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Some more weak Hilbert spaces

by

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Abstract. We construct, by a variation of the method used to construct the Tsirelson spaces, a new family of weak Hilbert spaces which contain copies of $l_2$ inside every subspace.

Preamble. The notions of weak type 2 and weak cotype 2 were introduced by Milman and Pisier in [3]; Pisier presented weak Hilbert spaces (the intersection of these two classes) as objects of study in [4]. That Tsirelson's space $T_2$ is a weak Hilbert space had essentially been proved in [1], where it was also shown that $T_2$ contains no isomorph of $l_2$. Recently the theory of weak Hilbert spaces has developed apace: yet the scarcity of known examples remains irritating—until now, the spaces $T_g(0 < \delta < 1)$ and their duals, Hilbert space itself, and all subspaces of quotients of finite $l_2$-sums of these, were essentially the only examples known. In this paper we present a few more. There still remains, however, the problem of finding examples (if there are any) without an unconditional basis, or even without a basis.

Definitions. Given a sequence $\alpha \in (\mathbb{R}^+)^N$ such that

$$\sum_{m} \alpha_m = 1,$$

and

$$\delta_\alpha \leq \alpha_{m+1}/\alpha_m \leq \delta_\beta$$

for all $\alpha$, where $0 < \delta_\alpha \leq \delta_\beta < 1$.

Write $\mathbb{R}^{N^*}$ for the vector lattice of all finitely supported vectors in $\mathbb{R}^{N^*}$.

If $x$ and $y$ are two vectors in $\mathbb{R}^{N^*}$ we denote by $x \cdot y$ the vector whose ith coordinate is $x_iy_i$.

Call a $k$-tuple $(y_i; i < k)$ acceptable (acc) if $y_i \in \mathbb{R}^{N^*}$, $supp(y_i) \subseteq [k, \infty)$, and

$$\sum_{i < k} ||y_i||^2 \leq 1.$$

For $x \in \mathbb{R}^{N^*}$ define

$$||x||_\alpha = \sup \{||x_n||; n \in \mathbb{N}^+\}, \quad ||x||_{\alpha + 1} = \sup \{\sum_{i < k} ||y_i||^2; (y_i) \text{ acc.}\}.$$

By induction we see that $|| \cdot ||_\alpha$ is a norm on $\mathbb{R}^{N^*}$ which is 2-convex (i.e. $||x||^2 + ||y||^2 \leq ||x + y||^2$) and satisfies $||x||_\alpha \leq ||x||_\beta$ for all $x$. The canonical vectors $e_n (n \in \mathbb{N}^*)$ constitute a normalised 1-unconditional basis for

$$(\mathbb{R}^{N^*}, || \cdot ||_\alpha).$$

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Finally, set
\[ \|x\| = \left(\sum_m \alpha_m^2 \|x\|_m^2\right)^{1/2}, \]
and let \(X = X_k\) be the completion of \(\mathbb{R}^{N^*}\) with respect to the norm \(\|\cdot\|\). Note that \((e_n : n \in \mathbb{N}^*)\) is a normalized 1-unconditional basis for \(X\), hence \(X\) is 2-convex, and \(\|x\| \leq \|x\|_1\) whenever \(x \in \mathbb{R}^{N^*}\).

**Lemma 0.** If \(x \in X\) and \((y_i)\) is acceptable then
\[ \|x\| \geq \delta_i \left(\sum_i \|y_i\| \right)^{1/2}. \]

**Proof.**
\[ \delta_i^2 \sum_i \|y_i\| = \delta_i \sum_m \alpha_m^2 \|y_i\|_m^2 = \delta_i \sum_m \alpha_m^2 \sum_i \|y_i\|_m^2 \leq \delta_i \sum_m \alpha_m^2 \|x\|_m^2 \leq \sum_m \alpha_m^2 \|x\|_m^2 \leq \|x\|^2. \]

**Theorem 1.** \(X\) is a weak Hilbert space.

**Proof.** (The following argument is based on that employed in [5] to show that Tsirelson's space is a weak Hilbert space.)

(I) \(X\) has weak cotype 2: Write \(X_k = [e_n : n > k]\). Suppose \(y_0, \ldots, y_{k-1} \in X_k\). By 2-convexity we have
\[ \left(\sum_i \|y_i\|^2\right)^{1/2} \leq \left(\sum_i \|y_i\|^2\right)^{1/2}. \]

Writing
\[ x = \left(\sum_i \|y_i\|^2\right)^{1/2}, \quad (z_i)_n = \begin{cases} 0, & x_n = 0, \\ (y_i)_n / x_n, & x_n \neq 0. \end{cases} \]
we have \(z_i \cdot x = y_i\) and \(\sum |z_i|^2 \leq 1\), so by the above Lemma
\[ \|x\| \geq \delta_i \left(\sum_i \|y_i\| \right)^{1/2}. \]

In other words,
\[ \delta_i \left(\sum_i \|y_i\| \right)^{1/2} \leq \left(\sum_i \|y_i\|^2\right)^{1/2} \leq \left(\sum_i \|y_i\|^2\right)^{1/2}. \]

This implies (see [5], Lemma 13.3) that every \(k\)-dimensional subspace \(K \subseteq X_k\) has \(d_p = d(E, K) \leq 4/\delta_k\).

Now suppose \(E \subseteq X\), dim \(E = 2k\): then dim \((E \cap X_k) \geq k\), so \(\exists F \subseteq E \cap X_k \subseteq E\) with dim \(F = k\) and \(d_p \leq 4/\delta_k\). This shows (see [5], Thm. 10.2) that \(X\) has weak cotype 2.

(II) \(X\) has type 2: \(X\) has weak cotype 2 \(\Rightarrow\) (see [5], Prop. 10.7) \(X\) has cotype \(q\) (\(\forall q > 2\)). But \(X\) is 2-convex, so (see [2], (1.f.3) (1.f.9)) \(X\) has type 2.

We will now show that two more (a priori different) definitions actually yield the same norms as the \(\|\cdot\|_m\) defined above.

**Definitions.** Suppose \((y_i) : i < k\) is acceptable. Then \((y_i)\) is *allowable* (all.) if \(y_i = y_k\), where the \((E_i)\) are disjoint and \(\bigcup_i E_i \subseteq [k, \infty)\).

For \(x \in \mathbb{R}^{N^*}\), define
\[ [x]_0 = \sup \{\|u_n\| : n \in \mathbb{N}^*\}, \quad [x]_m = \sup \{\sum_i \|y_i\|_m : (y_i)\ \text{all.}\}. \]

**Proposition 2.** \([x]_m = \|x\|_m\) for all \(m\).

**Proof.** (Again, the argument given here is based on the proof in [5] of the corresponding result in Tsirelson's space.) It is clear that \([x]_m \leq \|x\|_m\) for all \(x\). By 2-convexity,
\[ |x|_m = \left(\sum_i \|y_i\|_m^2\right)^{1/2} \]
defines a norm on \(\mathbb{R}^{N^*}\). If \((y_i)\) is allowable then
\[ \|x|_m = \left(\sum_i \|y_i\|_m^2\right)^{1/2} \leq \sum_i \|y_i\|_m^2 = \sum_i \|y_i\|_m^2, \]
since \(y_i = y_k\). For \(N \in \mathbb{N}\) set
\[ C_i^N = \text{conv} \{(y_i) : i < k\}, \quad \text{supp}(y) \subseteq [k, k+N) \]
\[ = \{z_i : i < k\}, \quad z_i \geq 0, \sum_i z_i < \infty, \text{supp}(z) \subseteq [k, k+N). \]

Therefore for all \((z_i) \in C^N_i\),
\[ \|x|_m = \sum_i \|z_i\|_m = \|x|_m, \]
so
\[ [x]_m = \left(\sum_i \|z_i\|_m^2\right)^{1/2} = \left(\sum_i \|z_i\|_m^2\right)^{1/2}. \]

But as \((z_i)\) runs over \(C^N_i\), \((z_i)\) runs over all (non-negative) acceptable \(k\)-tuples with supports inside \([k, k+N)\). Since \(N\) is arbitrary it follows that whenever \((y_i)\) is acceptable,
\[ [x]_{m+1} = \sum_i \|y_i\|_m^2 = \sum_i \|y_i\|_m^2, \]
so \([x]_{m+1} = \|x\|_{m+1}\).
DEFINITIONS. An allanthus is a tree with a positive integer at each node, such that

(a) the number of branches emanating from a given node is less than or equal to the number at the node, and
(b) the number at the end of a branch is greater than or equal to the number at the node from which the branch emanates.

For example,

```
    2
   / \
  4   3
   \  /\
    6 4 5 7
```

is the beginning of an allanthus.

A subset $S \subseteq \mathbb{N}^+$ is $m$-OK if it occurs as the $m$th row of some allanthus. Thus $\{2\}$ is 0-OK, $\{2, 3\}$ is 1-OK, $\{3, 4, 5, 8\}$ is 2-OK, and $\{3, 4, 5, 6, 7\}$ is 3-OK. Note that if $f: \mathbb{N}^+ \rightarrow \mathbb{N}^+$ satisfies $f(n) \geq n \forall n$ and $S$ is $m$-OK then $f(S)$ is $m$-OK, and any subset of an $m$-OK set ($m \geq 1$) is $m$-OK.

**Notation.** If $E \subseteq \mathbb{N}^+$ we write $Ex$ for $\chi_{E}(x)$.

**Proposition 3.** If $x \in \mathbb{R}(\mathbb{N}^*)$ and $m \in \mathbb{N}$ then

$$[x]_m = \|x\|_m = \sup \{ \|S_{m+1}\|_2 : S \text{ m-OK} \}.$$ 

**Proof.** We proceed by induction, the case $m = 0$ being clear since the 0-OK sets are just the singletons. Assume the result for $m$, and suppose $x \in \mathbb{R}(\mathbb{N}^*)$.

Clearly $[x]_{m+1} \geq \|x\|_{m+1}$.

(I) $[x]_{m+1} \leq \sup \{ \|S_{m+1}\|_2 : S \text{ (m+1)-OK} \}$: Suppose

$$[x]_{m+1} = \sum_{i \in k} |E_i x|_m^2$$

where the $(E_i)$ are disjoint and $E_i \subseteq [k, \infty)$. By hypothesis we can write

$$[E_i x]_m = \|S_{i, E_i x}\|_2$$

where $S_i$ is the $m$th row of an allanthus $A_i$. We can assume that every node of $A_i$ is $\geq k$. (Replace each node which is $< k$ by $k$, and we still have an allanthus.) In particular, the top node of each $A_i$ is $\geq k$, so that

$$A_0 A_1 \ldots A_{k-1}$$

is an allanthus. Let $S$ be its $(m+1)$th row, i.e. $S = \bigcup_{i} S_i$. Then

$$[x]_{m+1} = \sum_{i} [E_i x]_m^2 = \sum_{i} \|S_{i, E_i x}\|_2^2 \leq \|S x\|_2^2.$$  

(II) Suppose $\{\|S_{m+1}\|_2 : S \text{ (m+1)-OK} \}$.

Let $S$ be the $(m+1)$th row of an allanthus $A$. Observe that if we put each row of $A$ in non-decreasing order from left to right we still have an allanthus. (Start at the top and work down.) Order each row by a succession of operations each of which replaces ... $b$ ... $a$ ... $b$ ... $a$ ... $b$ ... where $a \leq b$.) Having done this, write $A$ in the form

```
   A_0 A_1 \ldots A_{k-1}
```

where each $A_i$ is an allanthus. Let $S_i$ be the $m$th row of $A_i$; so $S = \bigcup_i S_i$. By deleting any repetitions among the $S_i$ we may assume that the $S_i$ are disjoint. Since the top node of $A$ must be $\geq k$, we have $k \leq S_0 \ldots \leq S_{k-1}$ and

$$\|S x\|_2^2 = \sum_{i} \|S_i x\|_2^2 \leq \sum_{i} \|S_i x\|_2^2.$$  

**Lemma 4.** Suppose $(u_i : i \in \mathbb{N})$ is a normalised block basic sequence in $X, N \in \mathbb{N}, e > 0$ and $M \in \mathbb{N}^*$. Then $\exists v \in \text{span}_{u_i} u_i$ such that

(a) $\|v\|_N = 1$,
(b) $\|v\|_m < \varepsilon \quad (\forall m < N)$, and
(c) $\sum_{i < M} \|E_{i, v}\|_2^2 < \varepsilon^2 \quad (\forall m < N)$ whenever $E_0 \ldots \leq E_{M-1}$.

**Proof.** We proceed by induction on $N$. If $N = 0$ just take $v = u_0/\|u_0\|_0$.

Now assume the result for $N$.

Take $k \in \mathbb{N}^*$, $k > 2M/e^2$, and choose $\varepsilon_i > 0$ ($i < k$) so that

$$\sum_{i} \varepsilon_i^2 < k e^2 / M - 2.$$  

Choose $r_0 \geq k$ and for $i < k$, step-by-step choose $v_i \in \text{span}_{u_i} u_i$ such that

(a) $\|v_i\|_N = 1$,
(b) $\|v_i\|_m < \varepsilon_i \quad (\forall m < N)$, and
(c) $\sum_{i < M} \|E_{i, v_i}\|_2^2 < \varepsilon_i^2 \quad (\forall m < N)$ whenever $E_0 \ldots \leq E_{M-1}$, where $\text{supp} v_i = [r_i, r_{i+1})$

where $M_i \geq r_{i+1}$ (i.e. $i = 2, \ldots, k - 1$) and $M_0 \leq M_1 \leq \ldots \leq M_{k-1} \leq M_k$.

Write $H_i = [r_i, r_{i+1})$. Let $w = k^{1/2} \sum v_i$. So $\|w\|_{N+1} \geq 1$.

Now suppose $m < N$ and $r_0 \leq i < q_0 \ldots < q_i = r_i$; write $F_i = [q_i, q_{i+1})$.

Let $i$ be such that $r_i \leq l < r_{i+1}$.

(I) $i = k - 1$: Then

$$\sum_{i < l} \|F_{i} w\|_m^2 = \frac{1}{k} \sum_{i < l} \|F_{i} v_{k-1}\|_m^2 \leq \frac{1}{k} \|v_{k-1}\|_{m+1}^2 \leq \frac{1}{k} \leq \frac{\varepsilon^2}{2M}.$$
(II) \( i < k - 1 \): Subdivide the \((F_0)\) at the \(k - i - 1\) points \( r_{i+1}, \ldots, r_{k-1} \) and relabel them as \( G_0 < \ldots < G_{r-1} \) where \( 1 < l + k - i - 1 < r_{i+1} + k \leq M_{l+2} \). Then

\[
\sum_{k \leq l} \|F_kw\|_{\ell_2}^2 = \frac{1}{k} \sum_{k \leq l} \|F_kv\|_{\ell_2}^2 \leq \frac{1}{k} \sum_{k \leq l} \|G_kv\|_{\ell_2}^2 \\
\leq \frac{1}{k} \left( \sum_{k \leq l} \|G_kv\|_{\ell_2}^2 + \sum_{k \leq l} \|G_{k+1}v\|_{\ell_2}^2 + \sum_{l+2 < k \leq M_{l+2}} \|v\|_{\ell_2}^2 \right) \\
\leq \frac{1}{k} \left( \sum_{k \leq l} \|G_kv\|_{\ell_2}^2 + \sum_{k \leq l} \|G_{k+1}v\|_{\ell_2}^2 + \frac{k^2}{M} - 2 \right) \\
\leq \frac{1}{k} \left( 1 + \frac{k^2}{M} - 2 \right) = \frac{e^2}{M}.
\]

Hence in any case we have \( \|w\|_{\ell_2}^2 < e^2/M \); also \( \|w\|_{\ell_3} = \frac{1}{k} \sup_{k \leq l} \|v\|_{\ell_2} \leq \frac{1}{k} \) \( < e^2/2M < e^2/\sqrt{M} \).

Thus for all \( m < N + 1 \), \( \|w\|_{\ell_m} < \sqrt{M} \leq e \), and for any \( E_0 < \ldots < E_{M-1} \), \( \sum_{k \leq l} \|E_kw\|_{\ell_m} \leq M \|w\|_{\ell_m} < e^2 \). Finally, set

\[
u = w/\|w\|_{\ell_{N+1}}.
\]

**Corollary 5.** If \((u_i) \in N \) is a block basic sequence on the \((e_n)\) \( \epsilon > 0 \) and \( N \in \mathbb{N} \), then \( \exists v \in \text{span} u_i \) with \( \|v\|_{\ell_1} = 1 \) and \( \sum_{n \leq N} \|u_n^* v\|_{\ell_2}^2 < e \).

**Proof.** By the Lemma we can obtain \( \nu \in \text{span} u_i \) with \( \|w\|_{\ell_m} = \alpha \nu \) (\( \forall m < N \)). Let \( v = \|w\| \). Then

\[
\left( \sum_{m < N} \frac{\alpha_m^2}{\|w\|_{\ell_m}^2} \right)^{1/2} \leq \frac{\sum_{m < N} \alpha_m^2}{\|w\|_{\ell_m}} < e.
\]

**Theorem 6.** Any infinite-dimensional subspace of \( X \) contains an isomorph of \( l_2 \).

**Proof.** Suppose \( Y \lesssim X \), \( \dim Y = \infty \), and \( 1 > \epsilon > 0 \). So \( Y \) has a subspace which is isomorphic to the closed span of a block basic sequence \((u_i)\) on the \((e_n)\). By the above Corollary we can obtain disjointly supported \((v_i)\) in \( \text{span} u_i \) such that \( \|v_i\| = 1 \) and

\[
\sum_{q \leq m < q_{i+1}} \alpha_m^2 \|v_k\|_{\ell_m}^2 \geq (1 - \epsilon^2)/2,
\]

where \( q_0 < q_1 \ldots \)

Suppose \( \lambda \in \mathbb{R}^N \). By 2-convexity we have

\[
\sum_{k \leq l} \|F_kw\|_{\ell_2}^2 \leq \sum_{k \leq l} \|F_kw\|_{\ell_2}^2 = \sum_{k \leq l} \|G_kw\|_{\ell_2}^2.
\]

On the other hand,

\[
\sum_{k} \|\lambda_k v_k\|_{\ell_2}^2 = \sum_{k} \|\lambda_k v_k\|_{\ell_2}^2 \geq \sum_{j, k \leq l} \|\lambda_k v_k\|_{\ell_2}^2 \geq \sum_{j, k \leq l} \|\lambda_k v_k\|_{\ell_2}^2 \geq (1 - \epsilon^2) \sum_{j, k \leq l} \|\lambda_k v_k\|_{\ell_2}^2.
\]

This theorem sharply distinguishes \( X \) from the Tsirelson spaces, which contain no isomorphs of \( l_2 \) (see \([0]\)). We now recall their definition.

**Definitions.** Given \( 0 < \delta < 1 \). For \( x \epsilon \mathbb{R}^{N_+} \) define

\[
\|x\|_{\ell_2,0} = \sup \{ \|x_i\| : n \epsilon N \},
\]

\[
\|x\|_{\ell_2,\delta} = \max \{ \|x\|_{\ell_2} / \delta \sup \{|y_i| / \|y\|_{\ell_2}^{1/2} : (y) \text{ acc} \} \},
\]

\[
\|x\|_{\ell_2} = \sup \{ \|x\|_{\ell_2,\delta} : n \epsilon N \}.
\]

**Tsirelson's space** \( T_\delta \) is the completion of \( \mathbb{R}^{N_+} \) with respect to the norm \( \| \cdot \|_{\ell_2} \).

Clearly if \( x \epsilon \mathbb{R}^{N_+} \) then \( \|x\|_{\ell_2,\delta} \lesssim \|x\|_{\ell_2} \), and \( \delta^n \|x\|_{\ell_2} \lesssim \|x\|_{\ell_2,\delta} \).

We can use the fact that \( T_\delta \not\cong l_2 \) to show:

**Theorem 7.** \( X \not\cong l_2 \).

**Proof.** Suppose \( X \cong l_2 \). Then the sequence \((e_n)\) is itself equivalent to the canonical basis of \( l_2 \). Now choose \( \delta < \delta_\| \), and observe that (for \( x \epsilon \mathbb{R}^{N_+} \))

\[
\|x\| = (\sum_{m < N} \|x_m\|^{1/2})^{1/2} \leq (1 - \delta_\|)^{1/2} \left( \sum_{m < N} \|x_m\|^{1/2} \right).
\]

\[
\|x\|_{\ell_2} \geq (1 - \delta_\|)^{1/2} \left( \sum_{m < N} \|x_m\|^{1/2} \right)^{1/2} \leq \left( \sum_{m < N} \|x_m\|^{1/2} \right)^{1/2} \|x\|_{\ell_2} \lesssim \|x\|_{\ell_2,\delta},
\]

a contradiction.

The above argument works equally well to show that no subsequence of the \((e_n)\) spans a Hilbert space. Thus we see that \( X \) is not isomorphic to an \( l_2 \)-sum of finite blocks taken from \( T_\delta \); indeed, \((e_n)\) is a normalised sequence tending weakly to zero which contains no Hilbertian subsequence; an easy compactness argument shows that this is impossible in such a space.

We now proceed to show that \( X \not\cong X_\alpha \) if \( \alpha \) and \( \alpha' \) are sufficiently different.

To this end, suppose that \( X \not\cong X_\alpha \). Then there is a normalised block basic sequence \((v_i)\) on the \((e_n)\), and a subsequence \( q_0 < r_1 < \ldots \) of \( N_+ \), such that \((v_i) \approx (e_{q_i})\). Now \( \|v_i\|_{\ell_2} \approx \alpha_{q_i}^{-1} \) for all \( i \), so by passing to a subsequence we may assume that, for some numbers \( \lambda_m \), \( \|v_i\|_{\ell_2} \rightarrow \lambda_m \) as \( i \rightarrow \infty \), for all \( m \).
We certainly have $\sum a_m^2 \leq 1$. In fact $\sum a_m^2 \leq 1$: for suppose not; then $\exists \epsilon > 0$, $M_0 < M_1 < \ldots$ and $i_0 < i_1 < \ldots$ such that

$$\sum_{M_0 \leq m < M_{i+1}} a_m^2 \|u_m\|_m^2 > \epsilon^2 \quad (\forall n).$$

Then for any $x \in \mathbb{R}^N$, $\|\sum_n x_n u_n\| > \sum_n |x_n|^2$ (by 2-convexity), and

$$\|\sum_n x_n u_n\| \geq \sum_k \sum_{M_k \leq m < M_{k+1}} a_k^2 \|u_m\|_m^2 > \epsilon^2 \sum |x_k|^2,$$

so that $(u_k) \not\to L$. But $(u_k)$ is equivalent to a subsequence of the $(e_i)$, so this is impossible.

We now prove a lemma which in both statement and proof is very similar to Lemma 4.

**Definition.** Given a subsequence $p = (p_0 < p_1 < \ldots)$ of $N^+$. Call a vector $x \in \mathbb{R}^N$ $(p, 0)$-good if $x_i = \delta_{p_i}$ for some $n$. Call $x$ $(p, N+1)$-good if $x = k^{-1/2} \sum_{k \leq n} x_n$ where $k \in N^+$, the $x_n$ are $(p, N)$-good and $l \leq \sum_{k \leq n} x_n < \sum_{k \leq n} x_{k+1}$ where $p_l \geq k$.

**Lemma 8.** Suppose $n_0 < n_1 < \ldots$ and $p_0 < p_1 < \ldots$ are two subsequences of $N^+, N \in N, \epsilon > 0$ and $M \in N^+$. For $x \in \mathbb{R}^N$ write

$$T(x) = \sum_i x_i e_{n_i}.$$

Then $\exists x \in \mathbb{R}^N$ such that

(a) $x$ is $(p, N)$-good,

(b) $\|T(x)\|_m < \epsilon$ $(\forall m < N)$, and

(c) $\sum_{k \leq M} E_k(x) \|x_k\|_k < \epsilon^2$ $(\forall m < N)$ whenever $E_0 < \ldots < E_{M-1}$.

**Proof.** We use induction; if $N = 0$ set $x_i = \delta_{p_i}$. Assume the result for $N$. Take $k \in N^+$ such that $k > 2M/\epsilon^2$ and $i_0 \geq 0$ (i < k) such that

$$\sum_i e_i^2 < \epsilon^2k^2/M - 2.$$

Choose $r_0$ such that $n_{r_0} \geq k$ and $p_{r_0} \geq k$, and for $i < k$ choose $x_i \in \mathbb{R}^N$ such that

(a) $x_i$ is $(p, N)$-good,

(b) $\|T(x_i)\|_m < \epsilon$ $(\forall m < N)$, and

(c) $\sum_{k \leq M} E_k(x\|x_k\|_k < \epsilon^2$ $(\forall m < N)$ whenever $E_0 < \ldots < E_{M-1}$.

where $\text{supp} x_i \subseteq [r_i, r_{i+1}], M_i \geq n_{r_i} + k$ for $i = 2, \ldots, k - 1$, and $M_0 \leq \ldots \leq M_k$. Write $H_i = [n_{r_i}, n_{r_{i+1}}].$

Finally, let $x = k^{-1/2} \sum_{i \leq n} x_n$. By construction $x$ is $(p, N+1)$-good.

Suppose $n < N$ and $r_0 \leq q_0 \leq \ldots \leq q_i = r_k$ where $n_{r_i} \leq i \leq n_{r_{i+1}}$; write $F_i = [n_{r_i}, n_{r_{i+1}}]$. Let $i$ be such that $n_{r_i} \leq i \leq n_{r_{i+1}}$.

(I) $i = k - 1$: Then

$$\sum_{s < i} \|F_s(T(x_i))\|_s^2 = \sum_{k \leq i} \|F_k(T(x_{k-1}))\|_k^2 \leq \frac{1}{k} \|T(x_{k-1})\|_k^2 \leq \frac{1}{k^2} \frac{\epsilon^2}{2M}.$$  

(II) $i < k - 1$: Subdivide the $F_s$ at the $k - i - 1$ points $r_{s_1}, \ldots, r_{s_{k-1}}$ and relabel them $G_0 < \ldots < G_{k-1}$ where $v < i + k - i < n_{r_{s_i}} + k \leq M_{i+2}$. Then

$$\sum_{s < i} \|F_s(T(x_i))\|_s^2 = \sum_{k < s \leq v} \|F_k(T(x_{k-1}))\|_k^2 \leq \frac{1}{k} \|F_k(T(x_{k-1}))\|_k^2 \leq \frac{1}{k} \|T(x_{k-1})\|_k^2 \leq \frac{1}{k^2} \frac{\epsilon^2}{2M}.$$  

Hence in any case $\|T(x_i)\|_M^2 < \epsilon^2/M$; also $\|T(x_i)\|_M^2 = k^{-1} \sum_{k \leq i} \|G_k(T(x_{k-1}))\|_k^2 \leq \epsilon^2/k^2 < \epsilon^2/2M < \epsilon^2/M$. Therefore for all $m < N + 1$, $\|T(x_m)\|_m^2 < \epsilon^2/\sqrt{M}$, and for any $E_0 < \ldots < E_{M-1}$, $\|E_m(x)\|_m \leq \|T(x_m)\|_m < \epsilon^2$. \n
**Lemma 9.** Given $N \in N, \epsilon > 0$ and $K \in N, \exists x \in \mathbb{R}^N$ such that $\|x\|_M = 1, \|x_i\|_m < \epsilon$ $(\forall m < N)$ and $\sum_i x_i e_{n_i} < \epsilon$ $(\forall k \leq K)$

**Proof.** By deleting the first few $u_i$ and $e_{n_i}$, we may assume that $\|u_i\|_k^2 < \lambda^2 - \epsilon^2$ for all $i$ and all $k < K$.

Suppose $\text{supp} u_i \subseteq [p_i, p_{i+1})$. Apply Lemma 8 (in the space $X_k$) to obtain a vector $x \in \mathbb{R}^N$ such that $x$ is $(p, N)$-good and $\|x_i e_{n_i} \|_m < \epsilon$ for all $m < N$. The former implies, firstly, that $\|x\|_M = 1$, and, secondly, that $(p_i: x_i \neq 0)$ is an $N$-OK set; hence for all $k < K$,

$$\|x_i u_i \|_{M+i} \geq \sum_{x_j \neq 0} |x_i u_i|_{M+i} \geq \lambda^2 - \epsilon^2.$$

**Definition.** Given $\alpha$ and $\alpha'$, write $\alpha \vdash \alpha'$ to mean that

$$\inf_{\alpha} \frac{\alpha}{\alpha_k} \inf_{\alpha' k} \frac{\alpha_k}{\alpha_n}$$

is unbounded as a function of $N$.

For example, if $\alpha_n = (1 - \delta)^{1/2} \delta^m$ and $\alpha' = (1 - \delta')^{1/2} (\delta')^m$ then $\alpha \vdash \alpha'$ provided $\delta > \delta'$. 

More generally, if \( N \in \mathbb{N}^+ \) and \( S, S' \) are subsets of \( \mathbb{N} \) such that
\[
\text{card}(S \cap [rN, (r+1)N)) = k, \quad \text{card}(S' \cap [rN, (r+1)N)) = k',
\]
for each \( r \), and if
\[
\frac{a_{m+1}}{a_m} = \begin{cases} \delta, & m \in S, \\ \delta', & m \notin S, \end{cases}
\]
\[
\frac{a_{m+1}}{a_m} = \begin{cases} \delta, & m \in S', \\ \delta', & m \notin S'. \end{cases}
\]
then \( \alpha \not\succ \alpha' \) provided \( \delta > \delta' \) and \( k > k' \).

**Theorem 10.** If \( \alpha \not\succ \alpha' \) then \( X_\alpha \not\cong X_{\alpha'} \).

**Proof.** Write
\[
f(N) = \inf_k \frac{a_{N+k}}{a_k} (\geq \delta_k^N); \quad g'(N) = \sup_k \frac{a_{N+k}}{a_k} (\leq (\delta_k^N)^N).
\]

Given \( N \), take \( \varepsilon > 0 \) so small that
\[
1 - 2\varepsilon^2 \leq \frac{1}{1 + Ne^2 / g'(N)^2} \geq \frac{1}{4},
\]
and take \( K \in \mathbb{N} \) so large that
\[
\sum_{k \leq K} a_k^2 > \varepsilon^2.
\]

Apply Lemma 9 with these values of \( N, \varepsilon, K \) to get a vector \( x \in \mathbb{R}^N \). Then
\[
\left\| \sum_i x_i u_i \right\|^2 \geq \sum_k a_{N+k}^2 \left\| \sum_i x_i u_i \right\|_{N+k}^2 \geq f(N)^2 \sum_k a_k^2 (a_k^2 - \varepsilon^2) \geq f(N)^2 (1 - 2\varepsilon^2),
\]
while
\[
\left\| \sum_i x_i e_i \right\|^2 \leq Ne^2 + g'(N)^2 \sum_k a_k^2 = g'(N)^2 (1 + Ne^2 / g'(N)^2),
\]
so that
\[
\left\| \sum_i x_i u_i \right\| / \left\| \sum_i x_i e_i \right\| \geq \frac{1}{2} f(N) / g'(N),
\]
which by hypothesis is unbounded as a function of \( N \), contradicting the fact that \( (u_i) \approx (e_i) \).

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