# Remark on measures in almost-independent fields

by

HENRY HELSON (Cambridge, Mass.).

Let $X$ be a set of points and let $\boldsymbol{M}_t$ be a family of fields of sub-sets of $X$, where $t$ runs through some set $S$. Suppose a measure $\mu_t$ is defined in each $\boldsymbol{M}_t$, such that $\mu_t(X)=1$ for each $t$.

The measure $\mu$ defined in the smallest field $\boldsymbol{M}$ containing all the $\boldsymbol{M}_t$ will be called a *stochastic extension* of the $\mu_t$ if

$$\mu(A_1\cdot\ldots\cdot A_n)=\mu_{t_1}(A_1)\cdot\ldots\cdot\mu_{t_1}(A_n) \quad \text{whenever} \quad A_i\,\epsilon\,\boldsymbol{M}_{t_i} \quad (i=1,\ldots,n).$$

It is easy to see that the stochastic extension is uniquely defined if there is one. In particular $\mu(A)=\mu_s(A)$ for $A\,\epsilon\,\boldsymbol{M}_s$, every $s$.

The following theorems have been proved [1]):

(i) *The $\mu_t$ have a stochastic extension $\mu$ if and only if the fields $M_t$ are almost-independent with respect to the measures $\mu_t$* [2]).

(ii) *If the $M_t$ are denumerably independent, and each $\boldsymbol{M}_t$ and $\mu_t$ is denumerably additive, then $\mu$ is denumerably additive* [3]).

Marczewski has shown [4]) that independence is not a necessary condition for the denumerable additivity of $\mu$, and that at least for the case of a finite number of fields, weaker conditions are sufficient. Two conditions are suggested for study [5]), both of which reduce to almost-independence for a finite number of fields.

The purpose of this note [6]) is to show that neither of these conditions is strong enough to replace independence in the hypothesis of (ii), by giving an example of two denumerably additive, almost-independent fields $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, whose measures $\mu_1$, $\mu_2$ are denumerably additive, such that the stochastic extension $\mu$ of $\mu_1$, $\mu_2$ is not denumerably additive.

The construction is based on a set $T$ constructed by Steinhaus, with the following properties: $T$ is a subset of the unit square $I^2$, has plane measure less than 1, but intersects every Cartesian product $A\times B$, where $A$ and $B$ are linear sets of positive measure. Let, namely, $T$ be the set of points $(x,y)$ of $I^2$ such that $y-x$ is rational. By a theorem of Steinhaus [6]), the linear set of points of the form $y-x$ for $x\,\epsilon\,A$, $y\,\epsilon\,B$, covers some interval, and hence contains a rational point, provided $A$ and $B$ have positive measure, so that $T$ actually intersects $A\times B$. (Steinhaus has used the same principle to construct a number of sets with additional interesting properties).

Let the elements of $\boldsymbol{M}_1$ be the plane sets of the form $T\cdot(A\times I)$, and the elements of $\boldsymbol{M}_2$ — the sets of the form $T\cdot(I\times B)$, where $I$ is the unit interval, and $A$ and $B$ are measurable subsets of $I$. Set $\mu_1[T\cdot(A\times I)]=m(A)$, $\mu_2[T\cdot(I\times B)]=m(B)$, where $m$ is linear Lebesgue measure. It is easy to see that the measures are defined uniquely, and are denumerably additive. Furthermore, $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are almost-independent.

Since the measure of $T$ is smaller than 1, we can find a sum $\sum_{i=1}^{\infty}A_i\times B_i$ of disjoint measurable terms, covering $T$ and of measure smaller than 1. Assuming $\mu$, the stochastic extension of $\mu_1,\mu_2$, denumerably additive, we should have

$$T\subset\sum_{i=1}^{\infty}A_i\times B_i=\sum_{i=1}^{\infty}(A_i\times B_i)\cdot T=\sum_{i=1}^{\infty}T\cdot(A_i\times I)\cdot(I\times B_i),$$

and hence

$$\mu(T)\leqslant\sum_{i=1}^{\infty}m(A_i)\cdot m(B_i)<1.$$

However, $T=T\cdot(I\times I)$, so that $\mu(T)=m(I)=1$. This contradiction shows that $\mu$ is not denumerably additive.

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1) Definitions are given by E. Marczewski, *Indépendance d'ensembles et prolongement de mesures*, Colloquium Mathematicum I (1948), pp. 122-132.

2) E. Marczewski, *Mesures dans les corps presque indépendants*. Fundamenta Mathematicae 36 (to appear).

3) Proved in the paper of S. Banach, this volume, p. 159-177.

4) E. Marczewski, Colloquium Mathematicum, loco cit., p. 130.

5) Cf. ibidem, p. 130, P 24.

6) Written at the University of Wrocław while the author held a Sheldon Travelling Fellowship from Harvard University.

6) H. Steinhaus, *Sur les distances des points des ensembles de mesure positive*, Fundamenta Mathematicae 1 (1920), pp. 93-104, Théorème VII.



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