Contents of Volume 100, Number 1

STUDIA MATHEMATICA

Managing Editors: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

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STUDIA MATHEMATICA

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Published by PWN – Polish Scientific Publishers

ISBN 83-01-10592-5 ISSN 0039-3223

PRINTED IN POLAND

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Some more weak Hilbert spaces

by

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Abstract. We construct, by a variation of the method used to construct the Tsirelson spaces, a new family of weak Hilbert spaces which contain copies of l_2 inside every subspace.

Preamble. The notions of weak type 2 and weak cotype 2 were introduced by Milman and Pisier in [3]; Pisier presented weak Hilbert spaces (the intersection of these two classes) as objects of study in [4]. That Tsirelson's space T_δ is a weak Hilbert space had essentially been proved in [1], where it was also shown that T_δ contains no isomorph of l_2. Recently the theory of weak Hilbert spaces has developed apace: yet the scarcity of known examples remains irritating – until now, the spaces T_δ $(0 < \delta < 1)$ and their duals, Hilbert space itself, and all subspaces of quotients of finite l_2-sums of these, were essentially the only examples known. In this paper we present a few more. There still remains, however, the problem of finding examples (if there are any) without an unconditional basis, or even without a basis.

DEFINITIONS. Given a sequence $\alpha \in (\mathbf{R}^+)^{\mathbf{N}}$ such that

$$\sum_m \alpha_m^2 = 1,$$

and

$$\delta_L \leqslant \alpha_{m+1}/\alpha_m \leqslant \delta_U$$

for all m, where $0 < \delta_L \leqslant \delta_U < 1$.

Write $\mathbf{R}^{(\mathbf{N}^+)}$ for the vector lattice of all *finitely supported* vectors in $\mathbf{R}^{\mathbf{N}^+}$.

If x and y are two vectors in $\mathbf{R}^{\mathbf{N}^+}$ we denote by $x \cdot y$ the vector whose ith coordinate is $x_i y_i$.

Call a k-tuple $(y_i\colon\ i < k)$ *acceptable* (acc.) if $y_i \in \mathbf{R}^{(\mathbf{N}^+)}$, $\operatorname{supp}(y_i) \subseteq [k, \infty)$, and $\sum_{i<k} |y_i|^2 \leqslant \mathbf{1}$.

For $x \in \mathbf{R}^{(\mathbf{N}^+)}$ define

$$\|x\|_0 = \sup\{|x_n|\colon\ n \in \mathbf{N}^+\}, \qquad \|x\|_{m+1}^2 = \sup\Big\{\sum_{i<k} \|y_i \cdot x\|_m^2\colon\ (y_i)\ \text{acc.}\Big\}.$$

By induction we see that $\|\cdot\|_m$ is a norm on $\mathbf{R}^{(\mathbf{N}^+)}$ which is 2-convex (i.e. $\|(|x|^2+|y|^2)^{1/2}\|_m \leqslant (\|x\|_m^2 + \|y\|_m^2)^{1/2}$) and satisfies $\|x\|_m \leqslant \|x\|_{l_2}$ for all x. The canonical vectors e_n $(n \in \mathbf{N}^+)$ constitute a normalised 1-unconditional basis for $(\mathbf{R}^{(\mathbf{N}^+)}, \|\cdot\|_m)$.

1980 *Mathematics Subject Classification* (1985 *Revision*): Primary 46B20, 46C99

Finally, set

$$\|x\| = \Big(\sum_m \alpha_m^2 \|x\|_m^2\Big)^{1/2},$$

and let $X = X_\alpha$ be the completion of $\mathbf{R}^{(\mathbf{N}^+)}$ with respect to the norm $\|\cdot\|$. Note that $(e_n\colon n\in\mathbf{N}^+)$ is a normalised 1-unconditional basis for X, X is 2-convex, and $\|x\| \leqslant \|x\|_{l_2}$ whenever $x\in\mathbf{R}^{(\mathbf{N}^+)}$.

LEMMA 0. *If $x\in X$ and (y_i) is acceptable then*

$$\|x\| \geqslant \delta_L\Big(\sum_i \|y_i\cdot x\|^2\Big)^{1/2}.$$

Proof.

$$\delta_L^2\sum_i \|y_i\cdot x\|^2 = \delta_L^2\sum_i\sum_m \alpha_m^2\|y_i\cdot x\|_m^2 = \delta_L^2\sum_m\alpha_m^2\sum_i\|y_i\cdot x\|_m^2$$
$$\leqslant \delta_L^2\sum_m \alpha_m^2\|x\|_{m+1}^2 \leqslant \sum_m \alpha_{m+1}^2\|x\|_{m+1}^2 \leqslant \|x\|^2. \ \blacksquare$$

THEOREM 1. *X is a weak Hilbert space.*

Proof. (The following argument is based on that employed in [5] to show that Tsirelson's space is a weak Hilbert space.)

(I) *X has weak cotype* 2: Write $X_k = [e_n\colon n>k]$. Suppose $y_0, \ldots, y_{k-1} \in X_k$. By 2-convexity we have

$$\Big\|\Big(\sum_i |y_i|^2\Big)^{1/2}\Big\| \leqslant \Big(\sum_i \|y_i\|^2\Big)^{1/2}.$$

Writing

$$x = \Big(\sum_i |y_i|^2\Big)^{1/2}, \qquad (z_i)_n = \begin{cases} 0, & x_n = 0,\\ (y_i)_n/x_n, & x_n \neq 0,\end{cases}$$

we have $z_i\cdot x = y_i$ and $\sum_i |z_i|^2 \leqslant \mathbf{1}$, so by the above Lemma

$$\|x\| \geqslant \delta_L\Big(\sum_i \|y_i\|^2\Big)^{1/2}.$$

In other words,

$$\delta_L\Big(\sum_i\|y_i\|^2\Big)^{1/2} \leqslant \Big\|\Big(\sum_i |y_i|^2\Big)^{1/2}\Big\| \leqslant \Big(\sum_i\|y_i\|^2\Big)^{1/2}.$$

This implies (see [5], Lemma 13.3) that every k-dimensional subspace $E \subseteq X_k$ has $d_E = d(E, l_2^k) \leqslant 4/\delta_L$.

Now suppose $E\subseteq X$, $\dim E = 2k$: then $\dim(E\cap X_k) \geqslant k$, so $\exists F \subseteq E\cap X_k \subseteq E$ with $\dim F = k$ and $d_F \leqslant 4/\delta_L$. This shows (see [5], Thm. 10.2) that X has weak cotype 2.

(II) *X has type* 2: X has weak cotype 2 $\Rightarrow$ (see [5], Prop. 10.7) X has cotype q $(\forall q>2)$. But X is 2-convex, so (see [2], (1.f.3) & (1.f.9)) X has type 2. $\blacksquare$

We will now show that two more (*a priori* different) definitions actually yield the same norms as the $\|\cdot\|_m$ defined above.

DEFINITIONS. Suppose $(y_i\colon i<k)$ is acceptable. Then (y_i) is *allowable* (all.) if $y_i = \chi_{E_i}$ where the (E_i) are disjoint and $\bigcup_i E_i \subseteq [k, \infty)$.

For $x\in\mathbf{R}^{(\mathbf{N}^+)}$ define

$$[x]_0 = \sup\{|x_n|\colon n\in\mathbf{N}^+\}, \qquad [x]_{m+1}^2 = \sup\Big\{\sum_i [y_i\cdot x]_m^2\colon (y_i) \text{ all.}\Big\}.$$

PROPOSITION 2. $[\cdot]_m = \|\cdot\|_m$ *for all m.*

Proof. (Again, the argument given here is based on the proof in [5] of the corresponding result in Tsirelson's space.) It is clear that $[x]_m \leqslant \|x\|_m$ for all x. By 2-convexity,

$$\wr x \wr_m = [|x|^{1/2}]_m^2$$

defines a *norm* on $\mathbf{R}^{(\mathbf{N}^+)}$.

To show $[x]_m \geqslant \|x\|_m$ for all x we use induction. We have the case $m=0$, so assume the result for m.

Suppose $x\in\mathbf{R}^{(\mathbf{N}^+)}$. If (y_i) is allowable then

$$\wr x\wr_{m+1} = [|x|^{1/2}]_{m+1}^2 \geqslant \sum_i [y_i\cdot|x|^{1/2}]_m^2 = \sum_i \wr y_i\cdot x\wr_m,$$

since $y_i = y_i^2$. For $N\in\mathbf{N}$ set

$$C_k^{(N)} = \operatorname{conv}\{(y_i\colon i<k)\colon (y_i) \text{ all., } \operatorname{supp}(y_i)\subseteq[k, k+N)\}$$
$$= \Big\{(z_i\colon i<k)\colon z_i \geqslant \mathbf{0}, \sum_i z_i \leqslant \mathbf{1},\ \operatorname{supp}(z_i)\subseteq[k, k+N)\Big\}.$$

Therefore for all $(z_i)\in C_k^{(N)}$,

$$\wr x\wr_{m+1} \geqslant \sum_i \wr z_i\cdot x\wr_m.$$

So

$$[x]_{m+1} = \wr x^2\wr_{m+1}^{1/2} \geqslant \Big(\sum_i \wr z_i\cdot x^2\wr_m\Big)^{1/2} = \Big(\sum_i [z_i^{1/2}\cdot x]_m^2\Big)^{1/2}.$$

But as (z_i) runs over $C_k^{(N)}$, $(z_i^{1/2})$ runs over all (non-negative) acceptable k-tuples with supports inside $[k, k+N)$. Since N is arbitrary it follows that whenever (y_i) is acceptable,

$$[x]_{m+1} \geqslant \Big(\sum_i [y_i\cdot x]_m^2\Big)^{1/2} \geqslant \Big(\sum_i \|y_i\cdot x\|_m^2\Big)^{1/2},$$

so $[x]_{m+1} \geqslant \|x\|_{m+1}$. $\blacksquare$

DEFINITIONS. Suppose $(y_i\colon i<k)$ is allowable. Then (y_i) is *admissible* (adm.) if $y_i = \chi_{E_i}$ where $k\leqslant E_0 < \ldots < E_{k-1}$. (Here and elsewhere the notation $E<F$ (where E and F are nonempty subsets of $\mathbf{N}^+$) means that $\max E < \min F$; we write $k\leqslant E$ meaning $k\leqslant \min E$.)

For $x\in\mathbf{R}^{(\mathbf{N}^+)}$ define

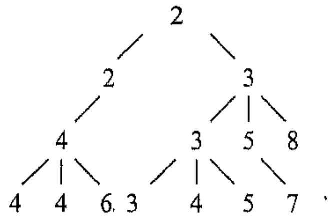
$$|||x|||_0 = \sup\{|x_n|\colon n\in\mathbf{N}^+\}, \qquad |||x|||_{m+1}^2 = \sup\Big\{\sum_i |||y_i\cdot x|||_m^2\colon (y_i) \text{ adm.}\Big\}.$$

DEFINITIONS. An *ailanthus* is a tree with a positive integer at each node, such that

(a). the number of branches emanating from a given node is less than or equal to the number at the node, and

(b) the number at the end of a branch is greater than or equal to the number at the node from which the branch emanates.

For example,



is the beginning of an ailanthus.

A subset $S \subseteq \mathbf{N}^+$ is *m-OK* if it occurs as the mth row of some ailanthus. Thus $\{2\}$ is 0-OK, $\{2, 3\}$ is 1-OK, $\{3, 4, 5, 8\}$ is 2-OK and $\{3, 4, 5, 6, 7\}$ is 3-OK. Note that if $f\colon \mathbf{N}^+ \to \mathbf{N}^+$ satisfies $f(n) \geqslant n$ $(\forall n)$ and S is m-OK then $f(S)$ is m-OK, and any subset of an m-OK set $(m \geqslant 1)$ is m-OK.

Notation. If $E \subseteq \mathbf{N}^+$ we write Ex for $\chi_E \cdot x$.

PROPOSITION 3. *If* $x \in \mathbf{R}^{(\mathbf{N}^+)}$ *and* $m \in \mathbf{N}$ *then*

$$[x]_m = |||x|||_m = \sup\{\|Sx\|_{l_2}\colon S \ m\text{-OK}\}.$$

Proof. We proceed by induction, the case $m = 0$ being clear since the 0-OK sets are just the singletons. Assume the result for m, and suppose $x \in \mathbf{R}^{(\mathbf{N}^+)}$.

Clearly $[x]_{m+1} \geqslant |||x|||_{m+1}$.

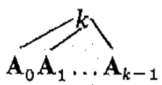
(I) $[x]_{m+1} \leqslant \sup\{\|Sx\|_{l_2}\colon S \ (m+1)\text{-OK}\}$: Suppose

$$[x]_{m+1}^2 = \sum_{i<k} [E_i x]_m^2$$

where the (E_i) are disjoint and $E_i \subseteq [k, \infty)$. By hypothesis we can write

$$[E_i x]_m = \|S_i E_i x\|_{l_2}$$

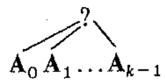
where S_i is the mth row of an ailanthus $\mathbf{A}_i$. We can assume that every node of $\mathbf{A}_i$ is $\geqslant k$. (Replace each node which is $< k$ by k, and we still have an ailanthus.) In particular, the *top* node of each $\mathbf{A}_i$ is $\geqslant k$, so that



is an ailanthus. Let S be its $(m+1)$th row, i.e. $S = \bigcup_i S_i$. Then

$$[x]_{m+1}^2 = \sum_i [E_i x]_m^2 = \sum_i \|S_i E_i x\|_{l_2}^2 \leqslant \|Sx\|_{l_2}^2.$$

(II) $\sup\{\|Sx\|_{l_2}\colon S \ (m+1)\text{-OK}\} \leqslant |||x|||_{m+1}$: Let S be the $(m+1)$th row of an ailanthus $\mathbf{A}$. Observe that if we put each row of $\mathbf{A}$ in non-decreasing order from left to right we still have an ailanthus. (Start at the top and work down. Order each row by a succession of operations each of which replaces $\ldots b \ldots a \ldots$ by $\ldots a \ldots b \ldots$ where $a \leqslant b$.) Having done this, write $\mathbf{A}$ in the form



where each $\mathbf{A}_i$ is an ailanthus. Let S_i be the mth row of $\mathbf{A}_i$; so $S = \bigcup_i S_i$. By deleting any repetitions among the S_i we may assume that the S_i are disjoint. Since the top node of $\mathbf{A}$ must be $\geqslant k$, we have $k \leqslant S_0 < \ldots < S_{k-1}$ and

$$\|Sx\|_{l_2}^2 = \sum_i \|S_i x\|_{l_2}^2 \leqslant \sum_i |||S_i x|||_m^2. \ \blacksquare$$

LEMMA 4. *Suppose* $(u_i\colon i \in \mathbf{N})$ *is a normalised block basic sequence in* X, $N \in \mathbf{N}$, $\varepsilon > 0$ *and* $M \in \mathbf{N}^+$. *Then* $\exists v \in \operatorname{span}_i u_i$ *such that*

(a) $\|v\|_N = 1$,

(b) $\|v\|_m < \varepsilon$ $(\forall m < N)$, *and*

(c) $\sum_{j<M} \|E_j v\|_m^2 < \varepsilon^2$ $(\forall m < N)$ *whenever* $E_0 < \ldots < E_{M-1}$.

Proof. We proceed by induction on N. If $N = 0$ just take $v = u_0/\|u_0\|_0$. Now assume the result for N.

Take $k \in \mathbf{N}^+$, $k > 2M/\varepsilon^2$, and choose $\varepsilon_i > 0$ $(i < k)$ so that

$$\sum_i \varepsilon_i^2 < k\varepsilon^2/M - 2.$$

Choose $r_0 \geqslant k$ and for $i < k$, step-by-step choose $v_i \in \operatorname{span}_n u_n$ such that

(a) $\|v_i\|_N = 1$,

(b) $\|v_i\|_m < \varepsilon_i$ $(\forall m < N)$, and

(c) $\sum_{j<M_i} \|E_j v_i\|_m^2 < \varepsilon_i^2$ $(\forall m < N)$ whenever $E_0 < \ldots < E_{M_i - 1}$,

where $\operatorname{supp} v_i \subseteq [r_i, r_{i+1})$,

$$M_i \geqslant r_{i-1} + k \qquad (i = 2, \ldots, k-1)$$

and $M_0 \leqslant M_1 \leqslant \ldots \leqslant M_{k-1} \leqslant M_k$.

Write $H_i = [r_i, r_{i+1})$. Let $w = k^{-1/2} \sum_i v_i$. So $\|w\|_{N+1} \geqslant 1$.

Now suppose $m < N$ and $r_0 \leqslant l \leqslant q_0 < \ldots < q_l = r_k$; write $F_s = [q_s, q_{s+1})$. Let i be such that $r_i \leqslant l < r_{i+1}$.

(I) $i = k-1$: Then

$$\sum_{s<l} \|F_s w\|_m^2 = \frac{1}{k} \sum_{s<l} \|F_s v_{k-1}\|_m^2 \leqslant \frac{1}{k} \|v_{k-1}\|_{m+1}^2 \leqslant \frac{1}{k} < \frac{\varepsilon^2}{2M}.$$

(II) $i < k-1$: Subdivide the (F_s) at the $k-i-1$ points $r_{i+1}, \ldots, r_{k-1}$ and relabel them as $G_0 < \ldots < G_{\nu-1}$ where $\nu \leqslant l+k-i-1 < r_{i+1}+k \leqslant M_{i+2}$. Then

$$\sum_{s<l} \|F_s w\|_m^2 = \frac{1}{k}\sum_{s<l} \|\sum_{t<k} F_s v_t\|_m^2 \leqslant \frac{1}{k}\sum_{s<\nu} \|\sum_{t<k} G_s v_t\|_m^2$$
$$= \frac{1}{k}\sum_{i\leqslant \gamma<k}\sum_{G_s\subseteq H_\gamma} \|\sum_{t<k} G_s v_t\|_m^2 = \frac{1}{k}\sum_{i\leqslant\gamma<k}\sum_{G_s\subseteq H_\gamma} \|G_s v_\gamma\|_m^2$$
$$\leqslant \frac{1}{k}\Big(\sum_{G_s\subseteq H_i} \|G_s v_i\|_m^2 + \sum_{G_s\subseteq H_{i+1}} \|G_s v_{i+1}\|_m^2 + \sum_{i+2\leqslant\gamma<k} \varepsilon_\gamma^2\Big)$$
$$< \frac{1}{k}\left(\|v_i\|_{m+1}^2 + \|v_{i+1}\|_{m+1}^2 + \frac{k\varepsilon^2}{M} - 2\right)$$
$$\leqslant \frac{1}{k}\left(1+1+\frac{k\varepsilon^2}{M}-2\right) = \frac{\varepsilon^2}{M}.$$

Hence in any case we have $\|w\|_{m+1}^2 < \varepsilon^2/M$; also $\|w\|_0^2 = k^{-1}\sup_i \|v_i\|_0^2 \leqslant k^{-1} < \varepsilon^2/2M < \varepsilon^2/M$.

Thus for all $m < N+1$, $\|w\|_m < \varepsilon/\sqrt{M} \leqslant \varepsilon$, and for any $E_0 < \ldots < E_{M-1}$, $\sum_{j<M} \|E_j w\|_m^2 \leqslant M\|w\|_m^2 < \varepsilon^2$. Finally, set

$$v = w/\|w\|_{N+1}. \ \blacksquare$$

COROLLARY 5. *If $(u_i\colon i\in\mathbf{N})$ is a block basic sequence on the (e_n), $\varepsilon > 0$ and $N\in\mathbf{N}$, then $\exists v\in \operatorname{span}_i u_i$ with $\|v\| = 1$ and $(\sum_{m<N}\alpha_m^2 \|v\|_m^2)^{1/2} < \varepsilon$.*

Proof. By the Lemma we can obtain $w\in\operatorname{span}_i u_i$ with $\|w\|_N = 1$ and $\|w\|_m < \alpha_N\varepsilon$ $(\forall m < N)$. Let $v = w/\|w\|$. Then

$$\Big(\sum_{m<N} \alpha_m^2 \|v\|_m^2\Big)^{1/2} \leqslant \frac{\alpha_N\varepsilon}{\|w\|}\Big(\sum_{m<N}\alpha_m^2\Big)^{1/2} < \varepsilon. \ \blacksquare$$

THEOREM 6. *Any infinite-dimensional subspace of X contains an isomorph of l_2.*

Proof. Suppose $Y\subseteq X$, $\dim Y = \infty$, and $1 > \varepsilon > 0$. So Y has a subspace which is isomorphic to the closed span of a block basic sequence (u_l) on the (e_n). By the above Corollary we can obtain disjointly supported (v_k) in $\operatorname{span}_i u_l$ such that $\|v_k\| = 1$ and

$$\Big(\sum_{q_j\leqslant m<q_{j+1}} \alpha_m^2 \|v_k\|_m^2\Big)^{1/2} \geqslant (1-\varepsilon^2)^{1/2},$$

where $q_0 < q_1 < \ldots$

Suppose $\lambda\in\mathbf{R}^{(\mathbf{N})}$. By 2-convexity we have

$$\|\sum_k \lambda_k v_k\|^2 \leqslant \sum_k |\lambda_k|^2 \|v_k\|^2 = \sum_k |\lambda_k|^2.$$

On the other hand,

$$\|\sum_k \lambda_k v_k\|^2 = \sum_m \alpha_m^2 \|\sum_k \lambda_k v_k\|_m^2 \geqslant \sum_j \sum_{q_j\leqslant m<q_{j+1}} \alpha_m^2 \|\sum_k \lambda_k v_k\|_m^2$$
$$\geqslant \sum_j \sum_{q_j\leqslant m<q_{j+1}} \alpha_m^2 |\lambda_j|^2 \|v_j\|_m^2 \geqslant (1-\varepsilon^2)\sum_j |\lambda_j|^2. \ \blacksquare$$

This theorem sharply distinguishes X from the Tsirelson spaces, which contain *no* isomorphs of l_2 (see [0]). We now recall *their* definition.

DEFINITIONS. Given $0 < \delta < 1$. For $x\in\mathbf{R}^{(\mathbf{N}^+)}$ define

$$\|x\|_{\delta,0} = \sup\{|x_n|\colon n\in\mathbf{N}^+\},$$
$$\|x\|_{\delta,m+1} = \max\Big(\|x\|_{\delta,m},\ \delta\sup\{(\sum_i \|y_i\cdot x\|_{\delta,m}^2)^{1/2}\colon (y_i) \text{ acc.}\}\Big),$$
$$\|x\|_\delta = \sup\{\|x\|_{\delta,m}\colon m\in\mathbf{N}\}.$$

Tsirelson's space T_δ is the completion of $\mathbf{R}^{(\mathbf{N}^+)}$ with respect to the norm $\|\cdot\|_\delta$.

Clearly if $x\in\mathbf{R}^{(\mathbf{N}^+)}$ then $\|x\|_{\delta,m} \leqslant \|x\|_{l_2}$, and $\delta^m\|x\|_m \leqslant \|x\|_{\delta,m}$. We can use the fact that $T_\delta \not\cong l_2$ to show:

THEOREM 7. $X \not\cong l_2$.

Proof. Suppose $X \cong l_2$. Then the sequence (e_n) is itself equivalent to the canonical basis of l_2. Now choose $\delta\in(\delta_U, 1)$ and observe that (for $x\in\mathbf{R}^{(\mathbf{N}^+)}$)

$$\|x\| = \Big(\sum_m \alpha_m^2\|x\|_m^2\Big)^{1/2} \leqslant (1-\delta_L^2)^{1/2}\Big(\sum_m \delta_U^{2m}\|x\|_m^2\Big)^{1/2}$$
$$= (1-\delta_L^2)^{1/2}\Big(\sum_m (\delta_U/\delta)^{2m}\delta^{2m}\|x\|_m^2\Big)^{1/2}$$
$$\leqslant (1-\delta_L^2)^{1/2}\big(1-(\delta_U/\delta)^2\big)^{-1/2}\sup_m(\delta^m\|x\|_m)$$
$$\leqslant (1-\delta_L^2)^{1/2}\big(1-(\delta_U/\delta)^2\big)^{-1/2}\|x\|_\delta \ll \|x\|_{l_2},$$

a contradiction. $\blacksquare$

The above argument works equally well to show that no *subsequence* of the (e_n) spans a Hilbert space. Thus we see that X is not isomorphic to an l_2-sum of finite blocks taken from T_δ: indeed, (e_n) is a normalised sequence tending weakly to zero which contains no Hilbertian subsequence; an easy compactness argument shows that this is impossible in such a space.

We now proceed to show that $X_\alpha \not\cong X_{\alpha'}$ if α and α' are sufficiently different. To this end, suppose that $X_\alpha \cong X_{\alpha'}$. Then there is a normalised block basic sequence (u_i) on the (e_n), and a subsequence $n_0 < n_1 < \ldots$ of $\mathbf{N}^+$, such that $(u_i) \approx (e'_{n_i})$. Now $\|u_i\|_m \leqslant \alpha_m^{-1}$ for all i, so by passing to a subsequence we may assume that, for some numbers λ_m, $\|u_i\|_m \to \lambda_m$ as $i\to\infty$, for all m.

We certainly have $\sum_m \alpha_m^2 \lambda_m^2 \leqslant 1$. In fact $\sum_m \alpha_m^2 \lambda_m^2 = 1$: for suppose not; then $\exists c > 0$, $M_0 < M_1 < \ldots$ and $i_0 < i_1 < \ldots$ such that

$$\sum_{M_n \leqslant m < M_{n+1}} \alpha_m^2 \|u_{i_n}\|_m^2 > c^2 \quad (\forall n).$$

Then for any $x \in \mathbf{R}^{(\mathbf{N})}$, $\|\sum_n x_n u_{i_n}\|^2 \leqslant \sum_n |x_n|^2$ (by 2-convexity), and

$$\Big\|\sum_n x_n u_{i_n}\Big\|^2 = \sum_m \alpha_m^2 \Big\|\sum_n x_n u_{i_n}\Big\|_m^2 \geqslant \sum_k \sum_{M_k \leqslant m < M_{k+1}} \alpha_m^2 \Big\|\sum_n x_n u_{i_n}\Big\|_m^2$$

$$\geqslant \sum_k \sum_{M_k \leqslant m < M_{k+1}} \alpha_m^2 |x_k|^2 \|u_{i_k}\|_m^2 \geqslant c^2 \sum_k |x_k|^2,$$

so that $(u_{i_n}) \approx l_2$. But (u_{i_n}) is equivalent to a subsequence of the (e_n'), so this is impossible.

We now prove a lemma which in both statement and proof is very similar to Lemma 4.

DEFINITION. Given a subsequence $\mathbf{p} = (p_0 < p_1 < \ldots)$ of $\mathbf{N}^+$. Call a vector $x \in \mathbf{R}^{(\mathbf{N})}$ $(\mathbf{p}, 0)$-*good* if $x_i = \delta_{in}$ for some n. Call x $(\mathbf{p}, N+1)$-*good* if $x = k^{-1/2} \sum_{i<k} x_i$ where $k \in \mathbf{N}^+$, the x_i are $(\mathbf{p}, N)$-good and $l \leqslant \operatorname{supp} x_0 < \ldots < \operatorname{supp} x_{k-1}$ where $p_l \geqslant k$.

LEMMA 8. *Suppose $n_0 < n_1 < \ldots$ and $p_0 < p_1 < \ldots$ are two subsequences of $\mathbf{N}^+$, $N \in \mathbf{N}$, $\varepsilon > 0$ and $M \in \mathbf{N}^+$. For $x \in \mathbf{R}^{(\mathbf{N})}$ write*

$$T(x) = \sum_i x_i e_{n_i}.$$

Then $\exists x \in \mathbf{R}^{(\mathbf{N})}$ such that

(a) *x is $(\mathbf{p}, N)$-good,*
(b) *$\|T(x)\|_m < \varepsilon$ $(\forall m < N)$, and*
(c) *$\sum_{j<M} \|E_j T(x)\|_m^2 < \varepsilon^2$ $(\forall m < N)$ whenever $E_0 < \ldots < E_{M-1}$.*

Proof. We use induction; if $N = 0$ set $x_i = \delta_{i0}$. Assume the result for N. Take $k \in \mathbf{N}^+$ such that $k > 2M/\varepsilon^2$, and $\varepsilon_i > 0$ $(i < k)$ such that

$$\sum_i \varepsilon_i^2 < k\varepsilon^2/M - 2.$$

Choose r_0 such that $n_{r_0} \geqslant k$ and $p_{r_0} \geqslant k$, and for $i < k$ choose $x_i \in \mathbf{R}^{(\mathbf{N})}$ such that

(a) x_i is $(\mathbf{p}, N)$-good,
(b) $\|T(x_i)\|_m < \varepsilon_i$ $(\forall m < N)$, and
(c) $\sum_{j<M_i} \|E_j T(x_i)\|_m^2 < \varepsilon_i^2$ $(\forall m < N)$ whenever $E_0 < \ldots < E_{M_i - 1}$.

where $\operatorname{supp} x_i \subseteq [r_i, r_{i+1})$, $M_i \geqslant n_{r_{i-1}} + k$ for $i = 2, \ldots, k-1$, and $M_0 \leqslant \ldots \leqslant M_k$. Write $H_i = [n_{r_i}, n_{r_{i+1}})$.

Finally, let $x = k^{-1/2} \sum_i x_i$. By construction x is $(\mathbf{p}, N+1)$-good.

Suppose $m < N$, and $r_0 \leqslant q_0 \leqslant \ldots \leqslant q_l = r_k$ where $n_{r_0} \leqslant l \leqslant n_{q_0}$; write $F_s = [n_{q_s}, n_{q_{s+1}})$. Let i be such that $n_{r_i} \leqslant l < n_{r_{i+1}}$.

(I) $i = k-1$: Then

$$\sum_{s<l} \|F_s T(x)\|_m^2 = \frac{1}{k} \sum_{s<l} \|F_s T(x_{k-1})\|_m^2 \leqslant \frac{1}{k} \|T(x_{k-1})\|_{m+1}^2 \leqslant \frac{1}{k} < \frac{\varepsilon^2}{2M}.$$

(II) $i < k-1$: Subdivide the F_s at the $k-i-1$ points $n_{r_{i+1}}, \ldots, n_{r_{k-1}}$ and relabel them as $G_0 < \ldots < G_{\nu-1}$, where $\nu \leqslant l + k - i - 1 < n_{r_{i+1}} + k \leqslant M_{i+2}$. Then

$$\sum_{s<l} \|F_s T(x)\|_m^2 = \frac{1}{k} \sum_{s<l} \Big\|\sum_{t<k} F_s T(x_t)\Big\|_m^2 \leqslant \frac{1}{k} \sum_{s<\nu} \Big\|\sum_{t<k} G_s T(x_t)\Big\|_m^2$$

$$= \frac{1}{k} \sum_{i \leqslant \gamma < k} \sum_{G_s \subseteq H_\gamma} \Big\|\sum_{t<k} G_s T(x_t)\Big\|_m^2$$

$$\leqslant \frac{1}{k}\Big(\sum_{G_s \subseteq H_i} \|G_s T(x_i)\|_m^2 + \sum_{G_s \subseteq H_{i+1}} \|G_s T(x_{i+1})\|_m^2 + \sum_{i+2 \leqslant \gamma < k-1} \varepsilon_\gamma^2 \Big)$$

$$< \frac{1}{k}\left(\|T(x_i)\|_{m+1}^2 + \|T(x_{i+1})\|_{m+1}^2 + \frac{k\varepsilon^2}{M} - 2 \right)$$

$$\leqslant \frac{1}{k}\left(1 + 1 + \frac{k\varepsilon^2}{M} - 2 \right) = \frac{\varepsilon^2}{M}.$$

Hence in any case $\|T(x)\|_{m+1}^2 < \varepsilon^2/M$; also $\|T(x)\|_0^2 = k^{-1} \sup_i \|T(x_i)\|_0^2 \leqslant k^{-1} < \varepsilon^2/2M < \varepsilon^2/M$. Therefore for all $m < N+1$, $\|T(x)\|_m < \varepsilon/\sqrt{M} \leqslant \varepsilon$, and for any $E_0 < \ldots < E_{M-1}$, $\sum_{j<M} \|E_j T(x)\|_m^2 \leqslant M \|T(x)\|_m^2 < \varepsilon^2$. ∎

LEMMA 9. *Given $N \in \mathbf{N}$, $\varepsilon > 0$ and $K \in \mathbf{N}$, $\exists x \in \mathbf{R}^{(\mathbf{N})}$ such that $\|x\|_{l_2} = 1$, $\|\sum_i x_i e_{n_i}'\|_m < \varepsilon$ $(\forall m < N)$ and $\|\sum_i x_i u_i\|_{N+k}^2 \geqslant \lambda_k^2 - \varepsilon^2$ $(\forall k < K)$.*

Proof. By deleting the first few u_i and e_{n_i}', we may assume that $\|u_i\|_k^2 \geqslant \lambda_k^2 - \varepsilon^2$ for all i and all $k < K$.

Suppose $\operatorname{supp} u_i \subseteq [p_i, p_{i+1})$. Apply Lemma 8 (in the space $X_{\alpha'}$) to obtain a vector $x \in \mathbf{R}^{(\mathbf{N})}$ such that x is $(\mathbf{p}, N)$-good and $\|\sum_i x_i e_{n_i}'\|_m < \varepsilon$ for all $m < N$. The former implies, firstly, that $\|x\|_{l_2} = 1$, and, secondly, that $\{p_i\colon x_i \neq 0\}$ is an N-OK set; hence for all $k < K$,

$$\Big\|\sum_i x_i u_i\Big\|_{N+k}^2 \geqslant \sum_{x_i \neq 0} \|x_i u_i\|_k^2 \geqslant \lambda_k^2 - \varepsilon^2. \;∎$$

DEFINITION. Given α and α', write $\alpha \vdash \alpha'$ to mean that

$$\inf_k \frac{\alpha_{N+k}}{\alpha_k} \times \inf_k \frac{\alpha_k'}{\alpha_{N+k}'}$$

is unbounded as a function of N.

For example, if $\alpha_m = (1-\delta^2)^{1/2} \delta^m$ and $\alpha_m' = (1-(\delta')^2)^{1/2} (\delta')^m$ then $\alpha \vdash \alpha'$ provided $\delta > \delta'$.

More generally, if $N \in \mathbf{N}^+$ and S, S' are subsets of $\mathbf{N}$ such that

$$\operatorname{card}(S \cap [rN, (r+1)N)) = k, \quad \operatorname{card}(S' \cap [rN, (r+1)N)) = k'$$

for each r, and if

$$\frac{\alpha_{m+1}}{\alpha_m} = \begin{cases} \delta, & m \in S, \\ \delta', & m \notin S, \end{cases} \qquad \frac{\alpha'_{m+1}}{\alpha'_m} = \begin{cases} \delta, & m \in S', \\ \delta', & m \notin S', \end{cases}$$

then $\alpha \models \alpha'$ provided $\delta > \delta'$ and $k > k'$.

THEOREM 10. *If $\alpha \models \alpha'$ then $X_\alpha \not\approx X_{\alpha'}$.*

Proof. Write

$$f(N) = \inf_k \frac{\alpha_{N+k}}{\alpha_k} \ (\geq \delta_L^N); \qquad g'(N) = \sup_k \frac{\alpha'_{N+k}}{\alpha'_k} \ (\leq (\delta'_U)^N).$$

Given N, take $\varepsilon > 0$ so small that

$$\frac{1-2\varepsilon^2}{1+N\varepsilon^2/g'(N)^2} \geq \frac{1}{4},$$

and take $K \in \mathbf{N}$ so large that

$$\sum_{k \geq K} \alpha_k^2 \lambda_k^2 < \varepsilon^2.$$

Apply Lemma 9 with these values of N, ε, K to get a vector $x \in \mathbf{R}^{(\mathbf{N})}$. Then

$$\Big\|\sum_i x_i u_i\Big\|^2 \geq \sum_k \alpha_{N+k}^2 \Big\|\sum_i x_i u_i\Big\|_{N+k}^2$$
$$\geq f(N)^2 \sum_{k<K} \alpha_k^2 (\lambda_k^2 - \varepsilon^2) \geq f(N)^2 (1 - 2\varepsilon^2),$$

while

$$\Big\|\sum_i x_i e'_{n_i}\Big\|^2 \leq N\varepsilon^2 + g'(N)^2 \sum_k \alpha_k^2 = g'(N)^2 (1 + N\varepsilon^2/g'(N)^2),$$

so that

$$\Big\|\sum_i x_i u_i\Big\| \Big/ \Big\|\sum_i x_i e'_{n_i}\Big\| \geq \tfrac{1}{2} f(N)/g'(N),$$

which by hypothesis is unbounded as a function of N, contradicting the fact that $(u_i) \approx (e'_{n_i})$. ■

Acknowledgements. I am extremely grateful to Peter Casazza, for acquainting me with so much of the weak Hilbert space folklore, and to Charles Read, for his indispensable mathematical guidance.

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Received January 19, 1990
Revised version December 7, 1990 (2643)