on a en vertu du théorème 5

$$\overline{\lim_{n\to\infty}}\,|a_n\cos\omega_n x+b_n\sin\omega_n x|=\overline{\lim_{n\to\infty}}\sqrt{a_n^2+b_n^2}$$

partout sauf un ensemble H_σ [7]), ce qui montre que la constante c dans la formule (12) est la meilleure possible.

6. L'hypothèse admise dans toutes les considérations de ce travail, à savoir que les fonctions envisagées sont des fonctions de Baire, n'est pas essentielle.

Le lecteur aura remarqué aisément qu'elle peut être remplacée aussi bien dans la définition de $\sup_B f(x)$ que dans les énoncés des théorèmes 1, 4, 5 et 6 par l'hypothèse que les fonctions en question satisfont à la condition de Baire. On n'a pas besoin de modifier la marche des démonstrations. Il faut seulement supprimer dans les thèses des théorèmes l'affirmation que les ensembles de I[e] catégorie dont il s'agit dans ces théorèmes sont boreliens.

[7]) C'est un théorème connu de H. Steinhaus; voir par exemple A. Zygmund, op. cit., p. 269.

(Reçu par la Rédaction le 10. III. 1948).

On measures in independent fields

by

S. BANACH †

Edited by S. HARTMAN.

Among the papers left by BANACH was found the incomplet Polish manuscript of this paper, written in 1940. § 1 is almost literally translated from the manuscript. The details of farther reasonings were elaborated by S. HARTMAN, who also supplied the paper with Appendices and adapted it for print, with some help of HENRY HELSON.

§ 1. Let T be an arbitrary space. A family $\mathfrak{K}$ of fields [1]) of sub-sets of T is said to be a family of *independent* fields if any finite number of non-empty sets, belonging to different fields of $\mathfrak{K}$, has a non-empty intersection. That is, $\mathfrak{K}$ is an independent family if the conditions $0\neq H_i\in A_i\in\mathfrak{K}$ and $A_i\neq A_j$ for $i\neq j$ $(i,j=1,\ldots,n)$ always imply $\prod_{i=1}^{n}H_i\neq 0$.

The family $\mathfrak{K}$ is called a family of *denumerably independent* fields if any sequence of non-empty sets, belonging to different fields of $\mathfrak{K}$, has a non-empty intersection; i. e. if $0\neq H_i\in A_i\in\mathfrak{K}$ and $A_i\neq A_j$ for $i\neq j$ $(i,j=1,2,\ldots)$ always imply $\prod_{i=1}^{\infty}H_i\neq 0$.

The concept of independence of fields of sets was introduced by MARCZEWSKI [2]), who also proved the following theorem [3]):

[1]) The class A of sub-sets of a space T is called a *field* if A contains with any set its complement and with any finite number of sets their sum. The field A is a *Borel field* if the sum of any denumerable number of sets of A belongs to A.

[2]) Cf. E. Marczewski, *Indépendance d'ensembles et prolongement de mesures (Résultats et problèmes)*, Colloquium Mathematicum I. 2, Wrocław 1948, p. 122-132, especially p. 125-127.

[3]) Ibidem, Théorème II, p. 126-127. For the proof of this theorem see E. Marczewski, *Mesures dans les corps presque indépendants*, Fundamenta Mathematicae 36 (to appear).

Let $\mathfrak{K}$ be a family of independent fields with a measure[4]) μ *defined in each field $\boldsymbol{A}\in\mathfrak{K}$, and let $\boldsymbol{U}(\mathfrak{K})$ be the smallest field containing all the fields of $\mathfrak{K}$. Then a measure μ^* can be defined in $\boldsymbol{U}(\mathfrak{K})$ with the following properties:*

(I) $\mu^*(H)=\mu(H)$ *if* $H\in\boldsymbol{A}\in\mathfrak{K}$,

(II) $\mu^*\left(\prod_{i=1}^{n}H_i\right)=\prod_{i=1}^{n}\mu^*(H_i)$ *if* $H_i\in\boldsymbol{A}_i\in\mathfrak{K}, \boldsymbol{A}_r\neq\boldsymbol{A}_s$ *for* $r\neq s$ *and* n *a natural number not greater than the power of* $\mathfrak{K}$ *(which can be finite)*[5]).

Marczewski has asked whether the following theorem is true[6]):

Theorem 1. *Let $\mathfrak{K}$ be a family of denumerably independent Borel fields with a denumerably additive measure μ defined in each field $\boldsymbol{A}\in\mathfrak{K}$; then a denumerably additive measure μ^* can be defined in the smallest Borel field containing all the fields of $\mathfrak{K}$, such that:*

(1) $\mu^*(H)=\mu(H)$ *if* $H\in\boldsymbol{A}\in\mathfrak{K}$,

(2) $\mu^*\left(\prod_{i=1}^{\infty}H_i\right)=\prod_{i=1}^{\infty}\mu^*(H_i)$ *if* $H_i\in\boldsymbol{A}_i\in\mathfrak{K}$, *and* $\boldsymbol{A}_i\neq\boldsymbol{A}_j$ *for* $i\neq j$ $(i,j=1,2,...)$.

The object of this paper is to answer the question affirmatively.

Theorem 1 was proved by Marczewski in the special case that every field $\boldsymbol{A}\in\mathfrak{K}$ contains just four sets[7]), viz. a set H, its complement, the empty set, and T. Then evidently every $\boldsymbol{A}\in\mathfrak{K}$ is a Borel field and any measure μ defined in $\boldsymbol{A}$ is denumerably additive. The theorem was enunciated by P. Lévy in another special case, namely when $\mathfrak{K}$ consists of two fields and the measures have a special form[8]).

[4]) A *measure* μ in a field $\boldsymbol{A}$ is a real function $\mu(H)\geqslant 0$ defined for every set $H\in\boldsymbol{A}$, such that $\mu(T)=1$ and $\mu(H_1+H_2)=\mu(H_1)+\mu(H_2)$ for any disjoint $H_1,H_2\in\boldsymbol{A}$. The measure is *denumerably* additive if $\mu\left(\sum_{i=1}^{\infty}H_i\right)=\sum_{i=1}^{\infty}\mu(H_i)$ for disjoint $H_1,H_2,...\in\boldsymbol{A}$.

[5]) Fields $\boldsymbol{A}_i$ for which condition (II) holds are said to be *stochastically independent* with respect to the measure μ^*.

[6]) loco cit[2], Théorème II_∞, especially p. 127.

[7]) Cf. E. Marczewski (Szpilrajn), *Ensembles indépendants et mesures non séparables*, Comptes rendus de l'Acad. des Sc. Paris 207 (1938), p. 768-770, especially Théorème II, p. 769, and E. Marczewski, *Ensembles indépendants et leurs applications à la théorie de la mesure*, Fundamenta Mathematicae 35 (1948), p. 13-28, especially II Théorème fondamental, p. 25.

§ 2. The smallest (finitely additive) field containing all the fields of $\mathfrak{K}$ will be denoted, as above, by $\boldsymbol{U}(\mathfrak{K})$. To prove Theorem 1 it is enough to define a measure μ^* in $\boldsymbol{U}(\mathfrak{K})$ satisfying (1), (2) and the following condition:

(3) *If* $H_i\in\boldsymbol{U}(\mathfrak{K})$ $(i=1,2,...)$ *are disjoint and if* $\sum_{i=1}^{\infty}H_i\in\boldsymbol{U}(\mathfrak{K})$, *then*

$$\mu^*\left(\sum_{i=1}^{\infty}H_i\right)=\sum_{i=1}^{\infty}\mu^*(H_i).$$

For then it is known that μ^* can be extended to a denumerably additive measure on the smallest Borel field over $\boldsymbol{U}(\mathfrak{K})$, i. e. the smallest Borel field containing all the fields of $\mathfrak{K}$.

Moreover, (3) can be replaced by the equivalent condition:

(4) *If* $H_i\in\boldsymbol{U}(\mathfrak{K})$, $H_i\supset H_{i+1}$ $(i=1,2,...)$ *and* $\lim_{i\to\infty}\mu^*(H_i)>0$, *then*

$$\prod_{i=1}^{\infty}H_i\neq 0.$$

Let $F[\boldsymbol{U}(\mathfrak{K})]$ be the family of all real functions defined for $t\in T$, assuming only a finite number of values, each value being assumed on a set belonging to $\boldsymbol{U}(\mathfrak{K})$. Every function $y\in F[\boldsymbol{U}(\mathfrak{K})]$ can be written in the following form (see Appendix I):

$$(5)\qquad y(t)=\sum a_{k_1...k_m}\prod_{j=1}^{m}z_{jk_j}(t).$$

Here the $z_{jk}(t)$ are the characteristic functions of non-empty sets Z_{jk}, which belong to different fields $\boldsymbol{A}_j\in\mathfrak{K}$ for different j, and which are disjoint in k for any field j. That is:

(α) $Z_{jk}\neq 0$,

(β) $Z_{jk}\in A_j\in K$, $\quad A_i\neq A_j$ for $i\neq j$,

(γ) $Z_{jk}\cdot Z_{jl}=0$, for $k\neq l$ and $j=1,2,...,m$.

[8]) See the book of P. Lévy, *Théorie de l'addition des variables aléatoires*, Paris 1937, p. 126 and 132.

The coefficients $a_{k_1\dots k_m}$ are real numbers, and the summation is extended over all systems $k_1\dots k_m$ which satisfy the conditions $1\leqslant k_j\leqslant r_j$ $(j=1,\dots,m)$.

§ 3. Uniformization. Let y_1 and y_2 be two (not necessarily different) functions from $F[\boldsymbol{U}(\mathfrak{K})]$. They are said to be *uniformized* when they are written

(6) $$y_1(t)=\sum a^{(1)}_{k_1\dots k_m}\prod_{j=1}^{m} z_{jk_j}(t),$$

(7) $$y_2(t)=\sum a^{(2)}_{k_1\dots k_m}\prod_{j=1}^{m} z_{jk_j}(t),$$

where the $z_{jk}(t)$ are the characteristic functions of sets Z_{jk} which satisfy conditions (α)-(γ). The summations in (6) and (7) are extended over the same systems of indices $k_1\dots k_m$ $(1\leqslant k_j\leqslant r_j)$. Thus the right sides of (6) and (7) can differ only in the coefficients $a_{k_1\dots k_m}$.

Refinement. Let y be a function given in the form (5), and let $\boldsymbol{B}_1,\boldsymbol{B}_2,\dots,\boldsymbol{B}_p$ $(p\geqslant m)$ be distinct fields of $\mathfrak{K}$, among which occur the $\boldsymbol{A}_j$. For each j suppose s_j non-empty disjoint sets $U_{ji}\in\boldsymbol{B}_j$ are given such that if $\boldsymbol{B}_j$ is identical with some $\boldsymbol{A}_l$; then every Z_{lk} is the sum of some of the U_{ji}; and if $\boldsymbol{B}_j$ is different from all the $\boldsymbol{A}_l$, the sum of the U_{ji} is T.

Then (5) can be transformed (see Appendix II, 1[0]) into the following:

(8) $$y(t)=\sum_{i_1\dots i_p} b_{i_1\dots i_p}\prod_{j=1}^{p} u_{ji_j}(t) \qquad (1\leqslant i_j\leqslant s_j),$$

where $u_{ji}(t)$ is the characteristic function of U_{ji}. We call (8) a *refinement* of (5) by the sets U_{ji}.

For two functions (not necessarily different) from $F[\boldsymbol{U}(\mathfrak{K})]$ given in the form (5), there always exists (see Appendix III) a system of sets U_{ji} by which both functions can be refined. Such a common refinement uniformizes the functions. Hence *any two functions of $F[\boldsymbol{U}(\mathfrak{K})]$ can be uniformized.*

Denumerable uniformization. If y_n is a sequence of functions belonging to $F[\boldsymbol{U}(\mathfrak{K})]$, in general no uniformization is possible for all the y_n in the sense defined above.

However the following representation can always be reached (see Appendix IV):

(9) $$y_n(t)=\sum a^{(n)}_{k_1\dots k_{m_n}}\prod_{j=1}^{m_n} z^{(n)}_{jk_j}(t) \quad \text{for} \quad n=1,2,\dots,$$

where $z^{(n)}_{jk}$ is the characteristic function of a non-empty set $Z^n_{jk}\in\boldsymbol{B}_j\in\mathfrak{K}$, with the sequence $\{\boldsymbol{B}_j\}$ (generally infinite) containing no field more than once. The numbering of the $\boldsymbol{B}_j$ is determined simultaneously for all n. Further, as usual, $Z^n_{jr}\cdot Z^n_{js}=0$ for $r\neq s$, and the summation in (9) is taken over all systems $k_1\dots k_{m_n}$ such that $1\leqslant k_j\leqslant r^{(n)}_j$. In general the sequence m_n is unbounded as n increases. The sequence y_n, given in the form (9), is said to be *denumerably uniformized.*

§ 4. Lemma 1. *Let two uniformized functions y_1 and y_2 of $F[\boldsymbol{U}(\mathfrak{K})]$ be given:*

$$y_1(t)=\sum a^{(1)}_{k_1\dots k_m}\prod_{j=1}^{m} z_{jk_j}(t), \qquad y_2(t)=\sum a^{(2)}_{k_1\dots k_m}\prod_{j=1}^{m} z_{jk_j}(t).$$

If for every t we have $y_1(t)=y_2(t)$ or $y_1(t)\geqslant y_2(t)$, then for every set of indices we have $a^{(1)}_{k_1\dots k_m}=a^{(2)}_{k_1\dots k_m}$ or $a^{(1)}_{k_1\dots k_m}\geqslant a^{(2)}_{k_1\dots k_m}$ respectively.

Proof. If we set $a^{(1)}_{k_1\dots k_m}-a^{(2)}_{k_1\dots k_m}=a_{k_1\dots k_m}$, then

$$\sum a_{k_1\dots k_m}\prod_{j=1}^{m} z_{jk_j}(t)=0 \text{ or } \geqslant 0, \text{ resp., for all } t.$$

From conditions (α) and (β) and from the independence of the fields of $\mathfrak{K}$ if follows that for every system of indices $\sigma_1,\dots,\sigma_m$ there is a $t\in T$ for which $\prod_{j=1}^{m} z_{j\sigma_j}(t)=1$.

From (γ), $\prod_{j=1}^{m} z_{jk_j}(t)=0$ for the same t and any other system $k_1\dots k_m$. So for this t, $\sum a_{k_1\dots k_m}\prod_{j=1}^{m} z_{jk_j}(t)=a_{\sigma_1\dots\sigma_m}$ and $a_{\sigma_1\dots\sigma_m}=0$ or $\geqslant 0$, according as $y_1(t)=y_2(t)$ or $y_1(t)\geqslant y_2(t)$ for all t. Since the set $\sigma_1\dots\sigma_m$ was arbitrary, the lemma is proved.

Remark. Only the finite independence of the fields of $\mathfrak{K}$ was used in the proof; their denumerable independence will be used later.

Lemma 2. *If* $y(t)\geqslant 0$, *or* $=0$, *or* $\leqslant 0$ *for all* t, *then for every system of indices*, $a_{k_1\ldots k_m}\geqslant 0$, *or* $=0$, *or* $\leqslant 0$ *respectively.*

This is an immediate consequence of Lemma 1.

§ 5. We now introduce three operations on the functions $y\in F[\boldsymbol{U}(\mathfrak{K})]$, called integration, contraction and separation.

Integration. If y is given by (5), write formally

$$\int_T y(t)\,dt=\sum a_{k_1\ldots k_m}\prod_{j=1}^{m}\mu Z_{jk_j}.$$

We show that the integral does not depend on the particular representation of y (always, of course, of the same type as (5)). Indeed, suppose

$$y(t)=\sum b_{k_1\ldots k_p}\prod_{j=1}^{p}v_{jk_j}(t), \tag{10}$$

where the $v_{jk}(t)$ are the characteristic functions of sets V_{jk}. Let U_{ji} be a system of sets which refines both representations, yielding

$$y(t)=\sum c^{(1)}_{k_1\ldots k_q}\prod_{j=1}^{q}u_{jk_j}(t),\qquad y(t)=\sum c^{(2)}_{k_1\ldots k_q}\prod_{j=1}^{q}u_{jk_j}(t)$$

respectively, where u_{jk} is the characteristic function of U_{jk}. By Lemma 1, $c^{(1)}_{k_1\ldots k_q}=c^{(2)}_{k_1\ldots k_q}$ for every set of indices; by Appendix II, 2^0 [7]) the integral is not changed by refinement. That is,

$$\sum a_{k_1\ldots k_m}\prod_{j=1}^{m}\mu(Z_{jk_j})=\sum c^{(1)}_{k_1\ldots c k_q}\prod_{j=1}^{q}\mu(U_{jk_j})=$$

$$=\sum c^{(2)}_{k_1\ldots k_q}\prod_{j=1}^{q}\mu(U_{jk_j})=\sum b_{k_1\ldots k_p}\prod_{j=1}^{p}\mu(V_{jk_j}),$$

which establishes the invariance of the definition.

Let y_1 and y_2 be functions from $F[\boldsymbol{U}(\mathfrak{K})]$, with uniformized representations

$$y_1(t)=\sum a^{(1)}_{k_1\ldots k_m}\prod_{j=1}^{m}u_{jk_j}(t),\qquad y_2(t)=\sum a^{(2)}_{k_1\ldots k_m}\prod_{j=1}^{m}u_{jk_j}(t).$$

[7]) Only finite additivity of the measure μ is assumed there. Denumerable additivity will be required later.

It is immediate that

$$\int_T (y_1(t)+y_2(t))\,dt=\int_T y_1(t)\,dt+\int_T y_2(t)\,dt, \tag{11}$$

$$\int_T ay(t)\,dt=a\int_T y(t)\,dt \tag{12}$$

for any number a. By Lemma 2,

$$\int_T y(t)\,dt\geqslant 0\quad\text{if}\quad y(t)\geqslant 0\quad\text{for all}\quad t, \tag{13}$$

Denote by $F(\boldsymbol{A})$ the subset of $F[\boldsymbol{U}(\mathfrak{K})]$ composed of all functions which assume each of their values in a set belonging to $\boldsymbol{A}\in\mathfrak{K}$, and let $y(t)\in F(\boldsymbol{A})$. The representation (5) becomes then

$$y(t)=\sum_{k=1}^{m}a_k z_k(t),$$

and the corresponding integral is $\sum_{k=1}^{m}a_k\mu(Z_k)$; this integral is identical with that in customary sense engendered by the measure μ. Hence follows

$$\int_T 1\,dt=\mu(T)=1, \tag{14}$$

the functions $y(t)=c$ being contained in all sets $F(\boldsymbol{A})$.

From (11) and (13) it follows that

$$\text{if}\quad y_1(t)\geqslant y_2(t)\quad\text{for all}\quad t,\quad\text{then}\quad\int_T y_1(t)\,dt\geqslant\int_T y_2(t)\,dt. \tag{15}$$

Finally suppose $U\in\boldsymbol{A}\in\mathfrak{K}$, $V\in\boldsymbol{B}\in\mathfrak{K}$, and let u, v be the characteristic functions of U, V resp. By the definition of the integral,

$$\int_T u(t)\cdot v(t)\,dt=\mu(U)\cdot\mu(V). \tag{16}$$

Contraction. Given $y(t)=\sum a_{k_1\ldots k_m}\prod_{j=1}^{m}z_{jk_j}(t)$, define:

$$W(y,t)=\sum a_{k_1\ldots k_m}z_{1k_1}(t)\prod_{j=2}^{m}\mu Z_{jk_j},\quad\text{if}\quad m\geqslant 2, \tag{17}$$

$$W(y,t)=y(t)\quad\text{if}\quad m=1. \tag{18}$$

We call (17) the *contraction* of y with respect to the field $\boldsymbol{A}_1$. The result of the operation evidently depends on the choice of $\boldsymbol{A}_1$; however, a proof entirely analogous to that given for integration and in Appendix II shows that $W(y,t)$ is not changed by refinement of y, and it follows that the definition does not depend on the representation of y, once $\mathbf{A}_1$ has been fixed. Exactly as for the integral one proves that contraction by a given field is a linear operation, and that if $y(t) \geqslant 0$ for all t, the same is true of $W(y,t)$. Hence

(19) if $y_1(t) \geqslant y_2(t)$ for all t, then $W(y_1,t) \geqslant W(y_2,t)$, for all t,

where contraction is performed with respect to the same field.

Separation. Again suppose $y(t) = \sum a_{k_1\ldots k_m} \prod_{j=1}^{m} z_{jk_j}(t)$. For any $t_1 \in T$ define

(20) $$S(y,t_1,t) = \sum a_{k_1\ldots k_m} z_{1k_1}(t_1) \prod_{j=2}^{m} z_{jk_j}(t).$$

We call this operation *separation* performed at t_1 with respect to the field $\mathbf{A}_1$. Again it is easy to show that $S(y,t_1,t)$ depends only on y, t_1, t and $\mathbf{A}_1$, and not on the representation of y once $\mathbf{A}_1$ has been fixed. For fixed t_1 and $\mathbf{A}_1$, separation is a linear operation; and if y is non-negative, so is $S(y,t_1,t)$. Hence

(21) if $y_1(t) \geqslant y_2(t)$ for all t, then $S(y_1,t_1,t) \geqslant S(y_2,t_1,t)$ for all t, t_1,

where separation is performed with respect to te same field.

Lemma 3. *If y is of the form* (5), *then* .

$$\int_T S(y,t_1,t)\,dt = W(y,t_1) \quad \text{and} \quad \int_T W(y,t)\,dt = \int_T y(t)\,dt.$$

These relations are evident on writing out the integrals explicitly.

§ 6. Measure. Let E belong to $\mathbf{U}(\mathfrak{R})$, with characteristic function y. Define

(22) $$\mu^*(E) = \int_T y(t)\,dt$$

It follows from (11) and (13) that μ^* is a (finitely additive) measure, and by the definition of the integral we have $\mu^*(E) = \mu(E)$ for $E \in \mathbf{A} \in \mathfrak{R}$. By (16) the fields $\mathbf{A} \in \mathfrak{R}$ are stochastically independent with respect to μ^*. So μ^* satisfies the conditions of the theorem of Marczewski. Since only the hypotheses of that theorem have been used, we have proved it on the way to the main result.

§ 7. Lemma 4. *Let y be of form* (5), *a a real number, and $t_1, \ldots, t_m$ points of T such that*

$$Y(t_1, \ldots, t_m) \underset{df}{=} \sum a_{k_1\ldots k_m} \prod_{j=1}^{m} z_{jk_j}(t_j) \geqslant a.$$

Then there are sets $H_j \in \mathbf{A}_j$ $(j=1,\ldots,m)$ for which

(23) $$t_j \in H_j,$$

(24) $$y(t) \geqslant a \quad \textit{for all} \quad t \in \prod_{j=1}^{m} H_j.$$

Proof. First assume that $t_j \in \sum_{k=1}^{r_j} Z_{jk}$ for each j; that is, there are indices $\sigma_1, \ldots, \sigma_m$ for which $t_j \in Z_{j\sigma_j}$. Let $H_j = Z_{j\sigma_j}$. Evidently $H_j \in \mathbf{A}_j$, so it remains to prove (24). Notice that $k_j \neq \sigma_j$ implies $t_j \operatorname{non} \in Z_{jk_j}$, or $z_{jk_j}(t_j) = 0$, so $Y(t_1, \ldots, t_m) = a_{\sigma_1\ldots\sigma_m}$; hence $a_{\sigma_1\ldots\sigma_m} \geqslant a$. Now if $t \in \prod_{j=1}^{m} H_j$ we have also $y(t) = a_{\sigma_1\ldots\sigma_m} \geqslant a$.

Now suppose $t_j \operatorname{non} \in \sum_{k=1}^{r_j} Z_{jk}$ for at least one j, say for $j = s_1, \ldots, s_p$ $(1 \leqslant p \leqslant m)$. For each such j, $z_{jk}(t_j) = 0$ $(k = 1, \ldots, r_j)$, so that $\prod_{j=1}^{m} z_{jk_j}(t) = 0$ for any indices $k_1, \ldots, k_m$. Hence $Y(t_1, \ldots, t_m) = 0$ and $a \leqslant 0$. Set $H_j = T - \sum_{k=1}^{r_j} Z_{jk}$ for $j = s_1, \ldots, s_p$, and $H_j = \sum_{k=1}^{r_j} Z_{jk}$ for other j. Then $t_j \in H_j \in \mathbf{A}_j$. If $t \in \prod_{k=1}^{m} H_j$, then $t \operatorname{non} \in Z_{jk}$ $(k = 1, \ldots, r_j)$ for at least one j (since $p \geqslant 1$), from which $\prod_{j=1}^{m} z_{jk_j}(t) = 0$ for any choice of the k_j. Hence $y(t) = 0 \geqslant a$, and (24) holds

Lemma 5. *Let $y_n(t) = \sum a^{(n)}_{k_1\ldots k_m} \prod_{j=1}^{m} z^{(n)}_{jk_j}(t)$ be a sequence of denumerably uniformized functions from $F[\mathbf{U}(\mathfrak{R})]$; suppose there is a finite or infinite sequence $t_1, t_2, \ldots$ of elements of T, such that for each n*

$$Y_n(t_1, \ldots, t_{m_n}) \geqslant a.$$

Then there is some $\vartheta \in T$ for which $y_n(\vartheta) \geqslant a$ $(n = 1, 2, \ldots)$.

Proof. By Lemma 4, for each n there are sets $H_j^n \epsilon A_j$ $(j=1,\ldots,m_n)$ with the properties:

$$t_j \epsilon H_j^n \quad \text{for} \quad j=1,\ldots,m_n,$$

$$y_n(t) \geqslant a \quad \text{for} \quad t \epsilon \prod_{j=1}^{m_n} H_j^n.$$

For $j > m_n$ set $H_j^n = T$ and define $W_j = \prod_{j=1}^{\infty} H_j^n$. Each W_j is non empty, since $t_j \epsilon W_j$; and because the $A \epsilon \mathfrak{K}$ are Borel fields (the first use of this hypothesis), $W_j \epsilon A_j$. Now set $H = \prod_{j=1}^{\infty} W_j$. This intersection is non empty, because the $A \epsilon \mathfrak{K}$ are denumerably independent (this is the only use of the hypothesis in the proof!). But if $\vartheta \epsilon H$, then $\vartheta \epsilon \prod_{j=1}^{m_n} H_j^n$ and $y_n(\vartheta) \geqslant a$ for all n. So the lemma is proved.

§ 8. Proof of Theorem 1. It only remains to prove that μ^* satisfies condition (4). Suppose

$$H_n \epsilon U(K), \quad H_n \supset H_{n+1}, \quad \mu^*(H_n) \geqslant a > 0, \quad n=1,2,\ldots$$

Let y_n be the characteristic function of H_n. Then $y_n \epsilon F[U(K)]$, $\int_T y_n(t)\,dt \geqslant a$, and for all t

$$y_n(t) \geqslant y_{n+1}(t) \quad (n=1,2,\ldots). \tag{25}$$

We take the y_n denumerably uniformized:

$$y_n(t) = \sum a_{k_1\ldots k_{m_n}}^{(n)} \prod_{j=1}^{m_n} z_{jk_j}^{(n)}(t).$$

Then

$$m_{n+1} \geqslant m_n \qquad (n=1,2,\ldots). \tag{26}$$

By (19), for all t

$$W(y_n,t) \geqslant W(y_{n+1},t) \qquad (n=1,2,\ldots), \tag{27}$$

and by Lemma 3

$$\int_T W(y_n,t)\,dt \geqslant a \qquad (n=1,2,\ldots). \tag{28}$$

Now every $W(y_n,t)$ belongs to $F(A_1)$. But μ^* is in A_1 a denumerably additive measure (this is the first use of the hypothesis), and the integral of $y_n(t)$ over T (in the sense given p. 164) is an integral generated by μ^* in the Lebesgue sense. Hence, applying (28),

$$\int_T \lim_{n\to\infty} W(y_n,t)\,dt = \lim_{n\to\infty} \int_T W(y_n,t)\,dt \geqslant a. \tag{29}$$

By (27) and (29) there is an element t_1 for which

$$W(y_n,t_1) \geqslant a \qquad (n=1,2,\ldots). \tag{30}$$

Now set $y_{n,1}(t) = S(y_n,t_1,t)$. These functions are already denumerably uniformized. By (21), $y_{n,1}(t) \geqslant y_{n+1,1}(t)$ for all t; by (30) and Lemma 3

$$\int_T y_{n,1}(t)\,dt \geqslant a \qquad (n=1,2,\ldots).$$

So the $y_{n,1}$ are like the y_n, and there is a t_2 for which

$$W(y_{n,1},t_2) \geqslant a \qquad (n=1,2,\ldots),$$

where contraction is performed with respect to A_2. Setting $y_{n,2}(t) = S(y_{n,1},t_2,t)$ we have $y_{n,2}(t) \geqslant y_{n+1,2}(t)$ for all t and

$$\int_T y_{n,2}(t)\,dt \geqslant a \qquad (n=1,2,\ldots).$$

This procedure can be repeated indefinitely by setting

$$y_{n,k+1}(t) = S(y_{n,k},t_{k+1},t).$$

If $m_n = l$ for some n (or several, or infinitely many n), say for n such that $p \leqslant n \leqslant r$, we have for these n after the $(l-1)^{\text{th}}$ step

$$y_{n,l-1}(t) = \sum a_{k_1\ldots k_l}^{(n)} \cdot z_{lk_l}^{(n)}(t) \prod_{j=1}^{l-1} z_{jk_j}^{(n)}(t_j). \tag{31}$$

The functions $y_{n,l-1}$ with $p \leqslant n \leqslant r$ each belongs to $F(A_l)$. Contraction with respect to A_l leaves by (18) the $y_{n,l-1}$ invariant. Having fixed l, the $W(y_{n,l-1},t)$ are a non-increasing sequence in the index n, and for every n

$$\int_T W(y_{n,l-1},t)\,dt \geqslant a.$$

It follows that there is a t_l such that

$$W(y_{n,l-1},t_l)\geqslant a \qquad (n=1,2,\ldots)$$

and in particular

(32) $$y_{n,l-1}(t_l)\geqslant a \quad \text{for} \quad p\leqslant n\leqslant r.$$

For these values of n, however, the left side of (32) is by (31) on one hand the function $y_{n,l}(t)=S(y_{n,l-1},t_l,t)$, which is a constant, and on the other hand identical with $Y_n(t_1,\ldots,t_{m_n})$.

In case $m_n=l$ for all $n\geqslant p$, the process is finished; otherwise continue with those $y_{n,l}(t)$ for which $m_n>l$, in other words for which $n>r$. Finally, a (possibly finite) sequence of elements $t_1,t_2,\ldots$ is at hand, which with the y_n satisfy the hypothesis of Lemma 5.

So there is a $\vartheta\epsilon T$ for which $y_n(\vartheta)\geqslant a$ $(n=1,2,\ldots)$. Since $a>0$ and the y_n are characteristic functions, $y_n(\vartheta)=1$, so that $\vartheta\epsilon H_n$ $(n=1,2,\ldots)$.

Hence $\prod_{n=1}^{\infty}H_n\neq 0$, (4) is shown, and Theorem 1 is proved.

Appendix I.

Representation of functions.

Let H_ν $(\nu=1,\ldots,p)$ be sets belonging to $\boldsymbol{U}(\mathfrak{R})$. For every ν

(1) $$H_\nu=\sum_{i=1}^{n_\nu}\prod_{j=1}^{m_{\nu i}}G_{ij}^{\nu},$$

where each set G_{ij}^{ν} belongs to a field $\boldsymbol{A}_{ij}^{\nu}\epsilon\mathfrak{R}$, with $\boldsymbol{A}_{ir}^{\nu}\neq\boldsymbol{A}_{is}^{\nu}$ if $r\neq s$. Renumber the fields $\boldsymbol{A}_{ij}^{\nu}$ in simple order: $\boldsymbol{A}_1,\ldots,\boldsymbol{A}_m$, where distinct indices belong to distinct fields. In each intersection of (1) write the factors G_{ij}^{ν} in the order of the indices of the fields to which they belong, supplying a T as the r^{th} factor if no G_{ij}^{ν} belongs to $\boldsymbol{A}_r$ $(\nu=1,\ldots,m)$. By the independence of the fields, no set except 0 and T can belong to more than one field, so this rearrangement can be carried out without ambiguity. By renoming the reordered factors we have a representation

(2) $$H_\nu=\sum_{i=1}^{n_\nu}\prod_{j=1}^{m}H_{ij}^{\nu},$$

where $H_{ij}^{\nu}\epsilon\boldsymbol{A}_j\epsilon\mathfrak{R}$ $(\boldsymbol{A}_r\neq\boldsymbol{A}_s$ for $r\neq s)$.

Now for each j put the sets H_{ij}^{ν} in a simple series H_{jk} $(k=1,\ldots,s_j;\ H_{jk}\neq H_{jl}$ for $k\neq l)$. For each j and every system $\sigma_1\ldots\sigma_{s_j}$ composed of zeros and ones, form the product

$$\prod_{k=1}^{s_j}(-1)^{\sigma_k}H_{jk},$$

where $-H_{jk}$ means $T-H_{jk}$.

Suppose r_j of these intersections are non-void; call them Z_{jk} $(k=1,\ldots,r_j)$. Evidently $Z_{jk}\epsilon\boldsymbol{A}_j$, $Z_{jk}\cdot Z_{jl}=0$ for $k\neq l$, and every H_{ij}^{ν} is the sum of some of the sets $Z_{j1},\ldots,Z_{jr_j}$. So there are numbers β_{ijk}^{ν} (each 0 or 1) such that

$$H_{ij}^{\nu}=\sum_{k=1}^{r_j}\beta_{ijk}^{\nu}Z_{jk}.$$

Setting this in (2) we obtain

$$H_\nu=\sum\sum_{i=1}^{n_\nu}[\beta^\nu_{i1k_1}\cdot\ldots\cdot\beta^\nu_{imk_m}\prod_{j=1}^{m}Z_{jk_j}],$$

where the outer summation is taken over all systems $k_1\ldots k_m$ $(1\leqslant k\leqslant r_j)$. Setting $a^\nu_{k_1\ldots k_m}$ equal to the smaller value of 1 and $\sum_{i=1}^{n_\nu}\beta^\nu_{i1k_1}\cdot\ldots\cdot\beta^\nu_{imk_m}$, we get

$$H_\nu=\sum a^\nu_{k_1\ldots k_m}\prod_{j=1}^{m}Z_{jk_j}. \tag{3}$$

The intersections in (3) are disjoint.

Let z_{jk} be the characteristic function of Z_{jk}, and y_ν the characteristic function of H_ν. Then

$$y_\nu(t)=\sum a^\nu_{k_1\ldots k_m}\prod_{j=1}^{m}z_{jk_j}(t). \tag{4}$$

Now if $y\,\epsilon\,F[\boldsymbol{U}(\mathfrak{R})]$ assumes its values $a_1,\ldots,a_p$ on the sets $H_1,\ldots,H_p$ respectively, $y(t)=\sum_{\nu=1}^{p}a_\nu y_\nu(t)$, i. e.

$$y(t)=\sum a_{k_1\ldots k_m}\prod_{j=1}^{m}z_{jk_j}(t),$$

where the summation is taken over systems $k_1\ldots k_m$ $(1\leqslant k_j\leqslant r_j)$, and

$$a_{k_1\ldots k_m}=\sum_{\nu=1}^{p}a_\nu a^\nu_{k_1\ldots k_m}.$$

Appendix II.

Refinement.

1° Let the function y be given in the form

$$y(t)=\sum a_{k_1\ldots k_m}\prod_{j=1}^{m}z_{jk_j}(t), \tag{1}$$

where each z_{jk} is the characteristic function of a non-empty set $Z_{jk}\,\epsilon\,\boldsymbol{A}_j\,\epsilon\,\mathfrak{R}$ $(1\leqslant k\leqslant r_j)$, with $\boldsymbol{A}_r\neq\boldsymbol{A}_s$ for $r\neq s$; and $Z_{jk}\cdot Z_{jl}=0$ for any j, when $k\neq l$.

Let $\boldsymbol{B}_1,\boldsymbol{B}_2,\ldots,\boldsymbol{B}_m,\ldots,\boldsymbol{B}_p$ be distinct fields of $\mathfrak{R}$, such that $\boldsymbol{B}_i=\boldsymbol{A}_i$ for $i\leqslant m$. Suppose further we are given sets $U_{ji}\,\epsilon\,\boldsymbol{B}_j$, non-empty, disjoint in i for fixed j, and such that for $j\leqslant m$

$$Z_{jk}=\sum_{i=1}^{s_j}\beta_{jki}U_{ji}\quad(\text{each }\beta_{jki}\text{ being either 0 or 1}), \tag{2}$$

and for $j>m$

$$T=\sum_{i=1}^{s_j}\beta_{j1i}U_{ji}\qquad(\text{each }\beta_{j1i}=1). \tag{3}$$

If u_{ji} is the characteristic function of U_{ji},

$$\sum_{i=1}^{s_j}\beta_{jki}u_{ji}(t)=z_{jk}(t)\ \text{ when } j\leqslant m, \tag{4}$$

$$\sum_{i=1}^{s_j}\beta_{j1i}u_{ji}(t)=1\ \text{ for all } t,\ \text{ when } j>m. \tag{5}$$

Write unit factors in each product of (1) so the index j runs from 1 to p, and then substitute formulas (4) and (5) for the functions z_{jk_j} and for the unit factors respectively. Then we have

$$y(t)=\sum a_{k_1\ldots k_m}\prod_{j=1}^{p}\sum_{i=1}^{s_j}\beta_{jk_ji}u_{ji}(t)=\sum a_{k_1\ldots k_m}\sum\beta_{1k_1i_1}\cdot\ldots\cdot\beta_{pk_pi_p}\prod_{j=1}^{p}u_{ji_j}(t), \tag{6}$$

where the inner summation is taken over all sets $i_1\ldots i_p$ such that $1\leqslant i_j\leqslant s_j$ for each j, and the outer summation as before over sets $k_1\ldots,k_m$ such that $1 \leqslant k_j \leqslant r_j$. Changing the order of summation and writing

$$b_{i_1\ldots i_p} = \sum_{k_1\ldots k_m} a_{k_1\ldots k_m} \beta_{1k_1i_1} \cdot \ldots \cdot \beta_{pk_pi_p}, \tag{7}$$

we have

$$y(t) = \sum_{i_1\ldots i_p} b_{i_1\ldots i_p} \prod_{j=1}^{p} u_{ji_j}(t).$$

This representation is a refinement of (1) by the sets U_{ji}.

2⁰ We show that

$$\sum a_{k_1\ldots k_m} \prod_{j=1}^{m} \mu(Z_{jk_j}) = \sum b_{i_1\ldots i_p} \prod_{j=1}^{p} \mu(U_{ji_j}). \tag{8}$$

Indeed, from (2)

$$\mu(Z_{jk}) = \sum_{i=1}^{s_j} \beta_{jki}\, \mu(U_{ji}) \quad \text{for } j \leqslant m, \tag{9}$$

and from (3)

$$1 = \sum_{i=1}^{s_j} \beta_{j1i}\, \mu(U_{ji}) \quad \text{for } j > m. \tag{10}$$

By the same algebraic procedure as before, i. e. by writing $p-m$ unit factors in each product on the left side of (8), and by substituting (9) and (10), we obtain

$$\sum a_{k_1\ldots k_m} \sum \beta_{1k_1i_1} \cdot \ldots \cdot \beta_{pk_pi_p} \prod_{j=1}^{p} \mu(U_{ji_j}),$$

where the sums are as in (6). By changing the order of summation and using (7) the right side of (8) appears.

Appendix III.

Common refinement.

Let functions y_1 and y_2 of $F[\mathbf{U}(\mathfrak{R})]$ be given, not necessarily distinct:

$$y_1(t) = \sum a^{(1)}_{k_1\ldots k_{m_1}} \prod_{j=1}^{m_1} z^{(1)}_{jk_j}(t), \tag{1}$$

$$y_2(t) = \sum a^{(2)}_{k_1\ldots k_{m_2}} \prod_{j=1}^{m_2} z^{(2)}_{jk_j}(t) \tag{2}$$

Here the $z^{(\nu)}_{jk}$ are characteristic functions of sets Z^ν_{jk} ($\nu = 1, 2$; $j = 1, \ldots, m_\nu$; $k = 1, \ldots, r^{(\nu)}_j$), such that $Z^\nu_{jk} \neq 0$, $Z^\nu_{jk} \in \mathbf{A}^\nu_j \in \mathfrak{R}(\mathbf{A}^\nu_r \neq \mathbf{A}^\nu_s$ for $r \neq s$), and $Z^\nu_{jr} \cdot Z^\nu_{js} = 0$ if $r \neq s$. There may or may not be fields $\mathbf{A}^1_i$ identical with fields $\mathbf{A}^2_j$. But form a series $\mathbf{B}_1, \ldots, \mathbf{B}_p$ out of the $\mathbf{A}^1_i$ and the $\mathbf{A}^2_j$ which includes each field just once, and for each of the $\mathbf{B}_j$, renumber in a series Z_{js} ($s = 1, \ldots, \varrho_j$) all the sets Z^ν_{rk} such that $\mathbf{A}^\nu_r = \mathbf{B}_j$. As in Appendix I form all the intersections

$$\prod_{s=1}^{\varrho_j} (-1)^{\sigma_s} Z_{js}, \tag{3}$$

where the σ_s assume the values 0, 1; and number the non-void intersections $U_{j1}, \ldots, U_{js_j}$. Evidently for each j, $U_{ji} \in \mathbf{B}_j$ and $U_{ji} \cdot U_{jk} = 0$ if $i \neq k$; each set Z^ν_{ik} belonging to $\mathbf{B}_j$ is the sum of some of the U_{ji}, and for fixed j the sum of the U_{ji} is T. Thus the sets U_{ji} yield a common refinement of (1) and (2).

Appendix IV.

Denumerable uniformization.

Let a sequence of functions from $F[\boldsymbol{U}(\mathfrak{R})]$ be given:

$$y_n(t)=\sum b^{(n)}_{i_1\ldots i_{p_n}}\prod_{j=1}^{p_n} u^{(n)}_{ji_j}(t), \tag{1}$$

where $u^{(n)}_{ji}(t)$ is the characteristic function of a non-empty set

$$U^n_{ji}\,\epsilon\,\boldsymbol{A}^n_j\,\epsilon\,\mathfrak{R} \quad \text{for} \quad n=1,2,\ldots;\quad j=1,\ldots,p_n;\quad i=1,\ldots,s^{(n)}_j.$$

If $r\neq s$, $\boldsymbol{A}^n_r\neq\boldsymbol{A}^n_s$ and $U^n_{jr}\cdot U^n_{js}=0$. We shall transform (1) by induction so as to obtain a representation of the following kind:

$$y_n(t)=\sum a^{(n)}_{k_1\ldots k_{m_n}}\prod_{j=1}^{m_n} z^{(n)}_{jk_j}(t), \tag{2}$$

where $z^{(n)}_{ji}(t)$ is the characteristic function of a set $Z^n_{jk}\neq 0$, $Z^n_{jk}\,\epsilon\,\boldsymbol{B}_j\,\epsilon\,\mathfrak{R}$ for $n=1,2,\ldots;$ $j=1,\ldots,m_n;$ $i=1,\ldots,r^{(n)}_j;$ if $r\neq s$, $\boldsymbol{B}_r\neq\boldsymbol{B}_s$ and $Z^n_{jr}Z^n_{js}=0$. Here the sequence $\{\boldsymbol{B}_j\}$ does not involve the index n.

Assume that $y_1,\ldots,y_{n-1}$ are already expressed in the form (2), with the fields $\boldsymbol{B}_1,\ldots,\boldsymbol{B}_{m_{n-1}}$ and the sets $Z^l_{jk}\,\epsilon\,\boldsymbol{B}_j$ determined $(l=1,\ldots,n-1;\ j=1,\ldots,m_{n-1};\ k=1,\ldots,r^{(l)}_j)$.

1° If a field $\boldsymbol{A}^n_j$ is identical with $\boldsymbol{B}_h$ for some $h\leqslant m_{n-1}$, set $U^n_{jk}=Z^n_{hk}$, and accordingly

$$u^{(n)}_{jk}(t)=z^{(n)}_{hk}(t),\quad s^{(n)}_j=r^{(n)}_h.$$

2° If there are fields $\boldsymbol{A}^n_j$ which are not among the $\boldsymbol{B}_i$ $(i=1,\ldots,m_{n-1})$, denote them by $\boldsymbol{B}_{m_{n-1}+1},\ldots,\boldsymbol{B}_{m_n}$, and write correspondingly:

$$U^n_{jk}=Z^n_{m_{n-1}+1\,k},\ldots,Z^n_{m_nk},$$

$$u^{(n)}_{jk}(t)=z^{(n)}_{m_{n-1}+1\,k}(t),\ldots,z^{(n)}_{m_nk}(t) \qquad (1\leqslant k\leqslant s^{(n)}_j),$$

$$s^{(n)}_j=r^{(n)}_{m_{n-1}+1},\ldots,r^{(n)}_{m_n}.$$

3° If some $\boldsymbol{B}_j$ $(j\leqslant m_{n-1})$ does not occur among the $\boldsymbol{A}^n_j$, set

$$Z^n_{j1}=T,\quad z^{(n)}_{j1}(t)=1 \text{ for all } t, \text{ and } r^{(n)}_j=1.$$

By this procedure each product

$$\prod_{j=1}^{p_n} u^{(n)}_{ji_j}(t) \tag{3}$$

of (1) is transformed into a product

$$\prod_{j=1}^{m_n} z^{(n)}_{jk_j}(t), \tag{4}$$

which differs from (3) only in the order of the factors and the presence of certain unit factors. Bearing in mind that $r^{(n)}_j=1$ for the j considered in 3°, each $b^{(n)}_{i_1\ldots i_{p_n}}$ can be rewritten $a^{(n)}_{k_1\ldots k_{m_n}}$, and y_n has been reduced to form (2).

(Reçu par la Rédaction le 4. 6. 1948).