i^k-\alpha_i| < r\varepsilon/2(r+1).$$

Therefore

$$\frac{1}{s}\sum_{k=0}^{s-1}\sum_{i=1}^{r} d_i^k(t)\alpha_i < \varepsilon/2. \tag{3}$$

Now we are in a position to prove (1) for t described above.

Let g: $\{0, \ldots, s-1\}\times\{1, \ldots, r\}\to\{1, \ldots, r\}$ be a function satisfying

$$d_i^k(t) = d(C_i^{k+t}, C_{g(k,i)}^t).$$

Take $y\in f(R_0)$. Then for n large enough there is a block $D_y\in A^n(B_1, \ldots, B_r)$ such that

$$d(N_{nh}^S(y, \mathbf{Q}), D_y) < \varepsilon/2(r+1).$$

We define a block $D\in A^{ns}(C_1^t, \ldots, C_r^t)$ in the following way:

If $D_y[mh+(k+t)\bar{h}, mh+(k+t+1)\bar{h}-1] = C_i^{k+t}$,

then $D[mh+(k+t)\bar{h}, mh+(k+t+1)\bar{h}-1] = C_{g(k,i)}^t$

where $k+t$ and $k+t+1$ are taken modulo s. Then

$$d(N_{ns\bar{h}}^S(y, \mathbf{Q}), D) \leqslant d(N_{ns\bar{h}}^S(y, \mathbf{Q}), D_y)+d(D, D_y) < \varepsilon/2(r+1)+d(D, D_y). \tag{4}$$

By the definition of D we have

$$d(D, D_y) = \frac{\sum_{k=0}^{s-1}\sum_{i=1}^{r} d_i^k(t)\gamma_i n\bar{h}}{n\bar{h}s} = \frac{1}{s}\sum_{k=0}^{s-1}\sum_{i=1}^{r} d_i^k(t)\gamma_i$$

where $\gamma_i n$ is just the number of occurrences of the block B_i in D_y. By Lemma 2 we can assume that $|\gamma_i-\alpha_i| < \varepsilon/2(r+1)$. Then by (3)

$$\begin{aligned} d(D, D_y) &< \frac{1}{s}\sum_{k=0}^{s-1}\sum_{i=1}^{r} d_i^k(t)(\alpha_i+\varepsilon/2(r+1)) \\ &= \frac{1}{s}\sum_{k=0}^{s-1}\sum_{i=1}^{r} d_i^k(t)\alpha_i+(\varepsilon/2(r+1))\cdot\frac{1}{s}\sum_{k=0}^{s-1}\sum_{i=1}^{r} d_i^k(t) \\ &< \varepsilon/2+(\varepsilon/2(r+1))r = (2r+1)\varepsilon/2(r+1). \end{aligned}$$

Putting this into (4) we obtain

$$d(N_{ns\bar{h}}^S(y, \mathbf{Q}), D) < \varepsilon/2(r+1)+(2r+1)\varepsilon/2(r+1) = \varepsilon$$

and the proof of (1) is complete.

By virtue of Lemma 1 and (1), $EUR(S)\leqslant r$.

Proof of Theorem 2. Assume that $\mathbf{P}^n$, $n = 1, 2, \ldots$, is a sequence of partitions of X such that $\mathbf{P}^n\to\mathscr{B}$. We define an infinite partition $\mathbf{Q}$ of X setting $\mathbf{Q} = \bigvee_{n=1}^{\infty}\mathbf{P}^n$. Let $\hat{X} = X/\mathbf{Q}$, this meaning that $\hat{X} = \{A\in\mathscr{B}\colon A\in\mathbf{Q}\}$. Let π: $X\to\hat{X}$ be the natural factor map. Denote by $\mathscr{C}$ the image via π of the family $\{B\in\mathscr{B}\colon B$ is $\mathbf{Q}$-measurable$\}$. Put $m = \mu\circ\pi^{-1}$, $S = \pi T\pi^{-1}$. Then $(\hat{X}, \mathscr{C}, m, S)$ is a factor of $(X, \mathscr{B}, \mu, T)$. Obviously S has rational discrete spectrum. By the results of [10] there exist a Lebesgue space $(V, \mathscr{F}, \nu)$ and a measurable family $f = \{f_y\colon y\in\hat{X}\}$ of automorphisms of $(V, \mathscr{F}, \nu)$ such that the automorphism S_f: $\hat{X}\times V\to\hat{X}\times V$ given by $S_f(y, v) = (Sy, f_y(v))$ is isomorphic to T. We will prove that $\mathrm{card}(V)\leqslant r = EUR(T)$.

Let U: $X\to\hat{X}\times V$, $U\circ T = S_f\circ U$, be an isomorphism. Define $\mathbf{R}^n = U(\mathbf{P}^n)$, $n = 1, 2, \ldots$ Then $\mathbf{R}^n\to U(\mathscr{B}) = \mathscr{C}\otimes\mathscr{F}$ and $\mathbf{R}^n\in H_r^{h(n)}(S_f)$. Since S is a canonical factor of T (see [7]), $U(\hat{\mathbf{P}}^n) = \hat{\mathbf{R}}^n$, $n = 1, 2, \ldots$ Moreover, if $\hat{\mathbf{R}}^n = (B_0, \ldots, B_{h(n)-1})$, then $B_k = \hat{B}_k\times V$, $k = 0, \ldots, h(n)-1$, where $(\hat{B}_0, \ldots, \hat{B}_{h(n)-1})$ is a partition of $\hat{X}$, $n = 1, 2, \ldots$

Assume that we can find $r+1$ pairwise disjoint subsets $A_1, \ldots, A_{r+1}$ of the space V such that $\nu(A_i) > 0$, $i = 1, \ldots, r+1$. Set $\tilde{m} = m\times\nu$. Let $\alpha = \min\{\tilde{m}(\hat{X}\times A_i)\colon i = 1, \ldots, r+1\} = \min\{\nu(A_i)\colon i = 1, \ldots, r+1\}$. Then $\alpha > 0$ and for some n there are disjoint $\mathbf{R}^n$-measurable subsets $W_1, \ldots, W_{r+1}$ of $\hat{X}\times V$ satisfying $\tilde{m}(W_i\triangle(\hat{X}\times A_i)) < \alpha\tilde{m}(\hat{X}\times A_i)$ (since $\mathbf{R}^n\to\mathscr{C}\otimes\mathscr{F}$). Then

(*) For each $k = 0, \ldots, h(n)-1$ there exists a $t(k)$, $1\leqslant t(k)\leqslant r+1$, such that $B_k\cap W_{t(k)} = \varnothing$ whenever B_k is the kth level of the partition $\mathbf{R}^n$.

Indeed, B_k is a union of r members of $\mathbf{R}^n$ and each of them can be included in at most one of the sets $W_1, \ldots, W_{r+1}$.

From (*) we have

$$\begin{aligned} \alpha &> \sum_{i=1}^{r+1}\tilde{m}(W_i\triangle(\hat{X}\times A_i)) \geqslant \sum_{i=1}^{r+1}\tilde{m}((\hat{X}\times A_i)\setminus W_i) \\ &= \sum_{i=1}^{r+1}\sum_{k=0}^{h(n)-1}\tilde{m}((B_k\cap(\hat{X}\times A_i))\setminus W_i) = \sum_{i=1}^{r+1}\sum_{k=0}^{h(n)-1}\tilde{m}(B_k\cap(\hat{X}\times A_i)\cap W_i^c) \\ &\geqslant \sum_{k=0}^{h(n)-1}\tilde{m}((\hat{X}\times A_{t(k)})\cap B_k\cap W_{t(k)}^c) = \sum_{k=0}^{h(n)-1}\tilde{m}((\hat{X}\times A_{t(k)})\cap B_k) \\ &= \sum_{k=0}^{h(n)-1}\tilde{m}(\hat{B}_k\times A_{t(k)}) = \sum_{k=0}^{h(n)-1} m(\hat{B}_k)\nu(A_{t(k)}) \geqslant h(n)\cdot\frac{1}{h(n)}\cdot\alpha = \alpha. \end{aligned}$$

This contradiction completes the proof.

Proof of Theorem 3. Assume that T^n is ergodic. We can assume that $n > 0$. Set $r = EUR(T)$. Let $\mathbf{Q} = (Q_0, \ldots, Q_{t-1})$ be a partition of X. Take $\varepsilon > 0$. Then there exists a partition $\mathbf{P}$ of X such that $\mathbf{P} \in H_r^h(T)$ for some h and $D(\mathbf{Q}, \mathbf{P}) < \varepsilon$. Let $\bar{\mathbf{P}} < \mathbf{P}$ be a partition of X satisfying $\bar{d}(\mathbf{Q}, \bar{\mathbf{P}}) = D(\mathbf{Q}, \mathbf{P})$. Then almost all points from the base of each stack in $\mathbf{P}$ have the same $(\bar{\mathbf{P}}, T)$-h names. Denote the names by $B_1, \ldots, B_r$, $B_i \in N_t^h$, $i = 1, \ldots, r$. Let $\mathscr{A}$ be the set of all $(\bar{\mathbf{P}}, T)$-hn names of points from the base of the partition $\hat{\mathbf{P}}$. Obviously $\operatorname{card}(\mathscr{A}) \leqslant r^n$. For $A \in \mathscr{A}$ we define a block $B_A \in N_t^h$ by $B_A[i] = A[ni]$, $i = 0, \ldots, h-1$. Let $\mathscr{A}_n = \{B \in N_t^h\colon B = B_A$ for some $A \in \mathscr{A}\}$. Clearly $\operatorname{card}(\mathscr{A}_n) \leqslant r^n$. Take $x \in X$. Assume that x belongs to the base of $\hat{\mathbf{P}}$. We will prove that $x \in E_\infty^{n\varepsilon}((\mathbf{Q}, T^n), \mathscr{A}_n)$. From the assumption of this theorem and from the proof of Lemma 1, $x \in E_\infty^{\varepsilon}((\mathbf{Q}, T), (B_1, \ldots, B_r))$. It follows that there is a number m_0 such that for $m \geqslant m_0$ we can find a block $A \in A^{mn}(B_1, \ldots, B_r)$ such that $d(N_{mnh}^T(x, \mathbf{Q}), A) < \varepsilon$. It is easy to see that $A \in A^m(\mathscr{A})$. Let $B \in A^m(\mathscr{A}_n)$, $B[i] = A[in]$, $i = 0, \ldots, mh-1$. Then $d(N_{mh}^{T^n}(x, \mathbf{Q}), B) \leqslant nd(N_{mnh}^T(x, \mathbf{Q}), A) < n\varepsilon$. Therefore $x \in E_\infty^{n\varepsilon}((\mathbf{Q}, T), \mathscr{A}_n)$. By Lemma 1, $EUR(T^n) \leqslant \operatorname{card}(\mathscr{A}_n) = r^n$.

Example. Now we give an example of an automorphism T such that $EUR(T) = 3$ and $EUR(T^2) = 9$. This automorphism comes from substitutions of constant length. For definition and basic properties of such systems we refer to [9, 1, 2].

Assume that γ is a primitive pure substitution of constant length p, where p is a prime number, defined on the set $N_r = \{0, 1, \ldots, r-1\}$. Denote by T the shift on $N_r^{\mathbf{Z}}$ and by $X(\gamma)$ the minimal (for T) γ-invariant subset of $N_r^{\mathbf{Z}}$. Let μ be the unique T-invariant probability measure on $X(\gamma)$. In what follows we will use the letters T and μ for the shift and for the T-invariant probability measure for any substitution. Therefore we will write $EUR(\gamma)$ instead of $EUR(T)$. Let x_0 be a fixed point of γ. Denote by $L(\gamma)$ the number L from Lemma 9 in [6]. Assume that $L(\gamma) = 1$. Using Lemma 1 from the present paper and Lemma 9 of [6] we obtain the following criterion:

CRITERION. $EUR(\gamma) \leqslant k$ iff there exist blocks $B_1, \ldots, B_k \in N_r^{p-1}$ such that $x_0[0, \infty] = AB_{i_1}a_{i_1}B_{i_2}a_{i_2}\ldots$ where the length of the block A is at most $p-1$, $i_t \in \{1, \ldots, k\}$, $a_{i_t} \in N_r$, $t = 1, 2, \ldots$

Now we define a pure substitution θ of length 3 on 3 symbols:

$$\theta\colon \quad \begin{array}{l} 0 \to 0\ 1\ 2 \\ 1 \to 1\ 2\ 0 \\ 2 \to 2\ 0\ 1 \end{array}$$

Obviously $L(\theta) = 1$. By the Criterion, $EUR(\theta) = 3$. It follows from [2] that the (ergodic) system $(X(\theta), \mu, T^2)$ is isomorphic to the system $(X(\eta), \mu, T)$, where η is a substitution of length 3 defined on the set of all pairs of symbols in N_3 that have positive measure $(\mu(ij) = \mu(\{x \in X(\theta)\colon x[0, 1] = ij\}))$. Define $a_j^i = ij$, $i, j \in N_3$. Then $\eta(a_0^0) = \theta(00) = 012012 = (01)(20)(12) = a_1^0 a_0^2 a_2^1$, etc.

We obtain

$$\begin{array}{lll} a_0^0 \to a_1^0 a_0^2 a_2^1 & a_0^1 \to a_2^1 a_0^0 a_2^1 & a_0^2 \to a_0^2 a_0^1 a_2^1 \\ a_1^0 \to a_1^0 a_1^2 a_0^2 & a_1^1 \to a_2^1 a_1^0 a_0^2 & a_1^2 \to a_0^2 a_0^1 a_2^1 \\ a_2^0 \to a_1^0 a_2^2 a_1^0 & a_2^1 \to a_2^1 a_2^0 a_1^0 & a_2^2 \to a_0^2 a_2^1 a_1^0 \end{array}$$

or shortly $a_j^i \to a_{i+1}^i a_j^{i+2} a_{j+2}^{j+1}$, $i, j \in N_3$.

Let $\mathscr{A} = \{a_j^i\colon i, j \in N_3\}$. We list the pairs of elements of $\mathscr{A}$ that have positive measure:

$$a_0^0 a_2^1,\ a_1^0 a_1^0,\ a_1^0 a_2^1,\ a_1^0 a_0^2,\ a_1^0 a_1^2,\ a_1^0 a_2^2,\ a_2^0 a_1^0,\ a_2^1 a_0^0,\ a_2^1 a_1^0,\ a_2^1 a_2^0,\ a_2^1 a_2^1,\ a_2^1 a_0^2,$$
$$a_0^1 a_2^1,\ a_0^2 a_1^0,\ a_0^2 a_0^1,\ a_0^2 a_1^1,\ a_0^2 a_2^1,\ a_0^2 a_0^2,\ a_1^0 a_0^2,\ a_1^2 a_0^2,\ a_2^2 a_1^0.$$

We will show that $L(\eta) = 1$.

Assume that $\eta(a_j^i) = \eta(bc)[1, 3]$ for some $b, c \in \mathscr{A}$. Since $\eta(a_j^i) = a_{i+1}^i a_j^{i+2} a_{j+2}^{j+1}$, from the definition of η we see that $i = j$. This implies $\eta(a_i^i) = a_{i+1}^i a_i^{i+2} a_{i+2}^{i+1}$. Therefore $\eta(b) = \sqcup a_{i+1}^i a_i^{i+2}$, $\eta(c) = a_{i+2}^{i+1} \sqcup \sqcup$, where by $\sqcup$ we denote a symbol that is either unknown or not interesting for us. It follows that $b = a_{i+1}^{i+1}$, $c = a_t^{i+1}$ for some $t \in N_3$. Therefore $\mu(a_{i+1}^{i+1} a_t^{i+1}) > 0$, a contradiction.

Assume that $\eta(a_j^i) = \eta(bc)[2, 4]$ for some $b, c \in \mathscr{A}$. In the same way we obtain $\mu(a_{i+2}^t a_{i+2}^{i+2}) > 0$, a contradiction. This implies $L(\eta) = 1$.

Now we prove that $EUR(\eta) = 9$. Let x_0 be a fixed point of η. By the Criterion, $x_0[0, \infty] = AB_{i_1} \sqcup B_{i_2} \sqcup \ldots$ where the length of the block A is at most 2, $B_{i_t} \in \{B_1, \ldots, B_s\}$ for some $s \leqslant 9$, the length of each B_j is 2.

If the length of A is either 0 or 1 then the set $\{B_{i_1}, B_{i_2}, \ldots\}$ consists of 9 blocks. Assume that the length of A is 2. Then each block B_t, $t = 1, \ldots, s$, is of the form $B_t = \eta(ab)[2, 3]$ for some $a, b \in \mathscr{A}$. In other words, B_t is a block such that its first symbol is equal to the last symbol of $\eta(a)$ and its second symbol is the first symbol of $\eta(b)$. But all pairs of positive measure appear in $x_0[0, \infty] = \eta(x_0[0, \infty])$. Thus s is at least the number of blocks of the form $\eta(ab)[2, 3]$, where $\mu(ab) > 0$. Simple calculations show that there are 9 such blocks. Therefore $EUR(\eta) = 9$.

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On separation theorems for subadditive and superadditive functionals

by

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Abstract. We generalize the well known separation theorems for subadditive and superadditive functionals to some classes of not necessarily Abelian semigroups. We also consider the problem of supporting subadditive functionals by additive ones in the not necessarily commutative case. Our results are motivated by similar extensions of the Hyers stability theorem for the Cauchy functional equation. In this context the so-called weakly commutative and amenable semigroups appear naturally. The relations between these two classes of semigroups are discussed at the end of the paper.

1. Introduction. In this paper we are concerned with the problem of separation of subadditive and superadditive functionals defined on not necessarily commutative semigroups. Results of this type, for Abelian semigroups, were first obtained by R. Kaufman [8] and P. Kranz [10]. They can also be derived from the celebrated separation theorem of G. Rodé [12] (cf. also H. König [9]) which represents a far-reaching generalization of the classical Hahn–Banach theorem. In spite of its highly abstract setting, Rodé's theorem does not yield any extensions of Kaufman's and Kranz's results beyond the class of Abelian semigroups (some special noncommutative versions of Rodé's theorem have recently been discussed by A. Chaljub-Simon and P. Volkmann [1]). The main purpose of the present work is to replace the commutativity assumption in separation theorems of Kranz's type by some essentially weaker conditions of algebraic or analytic nature. In this regard, we follow the lines along which the Hyers stability theorem for the Cauchy functional equation (see D. H. Hyers [7]) was generalized to certain classes of not necessarily commutative semigroups.

In what follows $\mathbf{R}$ and $\mathbf{N}$ denote the sets of all reals and positive integers, respectively, whereas $(S, \cdot)$ stands for a semigroup or, occasionally, a group. To emphasize the fact that the binary operation in S does not have to be commutative we use for it the multiplicative notation.

We recall that a functional $f\colon S \to \mathbf{R}$ is said to be *subadditive* iff

$$f(xy) \leqslant f(x)+f(y), \quad x, y \in S. \tag{1}$$

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