Si l'on avait $\frac{\psi''(x)}{\psi'(x)} \equiv \frac{\chi''(x)}{\chi'(x)}$ dans un intervalle $x' \leqslant x \leqslant x''$, on aurait par suite $\psi(x) = \alpha\chi(x) + \beta$ et les fonctions $\psi(x)$ et $\chi(x)$ seraient également convexes en logarithme dans cet intervalle. Cela prouve que la condition est nécessaire.

Admettons maintenant que l'inégalité (5) a lieu dans un ensemble partout dense. En intégrant cette inégalité deux fois, il vient

$$\frac{\psi(x)-\psi(x_1)}{\psi'(x_1)} < \frac{\chi(x)-\chi(x_1)}{\chi'(x_1)} \qquad \text{pour } x > x_1.$$

Remarquons que si les dérivées $\psi'(x)$ et $\chi'(x)$ sont positives et l'on a $\psi(x_1) = \chi(x_1)$ et $\psi(x_2) = \chi(x_2)$ où $x_1 < x_2$, alors $\chi'(x_1) < \psi'(x_1)$, ce qui entraîne $\psi(x) < \chi(x)$ au voisinage gauche de x_1 et $\chi(x) < \psi(x)$ au voisinage droit de ce point. Cette remarque nous sera utile tout à l'heure.

Fixons arbitrairement deux points x_1, x_2 (où $x_1 < x_2$) et désignons par $\bar\psi(x)$ et $\bar\chi(x)$ les fonctions correspondantes, normées dans ces points (voir N° 4). Il suffit de montrer que

(6) $$\bar\chi(x) < \bar\psi(x)$$

entre x_1 et x_2. Or, d'après la remarque que nous venons de faire, cette inégalité est vraie au voisinage droit de x_1. Désignons par (x_1, x_1') le plus grand intervalle, où l'inégalité (6) est encore satisfaite. En vertu de la continuité des fonctions considérées on a $(\bar\chi x_1') = \bar\psi(x_1')$. Si $x_1' < x_2$, on aurait, en appliquant aux points x_1' et x_2 la remarque précédente, $\bar\psi(x) < \bar\chi(x)$ au voisinage gauche de x', ce qui est impossible. On a donc $x_1' = x_2$ et la suffisance de la condition est aussi démontrée.

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On certain methods of summability associated with conjugate trigonometric series

by

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1. A number of methods of summability of numerical series

(1) $$u_0 + u_1 + u_2 + \ldots + u_n + \ldots$$

have their origin in the theory of trigonometric series. The most familiar of these methods is the method of Riemann. It consists in treating (1) as the series

(2) $$u_0 + \sum_{n=1}^{\infty} u_n \cos nx$$

at the point $x=0$. If we integrate (2) termwise k times and take the generalised k-th symmetric derivative at the point $x=0$ of the resulting function, the value of this derivative, if it exists, equals

(3) $$\lim_{\alpha\to 0} \left[u_0 + \sum_{n=1}^{\infty} u_n \left(\frac{\sin n\alpha}{n\alpha} \right)^k \right].$$

Correspondingly, we say that the series (2) is summable by the method (R,k), $k=1,2,3,\ldots$, to sum s, if the series in (3) converges for $|\alpha|$ small enough, and if the limit (3) exists and equals s. The cases $k=1,2$ are the most familiar ones.

In this note we are going to discuss another method of summability suggested by trigonometric series. As in the Riemann case, we have a whole sequence of methods corresponding to $k=1,2,\ldots$, but in this note we shall confine our attention to the cases $k=1$ and $k=2$ only.

Given a trigonometric series

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\equiv\sum_{n=0}^{\infty}A_n(x), \tag{4}$$

we also consider its conjugate series

$$0+\sum_{n=1}^{\infty}(a_n\sin nx-b_n\cos nx)\equiv\sum_{0}^{\infty}B_n(x). \tag{5}$$

If (4) is the Fourier series of a function f, the series (5) represents under certain conditions the conjugate function

$$F(x)=-\frac{1}{\pi}\int_0^{\pi}\frac{f(x+t)-f(x-t)}{2\tan\frac{1}{2}t}\,dt=-\frac{1}{\pi}\lim_{\alpha\to+0}\int_{\alpha}^{\pi}. \tag{6}$$

The existence of this (improper) integral can be interpreted as a method of summability of the series (5). If we replace f in (6) by its Fourier series and observe that Fourier series can be integrated term by term after having been multiplied by any function of bounded variation, we see that the existence of $f(x)$ is equivalent to the existence of

$$\lim_{\alpha\to+0}\sum_{n=1}^{\infty}B_n(x)\frac{2}{\pi}\int_{\alpha}^{\pi}\frac{\sin nt}{2\tan\frac{1}{2}t}\,dt. \tag{7}$$

If (4) is a Fourier-Lebesgue series, the series in (7) converges for every $0<\alpha\leqslant\pi$.

Let us now identify (5) with (2). We may say that the series (1) is summable by the method $(K,1)$ to sum s, if

(i) the series

$$\sum_{n=1}^{\infty}u_n\frac{2}{\pi}\int_{\alpha}^{\pi}\frac{\sin nt}{2\tan\frac{1}{2}t}\,dt$$

converges for α positive and small enough;

(ii) the limit of

$$u_0+\sum_{n=1}^{\infty}u_n\frac{2}{\pi}\int_{\alpha}^{\pi}\frac{\sin nt}{2\tan\frac{1}{2}t}\,dt \tag{8}$$

for $\alpha\to+0$ exists and equals s.

The Second Mean-Value Theorem shows that the coefficient of u_n in (8) is $O(1/n\alpha)$, which suggests that the range of applicability of the method $(K,1)$ may resemble that of $(R,1)$. In particular, the convergence of (1) need not imply its summability $(K,1)$.

Since

$$\int_0^{\pi}\frac{\sin nt}{2\tan\frac{1}{2}t}\,dt=\int_0^{\pi}\left(\frac{1}{2}+\cos t+\cos 2t+\ldots+\frac{1}{2}\cos nt\right)dt=\frac{1}{2}\pi, \tag{9}$$

the coefficient of every u_n in (8) approaches 1 as $\alpha\to+0$.

It must be added that summability $(K,1)$ is implicit in Hardy and Littlewood [1]. Their result (although it was given for a slightly different definition of summability $(K,1)$) can be stated as follows:

If (1) is summable $(K,1)$ to sum s, and if $u_n>-A/n$, then (1) converges to sum s.

Let us merely assume that (1) has terms tending to 0.

Integration by parts gives

$$\int_{\alpha}^{\pi}\frac{\sin nt}{2\tan\frac{1}{2}t}\,dt=\frac{\cos n\alpha}{n\,2\tan\frac{1}{2}\alpha}-n^{-1}\int_{\alpha}^{\pi}\frac{\cos nt}{(2\sin\frac{1}{2}t)^2}\,dt=$$

$$=\frac{\cos n\alpha}{n\,2\tan\frac{1}{2}\alpha}+O(n^{-2}\alpha^{-2}),$$

and since trigonometric series with coefficients $o(1/n)$ converge almost everywhere, the series (8) converges for almost every α in the interval $0<\alpha<\pi$. Let us consider any α for which (8) has meaning. Summing by parts and taking into account that $s_n=o(n)$, we find that the sum of (8) is

$$\lim_{N\to\infty}\left[s_0+\sum_{n=1}^{N}(s_n-s_{n-1})\frac{2}{\pi}\int_{\alpha}^{\pi}\frac{\sin nt}{2\tan\frac{1}{2}t}\,dt\right]=$$

$$=\lim_{N\to\infty}\left[\frac{s_0}{\pi}(\alpha+\sin\alpha)-\sum_{n=1}^{N}s_n\frac{2}{\pi}\int_{\alpha}^{\pi}\frac{\sin nt-\sin(n+1)t}{2\tan\frac{1}{2}t}\,dt\right]=$$

$$=\lim_{N\to\infty}\left[\frac{s_0}{\pi}(\alpha+\sin\alpha)-\sum_{n=1}^{N}s_n\frac{1}{\pi}\int_{\alpha}^{\pi}(\cos nt+\cos(n+1)t)\,dt\right]$$

or is

$$\frac{s_0}{\pi}(\alpha+\sin\alpha)+\sum_{n=1}^{\infty}\frac{s_n}{\pi}\left[\frac{\sin n\alpha}{n}+\frac{\sin(n+1)\alpha}{n+1}\right]. \tag{10}$$

If all the s_n are replaced by s, the last sum is identically s.

Let us now assume that (1) converges to sum s. By the last remark we may assume without loss of generality that $s=0$, i. e. that $s_n\to 0$. It is howerer, known that in this case the expression

$$\sum_{n=1}^{\infty} s_n\frac{\sin n\alpha}{n},\qquad \text{and so also}\qquad \sum_{n=1}^{\infty} s_n\frac{\sin(n+1)\alpha}{n+1},$$

tends asymptotically to 0 as $\alpha\to 0$ (See Rajchmann and Zygmund [3]). This means that the expressions tend to 0 as α approches 0 remaining in a set having 0 as a point of density 1. Hence,

Theorem 1. *If the series* (1) *converges to sum* s, *it is also summable* $(K_{as}, 1)$ *to sum* s.

In particular,

Theorem 2. *If* (5) *is the series conjugate to the Fourier series of a function* f, *and if it converges at the point* x *to sum* s, *then*

$$-\frac{1}{\pi}\int_{\alpha}^{\pi}\frac{f(x+t)-f(x-t)}{2\tan\frac{1}{2}t}\,dt$$

tends to s *as* α *tends to* $+0$ *trough a set of values having density* 1 *at* 0.

2. Let us consider a trigonometric series (4) with constant term a_0 equal to 0, and let us assume that the series

$$\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{n}=\sum_{n=1}^{\infty}\frac{A_n(x)}{n} \tag{11}$$

is the Fourier series of a function $F(x)$. Thus (4) is the conjugate of the termwise differentiated Fourier series of F. It is well known (see e. g. Zygmund [4], p. 62, Ex. 13) that under certain conditions the sum (ordinary or generalized) of the series conjugate to the formally differentiated Fourier series of a function $F(x)$ is

$$-\frac{1}{\pi}\int_{0}^{\pi}\frac{F(x+t)+F(x-t)-2F(x)}{(2\sin\frac{1}{2}t)^2}\,dt=-\frac{1}{\pi}\lim_{\alpha\to+0}\int_{\alpha}^{\pi}, \tag{12}$$

and the existence of this integral will, by definition, mean summability $(K, 2)$ of the series (4). Substituting for F into the second integral (12) its Fourier series, we obtain the expression

$$\sum_{n=1}^{\infty} A_n(x)\frac{1}{\pi n}\int_{\alpha}^{\pi}\frac{4\sin^2\frac{1}{2}nt}{4\sin^2\frac{1}{2}t}\,dt$$

converging (if (11) does) for every $0<\alpha<\pi$. One notices a resemblance of the last integrand to Fejér's kernel K_n defined by the formula

$$K_n(t)=\frac{1}{n+1}\sum_{\nu=0}^{n}D_\nu(t)=\frac{1}{n+1}\frac{\sin^2\frac{1}{2}(n+1)t}{2\sin^2\frac{1}{2}t},$$

where

$$D_\nu(t)=\frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}$$

is Dirichlet's kernel.

In accordance with this, we say that the series (1) is summable $(K, 2)$ to sum s, if:

(i) the series

$$u_0+\sum_{n=1}^{\infty}u_n\frac{2}{\pi n}\int_{\alpha}^{\pi}\frac{\sin^2\frac{1}{2}nt}{2\sin^2\frac{1}{2}t}\,dt=u_0+\sum_{n=1}^{\infty}u_n\frac{2}{\pi}\int_{\alpha}^{\pi}K_{n-1}(t)\,dt \tag{13}$$

converges for α positive and small enough;

(ii) the expression (13) tends to s as $\alpha\to+0$.

Thus summability $(K, 2)$ of the series (2), which for $x=0$ reduces to (1), consists in treating it as the conjugate of a termwise differentiated, Fourier series. Since the integral of $K_n(t)$ over $(0, \pi)$ is $\pi/2$, the factor of each u_n in (13) tends to 1 as $\alpha\to+0$.

One easily sees that

$$\int_{\alpha}^{\pi}\frac{\sin^2\frac{1}{2}nt}{2\sin^2\frac{1}{2}t}\,dt=\int_{\alpha}^{\pi}\frac{1-\cos nt}{(2\sin\frac{1}{2}t)^2}\,dt=\frac{1}{2\tan\frac{1}{2}\alpha}+O\left(\frac{1}{n\alpha^2}\right). \tag{14}$$

Since the mere condition $u_n=o(1)$ need not imply the convergence of $\sum u_n/n$, condition (i) need not be satisfied by the general series $\sum u_n$ with terms tending to 0. Nevertheless.

Theorem 3. *Every series convergent to sum s is summable $(K,2)$ to s.*

In particular,

Theorem 4. *If the series (4) converges at a point x, to sum s, and if (11) is the Fourier series of a function F, then the integral (12) exists and equals s.*

Proof. Since the convergence of $\sum u_n$ implies that of $\sum n^{-1}u_n$, condition (i) for summability $(K,2)$ is satisfied (see (14)). Moreover,

$$\lim_{N\to\infty}\left\{u_0+\sum_{n=1}^{N}u_n\frac{2}{\pi}\int_\alpha^\pi K_{n-1}(t)\,dt\right\}=$$

$$=\lim_{N\to\infty}\left\{u_0+\sum_{n=1}^{N}(s_n-s_{n-1})\frac{2}{\pi}\int_\alpha^\pi K_{n-1}(t)\,dt\right\}=$$

$$=u_0\cdot\left(1-\frac{2}{\pi}\int_\alpha^\pi K_0(t)\,dt\right)+\frac{2}{\pi}\sum_{n=1}^{\infty}s_n\int_\alpha^\pi(K_{n-1}-K_n)\,dt.$$

Thus we get a linear transformation of $\{s_n\}$, and it is enough to verify the three conditions of Toeplttz. That the elements in each column tend to 0 is obvious. It is also immediate that the sum of elements in each row is 1. It is therefore enough to show that the sum of the absolute values of the elements in each row remains bounded as $\alpha\to+0$. Let A denote a positive obsolute constant not necessarily always the same, and let $\Delta_n=K_{n-1}-K_n$. From the formula for K_n we get

$$\Delta_n=\frac{K_{n-1}}{n+1}-\frac{D_n}{n}. \tag{15}$$

Since $D_n(t)=O(n)$, $K_n(t)=O(n)$, uniformly in t, we immediately obtain

$$\left|\int_\alpha^\pi\Delta_n\,dt\right|=\left|\int_0^\alpha\Delta_n\,dt\right|\leqslant A\alpha \qquad (0<\alpha\leqslant\pi). \tag{16}$$

Also, from (15) and from the Second and First Mean-Value Theorems we obtain

$$\left|\int_\alpha^\pi\Delta_n\,dt\right|<\frac{A}{n^2\alpha}. \tag{17}$$

Using (16) and (17) we see that

$$\sum_{n=1}^{\infty}\left|\int_\alpha^\pi\Delta_n\,dt\right|=\sum_{n\leqslant 1/\alpha}+\sum_{n>1/\alpha}=\Sigma'+\Sigma'',$$

$$\Sigma'\leqslant\sum_{n\leqslant 1/\alpha}A\alpha<A, \qquad \Sigma''\leqslant\sum_{n>1/\alpha}\frac{A}{n^2\alpha}<A.$$

Thus the required condition of Toeplitz is satisfied and Theorem 3 follows.

3. Theorem 5. *If (1) is summable $(C,-\delta)$, $0<\delta<1$, to sum s, it is also summable $(K,1)$ to s.*

Theorem 6. *If (1) is summable $(C,1-\delta)$, $0<\delta<1$, to sum s, it is also summable $(K,2)$ to s.*

Theorem 7. *If (1) is summable $(C,1)$ to sum s, it is also summable $(K_{as},2)$ to s.*

These results are stated without proofs since, though not immediate, these are similar to the proofs of the corresponding results for summabilities $(R,1)$ and $(R,2)$. See Hardy and Littlewood [2] (Theorem 2 there) and Rajchman and Zygmund [3].

To the problem of summability (K,r) for $r>2$ we shall return on another occasion.

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