On multiplication of infinite series

by

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The most general definition of the product of two infinite series can be obtained as follows: denote by N the set of all pairs (i,k) of positive integers, and let $\mathfrak{N}$ be a sequence $N_1, N_2, \ldots$ of finite, mutually disjoint subsets of N such that $N=\sum_{i=1}^{\infty} N_i$. Given two infinite series

$$(1) \qquad \sum_{n=1}^{\infty} \xi_n,$$

$$(2) \qquad \sum_{n=1}^{\infty} \eta_n,$$

consider the series

$$(3) \qquad \sum_{n=1}^{\infty} \zeta_n$$

where $\zeta_j=\sum_{(i,k)\in N_j} \xi_i \eta_k$; the series (3) will be called the *product* of the series (1) and (2) *obtained by the method corresponding to the sequence* $\mathfrak{N}$ or simply *by the method* $(\mathfrak{N})$.

The method $(\mathfrak{N})$ will be called *perfect* if

1⁰ for any series (1) and (2) the convergence of these two series implies that of the series (3),

2⁰ the sum of the series (3) is equal to the product of the sums (1) and (2).

The method $(\mathfrak{N})$ will be said to have the *property* (m_1) (resp. (m_2)) if for any series (1) and (2) the convergence of the series (1) (resp. (2)) and the absolute convergence of the series (2) (resp. (1)) implies the convergence of the series (3) to a sum equal to the product of the sums of the series (1) and (2).

If a method $(\mathfrak{N})$ has both properties (m_1) and (m_2) we will say that it has *Mertens' property.*

R. Rado [1]) has given necessary and sufficient conditions for a method $(\mathfrak{N})$ to be perfect. Rado considers only methods with sets N_j consisting of one element; the necessary and sufficient conditions for the general case may be obtained easily from his results. In this paper we give a simpler proof of Rado's theorem, based on a different idea, and besides we shall characterize methods having Mertens' property. Our methods are based on the consideration of certain bilinear functionals in Banach spaces.

1. Let X and Y be two Banach spaces [2]); a functional $f(x,y)$ defined in the combinatorial product $X\times Y$ of these spaces is called *bilinear* if it is additive and continuous in each variable separately. Many theorems concerning linear functionals in Banach spaces may be extended without difficulty to bilinear functionals. For instance [3])

Lemma 1. *For any bilinear functional $f(x,y)$ there exists a constant A with*

$$(1.1) \qquad |f(x,y)| \leqslant A\|x\|\cdot\|y\|$$

for all values of x,y.

The greatest lower bound of the numbers A in formula (1.1) is called the *norm* of the functional $f(x,y)$ and will be denoted by $\|f(\cdot,\cdot)\|_{X,Y}$. The norm may be defined also by the formula

$$(1.2) \qquad \|f(\cdot,\cdot)\|_{X,Y}=\sup_{\|x\|\leqslant 1,\ \|y\|\leqslant 1}|f(x,y)|=$$
$$=\sup_{\|x\|\leqslant 1}[\sup_{\|y\|\leqslant 1}|f(x,y)|]=\sup_{\|y\|\leqslant 1}[\sup_{\|x\|\leqslant 1}|f(x,y)|].$$

Lemma 2. *If $\{f_n(x,y)\}$ is a sequence of bilinear functionals convergent in $X\times Y$, then the limit-functional is also bilinear.*

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[1]) R. Rado, *The distributive law for products of infinite series*, Quart. Journ. of Math., Oxford Ser. 11 (1940), p. 229-242.

[2]) For the definition of these spaces and some other concepts used in this paper the reader may consult S. Banach, *Théorie des opérations linéaires*, Monografie Matematyczne 1, Warszawa 1932.

[3]) Lemmas 1-4 are well known. For their generalization see S. Mazur and W. Orlicz, *Einige Eigenschaften der polynomischen Operationen*, Erste Mitt., Studia Math. 5 (1935) p. 50-68, Zweite Mitt., ibidem, p. 179-189.

Lemma 3. *If $\{f_n(x,y)\}$ is a sequence of bilinear functionals such that* $\overline{\lim\limits_{n\to\infty}} f_n(x,y) < +\infty$ *in* $X\times Y$, *then* $\overline{\lim\limits_{n\to\infty}} \|f_n(\cdot,\cdot)\|_{X,Y} < +\infty$.

Lemma 4. *If the sets X_0 and Y_0 are dense in the spaces X and Y respectively, and $\{f_n(x,y)\}$ is a sequence of bilinear functionals convergent in the set $X_0\times Y_0$, such that*

$$\overline{\lim_{n\to\infty}} \|f_n(\cdot,\cdot)\|_{X,Y} < +\infty, \tag{1.3}$$

then the sequence $\{f_n(x,y)\}$ is convergent everywhere.

It follows that a sequence of bilinear functionals is convergent in the whole of $X\times Y$ if, and only if it is convergent in a set $X_0\times Y_0$ (where X_0 and Y_0 have the same meaning as in lemma 4) and the inequality (1.3) holds.

We will deal in the sequel with the following functional spaces:

1⁰ the space (l) of sequences $x=\{\xi_n\}$ of real numbers such that $\sum\limits_{n=1}^{\infty}|\xi_n|<+\infty$, the norm being defined by the formula

$$\|x\|=\sum_{n=1}^{\infty}|\xi_n|,$$

2⁰ the space (r) of sequences $x=\{\xi_n\}$ of real numbers such that the series $\sum\limits_{n=1}^{\infty}\xi_n$ is convergent, the norm being defined by the formula

$$\|x\|=\sup_{n=1,2,\dots}\Big|\sum_{\nu=1}^{n}\xi_\nu\Big|.$$

If we define the addition of elements and multiplication by real numbers as usual, both spaces (l) and (r) are Banach spaces. All the elements of the space (l) form a set which is dense in (r).

The functional $g(x)=\sum\limits_{n=1}^{\infty}\xi_n$ is linear in both spaces (l) and (r). It is easy to show that every functional of the form

$$f(x)=\sum_{n=1}^{\infty}a_n\xi_n$$

where $a_n=0$ for $n>m$ is linear in (r); moreover, denoting by $\|f(\cdot)\|_{(r)}$ its norm, we have

$$\|f(\cdot)\|_{(r)}=|a_m|+\sum_{n=1}^{m-1}|a_n-a_{n+1}|=\sum_{n=1}^{\infty}|a_n-a_{n+1}|. \tag{1.4}$$

Suppose that $\{a_n\}$ is a sequence whose terms are equal to 0 or 1; a segment $a_k, a_{k+1},\dots,a_l$ of this sequence will be called a *gap* of $\{a_n\}$ if $a_k=a_{k+1}=\dots=a_l=0$, $a_{l+1}=1$, and either $k=1$ or $k>1$ and $a_{k-1}=1$. Let $f(x)=\sum\limits_{n=1}^{\infty}a_n\xi_n$ be a linear functional in (r) such that $a_n=0$ or 1 and almost all $a_n=0$; denote by ϱ the number of gaps in the sequence $\{a_n\}$. It is obvious that $2\varrho\leqslant\|f(\cdot)\|_{(r)}\leqslant 2\varrho+1$.

2. Put $P_n=N_1+N_2+\dots+N_n$; we may consider the set P_n as a set of points of the plane with positive integer coordinates. Write also

$$\varepsilon_{ik}^{(n)}=\begin{cases}1 & \text{if } (i,k)\,\epsilon\, P_n,\\ 0 & \text{if } (i,k) \text{ non } \epsilon\, P_n;\end{cases}$$

then the n-th partial sum of the series (3) may be written in the form

$$f_n(x,y)=\sum_{i,k=1}^{\infty}\varepsilon_{ik}^{(n)}\xi_i\eta_k, \tag{2.1}$$

where $\varepsilon_{ik}^{(n)}=0$ for i,k sufficiently large. The formula (2.1) defines a functional $f_n(x,y)$ which is evidently bilinear in each of the spaces $(r)\times(r)$, $(r)\times(l)$, $(l)\times(r)$. The sequence $\{f_n(x,y)\}$ is convergent in the set $(l)\times(l)$ to the functional $f(x,y)=g(x)g(y)$ which is bilinear in each of the spaces $(r)\times(r)$, $(r)\times(l)$, $(l)\times(r)$. Since the elements of the space (l) form a dense set in (r), using lemmas 2—4 we get the following

Theorem 1. *The method* $(\mathfrak{N})$ *has the property* (m_1) *(resp.* (m_2)*) if and only if for any convergent series* (1) *(resp.* (2)*) and any absolutely convergent series* (2) *(resp.* (1)*) the series* (3) *has bounded partial sums.*

The method $(\mathfrak{N})$ *is perfect if and only if,* (1) *and* (2) *being any convergent series, the series* (3) *has bounded partial sums*[4].

[4]) This part of theorem 3 has been proved in a slightly different form by Rado, loc. cit.

3. By lemma 3 and theorem 1, the necessary and sufficient condition for the method $(\mathfrak{N})$ to have the property (m_1) is the boundedness of the sequence of norms $\{\|f_n(\cdot,\cdot)\|_{(r),(l)}\}$.

Let x be fixed. Then $\varphi_n(y)=f_n(x,y)$ is a linear functional in (l); denote by

$$\|f_n(x,\cdot)\|_{(l)}=\sup_{\|y\|\leqslant 1}|f_n(x,y)|$$

the norm of this functional; by formula (1.2)

$$\|f_n(\cdot,\cdot)\|_{(r),(l)}=\sup_{\|x\|\leqslant 1}\|f_n(x,\cdot)\|_{(l)};$$

computing the norm $\|f_n(x,\cdot)\|_{(l)}$ we obtain

$$\|f_n(x,\cdot)\|_{(l)}=\sup_{k=1,2,\dots}\Big|\sum_{i=1}^{\infty}\varepsilon_{ik}^{(n)}\xi_i\Big|;$$

hence

$$\|f_n(\cdot,\cdot)\|_{(r),(l)}=\sup_{\|x\|\leqslant 1}\ \sup_{k=1,2,\dots}\Big|\sum_{i=1}^{\infty}\varepsilon_{ik}^{(n)}\xi_i\Big|=\sup_{k=1,2,\dots}\ \sup_{\|x\|\leqslant 1}\Big|\sum_{i=1}^{\infty}\varepsilon_{ik}^{(n)}\xi_i\Big|;$$

since by formula (1.4)

$$\sup_{\|x\|\leqslant 1}\Big|\sum_{i=1}^{\infty}\varepsilon_{ik}^{(n)}\xi_i\Big|=\sum_{i=1}^{\infty}|\varepsilon_{ik}^{(n)}-\varepsilon_{i+1\,k}^{(n)}|,$$

we obtain

$$\|f_n(\cdot,\cdot)\|_{(r),(l)}=\sup_{k=1,2,\dots}\sum_{i=1}^{\infty}|\varepsilon_{ik}^{(n)}-\varepsilon_{i+1\,k}^{(n)}|.$$

This fact can be expressed as follows:

The necessary and sufficient condition for the method $(\mathfrak{N})$ to have the property (m_1) is

$$\overline{\lim_{n\to\infty}}\ \sup_{k=1,2,\dots}\sum_{i=1}^{\infty}|\varepsilon_{ik}^{(n)}-\varepsilon_{i+1\,k}^{(n)}|<+\infty.$$

This statement may also be expressed in an another form: Consider the set P_n and denote by ϱ_{nk} the number of gaps in the sequence $\varepsilon_{1k}^{(n)},\varepsilon_{2k}^{(n)},\dots$ Put

$$\varrho_n=\max_{k=1,2,\dots}\varrho_{nk},$$

thus ϱ_n denoting the greatest number of gaps in all the columns of the set P_n. Since $2\varrho_{nk}\leqslant\sum_{i=1}^{\infty}|\varepsilon_{ik}^{(n)}-\varepsilon_{i+1\,k}^{(n)}|\leqslant 2\varrho_{nk}+1$, we get

Theorem 2. *The necessary and sufficient condition for the method $(\mathfrak{N})$ to have the property (m_1) is the boundedness of the sequence $\{\varrho_n\}$.*

Similarly, denote by σ_{ni} the number of gaps in the sequence $\varepsilon_{i1}^{(n)},\varepsilon_{i2}^{(n)},\dots$ and put

$$\sigma_n=\max_{i=1,2,\dots}\sigma_{ni}.$$

By symmetry we get

Theorem 3. *The necessary and sufficient condition for the method $(\mathfrak{N})$ to have Mertens' property is the boundedness of the sequences $\{\varrho_n\}$ and $\{\sigma_n\}$.*

4. A set Q of points (i,k) where i,k are positive integers satisfying the inequalities $\alpha\leqslant i\leqslant\beta$, $\lambda\leqslant k\leqslant\mu$ is called a *rectangle*. By $\vartheta(P_n)$ we shall denote the minimal number of non-overlapping rectangles into which the set P_n may be decomposed.

Consider the functional $f_n(x,y)=\sum_{i,k=1}^{\infty}\varepsilon_{ik}^{(n)}\xi_i\eta_k$, and let x be fixed; then the functional $\varphi_n(y)=f_n(x,y)$ is linear in (r) and its norm is

$$\sum_{\nu=1}^{\infty}\Big|\sum_{i=1}^{\infty}(\varepsilon_{i\nu}^{(n)}-\varepsilon_{i\nu+1}^{(n)})\xi_i\Big|=\sum_{\nu=1}^{\infty}\Big|\sum_{i=1}^{\infty}\delta_{i\nu}^{(n)}\xi_i\Big|$$

where $\delta_{i\nu}^{(n)}=\varepsilon_{i\nu}^{(n)}-\varepsilon_{i\nu+1}^{(n)}$. Hence by formula (1.2)

$$\|f_n(\cdot,\cdot)\|_{(r),(r)}=\sup_{\|x\|\leqslant 1}\sum_{\nu=1}^{\infty}\Big|\sum_{i=1}^{\infty}\delta_{i\nu}^{(n)}\xi_i\Big|. \tag{4.1}$$

Let the set P_n be a rectangle: $\alpha\leqslant i\leqslant\beta$, $\lambda\leqslant k\leqslant\mu$. Then: for $\lambda=1$ we have

$$\delta_{i\mu}^{(n)}=1 \quad\text{for}\quad i=\alpha,\alpha+1,\dots,\beta,$$
$$\delta_{ik}^{(n)}=0 \quad\text{for}\quad k\neq\mu,$$

and if $\lambda>1$

$$\delta_{i\lambda-1}^{(n)}=-1,\qquad \delta_{i\mu}^{(n)}=1\quad\text{for}\quad i=\alpha,\alpha+1,\dots,\beta,$$
$$\delta_{ik}^{(n)}=0\quad\text{for}\quad k\neq\lambda-1,\ k\neq\mu.$$

The formula (4.1) gives

$$\|f_n(\cdot,\cdot)\|=\sup_{\|x\|\leqslant 1}\sum_{\nu=1}^{\infty}\Big|\sum_{i=1}^{\infty}\delta_{i\nu}^{(n)}\xi_i\Big|\leqslant 2\sup_{\|x\|\leqslant 1}\Big|\sum_{i=\alpha}^{\beta}\xi_i\Big|\leqslant 4. \tag{4.2}$$

Suppose now that the set P_n is decomposed into $m=\vartheta(P_n)$ non-overlapping rectangles $Q_1, Q_2, \ldots, Q_m$. Write $\varepsilon_{ik}^{(nj)}=\varepsilon_{ik}^{(n)}$ for $(i,k)\in Q_j$ and $\varepsilon_{ik}^{(nj)}=0$ elsewhere. The formula $f_{nj}(x,y)=\sum_{i,k=1}^{\infty}\varepsilon_{ik}^{(nj)}\xi_i\eta_k$ defines a bilinear functional in $(r)\times(r)$ and we have

$$f_n(x,y)=\sum_{j=1}^{m}f_{nj}(x,y).$$

Since by (4.2) $\|f_{nj}(\cdot,\cdot)\|_{(r),(r)}\leqslant 4$, we get $\|f_n(\cdot,\cdot)\|_{(r),(r)}\leqslant 4m$. Hence by theorem 1 and lemma 4 we obtain the following statement:

If $\overline{\lim_{n\to\infty}}\,\vartheta(P_n)<+\infty$ *then the method* $(\mathfrak{R})$ *is perfect.*

Now we establish the converse proposition.

Denote by $\boldsymbol{D}$ the set of all infinite matrices $\Delta=(\delta_{ik})$ such that $\delta_{ik}=0,\pm 1$ and almost all $\delta_{ik}=0$, and consider the family of all functionals in (r) of the form

$$\varphi(x)=\varphi_\Delta(x)=\sum_{k=1}^{\infty}\Big|\sum_{i=1}^{\infty}\delta_{ik}\xi_i\Big|$$

where $\Delta\in\boldsymbol{D}$. Given any $\Delta=(\delta_{ik})\in\boldsymbol{D}$, let $\mu(\Delta)$ denote the number of those k for which there exists an i such that $\delta_{ik}\neq 0$; put also

$$\omega_i(\Delta)=|\delta_{i1}|+|\delta_{i2}|+\ldots,\qquad \omega(\Delta)=\max_{i=1,2,\ldots}\omega_i(\Delta).$$

Lemma 5. *Given any numbers n and p there exists a number $\varrho(n,p)$ satisfying the following condition:*

If Δ is an element of $\boldsymbol{D}$ such that $\mu(\Delta)\geqslant\varrho(n,p)$ and $\omega(\Delta)\leqslant p$, then there exists an element $x=\{\xi_i\}\in(r)$ for which $\xi_i=0,\pm 1$; $\|x\|\leqslant 1$ and $\varphi_\Delta(x)\geqslant n$.

Proof. We will prove this lemma by induction. It is obvious that for $n=1$ and for arbitrary p we may put $\varrho(1,p)=1$. Suppose the lemma to be true for $n-1$ and for arbitrary p. Write $\varrho(n,p)=\varrho(n-1,p)+p$ and suppose that $\mu(\Delta)\geqslant\varrho(n,p)$, $\omega(\Delta)\leqslant p$. Let i_0 be the least integer i such that $\omega_i(\Delta)\neq 0$; thus $\delta_{ik}=0$ for $i<i_0$. Since $\omega_i(\Delta)\leqslant p$ there exist at most p positive integers $k_1, k_2, \ldots, k_{p'}$ $(p'\leqslant p)$ with $\delta_{i_0k_1}\neq 0$, $\delta_{i_0k_2}\neq 0, \ldots, \delta_{i_0k_{p'}}\neq 0$ and $\delta_{i_0k}=0$ for $k\neq k_1, k_2, \ldots, k_{p'}$. Let Δ_1 be the matrix obtained from Δ by removing the rows $1,2,\ldots,i_0-1$ and the columns $k_1, k_2, \ldots, k_{p'}$. It is clear that $\mu(\Delta_1)\geqslant\varrho(n-1,p)$, $\omega(\Delta_1)\leqslant p$. By the assumption

there exists an element $x'=\{\xi_i'\}\in(r)$ such that $\|x'\|\leqslant 1$, $\xi_i'=0,\pm 1$ and $\varphi_{\Delta_1}(x')\geqslant n-1$, i. e.

$$\sum_{k\neq k_j}\Big|\sum_{i=i_0+1}^{\infty}\delta_{ik}\xi_i'\Big|\geqslant n-1.$$

We may suppose that $\xi_1'=\xi_2'=\ldots=\xi_{i_0}'=0$. If there exists a k_j $(1\leqslant j\leqslant p)$ for which $a=\big|\sum_{i=1}^{\infty}\delta_{ik_j}\xi_i'\big|\neq 0$, we have $a\geqslant 1$, from which $\varphi_\Delta(x')\geqslant n$. Suppose now that $\big|\sum_{i=1}^{\infty}\delta_{ik_j}\xi_i'\big|=0$ for $j=1,2,\ldots,p'$. For any $x=\{\xi_i\}\in(r)$ we have

$$\sum_{k\neq k_j}\Big|\sum_{i=i_0}^{\infty}\delta_{ik}\xi_i\Big|=\sum_{k\neq k_j}\Big|\delta_{i_0k}\xi_{i_0}+\sum_{i=i_0+1}^{\infty}\delta_{ik}\xi_i\Big|=\sum_{k\neq k_j}\Big|\sum_{i=i+1}^{\infty}\delta_{ik}\xi_i\Big|. \tag{4.3}$$

Denote by ξ_α' the first non-vanishing element in the sequence $\{\xi_i'\}$. Put $\xi_{i_0}=1$ if $\xi_\alpha'=-1$, $\xi_{i_0}=-1$ if $\xi_\alpha'=+1$, $\xi_1=\xi_2=\ldots=\xi_{i_0-1}=0$, and $\xi_i=\xi_i'$ for $i=i_0+1, i_0+2, \ldots$ We see that $\|x\|\leqslant 1$ for $x=\{\xi_n\}$, and since $\delta_{i_0k_1}\neq 0$, we have

$$\Big|\sum_{i=1}^{\infty}\delta_{ik_1}\xi_i\Big|=\Big|\delta_{i_0k_1}\xi_{i_0}+\sum_{i=i_0+1}^{\infty}\delta_{ik_1}\xi_i\Big|=1.$$

By (4.3) we get

$$\varphi_\Delta(x)=\sum_{k=1}^{\infty}\Big|\sum_{i=1}^{\infty}\delta_{ik}\xi_i\Big|\geqslant\Big|\sum_{i=1}^{\infty}\delta_{ik_1}\xi\Big|+\sum_{k\neq k_j}\Big|\sum_{i=1}^{\infty}\delta_{ik}\xi_i\Big|=$$

$$=1+\sum_{k\neq k_j}\Big|\sum_{i=i_0+1}^{\infty}\delta_{ik}\xi_i\Big|\geqslant 1+(n-1)=n.$$

Thus the lemma is proved.

Theorem 4. *The method* $(\mathfrak{R})$ *is perfect if, and only if*

$$\overline{\lim_{n\to\infty}}\,\vartheta(P_n)<+\infty\ [5].$$

Proof. The sufficiency of this condition has already been proved. Now let the method $(\mathfrak{R})$ be perfect. Suppose that

$$\overline{\lim_{n\to\infty}}\,\vartheta(P_n)=+\infty. \tag{4.4}$$

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[5]) This theorem was proved by Rado, loc. cit.

Put $\delta_{ik}^{(n)}=\varepsilon_{ik}^{(n)}-\varepsilon_{ik+1}^{(n)}$, $\delta'^{(n)}_{ik}=\varepsilon_{ik}^{(n)}-\varepsilon_{i+1\,k}^{(n)}$, $\Delta_n=(\delta_{ik}^{(n)})$, $\Delta'_n=(\delta'^{(n)}_{ik})$.

By formula (4.1) we have

$$\|f_n(\cdot,\cdot)\|_{(r),(r)}=\sup_{\|x\|\leqslant 1}\varphi_{\Delta_n}(x). \tag{4.5}$$

We observe easily that the inequalities $\mu(\Delta_n)\leqslant c$ and $\mu(\Delta'_n)\leqslant c$ imply $\vartheta(P_n)\leqslant 4c^2$; thus the supposition (4.4) implies $\overline{\lim\limits_{n\to\infty}}\mu(\Delta_n)+\overline{\lim\limits_{n\to\infty}}\mu(\Delta'_n)=+\infty$. We may suppose without loss of generality that $\overline{\lim\limits_{n\to\infty}}\mu(\Delta_n)=+\infty$. Any perfect method having Mertens' property, we see that there exists a p such that $\omega(\Delta_n)\leqslant p$ for $n=1,2,\ldots$ Hence by lemma 5 and (4.5) we get

$$\overline{\lim_{n\to\infty}}\|f_n(\cdot,\cdot)\|_{(r),(r)}=+\infty.$$

This is, however, impossible since by lemma 3 the sequence $\{\|f_n(\cdot,\cdot)\|_{(r),(r)}\}$ must be bounded. Thus, we have shown that $\overline{\lim\limits_{n\to\infty}}\vartheta(P_n)<+\infty$.

It is quite obvious that all the considerations of this paper are valid also for series with complex elements.

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An example in Fourier series

by

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1. Let $a_1, a_2, \ldots$ be a sequence of real numbers such that $\sum a_k^2$ converges, and let $n_1, n_2, \ldots$ be a sequence of positive integers such that $n_{k+1}/n_k\geqslant 3$ for all k. The partial products $p_k(x)$ of the infinite product

$$\prod_{\nu=1}^{\infty}(1+ia_\nu\cos n_\nu x) \tag{1}$$

are trigonometric polynomials,

$$p_k=\prod_{\nu=1}^{k}(1+ia_\nu\cos n_\nu x)=1+\sum_{\nu=1}^{\mu_k}\gamma_\nu\cos\nu x, \qquad \mu_k=n_k+\ldots+n_2+n_1,$$

where the γ_ν are either real or purely imaginary, since all the terms obtained from multiplying out p_k are distinct. The passage from p_k to p_{k+1} consists in adding to p_k the polynomial $p_{k+1}-p_k$ whose all terms are of rank $>\mu_k$. Hence, making $k\to\infty$, we obtain, formally, a trigonometric series

$$1+\sum_{\nu=1}^{\infty}\gamma_\nu\cos\nu x, \tag{2}$$

the partial sums $s_n(x)$ of which have the property that $s_{\mu_k}=p_k$.

The series (2) may be said to *represent* the product (1).

Since

$$|p_k(x)|\leqslant\prod_{\nu=1}^{k}(1+a_\nu^2)^{\frac{1}{2}}\leqslant\prod_{\nu=1}^{\infty}(1+a_\nu^2)^{\frac{1}{2}}<+\infty,$$

a sequence of the partial sums of the series (2) is uniformly bounded. This shows that (2) is the Fourier series of a bounded function $f(x)$.