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On nonstationary motion of a compressible barotropic viscous fluid bounded by a free surface

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CONTENTS

Abstract

We consider the motion of a viscous compressible barotropic fluid in $\mathbb{R}^3$ bounded by a free surface which is under constant exterior pressure. For a given initial density, initial domain and initial velocity we prove the existence of local-in-time highly regular solutions. Next assuming that the initial density is sufficiently close to a constant, the initial pressure is sufficiently close to the external pressure, the initial velocity is sufficiently small and the external force vanishes we prove the existence of global-in-time solutions which satisfy, at any moment of time, the properties prescribed at the initial moment.

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1. Introduction

We consider the motion of a viscous compressible barotropic fluid in a bounded domain $\Omega_t \subset \mathbb{R}^3$ which depends on time t. The free boundary S_t of Ω_t is built up of the same fluid particles for all time. Let $v = v(x,t)$ be the velocity of the fluid, $\varrho = \varrho(x,t)$ the density, $f = f(x,t)$ the external force field per unit mass, $p = p(\varrho)$ the pressure, μ and ν the viscosity coefficients, p_0 the external constant pressure. Then the problem is described by the following system (see [7], Chs. 1, 2, 7):

$$
\begin{aligned}
&\varrho(v_t + v\cdot\nabla v) + \nabla p(\varrho) - \mu\Delta v - \nu\nabla \operatorname{div} v = \varrho f && \text{in } \widetilde{\Omega}^T,\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \widetilde{\Omega}^T,\\
&\varrho|_{t=0} = \varrho_0, \quad v|_{t=0} = v_0 && \text{in } \Omega,\\
&\mathbb{T}\overline{n} = -p_0\overline{n} && \text{on } \widetilde{S}^T,\\
&v\cdot\overline{n} = -\phi_t/|\nabla\phi| && \text{on } \widetilde{S}^T,
\end{aligned}
\tag{1.1}
$$

where $\phi(x,t) = 0$ describes S_t, $\widetilde{\Omega}^T = \bigcup_{t\in(0,T)} \Omega_t \times \{t\}$, Ω_t is the domain of the drop at time t, $\Omega_0 = \Omega$ is its initial domain, $\widetilde{S}^T = \bigcup_{t\in(0,T)} S_t$, $\overline{n}$ is the unit outward vector normal to the boundary ($\overline{n} = \nabla\phi/|\nabla\phi|$), μ, ν are the constant viscosity coefficients. Moreover, thermodynamic considerations imply $\nu \ge \frac{1}{3}\mu > 0$. The last condition $(1.1)_5$ means that the free boundary S_t is built up of moving fluid particles. Finally, $\mathbb{T} = \mathbb{T}(v,p)$ denotes the stress tensor of the form

$$
T_{ij} = -p\delta_{ij} + \mu(\partial_{x^i} v^j + \partial_{x^j} v^i) + (\nu - \mu)\delta_{ij} \operatorname{div} v \equiv -p\delta_{ij} + D_{ij}(v),
\tag{1.2}
$$

where $i,j = 1,2,3$, and $\mathbb{D} = \mathbb{D}(v)$ is the deformation tensor. In this paper we restrict our considerations to the barotropic case, so $p = A\varrho^\kappa$, $A > 0$, $\kappa > 1$.

Let the domain Ω be prescribed. Then, by $(1.1)_5$, $\Omega_t = \{x \in \mathbb{R}^3 : x = x(\xi,t),\ \xi \in \Omega\}$, where $x = x(\xi,t)$ is the solution of the Cauchy problem

$$
\frac{\partial x}{\partial t} = v(x,t), \qquad x|_{t=0} = \xi \in \Omega, \qquad \xi = (\xi^1,\xi^2,\xi^3)\,.
\tag{1.3}
$$

The transformation $x = x(\xi,t)$ connects the Eulerian x and the Lagrangian ξ coordinates of the same fluid particle. Hence

$$
x = \xi + \int_0^t u(\xi,s)\,ds \equiv X_u(\xi,t)\,,
\tag{1.4}
$$

where $u(\xi,t)=v(X_u(\xi,t),t)$. Moreover, the kinematic boundary condition $(1.1)_5$ implies that the boundary S_t is a material surface, so if $\xi\in S=S_0$ then $X_u(\xi,t)\in S_t$ and $S_t=\{x : x=X_u(\xi,t),\ \xi\in S\}$.

By the continuity equation $(1.1)_2$ and $(1.1)_5$ the total mass M is conserved and

$$\int\limits_{\Omega_t}\varrho(x,t)\,dx=M\,, \tag{1.5}$$

which is a relation between ϱ and Ω_t.

Definition 1.1. Let us introduce a constant state which is a solution of (1.1) for $f=0$ such that

$$v=0,\quad \varrho=\varrho_e\,,\quad \Omega_t=\Omega_e \text{ for all } t\in\mathbb{R}^1\,,\quad \varrho_e\,,\ \Omega_e \text{ are constants}\,, \tag{1.6}$$

where the index e denotes the parameters of the state and $|\Omega|=\operatorname{vol}\Omega$. Then (1.5) implies that $M=\varrho_e|\Omega_e|$ and $(1.1)_4$ yields that ϱ_e is a solution to the equation

$$p(\varrho_e)=p_0\,, \tag{1.7}$$

so $p_e=p(\varrho_e)=p_0$.

The aim of this paper is to prove the existence of global-in-time solutions of (1.1). It cannot be expected that this can be proved for large data because up to now even in the case of a fixed domain the global existence of solutions for the compressible Navier–Stokes equations is known only for small data (see [8–12, 32, 33]).

The paper is divided into six sections. In the second part of Section 1 we review the previous work on free boundary problems for nonstationary, both incompressible and compressible, Navier–Stokes equations in the case of the drop problems only. Moreover, necessary notation is introduced. In Section 2 global conservation laws are found for sufficiently smooth solutions of (1.1). The most important result of the section is that under a proper choice of magnitudes of the parameters $(\mu,\nu,\varrho_0,v_0,\Omega,S,p_0,A,\kappa)$ of (1. 1), $\operatorname{var}_{t\in\mathbb{R}^1_+}|\Omega_t|$ is as small as we need. This is one of the main differences with the incompressible case where $|\Omega_t|$ is constant (see [23, 25, 27]). This fact implies that $\|v\|_{L_2(\Omega_t)}$ can be sufficiently small, which is necessary to prove the global existence. Moreover, conservation laws are found (see (2.2), (2.3)) which are necessary for the proof of the modified Korn inequalities (see Remark 2.4 and Section 5). The latter are then used to prove the main differential inequality in this paper (see (4.166)), which implies the global existence. In the case of constant density the conservation laws reduce to those shown by V. A. Solonnikov (see [23, 25, 27]) which are also used to prove the Korn inequalities.

In Section 3 the local existence of solutions of (1.1) is proved. To do this we use the Lagrangian coordinates so the transformed problem (1.1) is considered in the fixed domain Ω (see equations (3.1)). Since $(1.1)_1$ is a parabolic system for a given ϱ we use the results of V. A. Solonnikov (see [24]) on the existence of solutions for linear parabolic systems. Therefore we have to prove the existence of solutions of the linear problem (3.3) such that $v \in W_r^{2l+2,l+1}(\Omega^T)$, $0 \le l \in \mathbb{Z}$, $r > 3$. The condition $r > 3$ is necessary because otherwise the coefficients by the lower derivatives in the boundary norm in (3.7) depend on $T^{-a},\ a > 0$ (see [24]), so we meet difficulties in proving local existence for the nonlinear case where T must be small. Having proved the existence of $v(x(\xi,t),t) \in W_r^{2l+2,l+1}(\Omega^T)$, by the continuity equation $(1.1)_2$ we have

$$1/\varrho,\varrho(x(\xi,t),t) \in W_r^{2l+1,l+1/2}(\Omega^T) \cap C([0,T];\Gamma_{0,r}^{2l+1,l+1/2}(\Omega))$$

(see notation below). Finally, by the method of successive approximations the existence of local solutions (v,ϱ) of (1.1) in the above classes and for sufficiently small time is proved (see Theorem 3.6).

In Section 3 we essentially use papers [22, 24, 30]. During the preparation of this paper the author obtained a manuscript of V. A. Solonnikov and A. Tani [31] on the local existence of solutions of the free boundary problem for a compressible viscous fluid. However, in our paper we need much more regular solutions than those found in [31].

In Section 4 the differential inequality (4.166) (see Theorem 4.13) is proved under the following assumptions:

a) there exists a sufficiently regular local solution,
b) the transformation (1.4) together with its inverse exist,
c) the Korn type inequalities (see Lemmas 5.1–5.6) are satisfied,
d) the shape of the domain does not change much with time.

In Section 5 the Korn type inequalities are shown. Finally, in Section 6 the existence of a global solution is proved under the assumptions that the inequalities

1) $\varphi(0) \equiv \sum_{i=1}^{3}(\|\partial_t^i v(0)\|_{H^{3-i}(\Omega)}^2 + \|\partial_t^i p_\sigma(0)\|_{H^{3-i}(\Omega)}^2) \le \gamma\,,$

2) $\psi(t) = \|v\|_{L_2(\Omega_t)}^2 + \|p_\sigma\|_{L_2(\Omega_t)}^2 \le \gamma_1$

hold with γ and γ_1 sufficiently small. The proof is done in the following steps. First we have to show that the local solution belongs to $\mathfrak{M}(T)$ with data in $\mathfrak{N}(0)$ (definitions of the spaces $\mathfrak{M}(T)$ and $\mathfrak{N}(0)$ are given at the beginning of Section 6), which naturally follows from the differential inequality (4.166) (see the proof of Lemma 6.1). Assuming that the initial data in $\mathfrak{N}(0)$ are sufficiently small implies that 1) is satisfied. Next Remark 2.3 and Lemma 6.2 imply 2). Then Lemma 6.3 yields that $\varphi(0) \le \gamma$ implies $\varphi(T) \le \gamma$, which enables one to prolong the local solution to the interval $[T,2T]$ under the assumption that (4.166) holds in $[T,2T]$. The last fact follows from Lemmas 6.4, 5.7 and (6.56); the latter implies that b), c) and d) are satisfied. In this way the existence of solutions for all $t > 0$ can be proved.

The local existence is proved in three steps. First, the existence of local solutions is proved by using the existence of solutions of parabolic equations in anisotropic Sobolev spaces $W_r^{2l,l}$ shown in [24] by potential techniques (see Theorem 3.6). Next, Remark 3.8 shows that the solution is such that

$$v \in W_2^{4,2}(\Omega^T), \quad p_\sigma \in W_2^{3,3/2}(\Omega^T) \cap L_\infty(0,T;\Gamma_{0,2}^{3,3/2}(\Omega)), \quad p_\sigma = p - p_0 .$$

Then by the energy inequality (4.166) it follows that the solution belongs to $\mathfrak{M}(t)$, $t \le T$.

The fact that we have to prove global existence by means of the inequality (4.166) distinguishes our paper sharply from the papers of V. A. Solonnikov [23, 27]; this comes from the fact that the hyperbolic continuity equation is taken into account.

The inequality (4.166) is proved for so highly regular solutions because the equations (3.1) written in Lagrangian coordinates are strongly nonlinear and the coefficients which depend on $\int v_\xi(x(\xi,\tau),\tau)d\tau$ should be estimated in L_∞ norm, so by imbedding theorems v must be estimated in H^3 norm. Then the structure of the inequality implies that v is in H^4. The inequalities of type (4.166) were also obtained in [32, 33]. The proof of global existence is very close to the proof of global existence in [33] but is much more complicated.

The main result of this paper is formulated in Theorem 6.5.

Now we make some comments on the literature concerning free boundary problems for the nonstationary incompressible Navier–Stokes system. Local existence of solutions in the case without surface tension is proved in Hölder and Sobolev anisotropic spaces by V. A. Solonnikov in [26, 27] (see also [20]). To prove the existence of solutions of corresponding linear problems in Hölder and in Sobolev spaces the potential theory techniques are used (see [28], [29], respectively). Local existence with surface tension is considered by G. Allain [2]. In a series of papers V. A. Solonnikov shows the existence of global motions of a viscous incompressible fluid bounded by a free surface, both with surface tension (see [23, 25]) and without it (see [26]). The latter case is based on the Korn inequality. To prove the existence of solutions in the case of surface tension V. A. Solonnikov uses the anisotropic Sobolev–Slobodetskiĭ spaces $W_2^{l,l/2}$ with noninteger positive l. In all papers by Solonnikov the Lagrangian coordinates are used. Global existence is also proved by J. T. Beale [3, 4], where the free boundary is infinite and gravitation is taken into account.

Local existence of solutions for compressible fluids without surface tension is proved by P. Secchi and A. Valli [19] and with surface tension by V. A. Solonnikov and A. Tani [31]. In the one-dimensional case there is a result on global existence by T. Nishida [14], who additionally takes gravitation into account.

Recently, P. Secchi has obtained the existence results for equations describing motions in viscous gaseous stars (see [16]–[18]).

References to the literature concerning stationary free boundary problems are given in [15]. Moreover, in [15] K. Pileckas and W. M. Zajączkowski prove the existence of stationary motion of a viscous compressible barotropic fluid bounded by a free surface governed by surface tension. To show the result they have to assume that the domain and the external force satisfy some extra symmetry conditions. Moreover, in [15] an a priori estimate necessary for the proof of existence is found by the energy method.

The author is very indebted to Prof. M. Niezgódka and Prof. R. Racke for very fruitful discussions and important comments during the preparation of this paper.

Now we introduce notation. In this paper we use the anisotropic Sobolev–Slobodetskiĭ spaces $W_r^{l,l/2}(\Omega^T)$, $l \in \mathbb{R}_+$, $1 \le r \in \mathbb{R}$ (see [5, Ch. 18]), of functions defined in $\Omega^T = \Omega \times (0,T)$. In fact, for noninteger l, $W_r^{l,l/2}$ are Besov spaces; the equivalence between $W_r^{l,l/2}$ and Besov spaces follows from the considerations in [1, Ch. 7]. In the case of noninteger l ($\Omega \subset \mathbb{R}^3$),

$$\|u\|_{W_r^{l,l/2}(\Omega^T)} = (\|u\|^r_{W_r^{l,0}(\Omega^T)} + \|u\|^r_{W_r^{0,l/2}(\Omega^T)})^{1/r},$$

where

$$\begin{aligned}
\|u\|_{W_r^{l,0}(\Omega^T)} &= \Big(\int_0^T \|u\|^r_{W_r^l(\Omega)} dt\Big)^{1/r},\\
\|u\|_{W_r^{0,l/2}(\Omega^T)} &= \Big(\int_\Omega \|u\|^r_{W_r^{l/2}((0,T))} dx\Big)^{1/2},\\
\|u\|^r_{W_r^l(\Omega)} &= \sum_{|\alpha|\le[l]} \|D_x^\alpha u\|^r_{L_r(\Omega)}\\
&\quad + \sum_{|\alpha|=[l]} \int_\Omega\int_\Omega \frac{|D_x^\alpha u(x,t) - D_y^\alpha u(y,t)|^r}{|x-y|^{3+r(l-[l])}}\,dx\,dy\\
&\equiv \sum_{|\alpha|\le[l]} \|D_x^\alpha u\|^r_{L_r(\Omega)} + \langle\!\langle u\rangle\!\rangle^r_{l,r,\Omega}, \qquad (1.8)\\
\|u\|^r_{W_r^{l/2}((0,T))} &= \sum_{j=0}^{[l/2]} \|\partial_t^j u\|^r_{L_r((0,T))}\\
&\quad + \int_0^T\int_0^T \frac{|\partial_t^{[l/2]} u(x,t) - \partial_\tau^{[l/2]} u(x,\tau)|^r}{|t-\tau|^{1+r(l/2-[l/2])}}\,dt\,d\tau\\
&\equiv \sum_{j=0}^{[l/2]} \|\partial_t^j u\|^r_{L_r((0,T))} + \langle\!\langle u\rangle\!\rangle^r_{l,r,(0,T)}.
\end{aligned}$$

where $D_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$, $\partial_x = \partial/\partial x$, and we use generalized (Sobolev) derivatives. We also introduce

$$\langle u\rangle_{t,\alpha,(0,T)} = \sup_{t,t'\in(0,T)} \frac{|u(t)-u(t')|}{|t-t'|^\alpha}, \qquad \langle u\rangle_{\xi,\alpha,\Omega} = \sup_{\xi,\xi'\in\Omega} \frac{|u(\xi)-u(\xi')|}{|\xi-\xi'|^\alpha},$$

and

$$\langle\!\langle u\rangle\!\rangle_{l,r,\Omega^T,x}=\Big(\int\limits_0^T\langle\!\langle u\rangle\!\rangle_{l,r,\Omega}^r\,dt\Big)^{1/r},$$

$$\langle\!\langle u\rangle\!\rangle_{l,r,\Omega^T,t}=\Big(\int\limits_\Omega\langle\!\langle u\rangle\!\rangle_{l,r,(0,T)}^r\,dx\Big)^{1/r}.$$

Similarly using local coordinates and a partition of unity we introduce the norm in the space $W_r^{l,l/2}(S^T)$ of functions defined on $S^T=S\times(0,T)$, where $S=\partial\Omega$. We also use the spaces $W_r^l(\Omega)$ with norm $(1.8)_3$ for functions defined in Ω. We do not distinguish the norms of scalar and vector-valued functions. To simplify notation we write

$$\|u\|_{l,r,Q}=\|u\|_{W_r^{l,l/2}(Q)}\quad\text{ if } Q=\Omega^T \text{ or } Q=S^T,\ l\ge 0,$$
$$\|u\|_{l,r,Q}=\|u\|_{W_r^l(Q)}\quad\text{ if } Q=\Omega \text{ or } Q=(0,T),\ l\ge 0,$$

and $W_r^{0,0}(Q)=W_r^0(Q)=L_r(Q)$. In the case $r=2$ we have $W_2^l(Q)=H^l(Q)$ and $\|u\|_{l,Q}=\|u\|_{l,2,Q}$. Moreover,

$$\|u\|_{L_p(Q)}=|u|_{p,Q},\quad 1\le p\le\infty,$$
$$\|u\|_{0,Q}=|u|_{2,Q},\quad \|u\|_{l,r,p,\Omega^T}=\|u\|_{L_p(0,T;W_r^l(\Omega))}.$$

We also introduce the spaces $\Gamma_k^l(\Omega)$ and $\Gamma_{k,r}^{l,l/2}(\Omega)$ with the norms

$$\|u\|_{\Gamma_{k,r}^l(\Omega)}=\sum_{i\le l-k}\|\partial_t^i u\|_{l-i,r,\Omega}\equiv|u|_{l,k,r,\Omega},\quad \|u\|_{\Gamma_{k,2}^l(\Omega)}\equiv|u|_{l,k,\Omega},$$
$$\|u\|_{\Gamma_{k,r}^{l,l/2}(\Omega)}=\sum_{2i\le l-k}\|\partial_t^i u\|_{l-2i,\Omega}\equiv \big|\!\big| u\big|\!\big|_{l,k,r,\Omega},$$

where $0<l,\ k\in\mathbb{R}$. We introduce

$$|u|_{l,k}=\sum_{0\le i\le l-k}\sum_{|\alpha|=l-i}|D_x^\alpha\partial_t^i u|,$$

where $|\ |$ is the Euclidean norm either of a vector or of a matrix.

Finally, we define the spaces $C(0,T;\Gamma_{k,r}^l(\Omega))$, $C(0,T;\Gamma_{k,r}^{l,l/2}(\Omega))$, $L_\infty(0,T;\Gamma_{k,r}^l(\Omega))$, $L_\infty(0,T;\Gamma_{k,r}^{l,l/2}(\Omega))$ with the norms

$$\|u\|_{L_\infty(0,T;\Gamma_{k,r}^l(\Omega))}=|u|_{l,k,r,\infty,\Omega^T},$$
$$\|u\|_{L_\infty(0,T;\Gamma_{k,r}^{l,l/2}(\Omega))}=\big|\!\big| u\big|\!\big|_{l,k,r,\infty,\Omega^T}.$$

Moreover, we shall use the imbedding (see [5, 13])

$$W_r^\delta(\Omega)\subset L_p^\alpha(\Omega),\quad \Omega\subset\mathbb{R}^3,\ \alpha+3/r-3/p\le\delta, \tag{1.9}$$

and the corresponding interpolation inequality

$$\sum_{|\beta|=\alpha} |D_x^\beta u|_{p,\Omega} \le \varepsilon^{1-\kappa} \sum_{|\gamma|=\delta} |D_x^\gamma u|_{r,\Omega} + c\varepsilon^{-\kappa}|u|_{r,\Omega}\,, \tag{1.10}$$

where $\kappa = \alpha/\delta + (3/\delta)(1/r - 1/p) < 1$, ε is an arbitrary parameter and

$$\|u\|_{L_p^k(\Omega)} = \sum_{|\beta|=k} |D_x^\beta u|_{p,\Omega}\,.$$

2. Global estimates and relations

Similarly to [35] we prove

LEMMA 2.1. *For a sufficiently smooth solution of* (1.1) *we have*

$$\frac{d}{dt}\Big[\int_{\Omega_t} (\tfrac{1}{2}\varrho v^2 + \varrho h(\varrho))\,dx + p_0|\Omega_t|\Big] + \frac{\mu}{2}E_{\Omega_t}(v) + (\nu-\mu)\|\operatorname{div} v\|_{0,\Omega_t}^2 = \int_{\Omega_t} \varrho f\cdot v\,dx\,, \tag{2.1}$$

where

$$h(\varrho) = \int \frac{p(\varrho)}{\varrho^2}\,d\varrho\,, \qquad E_{\Omega_t}(v) = \int_{\Omega_t} (\partial_{x^i} v^j + \partial_{x^j} v^i)^2\,dx\,,$$

with summation over repeated indices. Moreover,

$$\frac{d}{dt}\int_{\Omega_t} \varrho v\cdot\eta\,dx = \int_{\Omega_t} \varrho f\cdot\eta\,dx\,, \tag{2.2}$$

where $\eta = a + b\times x$, with a, b arbitrary constant vectors, *is a vector such that $E_{\Omega_t}(\eta) = 0$. Finally*,

$$\frac{d}{dt}\int_{\Omega_t} \varrho x\,dx = \int_{\Omega_t} \varrho v\,dx\,. \tag{2.3}$$

From the thermodynamically justified inequality

$$\nu - \frac{1}{3}\mu \ge 0\,, \tag{2.4}$$

we obtain (see [35])

$$\frac{\mu}{2}E_{\Omega_t}(v) + (\nu-\mu)\|\operatorname{div} v\|_{0,\Omega_t}^2 \ge 0\,. \tag{2.5}$$

Hence for $f = 0$ we obtain from (2.1) the inequality

$$\tag{2.6} \frac{1}{2}\int_{\Omega_t} \varrho v^2\,dx + \int_{\Omega_t} \varphi(\varrho)\,dx + p_0|\Omega_t| \le \frac{1}{2}\int_{\Omega} \varrho_0 v_0^2\,dx + \int_{\Omega} \varphi(\varrho_0)\,dx + p_0|\Omega| \equiv d\,,$$

where $\varphi(\varrho) = \frac{A}{\kappa-1}\varrho^\kappa$, $\kappa > 1$, since $p(\varrho) = A\varrho^\kappa$.

In the same way as in [35] we obtain the relations

$$\tag{2.7} \left(\frac{A}{\kappa-1}\frac{M^\kappa}{d}\right)^{1/(\kappa-1)} \le |\Omega_t| \le \frac{d}{p_0}\,,$$

$$\tag{2.8} \frac{Mp_0}{d} \le \overline{\varrho}_t \le \left(\frac{(\kappa-1)d}{AM}\right)^{1/(\kappa-1)}\,,$$

$$\tag{2.9} (p_0/d)^{\kappa-1}M^\kappa \le \psi_t \le (\kappa-1)d/A\,,$$

where

$$\overline{\varrho}_t = \frac{M}{|\Omega_t|}\,, \qquad \psi_t = \frac{A}{\kappa-1}\int_{\Omega_t} \varrho^\kappa(x,t)\,dx\,.$$

Thus $|\Omega_t|$ and ψ_t are bounded from below and from above for all $t \in \mathbb{R}_+$.

Multiplying (2.6) by $|\Omega_t|^{\kappa-1}$, using (1. 5) and the Hölder inequality $(\int_{\Omega_t} \varrho\,dx)^\kappa \le |\Omega_t|^{\kappa-1}\int_{\Omega_t} \varrho^\kappa\,dx$ we obtain

$$\tag{2.10} y(|\Omega_t|) + |\Omega_t|^{\kappa-1}\frac{1}{2}\int_{\Omega_t} \varrho v^2\,dx + \frac{A}{\kappa-1}\Big(|\Omega_t|^{\kappa-1}\int_{\Omega_t} \varrho^\kappa\,dx - \Big(\int_{\Omega_t} \varrho\,dx\Big)^\kappa\Big) \le 0\,,$$

where

$$\tag{2.11} y(x) = p_0 x^\kappa - dx^{\kappa-1} + \frac{AM^\kappa}{\kappa-1}\,, \qquad \kappa > 1\,.$$

Since the last two terms in (2.10) are positive, (2.10) implies that for physical motions

$$\tag{2.12} y(|\Omega_t|) \le 0\,.$$

Our aim is to find restrictions on the coefficients of the polynomial $y = y(x)$ which guarantee that $\operatorname{var}_{t\in\mathbb{R}^1}|\Omega_t|$ is sufficiently small for all $|\Omega_t|$ satisfying (2.12). The function $y = y(x)$ has only one extremum point determined by the equation

$$\tag{2.13} y'(x) \equiv [p_0\kappa x - d(\kappa-1)]x^{\kappa-2} = 0\,, \quad \text{so} \quad x_0 = \frac{d(\kappa-1)}{p_0\kappa}\,,$$

which is a minimum because $y''(x_0) = d(\kappa-1)x_0^{\kappa-3} > 0$.

Since $y(0)=\frac{A}{\kappa-1}M^\kappa$ and $y(\infty)=\infty$ we wish to find where $y(x)\le 0$ and find conditions implying that $-y(x_0)$ is small. Using (2.13) we examine the quantity

$$-y(x_0)=\left[\left(\frac{(\kappa-1)d}{\kappa p_0}\right)^\kappa-\frac{AM^\kappa}{p_0}\right]\frac{p_0}{\kappa-1}\,. \tag{2.14}$$

In order to show that $-y(x_0)$ is positive and small we consider the difference

$$\left(\frac{\kappa-1}{\kappa}\frac{d}{p_0}\right)^\kappa-\left(\left(\frac{A}{p_0}\right)^{1/\kappa}M\right)^\kappa.$$

By the Hölder and Young inequalities we have

$$\begin{aligned}\left(\frac{A}{p_0}\right)^{1/\kappa}M&=\left(\frac{A}{p_0}\right)^{1/\kappa}\int_\Omega\varrho_0\,dx\le\left(\frac{A}{p_0}\right)^{1/\kappa}|\Omega|^{(\kappa-1)/\kappa}\Big(\int_\Omega\varrho_0^\kappa\,dx\Big)^{1/\kappa}\\&\le\frac{\kappa-1}{\kappa}|\Omega|+\frac{A}{\kappa p_0}\int_\Omega\varrho_0^\kappa\,dx=\frac{\kappa-1}{\kappa}\frac{d}{p_0}-\frac{\kappa-1}{2\kappa p_0}\int_\Omega\varrho_0v_0^2\,dx\,,\end{aligned}\tag{2.15}$$

where the last equality follows from the definition of d.

Using (2.15) in (2.14) yields

$$\begin{aligned}-y(x_0)&\ge\left[\left(\frac{(\kappa-1)d}{\kappa p_0}\right)^\kappa-\left(\frac{(\kappa-1)d}{\kappa p_0}-\frac{\kappa-1}{2\kappa p_0}\int_\Omega\varrho_0v_0^2\,dx\right)^\kappa\right]\frac{p_0}{\kappa-1}\\&=\left(\frac{\kappa-1}{\kappa p_0}\right)^{\kappa-2}(d-\tfrac12x)^{\kappa-1}\int_\Omega\varrho_0v_0^2\,dx>0\,,\end{aligned}\tag{2.16}$$

where $x\in(0,\int_\Omega\varrho_0v_0^2\,dx)$.

Let the initial state be the constant state described by (1.6). Then

$$v_0=0\,,\qquad\varrho_0=\varrho_e=\text{const}\,,\qquad p_0=A\varrho_0^\kappa=A\varrho_e^\kappa\,.\tag{2.17}$$

Hence $\frac{\kappa-1}{\kappa}\frac{d}{p_0}=|\Omega|$ and (2.16) becomes

$$(A/p_0)^{1/\kappa}M=(A/p_0)^{1/\kappa}\varrho_0|\Omega|=|\Omega|\,,$$

so $y(|\Omega|)=0$. Therefore, (2.15) and (2.16) imply that for a state much different from the constant state (2.17) the quantity $-y(x_0)$ must be large.

Now we estimate $-y(x_0)$ in terms of the quantities which measure the difference between the constant state (1.6) and the considered initial state. In view of Definition 1.1 we have $p_0=A\varrho_e^\kappa$ and $M=|\Omega_e|\varrho_e$. We write (2.14) in the form

$$\begin{aligned}-y(x_0)=\bigg[\bigg(|\Omega|+\frac{A}{\kappa p_0}\int_\Omega(\varrho_0^\kappa-\varrho_e^\kappa)\,dx+\frac{\kappa-1}{2\kappa p_0}\int_\Omega\varrho_0v_0^2\,dx\bigg)^\kappa\\-\left(\frac{\overline{\varrho}_0}{\varrho_e}\right)^\kappa|\Omega|^\kappa\bigg]\frac{p_0}{\kappa-1}\,,\end{aligned}$$

where $\overline{\varrho}_0 = (1/|\Omega|)\int_\Omega \varrho_0\,dx = M/|\Omega|$, so using the Taylor formula we have

$$
-y(x_0) = \frac{(|\Omega|+\delta)^{\kappa-1}}{\kappa-1}\bigg(\int\limits_\Omega (A\varrho_0^\kappa - p_0)\,dx + \frac{\kappa-1}{2}\int\limits_\Omega \varrho_0 v_0^2\,dx + \kappa A(\varrho_e^\kappa - \overline{\varrho}_0^\kappa)|\Omega|^\kappa\bigg), \tag{2.18}
$$

where

$$
0 \le \delta \le \frac{A}{\kappa p_0}\int\limits_\Omega (\varrho_0^\kappa - \varrho_e^\kappa)\,dx + \frac{\kappa-1}{2\kappa p_0}\int\limits_\Omega \varrho_0 v_0^2\,dx\,.
$$

Hence in general $y(x_0) < 0$ so the equation $y(x) = 0$ has two different solutions. Denote them by w_1, w_2. From $y''(x_0) > 0$ it follows that there exists an interval $(x_0 - h_*, x_0 + h_*)$ such that $y''(x) > 0$ is positive and is separated from zero for $x \in (x_0 - h_*, x_0 + h_*)$. Moreover, expanding $y = y(x)$ in a Taylor series in a neighbourhood of x_0 we obtain

$$
y(x) = y(x_0) + \tfrac{1}{2}y''(x_0+\theta h)h^2\,,
$$

hence for $x \in (w_1, w_2) \setminus (x_0 - h_*, x_0 + h_*)$ we have

$$
0 < y(x) - y(x_0) = \tfrac{1}{2}y''(x_0+\theta h)h^2\,, \quad \theta = \theta(x)\,, \quad x_0 + \theta h \in (w_1, w_2)\,,
$$

so $y''(x_0+\theta h)$ is bounded from below because $y''(x_0+\theta h) \ge (y_* - y(x_0))/h^{*2}$, where $y_* = \min\{y(x_0-h_*), y(x_0+h_*)\}$ and $h^* = \max\{x_0-w_1, w_2-x_0\}$. Therefore,

$$
h \le \left(\frac{-2y(x_0)}{y''(x_0+\theta h)}\right)^{1/2}. \tag{2.19}
$$

Thus, assuming that $-y(x_0) \le \varepsilon^2$, where ε is sufficiently small, the above arguments are valid and $y''(x) \ge y''_* > 0$ for $x \in (x_0 - h, x_0 + h) \subset (w_1, w_2)$.

Now we find an explicit bound from below for $y''(x)$, where $x \in (w_1, w_2)$ and the latter interval is assumed to be small. We also assume that initially the drop is very close to the constant state, so $|\varrho_0 - \varrho_e| \le \overline{\varepsilon}$, $\int_\Omega \varrho_0 v_0^2\,dx \le \overline{\varepsilon}$, where $\overline{\varepsilon}$ is small. We have

$$
y''(x) = (\kappa-1)x^{\kappa-3}[\kappa p_0 x - (\kappa-2)d]\,.
$$

From the conservation of mass we have

$$
|\Omega_e| - |\Omega| = \frac{1}{\varrho_e}\int\limits_\Omega (\varrho_0 - \varrho_e)\,dx\,.
$$

Moreover, for the constant state $x_0 = |\Omega_e|$. Let h be so small that $h \le \overline{\varepsilon}$. Hence $y''(|\Omega_e| + \theta h) = \kappa p_0 |\Omega_e|^{\kappa-2} + O(\overline{\varepsilon})$, so taking $\overline{\varepsilon}$ sufficiently small we get $y''(|\Omega| + \theta h) \ge \frac{1}{2}\kappa p_0 |\Omega_e|^{\kappa-2}$. Thus

$$
\sup_t \operatorname{var}|\Omega_t| \le c(-y(x_0))^{1/2}\,. \tag{2.20}
$$

Moreover, since $\psi_t = \frac{A}{\kappa-1}\int_{\Omega_t} \varrho^\kappa\, dx$ is bounded (see (2.9)) we also have

$$\sup_t \operatorname{var} \psi_t \le c(-y(x_0))^{1/2}\,. \tag{2.21}$$

Thus, we have proved

LEMMA 2.2. *Let v, ϱ and Ω_t be a sufficiently smooth solution of* (1.1), *for $f = 0$. Let $\varepsilon > 0$. Then there exist ϱ_0, v_0 and $\varepsilon_* = \varepsilon_*(\varepsilon) = O(\varepsilon)$ such that if $|\varrho_0 - \varrho_e| \le \varepsilon_*$, $|A\varrho_0^\kappa - p_0| \le \varepsilon_*$, $|v_0| \le \varepsilon_*$ then $-y(x_0) \le \varepsilon^2$, so by* (2.20) *and* (2.21),

$$\sup_t \operatorname{var} |\Omega_t| \le c_1\varepsilon\,, \qquad \sup_t \operatorname{var} \psi_t \le c_2\varepsilon\,. \tag{2.22}$$

Moreover, if the considered drop is initially in the constant state (2.17) then it remains in the constant state for all time, because (2.10) implies that $v = 0$ and ϱ must be a constant.

To prove global existence we need

Remark 2.3. Let the assumptions of Lemma 2.2 be satisfied. Then the following minima and maxima are attained:

$$|\Omega_*| = \min_t |\Omega_t|, \quad \psi_* = \min_t \psi_t, \quad |\Omega^*| = \max_t |\Omega_t|, \quad \psi^* = \max_t |\psi_t|\,;$$

moreover, $|\,|\Omega^*| - |\Omega_*|\,| \le c_1\varepsilon$, $\psi^* - \psi_* \le c_2\varepsilon$. Then writing (2.1) in the form

$$\frac{d}{dt}\Big(\frac12 \int_{\Omega_t} \varrho v^2\, dx + \psi_t + p_0|\Omega_t|\Big) + \frac{\mu}{2}E_{\Omega_t}(v) + (\nu-\mu)\|\operatorname{div} v\|^2_{0,\Omega_t} = 0\,, \tag{2.23}$$

we obtain

$$\begin{aligned}\frac12 \int_{\Omega_t} \varrho v^2\, dx + \psi_t - \psi_* + p_0(|\Omega_t| - |\Omega_*|) \\ + \int_0^t \Big[\frac{\mu}{2}E_{\Omega_\tau}(v) + (\nu-\mu)\|\operatorname{div} v\|^2_{0,\Omega_\tau}\Big]\, d\tau \\ = \frac12 \int_{\Omega} \varrho_0 v_0^2\, dx + \psi - \psi_* + p_0(|\Omega| - |\Omega_*|) \le \kappa_0\varepsilon_0\,,\end{aligned} \tag{2.24}$$

where $\psi = \psi_0$ and $\varepsilon_0 = \varepsilon_0(\varepsilon) = O(\varepsilon)$.

Remark 2.4. Assume $f = 0$ and

$$\int_{\Omega} \varrho_0 v_0 \cdot \eta\, d\xi = 0\,, \qquad \int_{\Omega} \varrho_0 \xi\, d\xi = 0\,. \tag{2.25}$$

Then (2.2) and $(2.25)_1$ imply

$$\int_{\Omega_t} \varrho v \cdot \eta\, dx = 0\,. \tag{2.26}$$

Moreover, (2.3) and $(2.25)_2$ give

$$\int\limits_{\Omega_t} \varrho x\,dx = 0\,. \tag{2.27}$$

The last condition guarantees that the barycentre of Ω_t coincides with the origin of coordinates.

3. Local existence

To prove local existence of solutions to (1.1) we write it in the Lagrangian coordinates introduced by (1.3) and (1.4):

$$\begin{aligned}
&\eta u_t - \mu\nabla_u^2 u - \nu\nabla_u\nabla_u\cdot u + \nabla_u q = \eta g && \text{in } \Omega^T,\\
&\eta_t + \eta\nabla_u\cdot u = 0 && \text{in } \Omega^T,\\
&\mathbb{T}_u(u,q)\overline{n} = -p_0\overline{n} && \text{on } S^T,\\
&u|_{t=0} = v_0 && \text{in } \Omega,\\
&\eta|_{t=0} = \varrho_0 && \text{in } \Omega,
\end{aligned} \tag{3.1}$$

where $\eta(\xi,t) = \varrho(X_u(\xi,t),t)$, $q(\xi,t) = p(X_u(\xi,t),t)$, $g(\xi,t) = f(X_u(\xi,t),t)$, $\nabla_u = \xi_x^i\nabla_{\xi^i}$, $\nabla_{\xi^i} = \partial_{\xi^i}$, $\mathbb{T}_u(u,q) = -q\delta + \mathbb{D}_u(u)$, $\xi_x^i = \partial_x\xi^i$, $\delta = \{\delta_{ij}\}$ is the identity matrix and

$$\mathbb{D}_u(u) = \{\mu(\xi_{x^i}^k\nabla_{\xi^k}u^j + \xi_{x^j}^k\nabla_{\xi^k}u^i) + (\nu-\mu)\delta_{ij}\nabla_u\cdot u\}, \qquad \nabla_u\cdot u = \xi_{x^i}^k\nabla_{\xi^k}u^i\,,$$

where the summation convention is understood. Let A be the Jacobi matrix of the transformation $x = x(\xi,t)$ with elements $a_{ij} = \delta_{ij} + \int_0^t \partial_{\xi^j}u^i(\xi,\tau)\,d\tau$. Assuming $|\nabla_\xi u|_{\infty,\Omega^T} \le M$ we obtain

$$0 < c_1(1-Mt)^3 \le \det\{\partial_\xi x\} \le c_2(1+Mt)^3\,, \qquad t \le T\,, \tag{3.2}$$

where c_1, c_2 are constants and T is sufficiently small. Moreover,

$$\det A = \exp\Big(\int\limits_0^t \nabla_u\cdot u\,d\tau\Big) = \varrho_0/\eta\,.$$

Let S_t be determined at least locally by the equation $\phi(x,t) = 0$. Then S is described by $\phi(x(\xi,t),t)\,|_{t=0} \equiv \widetilde{\phi}(\xi) = 0$ and we have

$$\overline{n}(x(\xi,t),t) = \frac{\nabla_x\phi(x,t)}{|\nabla_x\phi(x,t)|}\bigg|_{x=x(\xi,t)}, \qquad \overline{n}_0(\xi) = \frac{\nabla_\xi\widetilde{\phi}(\xi)}{|\nabla_\xi\widetilde{\phi}(\xi)|}\,.$$

First we consider the problem

$$\begin{aligned}
&u_t - \mu\nabla_\xi^2 u - \nu\nabla_\xi\nabla_\xi\cdot u = F_1 && \text{in } \Omega^T,\\
&\mathbb{D}_\xi(u)\overline{n}_0 = G_1 && \text{on } S^T,\\
&u|_{t=0} = u_1 && \text{in } \Omega,
\end{aligned} \tag{3.3}$$

where $\mathbb{D}_\xi(u) = \{\mu(\partial_{\xi^i}u^j + \partial_{\xi^j}u^i) + (\nu-\mu)\delta_{ij}\partial_{\xi^k}u^k\}$.

From [24, 36] we obtain

LEMMA 3.1. *Let* $S \in W_r^{2l+2-1/r}$, $u_1 \in \Gamma_{0,r}^{2l+2-2/r,l+1-1/r}(\Omega)$, $F_1 \in W_r^{2l,l}(\Omega^T)$, $G_1 \in W_r^{2l+1-1/r,l+1/2-1/(2r)}(S^T)$, $3 < r \in \mathbb{R}$, $0 \le l \in \mathbb{Z}$, $l + 1/2 - 3/(2r) \notin \mathbb{Z}$, $T < \infty$. *Then there exists a unique solution to problem* (3.3) *such that* $u \in W_r^{2l+2,l+1}(\Omega^T)$ *and*

$$\|u\|_{2l+2,r,\Omega^T} \le c(T)(\|F_1\|_{2l,r,\Omega} + \|G_1\|_{2l+1-1/r,r,S^T} + |\!|\!| u_1 |\!|\!|_{2l+2-2/r,0,r,\Omega}), \tag{3.4}$$

where $c(T)$ *is an increasing function.*

The condition $r > 3$ is assumed to omit coefficients of type T^{-a}, $a > 0$, by the lower derivatives in the boundary norm (see [24], (5.11) and the following considerations, and [36]).

Remark 3.2. Let $(3.3)_1$ be written in the form $u_t = Au + F_1$. Then the compatibility conditions for system (3.3) are

$$\mathbb{D}_\xi(\partial_t^i u|_{t=0})\overline{n}_0 = \partial_t^i G_1|_{t=0} \quad \text{on } S\,,\ i \le s = [l + 1/2 - 3/(2r)]\,, \tag{3.5}$$

where $\partial_t^i u|_{t=0} = (A\partial_t^{i-1}u + \partial_t^{i-1}F_1)|_{t=0}$ are calculated inductively ($[\sigma]$ is the integer part of σ). The number s is such that $\partial_t^s \mathbb{D}_\xi(u)|_{S,t=0}$ is meaningful by imbedding theorems. Therefore at step s we have a relation between the derivatives

$$D_\xi^\alpha u_1|_S\,, \quad |\alpha| = 2s+1, \quad D_\xi^\beta \partial_t^i F_1|_{S,t=0}, \quad |\beta| + 2i = 2s-1, \quad \partial_t^s G_1|_{t=0}\,.$$

Now we consider the following problem:

$$\begin{aligned} &\eta u_t - \mu\nabla_\xi^2 u - \nu\nabla_\xi\nabla_\xi\cdot u = F_2 && \text{in } \Omega^T\,,\\ &\mathbb{D}_\xi(u)\cdot\overline{n}_0 = G_2 && \text{on } S^T\,,\\ &u|_{t=0} = u_2 && \text{in } \Omega\,. \end{aligned} \tag{3.6}$$

Lemma 3.1 implies

LEMMA 3.3. *Assume that* $F_2 \in W_r^{2l,l}(\Omega^T)$, $G_2 \in W_r^{2l+1-1/r,l+1/2-1/(2r)}(S^T)$, $S \in W_r^{2l-1/r}$, $u_2 \in \Gamma_{0,r}^{2l+2-2/r,l+1-1/r}(\Omega)$, $\eta \in W_r^{2l+1,l+1/2}(\Omega^T) \cap L_\infty(0,T;$ $\Gamma_{0,r}^{2l+1,l+1/2}(\Omega))$, $1/\eta \in L_\infty(\Omega^T)$, $\eta \in C^\alpha(\Omega^T)$, $\alpha \in (0,1)$, $(2l+1)r > 3$ *and* $0 \le l \in \mathbb{Z}$, $3 < r \in \mathbb{R}$, $l + 1/2 - 3/(2r) \notin \mathbb{Z}$, $T < \infty$. *Let the compatibility conditions up to order* $s < l + 1/2 - 3/(2r)$ *hold. Then there exists a unique solution to problem* (3.6) *such that* $u \in W_r^{2l+2,l+1}(\Omega^T)$, *and*

$$\begin{aligned} \|u\|_{2l+2,r,\Omega^T} &\le \varphi_1(|1/\eta|_{\infty,\Omega^T}, |\!|\!|\eta|\!|\!|_{2l+1,0,r,\infty,\Omega^T}, T)\\ &\quad\times[\|F_2\|_{2l,r,\Omega^T} + \|G_2\|_{2l+1-1/r,r,S^T} + |\!|\!|u_2|\!|\!|_{2l+2-2/r,0,r,\Omega} + \|u\|_{2l,r,\Omega^T}]\,, \end{aligned} \tag{3.7}$$

where φ_1 *is a positive increasing function of its arguments.*

In the sequel we assume the two conditions $(2l+1)r > 3$ and $r > 3$ which have different origin. The first follows from imbedding theorems used to estimate the nonlinear terms and the second implies that (3.4) holds for small T with a constant independent of T (see explanation after (3.4)).

Proof. The existence follows from [24]. The compatibility conditions follow from the considerations of Remark 3.2 applied to problem (3.6). Therefore we only have to show the estimate.

Let us introduce a partition of unity $\{\zeta_k(\xi,t),Q_k\}$ (see [22]), $Q_k=\operatorname{supp}\zeta_k$, $k=1,\dots,N$, such that $\sum_{k=1}^N\zeta_k(\xi,t)=1$, $\xi\in\Omega$, $t\in(0,T)$, $\lambda=\max\operatorname{diam}Q_k$, $\zeta_k\ge 0$, $0<\mu_0\le\sum_{k=1}^N\zeta_k^2(\xi,t)\le N_0$ and $|D^\alpha_{\xi,t}\zeta_k(\xi,t)|\le c|\lambda|^{-|\alpha|}$, where $D^\alpha_{\xi,t}=\partial_t^{\alpha_0}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}$, $|\alpha|=\alpha_0+\alpha_1+\alpha_2+\alpha_3$. Let $u_k=u\zeta_k$, $F_{2k}=F_2\zeta_k$, $G_{2k}=G_2\zeta_k$, $u_{2k}=u_2\zeta_k$. Thus (3.6) yields a system of problems

$$
\begin{aligned}
&\eta_k u_{kt}-\mu\nabla^2_\xi u_k-\nu\nabla_\xi\nabla_\xi\cdot u_k=F_{2k}+(\eta_k-\eta)u_{kt}-\mu[\nabla^2_\xi,\zeta_k]u\\
&\qquad\qquad -\nu[\nabla_\xi\nabla_\xi\cdot,\zeta_k]u+\eta\zeta_{kt}u\equiv F'_{2k},\\
&\mathbb{D}_\xi(u_k)\cdot\overline{n}_0=G_{2k}+u\mathbb{D}_\xi(\zeta_k)\cdot\overline{n}_0\equiv G'_{2k},\\
&u_k|_{t=0}=u_{2k},
\end{aligned}\tag{3.8}
$$

where $\eta_k=\eta(\xi_k,t_k)$, $(\xi_k,t_k)\in Q_k$, $[L,u]v=L(uv)-uL(v)$ and L is an operator.

Replace t by $\tau=\eta_k^{-1}t$; then $\widetilde{u}_k=u_k|_{t=\tau}$ satisfies

$$
\begin{aligned}
&\widetilde{u}_{k\tau}-\mu\nabla^2_\xi\widetilde{u}_k-\nu\nabla_\xi\nabla_\xi\cdot\widetilde{u}_k=\widetilde{F}_{2k}+(1-\widetilde{\eta}/\eta_k)|_{t=\tau}\widetilde{u}_{k\tau}\\
&\qquad\qquad +\widetilde{\eta}\widetilde{u}\widetilde{\zeta}_{k\tau}-\mu[\nabla^2_\xi,\widetilde{\zeta}_k]\widetilde{u}-\nu[\nabla_\xi\nabla_\xi\cdot,\widetilde{\zeta}_k]\widetilde{u}\equiv\widetilde{F}'_{2k},\\
&\mathbb{D}_\xi(\widetilde{u}_k)\cdot\overline{n}_0=\widetilde{G}_{2k}+\widetilde{u}\mathbb{D}_\xi(\widetilde{\zeta}_k)\cdot\overline{n}_0\equiv\widetilde{G}'_{2k},\\
&\widetilde{u}_k|_{\tau=0}=\widetilde{u}_{2k},
\end{aligned}\tag{3.9}
$$

where we have used the fact that $\widetilde{y}=y(\xi,t)|_{t=\tau}$.

Applying Lemma 3.1 to problem (3.9) we obtain

$$
\begin{aligned}
&\Big(\sum_{2i+|\alpha|\le 2l+2}\int_0^T\int_\Omega|D^\alpha_\xi\partial_t^iu_k|^r\eta_k^{ir-1}\,d\xi\,dt\Big)^{1/r}\\
&\le c(T)\bigg[\Big(\sum_{2i+|\alpha|\le 2l}\int_0^T\int_\Omega|D^\alpha_\xi\partial_t^iF'_{2k}|^r\eta_k^{ir-1}\,d\xi\,dt\Big)^{1/r}\\
&\quad+\Big(\sum_{2i+|\alpha|\le 2l}\int_0^T\int_S|D^\alpha_{\xi'}\partial_t^iG'_{2k}|^r\eta_k^{ir-1}\,d\xi'\,dt\\
&\quad+\sum_{|\alpha|=2l}\int_0^T\int_S\int_S\frac{|D^\alpha_\xi G'_{2k}(\xi,t)-D^\alpha_{\xi'}G'_{2k}(\xi',t)|^r}{|\xi-\xi'|^{1+r}}\eta_k^{-1}\,d\xi\,d\xi'\,dt\\
&\quad+\int_S d\xi'\int_0^T\int_0^T\frac{|\partial_t^lG'_{2k}(\xi',t)-\partial_{t'}^lG'_{2k}(\xi',t')|^r}{|t-t'|^{1/2+r/2}}\eta^{lr+r/2-3/2}d\xi'\,dt\,dt'\Big)^{1/r}\\
&\quad+|u_{2k}|_{2l+2-2/r,0,r,\Omega}\bigg].
\end{aligned}\tag{3.10}
$$

Since $\min \eta(\xi,t)$ and $\max \eta(\xi,t)$ are attained, from (3.10) we obtain

$$\begin{aligned}(3.11)\qquad \|u_k\|_{2l+2,r,\Omega^T} &\le c(T,|\eta|_{\infty,\Omega^T},|1/\eta|_{\infty,\Omega^T})\\ &\quad\times[\|F'_{2k}\|_{2l,r,\Omega^T}+\|G'_{2k}\|_{2l+1-1/r,r,\Omega^T}\\ &\quad+\| u_{2k}\|_{2l+2-2/r,0,r,\Omega^T}].\end{aligned}$$

Now we shall estimate the norms of F'_{2k} and G'_{2k}. First we estimate the second term in F'_{2k}. We have

$$(3.12)\qquad \|(\eta_k-\eta)u_{kt}\|_{2l,r,\Omega^T}\le c\lambda^\alpha|\eta|_{C^\alpha(\Omega^T)}\|u_{kt}\|_{2l,r,\Omega^T}+I_1\,,$$

where to estimate I_1 it is sufficient to consider the expression

$$\sum_{1\le s\le 2l}\sum_{|\alpha-\beta|=2l-s}\sum_{|\beta|=s}c_s|D_\xi^\beta\eta D_\xi^{\alpha-\beta}u_{2kt}|_{r,\Omega^T}+\sum_{1\le s\le l}c_s|\partial_t^s\eta\partial_t^{l-s}u_{kt}|_{r,\Omega^T}\equiv I_2+I_3\,.$$

By the Hölder inequality,

$$I_2\le\sum_{1\le s\le 2l}c_s\Big(\int_0^T|D_\xi^{\gamma_s}\eta|^r_{rp_s,\Omega}|D_\xi^{\gamma_{2l-s}}u_{kt}|^r_{rp'_s,\Omega}\,dt\Big)^{1/r}\equiv I_4\,,$$

where $1/p_s+1/p'_s=1$ and γ_s is a multiindex such that $|\gamma_s|=s$. By the imbedding (1.9) and the interpolation inequality (1.10) we have

$$I_4\le\sum_{1\le s\le 2l}c_s\Big(\int_0^T\|\eta\|^r_{2l+1,r,\Omega}(\varepsilon_1^{1-\kappa_s}|D_\xi^{\gamma_{2l}}u_{kt}|^r_{r,\Omega}+\varepsilon_1^{-\kappa_s}|u_{kt}|^r_{r,\Omega})\,dt\Big)^{1/r}\equiv I_5\,,$$

provided $(3/r)(1-1/p_s)\le 2l+1-s$ and $\kappa_s=(1/(2l))(2l-s+(3/r)(1-1/p'_s))<1$ so the last inequality holds for $3/r<2l+1$. Continuing we get

$$\begin{aligned}I_5&\le\varepsilon|D_\xi^{\gamma_{2l}}u_{kt}|_{r,\Omega^T}+c(\varepsilon)\Big(\sum_s\sup_t\|\eta\|^{1/(1-\kappa_s)}_{2l+1,r,\Omega}\Big)|u_{kt}|_{r,\Omega^T}\\&\le\varepsilon\|u_k\|_{2l+2,r,\Omega^T}+\varphi'_1(1/\varepsilon,\sup_t\|\eta\|_{2l+1,r,\Omega})\|u_k\|_{2l,r,\Omega^T}\,,\end{aligned}$$

where $c(\varepsilon)$ increases as ε decreases and φ'_1 is an increasing function of its arguments.

Now consider

$$\begin{aligned}I_3&\equiv\sum_{1\le s\le l}c_s|\partial_t^s\eta\partial_t^{l-s}u_{kt}|_{r,\Omega^T}\le\sum_{1\le s\le l}c_s\Big(\int_0^T|\partial_t^s\eta\partial_t^{l-s+1}u_k|^r_{r,\Omega}\,dt\Big)^{1/r}\\&\le\sum_{1\le s\le l}c_s\Big(\int_0^T|\partial_t^s\eta|^r_{p_1r,\Omega}|\partial_t^{l-s+1}u_k|^r_{p_2r,\Omega}\,dt\Big)^{1/r}\equiv I_6\,,\end{aligned}$$

where $1/p_1+1/p_2=1$. By the imbedding theorem (1.9) we have the estimates

$$|\partial_t^s\eta|_{rp_1,\Omega}\le c\|\partial_t^s\eta\|_{2l+1-2s,r,\Omega}\qquad\text{for } 3/r-3/(rp_1)<2l+1-2s\,,$$

$$|\partial_t^{l-s+1}u_k|_{rp_2,\Omega}\le c\|\partial_t^{l-s+1}u_k\|_{2s,r,\Omega}\quad \text{for } 3/r-3/(rp_2)<2l+2-2(l-s+1)\,,$$

which hold for $2l+1>3/r$, where $p_i=p_i(s)$, $i=1,2$. By means of these inequalities I_6 is estimated as follows:

$$\begin{aligned}I_6\le{}&\sup_t \| \eta \|_{2l+1,0,r,\Omega}\\ &\times\sum_{1\le s\le l}cc_s(\varepsilon_1^{1-\kappa_s}|D_\xi^{\gamma_{2s}}\partial_t^{l-s+1}u_k|_{r,\Omega^T}+c\varepsilon_1^{-\kappa_s}|\partial_t^{l-s+1}u_k|_{r,\Omega^T})\equiv I_7\,.\end{aligned}$$

Hence, exactly in the same way as in the case of I_5, we obtain

$$\begin{aligned}I_7&\le\varepsilon\|u_k\|_{2l+2,r,\Omega^T}+c(\varepsilon)\Big(\sum_s\sup_t \| \eta \|_{2l+1,0,r,\Omega}^{1/(1-\kappa_s)}\Big)\|u_k\|_{2l,r,\Omega^T}\\ &\equiv\varepsilon\|u_k\|_{2l+2,r,\Omega^T}+\varphi_2'(1/\varepsilon,\sup_t \| \eta \|_{2l+1,0,r,\Omega})\|u_k\|_{2l,r,\Omega^T}\,,\end{aligned}$$

where φ_2' is an increasing function of its arguments.

Assume λ in (3.12) is such that

$$c\lambda^\alpha|\eta|_{C^\alpha(\Omega^T)}=\varepsilon\,. \tag{3.13}$$

Then by continuing the above considerations, (3.12) gives

$$\begin{aligned}\|(\eta_k-\eta)u_{kt}\|_{2l,r,\Omega^T}\le{}&\varepsilon\|u\|_{2l+2,r,\Omega^T}+\varphi_3'(1/\varepsilon,\sup_t \| \eta \|_{2l+1,0,r,\Omega},\\ &|\eta|_{C^\alpha(\Omega^T)},|1/\eta|_{\infty,\Omega^T},T)[\|F_2\|_{2l,r,Q_k}+\|u_k\|_{2l,r,\Omega^T}]\,,\end{aligned}\tag{3.14}$$

where φ_3' is an increasing function. Employing the same considerations for other terms on the right-hand side of (3.11) we get

$$\begin{aligned}&\|u_k\|_{2l+2,r,\Omega^T}\\ &\quad\le\varepsilon\|u\|_{2l+2,r,Q_k}+\varphi_4'(1/\varepsilon,\sup_t \| \eta \|_{2l+1,0,r,\Omega},|\eta|_{C^\alpha(\Omega^T)},|1/\eta|_{\infty,\Omega^T},T)\\ &\qquad\times[\|F_2\|_{2l,r,Q_k}+\|G_2\|_{2l+1-1/r,r,Q_k\cap S^T}+ \| u_{2k} \|_{2l+2-2/r,0,r,\Omega}\\ &\qquad+\|u_k\|_{2l,r,\Omega^T}]\,,\end{aligned}\tag{3.15}$$

where φ_4' is an increasing function of its arguments.

Summing (3.15) over all neighbourhoods of the partition of unity and assuming that ε is sufficiently small we obtain (3.7). This concludes the proof.

Now we consider the problem

$$\begin{aligned}&\eta u_t-\mu\nabla_w^2u-\nu\nabla_w\nabla_w\cdot u=F_3 &&\text{in } \Omega^T\,,\\ &\mathbb{D}_w(u)\overline{n}=G_3 &&\text{on } S^T\,,\\ &u|_{t=0}=u_3 &&\text{in } \Omega\,,\end{aligned}\tag{3.16}$$

where $\overline{n}=\overline{n}(X_w(\xi,t),t)$.

Lemma 3.4. *Let* $F_3 \in W_r^{2l,l}(\Omega^T)$, $G_3 \in W_r^{2l+2-1/r,l+1-1/(2r)}(S^T)$, $u_3 \in \Gamma_{0,r}^{2l+2-2/r,l+1-1/r}(\Omega)$, $S \in W_r^{2l-1/r}$, $w \in W_r^{2l+2,l+1}(\Omega^T) \cap L_\infty(0,T;\Gamma_{0,r}^{2l+2-2/r,l+1-1/r}(\Omega))$, $(2l+1)r > 3$, $r > 3$, $l + 1/2 - 3/(2r) \notin \mathbb{Z}$,$T < \infty$ *and suppose* η *satisfies the same assumptions as in Lemma* 3.3. *Let*

$$
\begin{aligned}
(3.17)\qquad & T^a(\|w\|_{2l+2,r,\Omega^T} + |\!|\!| w |\!|\!|_{2l+2-2/r,0,r,\infty,\Omega^T}) \\
&\times\varphi_2(T, \|w\|_{2l+2,r,\Omega^T}, |\!|\!| w |\!|\!|_{2l+2-2/r,0,r,\infty,\Omega^T}, \|\eta\|_{2l+1,r,\Omega^T}, \\
&\qquad \|1/\eta\|_{2l+1,r,\Omega^T}, |\!|\!| \eta |\!|\!|_{2l+1,0,r,\infty,\Omega^T}) \le \delta_*\,,
\end{aligned}
$$

where φ_2 *is an increasing function of its arguments*, $a > 0$ *and* $0 < \delta_*$ *is sufficiently small. Then there exists a unique solution to problem* (3.16) *such that* $u \in W_r^{2l+2,l+1}(\Omega^T)$, *and*

$$
\begin{aligned}
(3.18)\qquad \|u\|_{2l+2,r,\Omega^T} \le{}& \varphi_3(T, |1/\eta|_{\infty,\Omega^T}, \sup_t |\!|\!| \eta |\!|\!|_{2l+1,0,r,\Omega}, |\eta|_{C^\alpha(\Omega^T)}) \\
&\times[\|F_3\|_{2l,r,\Omega^T} + \|G_3\|_{2l+1-1/r,r,S^T} + |\!|\!| u_3 |\!|\!|_{2l+2-2/r,0,r,\Omega} + \|u\|_{2l,r,\Omega^T}]\,,
\end{aligned}
$$

where φ_3 *is a positive increasing function.*

Proof. To prove the existence of solutions to (3.16) and to find an a priori estimate we use Lemma 3.3. We write (3.16) in the form

$$
(3.19)\qquad
\begin{aligned}
&\eta u_t - \mu\nabla_\xi^2 u - \nu\nabla_\xi\nabla_\xi\cdot u = F_3 + \mu(\nabla_w^2 u - \nabla_\xi^2 u) \\
&\qquad\qquad + \nu(\nabla_w\nabla_w\cdot u - \nabla_\xi\nabla_\xi\cdot u) \equiv F_3 + \widetilde{F} && \text{in } \Omega^T, \\
&\mathbb{D}_\xi(u)\overline{n}_0 = G_3 + (\mathbb{D}_\xi(u)\overline{n}_0 - \mathbb{D}_w(u)\overline{n}) \equiv G_3 + \widetilde{G} && \text{on } S^T, \\
&u|_{t=0} = u_3 && \text{in } \Omega.
\end{aligned}
$$

Now we estimate $\widetilde{F}$ and $\widetilde{G}$. By the form of ∇_w,

$$
\begin{aligned}
(3.20)\qquad \|\widetilde{F}\|_{2l,r,\Omega^T} &\le c\|\xi_x\nabla_\xi(\xi_x\nabla_\xi u) - \nabla_\xi^2 u\|_{2l,r,\Omega^T} \\
&\le c\|(\xi_x^2 - \delta)\nabla_\xi^2 u\|_{2l,r,\Omega^T} + c\|\xi_x\nabla_\xi(\xi_x)\nabla_\xi u\|_{2l,r,\Omega^T} \equiv J_1 + J_2\,,
\end{aligned}
$$

where ξ_x is the matrix $\{\partial_{x^i}\xi^j\}$, $i,j = 1,2,3$, which can be expressed in the form $\xi_x = x_\xi^{-1} = \widetilde{x}_\xi/\det\{x_\xi\}$, where $\{\widetilde{x}_\xi\}$ is the matrix of algebraic complements for $\{x_\xi\}$ and $x_\xi = \delta + \int_0^T w_\xi\,d\tau$, where δ is the unit matrix. Hence we can write $\xi_x^2 - \delta = f(\delta + \int_0^T w_\xi\,d\tau)\int_0^T w_\xi\,d\tau$, so to estimate J_1 it is sufficient to consider the highest derivatives. Therefore we examine

$$
\begin{aligned}
(3.21)\qquad J_3 \equiv{}& \Big|D_\xi^{\gamma_{2l}}\Big(f\Big(\delta + \int_0^t w_\xi\,d\tau\Big)\int_0^t w_\xi\,d\tau\,u_{\xi\xi}\Big)\Big|_{r,\Omega^T} \\
&+\Big|\partial_t^l\Big(f\Big(\delta + \int w_\xi\,d\tau\Big)\int w_\xi\,d\tau\,u_{\xi\xi}\Big)\Big|_{r,\Omega^T} \equiv J_4 + J_5\,.
\end{aligned}
$$

First we estimate J_4. In view of the Leibniz formula,

$$J_4 \le \sum_{\varrho+\sigma\le 2l}\sum_{\{\alpha_s\}} c_{\varrho\sigma s}\Big(\int_0^T dt \int_\Omega \Big(\Big|D_\xi^{\gamma_1}\int_0^t w_\xi\,d\tau\Big|^{\alpha_1}\dots$$
$$\dots\Big|D_\xi^{\gamma_s}\int_0^t w_\xi\,d\tau\Big|^{\alpha_s}\Big|D_\xi^{\gamma_\varrho}\int_0^t w_\xi\,d\tau\Big|\,|D_\xi^{\gamma_{\sigma+2}}u|\Big)^r d\xi\Big)^{1/r} \equiv J_6\,,$$

where the summation is taken over s and $\alpha_i,\ i=1,\dots,s$, such that

$$\alpha_1+\dots+\alpha_s = 2l-\varrho-\sigma+1-s,\qquad \alpha_1+2\alpha_2+\dots+s\alpha_s = 2l-\varrho-\sigma\,, \tag{3.22}$$

$0\le s\le 2l-\sigma-\varrho$, and we recall that γ_i is a multiindex such that $|\gamma_i|=i$. Using the Hölder and Minkowski inequalities in J_6 implies

$$J_6 \le \sum_{\varrho+\sigma\le 2l}\sum_{\{\alpha_s\}} c_{\varrho\sigma s}\Big(\int_0^T dt\Big(\int_0^t |w_{\xi\xi}|_{\alpha_1 r r_1,\Omega}\,d\tau\Big)^{\alpha_1 r}\dots$$
$$\dots\Big(\int_0^t |D_\xi^{\gamma_{s+1}}w|_{\alpha_s r r_s,\Omega}\,d\tau\Big)^{\alpha_s r}\Big(\int_0^t |D_\xi^{\gamma_{\varrho+1}}w|_{pr,\Omega}\,d\tau\Big)^r |D_\xi^{\gamma_{\sigma+2}}u|^r_{qr,\Omega}\Big)^{1/r} \equiv J_7\,,$$

where

$$1/r_1+\dots+1/r_s+1/p+1/q = 1\,. \tag{3.23}$$

In view of the imbedding (1.9) we have

$$J_7 \le c\Big(\int_0^T dt\Big(\int_0^t \|w\|_{2l+2,r,\Omega}\,d\tau\Big)^{\alpha_1 r}\dots\Big(\int_0^t \|w\|_{2l+2,r,\Omega}\,d\tau\Big)^{\alpha_s r}$$
$$\times\Big(\int_0^t \|w\|_{2l+2,r,\Omega}\,d\tau\Big)^r \|u\|^r_{2l+2,r,\Omega}\Big)^{1/r} \equiv J_8\,,$$

provided the following inequalities hold:

$$\begin{aligned} &i+1+3/r-3/(\alpha_i r_i r)\le 2l+2,\qquad i=1,\dots,s\,,\\ &\varrho+1+3/r-3/(pr)\le 2l+2\,,\\ &\sigma+2+3/r-3/(qr)\le 2l+2\,. \end{aligned} \tag{3.24}$$

Multiplying $(3.24)_1$ by α_i, summing over $i=1,\dots,s$, adding to $(3.24)_{2,3}$, and using (3.22) yields

$$(2l+1-3/r)\sum\alpha_i + (2l+1-3/r)\ge 0\,,$$

which holds for $2l+1-3/r\ge 0$, because $\sum\alpha_i\ge 1$.

Since $w\in W_r^{2l+2,l+1}(\Omega^T)$, by the Hölder inequality,

$$J_8 \le cT^{(1-1/r)(\Sigma\alpha_i+1)}\|w\|^{\Sigma\alpha_i+1}_{2l+2,r,\Omega^T}\|u\|_{2l+2,r,\Omega^T}\,.$$

Now we consider J_5. By the Leibniz formula,

$$J_5 \le c \sum_{\varrho+\sigma\le l} \sum_{\{\alpha_s\}} c_{\varrho\sigma\alpha_s} \Big(\int_0^T \int_\Omega \Big(|w_\xi|^{\alpha_1} |\partial_t w_\xi|^{\alpha_2} \ldots$$
$$\ldots |\partial_t^{s-1} w_\xi|^{\alpha_s} |\partial_t^{\varrho-1} w_\xi|\, |\partial_t^\sigma u_{\xi\xi}| \Big)^r \Big)^{1/r} \equiv J_9 \,,$$

where the summation is taken over s and α_i, $i = 1, \ldots, s$, such that

$$(3.25) \qquad \alpha_1 + \ldots + \alpha_s = l - \varrho - \sigma + 1 - s, \qquad \alpha_1 + 2\alpha_2 + \ldots + s\alpha_s = l - \varrho - \sigma$$

for $1 \le s \le l - \varrho - \sigma$. By the Hölder inequality,

$$J_9 \le c \sum_{\varrho+\sigma\le l} \sum_{\{\alpha_s\}} c_{\varrho\sigma\alpha_s} \Big(\int_0^T |w_\xi|^{\alpha_1 r}_{\alpha_1 r r_1,\Omega} |\partial_t w_\xi|^{\alpha_2 r}_{\alpha_2 r r_2,\Omega} \ldots$$
$$\ldots |\partial_t^{s-1} w_\xi|^{\alpha_s r}_{\alpha_s r r_s,\Omega} |\partial_t^{\varrho-1} w_\xi|^r_{pr,\Omega} |\partial_t^\sigma u_{\xi\xi}|^r_{qr,\Omega}\, dt \Big)^{1/r} \equiv J_{10} \,,$$

where

$$(3.26) \qquad 1/r_1 + \ldots + 1/r_s + 1/p + 1/q = 1 \,.$$

Now using the imbedding (1.9) with noninteger δ yields ($\sigma < l$)

$$J_{10} \le c\|w\|^{\alpha_1}_{2l+2-2/r,r,\Omega} \|\partial_t w\|^{\alpha_2}_{2l-2/r,r,\Omega} \ldots \|\partial_t^{s-1} w\|^{\alpha_s}_{2l+2-2(s-1)-2/r,r,\Omega}$$
$$\times \|\partial_t^{\varrho-1} w\|_{2l+2-2(\varrho-1)-2/r,r,\Omega} \Big(\int_0^T \|\partial_t^\sigma u\|^{\theta r}_{2l+2-2\sigma,r,\Omega} \|\partial_t^\sigma u\|^{(1-\theta)r}_{0,r,\Omega}\, dt \Big)^{1/r},$$

provided

$$(3.27) \qquad \begin{aligned} &2(i-1) + 1 + 3/r - 3/(\alpha_i r r_i) \le 2l + 2 - 2/r, \qquad i = 1, \ldots, s, \\ &2(\varrho - 1) + 1 + 3/r - 3/(pr) \le 2l + 2 - 2/r, \\ &\theta = (2 + 3/r - 3/(qr))/(2l + 2 - 2\sigma) < 1; \end{aligned}$$

the last factor in J_{10} was estimated by using the interpolation inequality (see [5]).

Here and in the sequel we frequently use the estimate

$$\|w|_{t=t}\|_{2s-2/r,r,\Omega} \le c(\|w\|_{2s,r,\Omega^T} + \|w|_{t=0}\|_{2s-2/r,r,\Omega}), \qquad t \le T \,,$$

with a constant c independent of T (see Theorem 2 in [30] and (3) in [22]).

Multiplying $(3.27)_1$ by α_i, summing over i, using $(3.27)_{2,3}$ and (3.26) we obtain the inequality

$$(2l + 3 - 5/r) \sum \alpha_i + (2l + 3 - 5/r) > 0 \,,$$

which is satisfied for $2l + 3 - 5/r > 0$ since $2l + 1 - 3/r > -(2 - 2/r)$, which holds because $2l + 1 - 3/r > 0$ and $r \ge 1$, so $2 - 2/r \ge 0$.

Thus we have obtained

$$J_{10} \le cT^{1-\theta} \lVert w \rVert^{\Sigma\alpha_i}_{2l+2-2/r,0,r,\Omega} \sup_t \lVert u \rVert^{(1-\theta)r}_{2l+2-2/r,0,r,\Omega} \|u\|_{2l+2,r,\Omega^T} \,.$$

Let us consider separately the case $s=0$, $\varrho=0$, $\sigma=l$. Then

$$J_5 \le c\Big(\int\limits_0^T\int\limits_\Omega \Big|\int\limits_0^t w_\xi\,d\tau\Big|^r |D_\xi^{\gamma_l}u_{\xi\xi}|^r\,d\xi\,dt\Big)^{1/r} \le c\sup_\Omega\Big|\int\limits_0^T w_\xi\,d\tau\Big|\|u\|_{2l+2,r,\Omega^T}$$
$$\le c\int\limits_0^T \|w\|_{2l+2,r,\Omega}\,d\tau\,\|u\|_{2l+2,r,\Omega^T}\,,$$

for $2l+1>3/r$, where $|\gamma_l|=l$. Finally, by the Hölder inequality the above expression is estimated by

$$cT^{1-1/r}\|w\|_{2l+2,r,\Omega^T}\|u\|_{2l+2,r,\Omega^T}\,.$$

In the same way the other terms in $\widetilde F$ and $\widetilde G$ can be estimated. Summarizing, we see that (3.17) implies (3.18).

To prove the existence of solutions to problem (3.16) we use the method of successive approximations. In (3.16), replace u by u_m in the right-hand side, and by u_{m+1} in the left-hand side. Let $u_0=u_3$. Then by the contraction theorem we have the existence of solutions to (3.16) for sufficiently small T. Therefore, the lemma is proved.

A solution of $(3.1)_{2,5}$ has the form

$$\eta(\xi,t)=\varrho_0(\xi)\exp\Big[-\int\limits_0^t \nabla_u\cdot u(\xi,\tau)\,d\tau\Big]\,. \tag{3.28}$$

Hence we have

Lemma 3.5. *Assume that* $\varrho_0\in W_r^{2l+1}(\Omega)$, $u\in W_r^{2l+2,l+1}(\Omega^T)$, $2l+1>r/3$, $r>1$, $T<\infty$. *Then the solution* (3.28) *of* $(3.1)_{2,5}$ *satisfies the estimate*

$$\begin{aligned}\|\eta\|_{2l+1,r,\Omega^T}+\|1/\eta\|_{2l+1,r,\Omega^T}&\\ \le T^{1/2}(\|\varrho_0\|_{2l+1,r,\Omega}&+\|1/\varrho_0\|_{2l+1,r,\Omega})\\ \times\varphi_4(T,T^a(\|u\|_{2l+2,r,\Omega^T}&+\sup_t |u|_{2l+2-2/r,0,r,\Omega}))\,,\end{aligned}\tag{3.29}$$

where φ_4 *is an increasing positive function*, *and*

$$\begin{aligned}&|\eta|_{2l+1,0,r,\Omega^T}\\ &\quad\le\|\varrho_0\|_{2l+1,r,\Omega}\,\varphi_5(T,\|u\|_{2l+2,r,\Omega^T}+\sup_t|u|_{2l+2-2/r,0,r,\Omega})\,,\\ &\|\eta\|_{C(0,T;\Gamma_r^{2l+1}(\Omega))}\le\|\varrho_0\|_{2l+1,r,\Omega}\|u\|_{2l+2,r,\Omega^T}\\ &\qquad\times\varphi_5(T,\|u\|_{2l+2,r,\Omega^T}+\sup_t|u|_{2l+2-2/r,0,r,\Omega})\,,\end{aligned}\tag{3.30}$$

where φ_5 *is an increasing positive function and* $a>0$. *Moreover*, $\eta\in C^{\alpha,\alpha/2}(\Omega^T)$, $\alpha/2\le 1-1/r$ *and*

$$|\eta|_{C^{\alpha,\alpha/2}(\Omega^T)}\le|\varrho_0|_{C^\alpha(\Omega)}\,\varphi_6(1+T^a\|u\|_{2l+2,r,\Omega^T})\,,\tag{3.31}$$

where $a=1-1/r-\alpha/2>0$ *and* φ_6 *is an increasing positive function.*

P r o o f. First we show that $\eta \in L_\infty(0,T;\Gamma^{2l+1}_{0,r}(\Omega))$. It is sufficient to consider the expression

$$\sup_t |D^{\gamma_{2l+1}}_{t,\xi}\eta|_{r,\Omega} \le c_1 \sum_\sigma \sum_{\{\alpha_s\}} \sup_t \Big| D^{\gamma_{2l+1-\sigma}}_{t,\xi} \varrho_0 \Big(D^{\gamma_1}_{t,\xi} \int_0^t \nabla_u \cdot u\, d\tau \Big)^{\alpha_1} \ldots \ldots \Big(D^{\gamma_s}_{t,\xi} \int \nabla_u \cdot u\, d\tau \Big)^{\alpha_s} \Big|_{r,\Omega} \equiv K_1 ,$$

where γ_i is a multiindex such that $|\gamma_i| = i$, $D^\gamma_{t,\xi} = \partial_t^{\gamma_0} D_\xi^{\gamma'}$, $|\gamma| = 2\gamma_0 + |\gamma'|$, $|\gamma'| = \gamma_1 + \gamma_2 + \gamma_3$, $\alpha_1 + \alpha_2 + \ldots + \alpha_s = \sigma + 1 - s$, $\alpha_1 + 2\alpha_2 + \ldots + s\alpha_s = \sigma$, and $c_1 = c_1(|\int_0^T u_\xi\, d\tau|_{\infty,\Omega})$.

Since $\int_0^T \nabla_u \cdot u\, dt = \int_0^T \xi_x \cdot \partial_\xi u\, dt = \int_0^T f(\delta + \int_0^T u_\xi\, d\tau) \cdot u_\xi\, dt$ we have to apply the Leibniz formula also to the function $f(\delta + \int_0^T u_\xi\, d\tau)$ (see the proof of Lemma 3.4). Then the above formula for K_1 remains similar; however, the function c_1 will be different. Thus

$$K_1 \le c_2 \sum_\sigma \sum_{\{\alpha_s\}} \sup_t \Big| D^{\gamma_{2l+1-\sigma}}_{t,\xi} \varrho_0 \Big(D^{\gamma_1}_{t,\xi} \int_0^t u_\xi\, d\tau \Big)^{\alpha_1} \ldots \ldots \Big(D^{\gamma_s}_{t,\xi} \int_0^t u_\xi\, d\tau \Big)^{\alpha_s} \Big|_{r,\Omega} \equiv K_2 . \tag{3.32}$$

To estimate K_2 we shall consider some particular cases. Consider the case where $D^{\gamma_{2l+1}}_{t,\xi} = \partial_t^l D_\xi^{\gamma_1}$. Then we examine

$$c_2 \sum_{\{\alpha_s\}} \sup_t \Big| D_\xi^{\gamma_1} \varrho_0 \Big(\partial_t \int_0^t u_\xi\, d\tau \Big)^{\alpha_1} \ldots \Big(\partial_t^s \int_0^t u_\xi\, d\tau \Big)^{\alpha_s} + \varrho_0 D_\xi^{\gamma_1} \Big(\Big(\partial_t \int_0^t u_\xi\, d\tau \Big)^{\alpha_1} \ldots \Big(\partial_t \int_0^t u_\xi\, d\tau \Big)^{\alpha_s} \Big) \Big|_{r,\Omega} , \tag{3.33}$$

where

$$\alpha_1 + \ldots + \alpha_s = l + 1 - s, \quad \alpha_1 + 2\alpha_2 + \ldots + s\alpha_s = l, \quad 1 \le s \le l . \tag{3.34}$$

By the Hölder inequality the first expression in (3.33) is estimated by

$$c\Big(\Big| \int_0^T u_\xi\, d\tau \Big|_{\infty,\Omega} \Big) |\varrho_{0\xi}|_{pr,\Omega} \sup_t |u_\xi|^{\alpha_1}_{\alpha_1 r r_1,\Omega} \ldots |\partial_t^{s-1} u_\xi|^{\alpha_s}_{\alpha_s r r_s,\Omega} ,$$

where $1/p + 1/r_1 + \ldots + 1/r_s = 1$. By the imbedding (1.9) this is bounded by

$$c\Big(\Big| \int_0^T u_\xi\, d\tau \Big|_{\infty,\Omega} \Big) \|\varrho_0\|_{2l+1,r,\Omega} \,\|\!| u |\!\|^{\Sigma\alpha_i}_{2l+2-2/r,0,r,\Omega} ,$$

provided

$$(3.35)\qquad \begin{aligned} &1+3/r-3/(pr)\le 2l+1,\\ &2(i-1)+1+3/r-3/(\alpha_i r r_i)\le 2l+2-2/r,\quad i=1,\dots,s,\ s\le l\,. \end{aligned}$$

Multiplying $(3.35)_i$ by α_i, summing over i from 1 to s, adding to $(3.35)_1$ we obtain $(2l+3-5/r)\sum\alpha_i\ge 0$, which is satisfied for $2l+3-5/r\ge 0$ because $\sum\alpha_i>1$. Similar considerations apply to the second term. Consider the second term in the case $s=l$, $\alpha_s=1$. Then it is estimated by

$$c\Big(\Big|\int_0^T u_\xi\,d\tau\Big|_{\infty,\Omega}\Big)|\varrho_0|_{\infty,\Omega}\sup_t|\partial_t^{l-1}u_{\xi\xi}|_{r,\Omega}$$
$$\le c\Big(\Big|\int_0^T u_\xi\,d\tau\Big|_{\infty,\Omega}\Big)\|\varrho_0\|_{2l+1,r,\Omega}\sup_t [\![u]\!]_{2l+2-2/r,0,r,\Omega}\,,$$

where we have used the fact that $2l+1-3/r>0$.

Consider the expression K_2 (see (3.32)) in the case when the ξ-derivatives appear only. Then it is estimated by

$$c\Big(\Big|\int_0^T u_\xi\,d\tau\Big|_{\infty,\Omega}\Big)\sum_\sigma\sum_{\{\alpha_s\}}\sup_t\Big|D_\xi^{\gamma_{2l+1-\sigma}}\varrho_0\Big(\int_0^t D_\xi^{\gamma_2}u\,d\tau\Big)^{\alpha_1}\dots$$
$$\dots\Big(\int_0^t D_\xi^{\gamma_{s+1}}u\,d\tau\Big)^{\alpha_s}\Big|_{r,\Omega}$$
$$\le c\Big(\Big|\int_0^t u_\xi\,d\tau\Big|_{\infty,\Omega}\Big)\sum_\sigma\sum_{\{\alpha_s\}}\sup|D_\xi^{\gamma_\sigma}\varrho_0|_{pr,\Omega}\Big|\int_0^t D_\xi^{\gamma_2}u\,d\tau\Big|^{\alpha_1}_{\alpha_1 r r_1,\Omega}\dots$$
$$\dots\Big|\int_0^t D_\xi^{\gamma_{s+1}}u\,d\tau\Big|^{\alpha_s}_{\alpha_s r r_s,\Omega}\,,$$

where $1/p+1/r_1+\ldots+1/r_s=1$. By $\sigma+\sum_{i=1}^s i\alpha_i=2l+1$ and the imbedding (1.9) this is estimated by

$$c\Big(\Big|\int_0^T u_\xi\,d\tau\Big|_{\infty,\Omega}\Big)\|\varrho_0\|_{2l+1,r,\Omega}\Big\|\int_0^T D_\xi^{\gamma_2}u\,d\tau\Big\|_{2l,r,\Omega}\dots\Big\|\int_0^T D_\xi^{\gamma_{s+1}}u\,d\tau\Big\|_{2l-2s,r,\Omega}$$
$$\le c\Big(\Big|\int_0^T u_\xi\,d\tau\Big|_{\infty,\Omega}\Big)\|\varrho_0\|_{2l+1,r,\Omega}T^{(1-1/r)s}\|u\|^s_{2l+2,r,\Omega^T}\,,$$

provided $\sigma+3/r-3/(pr)\le 2l+1$, $i+1+3/r-3/(\alpha_i r r_i)\le 2l+2$, $i=1,\dots,s$, $s\le 2l+1$; these are satisfied if $(2l+1-3/r)\sum\alpha_i\ge 0$, and the latter follows from the assumptions of the lemma. Thus, we have shown

$$[\![\eta]\!]_{2l+1,0,\infty,\Omega^T}\le c\|\varrho_0\|_{2l+1,r,\Omega}\,\varphi_1'(T,\|u\|_{2l+2,r,\Omega^T},\sup_t[\![u]\!]_{2l+2-2/r,0,r,\Omega})\,.$$

Similarly taking $|\eta(t)-\eta(t')|$ instead of the L_∞ norm we obtain $(3.30)_2$.

Now we show that $\eta\in W_r^{2l+1,l+1/2}(\Omega^T)$. It is sufficient to consider the highest derivatives. By the previous considerations the term $\|D_\xi^{\gamma_{2l+1}}\eta\|_{0,r,\Omega^T}$ is estimated by the right-hand side of (3.29). Hence it remains to estimate

$$K_3\equiv\Big(\int\limits_\Omega d\xi\int\limits_0^T dt\int\limits_0^T dt'\,|\partial_t^l\eta(\xi,t)-\partial_{t'}^l\eta(\xi,t')|^r/|t-t'|^{1+r/2}\Big)^{1/r}.$$

Employing the form (3.28) of η we obtain

$$\begin{aligned}K_3&\le c\|\varrho_0\|_{2l+1,r,\Omega}\Big(\int\limits_\Omega d\xi\int\limits_0^T dt\int\limits_0^T dt'\,\Big|f\Big(\int\limits_0^t u_\xi\,d\tau\Big)u_\xi(t)^{\alpha_1}\dots(\partial_t^{s-1}u_\xi(t))^{\alpha_s}\\
&\quad-f\Big(\int\limits_0^{t'}u_\xi\,d\tau\Big)u_\xi(t')^{\alpha_1}\dots(\partial_{t'}^{s-1}u_\xi(t'))^{\alpha_s}|^r/|t-t'|^{1+r/2}\Big)^{1/r}\\
&\le c\|\varrho_0\|_{2l+1,r,\Omega}\sum_{i=1}^s\Big(\int\limits_\Omega d\xi\int\limits_0^T dt\int\limits_0^T dt'\,\Big|f\Big(\int\limits_0^t u_\xi\,d\tau\Big)\\
&\quad\times u_\xi(t)^{\alpha_1}\dots(\partial_t^{i-2}u_\xi(t))^{\alpha_{i-1}}\\
&\quad\times((\partial_t^{i-1}u_\xi(t))^{\alpha_i}-(\partial_{t'}^{i-1}u_\xi(t'))^{\alpha_i})\\
&\quad\times(\partial_{t'}^i u_\xi(t'))^{\alpha_{i+1}}\dots(\partial_{t'}^{s-1}u_\xi(t'))^{\alpha_s}\Big|^r/|t-t'|^{1+r/2}\Big)^{1/r}\equiv K_4\,,\end{aligned}$$

where $i=0$ corresponds to the function f. The difference factor in K_4 can be written in the form

$$(\partial_t^{i-1}u_\xi(\sigma))^{\alpha_{i-1}}\partial_t^i u_\xi(\sigma)(t-t')\,,\quad\text{where}\quad\sigma\in[t,t']\,.$$

Then by the Hölder inequality we get

$$\begin{aligned}K_4&\le c\Big(\Big|\int\limits_0^T u_\xi dt\Big|_{\infty,\Omega}\Big)\|\varrho_0\|_{2l+1,r,\Omega}\Big(\int\limits_0^T dt\int\limits_0^T dt'\,|t-t'|^{r/2-1}|u_\xi(t)|^{\alpha_1 r}_{\alpha_1 rr_1,\Omega}\cdots\\
&\quad\dots|\partial_t^{i-2}u_\xi(t)|^{\alpha_{i-1}r}_{\alpha_{i-1}rr_{i-1},\Omega}|\partial_t^{i-1}u_\xi(\sigma)|^{\alpha_i-1}_{(\alpha_i-1)rr_i,\Omega}|\partial_t^i u_\xi(\sigma)|_{rr_{i+1},\Omega}\\
&\quad\times|\partial_t^i u_\xi(t')|^{\alpha_{i+1}}_{\alpha_{i+1}rr_{i+1},\Omega}\dots|\partial_{t'}^{s-1}u_\xi(t')|^{\alpha_s}_{\alpha_s rr_s,\Omega}\Big)^{1/r}\equiv K_5\,,\end{aligned}$$

where $1/r_1+\ldots+1/r_s=1$.

Assume that $s<l$. Then by the imbedding (1.9),

$$\begin{aligned}K_5&\le c(T^{1-1/r}\|u\|_{2l+2,r,\Omega^T})\|\varrho_0\|_{2l+1,r,\Omega}\\
&\quad\times\Big(\int\limits_0^T dt\int\limits_0^T dt'\,|t-t'|^{r/2-1}\sup_t\|u\|^{r\Sigma\alpha_i}_{2l+2-2/r,0,r,\Omega}\Big)^{1/r}\equiv K_6\,,\end{aligned}$$

provided

$$\begin{aligned}&1+3/r-3/(\alpha_1rr_1)\le 2l+2-2/r\,,\\&2(i-2)+1+3/r-3/(\alpha_{i-1}rr_{i-1})\le 2l+2-2/r\,,\\&2(i-1)+1+3/r-3/((\alpha_i-1)rr_i)\le 2l+2-2/r\,,\\&2i+1+3/r-3/((\alpha_{i+1}+1)rr_{i+1})\le 2l+2-2/r\,,\dots,\\&2(s-1)+1+3/r-3/(\alpha_srr_s)\le 2l+2-2/r\,,\end{aligned}\tag{3.36}$$

and $\alpha_1+2\alpha_2+\ldots+s\alpha_s=l$, $\alpha_1+\alpha_2+\ldots+\alpha_s=l+1-s$.

Inequalities (3.36) are satisfied if $2\sum i\alpha_i+2-3/r\le(2l+3-5/r)\sum\alpha_i$ and

$$2l-3/r-3/(rr_i(\alpha_i-1))+3/(rr_{i+1}(\alpha_{i+1}+1))\le\sum\alpha_i(2l+3-5/r)\,,\tag{3.37}$$

where $\sum\alpha_i\ge 2$. Now, the inequality (3.37) is satisfied because $2l+1-3/r>0$ and $r>3/2$. Hence we have obtained the estimate

$$K_6\le c(T^{1-1/r}\|u\|_{2l+2,r,\Omega})\|\varrho_0\|_{2l+1,r,\Omega}T^{1/2+1/r}\sup_t |\!|\!| u |\!|\!|^{\Sigma\alpha_i}_{2l+2-2/r,0,r,\Omega}\,.$$

Finally, we consider the case $s=l$, so $\alpha_l=1$. Then the following term should be estimated:

$$\begin{aligned}&\Big(\int_\Omega d\xi\int_0^T dt\int_0^T dt'\,|\partial_t^{l-1}u_\xi(t)-\partial_{t'}^{l-1}u_\xi(t')|^r/|t-t'|^{1+r/2}\Big)^{1/r}\\&\qquad\le c\Big(\int_\Omega d\xi\int_0^T dt\int_0^T dt'\,|\partial_t^l u_\xi(t+\vartheta(t'-t))|^r|t-t'|^{r/2-1}\Big)^{1/r}\\&\qquad\le cT^{1/2}\|u\|_{2l+2,r,\Omega^T}\,.\end{aligned}$$

In this way we have proved the estimate

$$\|\eta\|_{2l+1,r,\Omega^T}\le cT^{1/2}\|\varrho_0\|_{2l+1,r,\Omega}\,\varphi_2'(\|u\|_{2l+2,r,\Omega^T},\sup_t|\!|\!| u|\!|\!|_{2l+2-2/r,0,r,\Omega})\,,$$

which is valid for bounded T and φ_2' is an increasing function of its arguments.

Finally, we show that $\eta\in C^{\alpha,\alpha/2}(\Omega^T)$, where $\alpha/2\le 1-1/r$. First we consider

$$\begin{aligned}|\langle\eta\rangle_{\xi,\alpha,\Omega}|_{\infty,(0,T)}&\le\Big|\Big\langle\varrho_0(\xi)\exp\int_0^t\nabla_u\cdot u(\xi)\,d\tau\Big\rangle_{\xi,\alpha,\Omega}\Big|_{\infty,(0,T)}\\&\le\langle\varrho_0\rangle_{\xi,\alpha,\Omega}\Big|\exp\int_0^t\nabla_u\cdot u\,d\tau\Big|_{\infty,\Omega^T}\\&\quad+|\varrho_0|_{\infty,\Omega}\Big|\Big\langle\exp\int_0^t\nabla_u\cdot u(\xi)\,d\tau\Big\rangle_{\xi,\alpha,\Omega}\Big|_{\infty,(0,T)}\equiv K_7\,.\end{aligned}$$

Using (3.2) and the form of the function $\xi_x=\xi_x(\delta+\int_0^t u_\xi\,d\tau)$, we have

$$|\xi_x|_{\infty,\Omega^T}\le h(1+T^{1-1/r}\|u\|_{2l+2,r,\Omega^T})\quad\text{for } 2l+1>3/r\,,\tag{3.38}$$

$$(3.39)\qquad |\langle \xi_x\rangle_{\xi,\alpha,\Omega}|_{\infty,(0,T)} \le ch_1(1+T^{1-1/r}\|u\|_{2l+2,r,\Omega^T})\int_0^T \langle u_\xi(\tau)\rangle_{\xi,\alpha,\Omega}\,d\tau\,,$$

where h, h_1 are polynomials. Next using the fact that

$$\langle u_\xi(t)\rangle_{\xi,\alpha,\Omega} \le c\|u\|_{2l+2,r,\Omega} \quad \text{for } 2l+1 > 3/r+\alpha\,,$$

we have

$$(3.40)\qquad \int_0^T \langle u_\xi(\tau)\rangle_{\xi,\alpha,\Omega}\,d\tau \le cT^{1-1/r}\|u\|_{2l+2,r,\Omega^T}\,.$$

From (3.37) we have

$$\Big|\exp\int_0^t \nabla_u\cdot u\,d\tau\Big|_{\infty,\Omega^T} \le \exp[h(1+T^{1-1/r}\|u\|_{2l+2,r,\Omega^T})T^{1-1/r}\|u\|_{2l+2,r,\Omega^T}]\,.$$

Similarly, from (3.39) and (3.40) we obtain

$$\Big|\Big\langle \exp\int_0^t \nabla_u\cdot u(\xi)\,d\tau\Big\rangle_{\xi,\alpha,\Omega}\Big|_{\infty,(0,T)}$$
$$\le c\Big|\exp\int_0^t \nabla_u\cdot u\,d\tau\Big|_{\infty,\Omega^T}\Big|\int_0^t \langle\nabla_u\cdot u\rangle_{\xi,\alpha,\Omega}\,d\tau\Big|_{\infty,(0,T)}\,.$$

The last factor in the above expression is estimated by

$$\sup_t \int_0^t (\langle\xi_x\rangle_{\xi,\alpha,\Omega}|u_\xi|_{\infty,\Omega} + |\xi_x|_{\infty,\Omega}\langle u_\xi\rangle_{\xi,\alpha,\Omega})\,d\tau$$
$$\le (\sup_t\langle\xi_x\rangle_{\xi,\alpha,\Omega} + \sup_t|\xi_x|_{\infty,\Omega})\int_0^T \|u\|_{2l+2,r,\Omega}\,d\tau$$
$$\times\, ch_2(1+T^{1-1/r}\|u\|_{2l+2,r,\Omega^T})T^{1-1/r}\|u\|_{2l+2,r,\Omega^T}\,,$$

where h_2 is a polynomial in h, h_1 and the exponential function.

Summarizing, we obtain

$$|\langle\eta\rangle_{\xi,\alpha,\Omega}|_{\infty,(0,T)} \le |\varrho_0|_{C^\alpha(\Omega)}h_3(1+T^{1-1/r}\|u\|_{2l+2,r,\Omega^T})\,,$$

where h_3 depends on exp, h, h_1.

Now we consider

$$|\langle\eta\rangle_{t,\alpha/2,(0,T)}|_{\infty,\Omega} \le |\varrho_0|_{\infty,\Omega}\Big|\Big\langle\exp\int_0^t \nabla_u\cdot u\,d\tau\Big\rangle_{t,\alpha/2,(0,T)}\Big|_{\infty,\Omega}$$
$$\le |\varrho_0|_{\infty,\Omega}\Big|\exp\int_0^t \nabla_u\cdot u\,d\tau\Big|_{\infty,\Omega^T}\Big|\Big\langle\int_0^t \nabla_u\cdot u\,d\tau\Big\rangle_{t,\alpha/2,(0,T)}\Big|_{\infty,\Omega}$$

$$\le c|\varrho_0|_{\infty,\Omega}\exp[h(1+T^{1-1/r}\|u\|_{2l+2,r,\Omega^T})T^{1-1/r}\|u\|_{2l+2,r,\Omega^T}]$$
$$\times\Big|\sup_{t,t'}\Big|\int_t^{t'}\nabla_u\cdot u\,d\tau\Big|/|t-t'|^{\alpha/2}\Big|_{\infty,\Omega}.$$

The last factor in the above expression is estimated by

$$|t-t'|^{1-1/r-\alpha/2}\Big(\int_0^T|\nabla_u\cdot u|^r_{\infty,\Omega}\,d\tau\Big)^{1/r}$$
$$\le|t-t'|^{1-1/r-\alpha/2}h(1+T^{1-1/r}\|u\|_{2l+2,r,\Omega^T})\|u\|_{2l+2,r,\Omega^T}.$$

Hence for $\alpha/2\le 1-1/r$ we have obtained the estimate

$$|\langle\eta\rangle|_{t,\alpha/2,(0,T)}|_{\infty,\Omega}$$
$$\le cT^{1-1/r-\alpha/2}|\varrho_0|_{\infty,\Omega}\,h_4(1+T^{1-1/r}\|u\|_{2l+2,r,\Omega^T})\|u\|_{2l+2,r,\Omega^T},$$

where h_4 depends on exp and h.

We have thus proved (3.31). This concludes the proof of the lemma.

Theorem 3.6. *Let* $v_0\in\Gamma_{0,r}^{2l+2-2/r,l+1-1/r}(\Omega)$, $\varrho_0,\ 1/\varrho_0\in W_r^{2l+1}(\Omega)$, $f\in C^{2l+2}(\mathbb{R}^3\times(0,T))$, $S\in W_r^{2l+2-1/r}$, $2l+1>3/r$, $r>3$, $l+1/2-3/(2p)\in\mathbb{Z}$. *Let G be defined by* (3.64) *and suppose that* $G(\gamma,0,0,A)\le\delta_0A$, $\delta_0>0$, *where γ is introduced by* (3.62), *and*

$$\|u_0\|_{2l+2,r,\Omega^t}+\|u_0\|_{2l+2-2/r,0,r,\Omega}\le A\quad\textit{for } t\le T,$$

where $u_0=\widetilde{u}|_\Omega$ *and $\widetilde{u}$ is a solution of the Cauchy problem* (3.44). *Let δ_* be sufficiently small. Let T_* be so small that*

$$T_*^aA\varphi_2(T,A)\le\delta_*,\quad 0\le c_1(1-\delta_1AT_*)^3\le\det\{\partial_\xi x\}\le c_2(1+\delta_1AT_*)^3,$$

where $\delta_1>\delta_0$ *and* $x(\xi,t)=\xi+\int_0^t\widetilde{u}(\xi,\tau)\,d\tau$, $t\le T_*$. *Then there exists T_{**} with $0<T_{**}\le T_*$ such that for $T\le T_{**}$ there exists a unique solution to problem* (3.1) *such that* $u\in W_r^{2l+2,l+1}(\Omega^T)$, $\eta\in W_r^{2l+1,l+1/2}(\Omega^T)\cap C([0,T];\Gamma_{0,r}^{2l+1,l+1/2}(\Omega))$ *and*

$$\text{(3.41)}\quad\begin{aligned}&\|u\|_{2l+2,r,\Omega^T}\le\delta_1A,\\&\|\eta\|_{2l+1,r,\Omega^T}+\|1/\eta\|_{2l+1,r,\Omega^T}+\|\eta\|_{2l+1,0,r,\infty,\Omega^T}\\&\qquad\le(\|\varrho_0\|_{2l+1,r,\Omega}+\|1/\varrho_0\|_{2l+1,r,\Omega})\varphi_7(T,A),\end{aligned}$$

where φ_7 is an increasing positive function of its arguments.

Proof. To prove the existence of solutions to (3.1) we use the following method of successive approximations:

$$
\begin{aligned}
&\eta_m\partial_t u_{m+1}-\mu\nabla^2_{u_m}u_{m+1}-\nu\nabla_{u_m}\nabla_{u_m}\cdot u_{m+1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad=-\nabla_{u_m}q(\eta_m)+\eta_m g \quad &&\text{in } \Omega^T\,,\\
&\mathbb{D}_{u_m}(u_{m+1})\overline{n}(u_m)=(q(\eta_m)-p_0)\overline{n}(u_m) &&\text{on } S^T\,,\\
&u_{m+1}|_{t=0}=v_0 &&\text{in } \Omega\,,
\end{aligned}\tag{3.42}
$$

and

$$
\begin{aligned}
&\partial_t\eta_m+\eta_m\nabla_{u_m}\cdot u_m=0 \quad &&\text{in } \Omega^T\,,\\
&\eta_m|_{t=0}=\varrho_0 &&\text{in } \Omega\,,
\end{aligned}\tag{3.43}
$$

where $m=0,1,\dots$ To construct the zero step function u_0 we calculate from (3.1) the functions $\varphi^i=\partial_t^i u|_{t=0}$, $i=1,\dots,l$, because we are looking for solutions of class $W_r^{2l+2,l+1}(\Omega^T)$. Next we extend each φ^i to a function $\widetilde{\varphi}_i$ on $\mathbb{R}^n$. Then we construct $\widetilde{u}$ as a solution of the Cauchy problem

$$
(\partial_t-\nabla^2)^{l-1}\widetilde{u}=0\,,\qquad \partial_t^i\widetilde{u}|_{t=0}=\widetilde{\varphi}^i\,,\qquad i=0,1,\dots,l\,,\tag{3.44}
$$

where $\widetilde{\varphi}^0=\widetilde{v}_0$ and $\widetilde{v}_0$ is an extension of v_0 to $\mathbb{R}^n$. Finally, $u_0=\widetilde{u}|_\Omega$.

The compatibility conditions for problems (3.42), (3.43) are the same as for problem (3.1). Namely, they have the form

$$
\partial_t^i[(\mathbb{D}_{u_m}(u_{m+1})-q(\eta_m)+p_0)\overline{n}(u_m)]|_{S,t=0}=0\tag{3.45}
$$

for $i=0,1,\dots,[l+1/2-3/(2r)]$, where $\partial_t u_m$, $\partial_t u_{m+1}$ and $\partial_t\eta_m$ are calculated from (3.42) and (3.43).

Assume that (3.17) with $w=u_m$, $\eta=\eta_m$ is satisfied with sufficiently small δ_*. Then by Lemma 3.4 there exists a unique solution to (3.42) such that $u_{m+1}\in W_r^{2l+2,l+1}(\Omega^T)$, where $T=T(\delta_*)$ is also small, and using also Lemma 3.5 we have

$$
\begin{aligned}
&\|u_{m+1}\|_{2l+2,r,\Omega^T}+\|u_{m+1}\|_{2l+2-2/r,0,r,\infty,\Omega^T}\\
&\le\varphi_1(\|\eta_m\|_{2l+1,r,\Omega^T},\|1/\eta_m\|_{2l+1,r,\Omega^T},|\eta_m|_{C^\alpha(\Omega^T)})\\
&\quad\times[\|\nabla_\xi q(\eta_m)\|_{2l,r,\Omega^T}+\|\eta_m g\|_{2l,r,\Omega^T}+\|(q(\eta_m)-p_0)\overline{n}(u_m)\|_{2l+1-1/r,r,S^T}\\
&\quad+\|v(0)\|_{2l+1-2/r,0,r,\Omega}+\|u_{m+1}\|_{2l,r,\Omega^T}]\,,
\end{aligned}\tag{3.46}
$$

where $\alpha<2l+1-3/r$.

Now we estimate the particular terms on the right-hand side of (3.46). First consider

$$
N\equiv\|\nabla_{u_m}q(\eta_m)\|_{2l,r,\Omega^T}+\|(q(\eta_m)-p_0)\overline{n}(u_m)\|_{2l+1,r,\Omega^T}\,.\tag{3.47}
$$

To estimate (3.47) it is sufficient to consider the norms

$$
\Big|D_\xi^{\gamma_{2l+1}}\Big(q(\eta)f\Big(\int_0^t u_\xi\,d\tau\Big)\Big)\Big|_{r,\Omega^T}+\Big|\partial_t^{l+1/2}\Big(q(\eta)f\Big(\int_0^t u_\xi\,d\tau\Big)\Big)\Big|_{r,\Omega^T}\equiv N_1+N_2\,,
$$

where f expresses the functional dependence of $\overline{n}=\overline{n}(\int u_\xi)$ and $\xi_x=\xi_x(\int u_\xi)$.

First we examine N_1. By the Leibniz formula and the Hölder inequality,

$$
\begin{aligned}
N_1 &\le \sum_{0\le\sigma\le 2l+1} c_\sigma \Big| D_\xi^{\gamma_\sigma} q(\eta) D_\xi^{\gamma_{2l+1-\sigma}} f\Big(\int_0^t u_\xi\,d\tau\Big)\Big|_{r,\Omega^T} \\
&\le c_1(|\eta|_{\infty,\Omega^T}, |1/\eta|_{\infty,\Omega^T}, T^{1-1/r}\|u\|_{2l+2,r,\Omega^T}) \\
&\quad \times \Big(\sum_{0\le\sigma\le 2l+1} \int_0^T |D_\xi^{\gamma_1}\eta|^{\alpha_1 r}_{\alpha_1 r r_1,\Omega} \dots |D_\xi^{\gamma_\mu}\eta|^{\alpha_\mu r}_{\alpha_\mu r r_\mu,\Omega} \Big| \int_0^t u_{\xi\xi}\,d\tau\Big|^{\beta_1 r}_{\beta_1 r p_1,\Omega} \dots \\
&\qquad \dots \Big| \int_0^t D_\xi^{\gamma_{\nu+1}} u\,d\tau\Big|^{\beta_\nu r}_{\beta_\nu r p_\nu,\Omega} dt\Big)^{1/r} \equiv N_3\,,
\end{aligned}
$$

where γ_i is a multiindex such that $|\gamma_i| = i$, and

$$
\begin{aligned}
&\alpha_1+\alpha_2+\dots+\alpha_\mu = \sigma+1-\mu, \quad \alpha_1+2\alpha_2+\dots+\mu\alpha_\mu = \sigma\,,\\
&\beta_1+\beta_2+\dots+\beta_\nu = 2l+1-\sigma+1-\nu\,,\\
&\beta_1+2\beta_2+\dots+\nu\beta_\nu = 2l+1-\sigma\,,
\end{aligned}
\tag{3.48}
$$

where $\mu\le\sigma$, $\nu\le 2l+1-\sigma$, and $1/r_1+\dots+1/r_\mu+1/p_1+\dots+1/p_\nu = 1$.

By the imbedding (1.9) we have

$$
N_3 \le cc_1\Big(\int_0^T \|\eta(t)\|^{r\Sigma\alpha_\mu}_{2l+1,r,\Omega}\Big(\int_0^T \|u(\tau)\|_{2l+2,r,\Omega}\,d\tau\Big)^{r\Sigma\beta_\nu} dt\Big)^{1/r} \equiv N_4\,,
$$

provided

$$
\begin{aligned}
&i+3/r-3/(\alpha_i r r_i) \le 2l+1, \quad i=1,\dots,\mu,\\
&(j+1)+3/r-3/(\beta_j r p_j) \le 2l+2, \quad j=1,\dots,\nu.
\end{aligned}
\tag{3.49}
$$

Multiplying $(3.49)_i$ by α_i, summing over $i=1,\dots,\mu$, and multiplying $(3.49)_j$ by β_j, summing over $j=1,\dots,\nu$, we get

$$
\begin{aligned}
&\sum \mu\alpha_\mu - (3/r)(1/r_1+\dots+1/r_\mu) \le (2l+1-3/r)\sum\alpha_\mu,\\
&\sum \nu\beta_\nu - (3/r)(1/p_1+\dots+1/p_\nu) \le (2l+1-3/r)\sum\beta_\nu.
\end{aligned}
\tag{3.50}
$$

Employing (3.48) in (3.50) and summing together we obtain

$$
2l+1-3/r \le \Big(\sum\alpha_\mu + \sum\beta_\nu\Big)(2l+1-3/r)\,,
$$

which is always satisfied because we have at least either $\sum\alpha_\mu \ge 1$ or $\sum\beta_\nu\ge 1$.

Therefore, the Hölder inequality yields

$$
N_4 \le cc_1 T^{1/r+(1-1/r)\Sigma\beta_\nu} \sup_t \|\eta\|^{\Sigma\alpha_\mu}_{2l+1,r,\Omega}\|u\|^{\Sigma\beta_\nu}_{2l+2,r,\Omega^T}\,.
\tag{3.51}
$$

For N_2, we consider

$$N_2 = \left\langle\!\!\left\langle \partial_t^l\Big(q(\eta)f\Big(\int\limits_0^t u_\xi\,d\tau\Big)\Big)\right\rangle\!\!\right\rangle_{1/2,r,\Omega^T,t}$$
$$\le \left\langle\!\!\left\langle \sum_{\sigma,\mu,\nu}\sum_{\{\alpha_\mu\}}\sum_{\{\beta_\nu\}} c_{\sigma\mu\nu}\partial^{\sigma+1-\mu}q\partial^{l-\sigma+1-\nu}f(\partial_t\eta)^{\alpha_1}\dots(\partial_t^\mu\eta)^{\alpha_\mu}\right.\right.$$
$$\left.\left.\times\Big(\partial_t\int\limits_0^t u_\xi\,d\tau\Big)^{\beta_1}\dots\Big(\partial_t^\nu\int\limits_0^t u_\xi\,d\tau\Big)^{\beta_\nu}\right\rangle\!\!\right\rangle_{1/2,r,\Omega^T,t},$$

where ∂ denotes the derivative with respect to an argument.

To estimate N_2 it is sufficient to consider

$$N_5 = \left\langle\!\!\left\langle (\partial_t\eta)^{\alpha_1}\dots(\partial_t\eta)^{\alpha_\mu}\Big(\partial_t\int\limits_0^t u_\xi\,d\tau\Big)^{\beta_1}\dots\Big(\partial_t^\nu\int\limits_0^t u_\xi\,d\tau\Big)^{\beta_\nu}\right\rangle\!\!\right\rangle_{1/2,r,\Omega^T,t}$$

with

$$\begin{aligned}&\alpha_1+\alpha_2+\dots+\alpha_\mu=\sigma+1-\mu, && \alpha_1+2\alpha_2+\dots+\mu\alpha_\mu=\sigma,\\ &\beta_1+\beta_2+\dots+\beta_\nu=l-\sigma+1-\nu, && \beta_1+2\beta_2+\dots+\nu\beta_\nu=l-\sigma,\end{aligned}\tag{3.52}$$

$\mu\le\sigma$, $\nu\le l-\sigma$, $\sigma=1,\dots,l$. We have

$$\begin{aligned}N_5 \le{}& \sum_{i=1}^{\mu}\Big(\int\limits_\Omega d\xi\int\limits_0^T dt\int\limits_0^T dt'\,|(\partial_t\eta)^{\alpha_1}\dots(\partial_t^{i-1}\eta)^{\alpha_{i-1}}((\partial_t^i\eta)^{\alpha_i}-(\partial_{t'}^i\eta)^{\alpha_i})\\ &\times(\partial_{t'}^{i+1}\eta)^{\alpha_{i+1}}\dots(\partial_{t'}^\mu\eta)^{\alpha_\mu}u_\xi(t')^{\beta_1}\dots(\partial_{t'}^{\nu-1}u_\xi)^{\beta_\nu}|^r|t-t'|^{-(1+r/2)}\Big)^{1/r}\\ &+\sum_{j=1}^{\nu}\Big(\int\limits_\Omega d\xi\int\limits_0^T dt\int\limits_0^T dt'\,|(\partial_t\eta)^{\alpha_1}\dots(\partial_t^\mu\eta)^{\alpha_\mu}(u_\xi)^{\beta_1}\dots(\partial_t^{j-2}u_\xi)^{\beta_{j-1}}\\ &\times((\partial_t^{j-1}u_\xi)^{\beta_j}-(\partial_{t'}^{j-1}u_\xi)^{\beta_j})(\partial_{t'}^{j}u_\xi)^{\beta_{j+1}}\dots\\ &\dots(\partial_{t'}^{\nu-1}u_\xi)^{\beta_\nu}|^r|t-t'|^{-(1+r/2)}\Big)^{1/r}\equiv N_6\,.\end{aligned}$$

Using the formula

$$(\partial_t^i\eta)^{\alpha_i}-(\partial_{t'}^i\eta)^{\alpha_i}=(\partial_t^i\eta(\widetilde{t}))^{\alpha_i-1}\partial_t^{i+1}\eta(\widetilde{t})(t-t'),\qquad\text{where }\widetilde{t}\in[t,t'],$$

a similar one for $(\partial_t^{j-1}u_\xi)^{\beta_j}$ and also the Hölder inequality we obtain

$$N_6\le\sum_{i=1}^{\mu}\Big(\int\limits_0^T dt\int\limits_0^T dt'\,|t-t'|^{r/2-1}|\partial_t\eta|^{\alpha_1 r}_{\alpha_1 r r_1,\Omega}\dots|\partial_t^{i-1}\eta|^{\alpha_{i-1}r}_{\alpha_{i-1}rr_{i-1},\Omega}$$

$$\times |\partial_t^i \eta(\widetilde{t})|^{(\alpha_i-1)r}_{(\alpha_i-1)rr_i,\Omega} |\partial_t^{i+1}\eta(\widetilde{t})|^r_{rr'_{i+1},\Omega} |\partial_{t'}^{i+1}\eta|^{\alpha_{i+1}r}_{\alpha_{i+1}rr_{i+1},\Omega} \cdots$$

$$\cdots |\partial_{t'}^{\mu}\eta|^{\alpha_\mu r}_{\alpha_\mu rr_\mu,\Omega} |u_\xi(t')|^{\beta_1 r}_{\beta_1 rp_1,\Omega} \cdots |\partial_{t'}^{\nu-1}u_\xi|^{\beta_\nu r}_{\beta_\nu rp_\nu,\Omega}\Big)^{1/r}$$

$$+ \sum_{j=1}^{\nu}\Big(\int_0^T dt \int_0^T dt'\, |t-t'|^{r/2-1} |\partial_t\eta|^{\alpha_1 r}_{\alpha_1 r\bar{r}_1,\Omega} \cdots |\partial_t^{\mu}\eta|^{\alpha_\mu r}_{\alpha_\mu r\bar{r}_\mu,\Omega}$$

$$\times |u_\xi(t)|^{\beta_1 r}_{\beta_1 r\bar{p}_1,\Omega} \cdots |\partial_t^{j-2}u_\xi|^{\beta_{j-1}r}_{\beta_{j-1}r\bar{p}_{j-1},\Omega} |\partial_t^{j-1}u_\xi(\widetilde{t})|^{(\beta_j-1)r}_{(\beta_j-1)r\bar{p}_j,\Omega}$$

$$\times |\partial_t^j u_\xi(\widetilde{t})|^r_{r\bar{p}'_{j+1},\Omega} |\partial_{t'}^j u_\xi|^{\beta_{j+1}r}_{\beta_{j+1}r\bar{p}_{j+1},\Omega} \cdots |\partial_{t'}^{\nu-1}u_\xi|^{\beta_\nu r}_{\beta_\nu r\bar{p}_\nu,\Omega}\Big)^{1/r} \equiv N_7\,,$$

where, for the first integral,

$$(3.53)\quad 1/r_1 + \ldots + 1/r_i + 1/r'_{i+1} + 1/r_{i+1} + \ldots + 1/r_\mu + 1/p_1 + \ldots + 1/p_\mu = 1\,,$$

and for the second

$$(3.54)\quad 1/\bar{r}_1 + \ldots + 1/\bar{r}_\mu + 1/\overline{p}_1 + \ldots + 1/\overline{p}_j + 1/\overline{p}'_{j+1} + 1/\overline{p}_{j+1} + \ldots + 1/\overline{p}_\nu = 1\,.$$

By the imbedding (1.9) we have the estimate

$$N_7 \le cT^{r/2+1}\sup_t \|\eta\|^{\Sigma\alpha_\mu}_{2l+1,0,r,\Omega} \sup_t \|u\|^{\Sigma\beta_\nu}_{2l+2-2/r,0,r,\Omega}$$

where we have used the fact that

$$\int_0^T dt \int_0^T dt'\, |t-t'|^{r/2-1} = (8/(r(r+2)))T^{r/2+1}$$

and the following inequalities for the first integral in N_7:

(3.55)

$$\begin{aligned}
&2+3/r-3/(\alpha_1 rr_1) \le 2l+1, \ldots, 2(i-1)+3/r-3/(\alpha_{i-1}rr_{i-1}) \le 2l+1\,,\\
&2i+3/r-3/((\alpha_i-1)rr_i) \le 2l+1, \quad 2(i+1)+3/r-3/(rr'_{i+1}) \le 2l+1\,,\\
&2(i+1)+3/r-3/(\alpha_{i+1}rr_{i+1}) \le 2l+1, \ldots, 2\mu+3/r-3/(\alpha_\mu rr_\mu) \le 2l+1\,,\\
&2(j-1)+1+3/r-3/(\beta_j rp_j) \le 2l+2-2/r, \quad j=1,\ldots,\nu\,,
\end{aligned}$$

where for $\alpha_i = 1$ the term with α_i does not appear.

Similar inequalities must also be satisfied for the second integral in N_7. We restrict our considerations to the first integral in N_7 only. Employing (3.52) and (3.53) in (3.55) we obtain

$$(3.56)\qquad 2l+2-3/r \le (2l+1-3/r)\sum\alpha_i + (2l+3-5/r)\sum\beta_j\,,$$

which is satisfied for $\sum\alpha_i \ge 1$, $\sum\beta_i \ge 1$ and $l \ge 2$, because it leads to the expression $1 \le 2l+1-3/r+(2-2/r)\sum\beta_i$.

However, in the above considerations the case $\sigma = l$, $\mu = l$ was not taken into account. In this case

$$N_2 \le \psi_1'(T^{1-1/r}\|u\|_{2l+2,r,\Omega^T})\langle\langle\partial_t^l\eta\rangle\rangle_{1/2,r,\Omega^T,t} \le \psi_1'(T^{1-1/r}\|u\|_{2l+2,r,\Omega^T})\|\eta\|_{2l+1,r,\Omega^T}\,.$$

Moreover, the case $\sigma=0$, $\nu=l$, $\beta_l=1$ leads to

$$N_2 \le \psi_2'(|\eta|_{\infty,\Omega^T})\langle\langle\partial_t^{l-1}u_\xi\rangle\rangle_{1/2,r,\Omega^T,t} \le c\psi_2' T^{1/2}\|u\|_{2l+2,r,\Omega^T}\,.$$

Finally, in the case $l=1$ we have

$$N_2 \le \psi_3'\Big(\Big|\int_0^t u_\xi\,d\tau\Big|_{\infty,\Omega^T}\Big)\langle\langle\partial_t\eta\rangle\rangle_{1/2,r,\Omega^T,t} + \psi_4'(|\eta|_{\infty,\Omega^T})\langle\langle u_\xi\rangle\rangle_{1/2,r,\Omega^T,t}$$
$$\le \psi_5'(T^{1-1/r}\|u\|_{2l+2,r,\Omega^T})(\|\eta\|_{3,r,\Omega^T} + T^{1/2}\|u\|_{4,r,\Omega^T})\,.$$

In this way we have shown that

$$N \le T^a\psi_1(\sup_t \| \eta_m \|_{2l+1,0,r,\Omega}, \|u_m\|_{2l+2,r,\Omega^T}, \sup_t \| u_m \|_{2l+2-2/r,0,r,\Omega}) + \psi_2(T^{1/2}\|u\|_{2l+2,r,\Omega^T})\|\eta_m\|_{2l+1,r,\Omega^T}\,, \tag{3.57}$$

where ψ_1, ψ_2 are increasing positive functions and $a>0$.

It remains to consider $\|\eta g\|_{2l,r,\Omega^T}$, where $g(\xi,t)=f(x(\xi,t),t)$. It is sufficient to consider the highest derivatives. First we examine

$$|D_\xi^{\gamma_{2l}}(\eta g)|_{r,\Omega^T} \le \sum_{0\le i\le 2l} c_i|D_\xi^{\gamma_i}\eta D_\xi^{\gamma_{2l-i}}g|_{r,\Omega^T} \equiv N_8\,.$$

Using the formula

$$D_\xi^{\gamma_k}g = \sum_{1\le s\le k}\sum_{\{\alpha_s\}} c_{s\alpha_s}D_x^{\gamma_{k+1-s}}f x_\xi^{\alpha_1}\ldots(D_\xi^{\gamma_s}x)^{\alpha_s}\,,$$

where γ_i are multiindices such that $|\gamma_i|=i$ and the summation is taken over $\{\alpha_s\}$ such that $\alpha_1+\ldots+\alpha_s=k+1-s$, $\alpha_1+2\alpha_2+\ldots+s\alpha_s=k$, we obtain

$$N_8 \le \sum_{0\le i\le 2l}\sum_{1\le s\le 2l-1}\sum_{\{\alpha_s\}} c_{is\alpha_s}\Big(\int_0^T dt\int_\Omega d\xi\,\Big|D_\xi^{\gamma_i}\eta D_x^{\gamma_{2l-i+1-s}}f\Big(\int_0^t u_\xi\,d\tau\Big)^{\alpha_1}\ldots$$
$$\ldots\Big(\int_0^t D_\xi^{\gamma_s}u\,d\tau\Big)^{\alpha_s}\Big|\Big)^{1/r} \equiv N_9\,.$$

By the Hölder inequality,

$$N_9 \le c\sup_t|f|_{C^{2l}(\Omega)}\sum_{0\le i\le 2l}\sum_{1\le s\le 2l-i}\sum_{\{\alpha_s\}}\Big(\int_0^T dt\,|D_\xi^{\gamma_i}\eta|^r_{pr,\Omega}\Big|\int_0^t u_\xi\,d\tau\Big|^{\alpha_1 r}_{\alpha_1 rr_1,\Omega}\ldots$$
$$\ldots\Big|\int_0^t D_\xi^{\gamma_s}u\,d\tau\Big|^{\alpha_s r}_{\alpha_s rr_s,\Omega}) \equiv N_{10}\,,$$

where $1/p+1/r_1+\ldots+1/r_s=1$. To use the imbedding (1.9) we need the restrictions

$$i+3/r-3/(pr)\le 2l+1\,,\qquad j+3/r-3/(\alpha_j r r_j)\le 2l+2$$

for $j=1,\ldots,s$, $s\le 2l-i$, $i\le 2l$, which hold because $1+(2l+2-3/r)\sum\alpha_s\ge 0$ is always satisfied. Therefore,

$$N_{10}\le c\sum_{s\le 2l}T^{1/r+(1-1/r)\Sigma\alpha_s}\sup_t|f|_{C^{2l}(\Omega)}\sup_t\|\eta\|_{2l+1,r,\Omega}\|u\|^{\Sigma\alpha_s}_{2l+2,r,\Omega^T}\,.$$

Finally, we consider

$$\begin{aligned}|\partial_t^l(\eta g)|_{r,\Omega^T}&\le\sum_{0\le i\le l}c_i|\partial_t^i\eta\partial_t^{l-i}g|_{r,\Omega^T}\\&\le\sum_{0\le i\le l}\sum_{1\le s\le l-i}\sum_{\{\alpha_s\}}c_{is\alpha_s}|\partial_t^i\eta D_{x,t}^{\gamma_{l-i+1-s}}fu^{\alpha_1}\ldots(\partial_t^{s-1}u)^{\alpha_s}|_{r,\Omega^T}\equiv N_{11}\,,\end{aligned}$$

where $\alpha_1+\alpha_2+\ldots+\alpha_s=l-i+1-s$, $\alpha_1+2\alpha_2+\ldots+s\alpha_s=l-i$. By the Hölder inequality,

$$\begin{aligned}N_{11}\le c|f|_{C^l(\Omega^T)}\sum_{0\le i\le l}\sum_{1\le s\le l-i}\sum_{\{\alpha_s\}}\Big(\int_0^T|\partial_t\eta|^r_{pr,\Omega}|u|^{\alpha_1 r}_{\alpha_1 rr_1,\Omega}\ldots\\\ldots|\partial_t^{s-1}u|^{\alpha_s r}_{\alpha_s rr_s,\Omega}\,dt\Big)^{1/r}\equiv N_{12}\,,\end{aligned}$$

where $1/p+1/r_1+\ldots+1/r_s=1$.

To use the imbedding (1.9) the following restrictions must be satisfied:

$$2i+3/r-3/(pr)\le 2l+1\,,\qquad 2(j-1)+3/r-3/(rr_j)\le 2l+2-2/r$$

for $j=1,\ldots,s$, $0\le s\le l-i$, $i\le l$, which hold because $0\le 1+\sum\alpha_s(2l+4-5/r)$ is always valid. Therefore,

$$N_{12}\le cT^{1/r}|f|_{C^l(\Omega^T)}\sup_t|||\eta|||_{2l+1,0,r,\Omega}\sup_t|||u|||^{\Sigma\alpha_s}_{2l+2-2/r,0,r,\Omega}\,.$$

Summarizing,

$$\begin{aligned}(3.58)\qquad\|\eta_m g\|_{2l,r,\Omega^T}\le cT^a|f|_{C^{2l,l}(\mathbb{R}^3\times(0,T))}\sup_t|||\eta_m|||_{2l+1,0,r,\Omega}\\\times\psi_3(\|u_m\|_{2l+2,r,\Omega^T},\sup_t|||u_m|||_{2l+2-2/r,0,r,\Omega})\,,\end{aligned}$$

where $a>0$.

Let us introduce the quantity

$$(3.59)\qquad y_m(t)=\|u_m\|_{2l+2,r,\Omega^t}+|||u_m|||_{2l+2-2/r,0,r,\infty,\Omega^t}\,.$$

Then using the above considerations in (3.46) we obtain

$$(3.60)\qquad y_{m+1}(t)=\varphi_1(\|\eta_m\|_{2l+1,r,\Omega^t},\|1/\eta_m\|_{2l+1,r,\Omega^t},t)$$

$$\times \Big[t^a\psi_1(t^a \sup_t \|\eta_m\|_{2l+1,0,r,\Omega}, y_m(t)) + \psi_2(y_m(t))\|\eta_m\|_{2l+1,r,\Omega^t}$$

$$+ t^a|f|_{C^{2l,l}(\mathbb{R}^3\times(0,t))} \sup_t \|\eta_m u\|_{2l+1,0,r,\Omega}\psi_3(y_m(t))$$

$$+ \int_0^t y_{m+1}(\tau)\,d\tau + \|v(0)\|_{2l+2-2/r,0,r,\Omega}\Big].$$

By Lemma 3.5 we have

$$\begin{aligned} &|\eta_m|_{2l+1,0,r,\infty,\Omega^t} \le c\|\varrho_0\|_{2l+1,r,\Omega}\,\varphi_5(t, y_m(t)),\\ &\|\eta_m\|_{2l+1,r,\Omega^t} + \|1/\eta_m\|_{2l+1,r,\Omega^t}\\ &\qquad \le ct^{1/2}(\|\varrho_0\|_{2l+1,r,\Omega} + \|1/\varrho_0\|_{2l+1,r,\Omega})\varphi_4(t, t^a y_m(t)). \end{aligned} \tag{3.61}$$

Let us introduce the quantity

$$\gamma = \|\varrho_0\|_{2l+1,r,\Omega} + \|1/\varrho_0\|_{2l+1,r,\Omega} + |f|_{C^{2l,l}(\mathbb{R}^3\times(0,T))}\,. \tag{3.62}$$

Then (3.60) and (3.61) yield

$$\begin{aligned} y_{m+1}(t) \le \varphi_8(\gamma, t^a y_m(t))\Big[&\varphi_{10}(\gamma, t^a y_m(t)) \int_0^t y_{m+1}(\tau)\,d\tau\\ &+ \varphi_9(\gamma, t^a y_m(t))t^a + \|v(0)\|_{2l+2-2/r,0,r,\Omega}\Big], \qquad a > 0\,. \end{aligned} \tag{3.63}$$

By the Gronwall lemma,

$$\begin{aligned} y_{m+1}(t) \le \varphi_8(\gamma, t^a y_m(t)) &\exp[t\varphi_8(\gamma, t^a y_m(t))\varphi_{10}(\gamma, t^a y_m(t))]\\ &\times [\varphi_9(\gamma, t^a y_m(t))t^a + y_0] \equiv G(\gamma, t, t^a y_m, y_0)\,. \end{aligned} \tag{3.64}$$

$G(\gamma, t, t^a y_m, y_0)$ is a continuous increasing function of its arguments. Let $y_0 \le A$. Then there exists $\delta_0 \ge 1$ such that

$$G(\gamma, 0, 0, A) = \varphi_8(\gamma, 0)A \equiv \delta_0 A\,.$$

Assume that $y_m(t) \le \delta_1 A$, $\delta_1 > \delta_0$. Then there exists $T_* \le T$ such that

$$G(\gamma, t, t^a\delta_1 A, A) \le \delta_1 A$$

for $t \le T_*$. In this way we have shown that

$$y_{m+1}(t) \le \delta_1 A\,, \qquad m = 0, 1, \ldots,\ t \le T_*\,. \tag{3.65}$$

Now we prove the convergence of the sequence $\{u_m, \eta_m\}$. To do this we consider the following system of problems for the differences $U_m = u_m - u_{m-1}$

and $H_m = \eta_m - \eta_{m-1}$:

$$
(3.66)\qquad
\begin{aligned}
&\eta_m\partial_t U_{m+1} - \mu\nabla^2_{u_m} U_{m+1} - \nu\nabla_{u_m}\nabla_{u_m}\cdot U_{m+1} = -H_m\partial_t u_m\\
&\qquad - \mu(\nabla^2_{u_m} - \nabla^2_{u_{m-1}})u_m - \nu(\nabla_{u_m}\nabla_{u_m}\cdot - \nabla_{u_{m-1}}\nabla_{u_{m-1}}\cdot)u_m\\
&\qquad\qquad + \nabla_{u_m} q(\eta_m) - \nabla_{u_{m-1}} q(\eta_{m-1}) + H_m g\,,\\
&\mathbb{D}_{u_m}(U_{m+1})\cdot\overline{n}(u_m) = -[\mathbb{D}_{u_m}(u_m)\cdot\overline{n}(u_m) - \mathbb{D}_{u_{m-1}}(u_m)\cdot\overline{n}(u_{m-1})]\\
&\qquad + [q(\eta_m)\overline{n}(u_m) - q(\eta_{m-1})\overline{n}(u_{m-1})] - p_0(\overline{n}(u_m) - \overline{n}(u_{m-1}))\,,\\
&U_{m+1}|_{t=0} = 0\,,
\end{aligned}
$$

and

$$
(3.67)\qquad
\begin{aligned}
&\partial_t H_m + H_m \operatorname{div}_{u_m} u_m = -\eta_{m-1}(\operatorname{div}_{u_m} u_m - \operatorname{div}_{u_{m-1}} u_{m-1})\,,\\
&\qquad H_m|_{t=0} = 0\,.
\end{aligned}
$$

Integrating (3.67) with respect to time one obtains

$$
\begin{aligned}
H_m(\xi,t) = &-\exp\Big[-\int_0^t \operatorname{div}_{u_m} u_m\,dt'\Big]\\
&\times \int_0^t \Big(\eta_{m-1}(\operatorname{div}_{u_m} u_m - \operatorname{div}_{u_{m-1}} u_{m-1})\exp\int_0^{t'} \operatorname{div}_{u_m} u_m\,dt''\Big)\,dt'\,,
\end{aligned}
$$

so we get

$$
(3.68)\qquad \|H_m\|_{2l+1,r,\Omega^t} + \|H_m\|_{2l+1,0,r,\infty,\Omega^t} \le \varphi_{11}(t,A)t^b\|U_m\|_{2l+2,r,\Omega^t}\,.
$$

Applying Lemma 3.3 to problem (3.66) and using (3.61) we obtain

$$
(3.69)\qquad
\begin{aligned}
&\|U_{m+1}\|_{2l+2,r,\Omega^t} + \|U_{m+1}\|_{2l+2-2/r,0,\infty,\Omega^t}\\
&\qquad \le \varphi_3(t,A)[\|U_{m+1}\|_{2l,r,\Omega^t} + \|\overline{F}\|_{2l,r,\Omega^t} + \|\overline{G}\|_{2l+1-1/r,r,S^t}]\,,
\end{aligned}
$$

where $\overline{F}$ and $\overline{G}$ are the right-hand sides in (3.66). By the form of $\overline{F}$ and $\overline{G}$,

$$
(3.70)\qquad
\begin{aligned}
\|\overline{F}\|_{2l,r,\Omega^t} \le c(&\|H_m\partial_t u_m\|_{2l,r,\Omega^t} + \|(\nabla^2_{u_m} - \nabla^2_{u_{m-1}})u_m\|_{2l,r,\Omega^t}\\
&+ \|\nabla_{u_m} q(\eta_m) - \nabla_{u_{m-1}} q(\eta_{m-1})\|_{2l,r,\Omega^t} + \|H_m g\|_{2l,r,\Omega^t})\,,
\end{aligned}
$$

and

$$
(3.71)\qquad
\begin{aligned}
&\|\overline{G}\|_{2l+1-1/r,r,S^t}\\
&\qquad \le c(\|\mathbb{D}_{u_m}(u_m)\cdot\overline{n}(u_m) - \mathbb{D}_{u_{m-1}}(u_m)\cdot\overline{n}(u_{m-1})\|_{2l+1-1/r,r,S^t}\\
&\qquad\quad + \|q(\eta_m)\overline{n}(u_m) - q(\eta_{m-1})\overline{n}(u_{m-1})\|_{2l+1-1/r,r,S^t}\\
&\qquad\quad + \|\overline{n}(u_m) - \overline{n}(u_{m-1})\|_{2l+1-1/r,r,S^t})\,.
\end{aligned}
$$

Now we estimate the terms on the right-hand side of (3.70). For the first term, we consider the highest derivatives only. We have

$$
|D_\xi^{\gamma_{2l}}(H_m u_{mt})|_{r,\Omega^t} \le \sum_{0\le i\le 2l} c_i |D_\xi^{\gamma_i} H_m D_\xi^{\gamma_{2l-i}} u_{mt}|_{r,\Omega^t}
$$

$$\le \sum_{0\le i\le 2l} c_i\Big(\int_0^t |D_\xi^{\gamma_i} H_m|^r_{rp_i,\Omega}|D_\xi^{\gamma_{2l-i}} u_{mt}|_{rq_i,\Omega}\,dt\Big)^{1/r} \equiv A_1\,,$$

where $1/p_i+1/q_i=1$. To use the imbedding (1.9) we need the inequalities

$$i+3/r-3/(rp_i)\le 2l+1\,,\qquad 2l-i+2+3/r-3/(rq_i)\le 2l+2$$

for $i=0,1,\dots,2l$. Summing the inequalities together we obtain $3/r\le 2l+1$, which is always satisfied. Therefore

$$A_1\le c\sup_t\|H_m\|_{2l+1,r,\Omega}\|u_m\|_{2l+2,r,\Omega^t}\,. \tag{3.72}$$

Considering the time derivatives we have

$$\begin{aligned}|\partial_t^l(H_m u_{mt})|_{r,\Omega^t} &\le \sum_{0\le i\le l} c_i|\partial_t^i H_m\partial_t^{l-i+1}u_m|_{r,\Omega^t}\\ &\le \sum_{0\le i\le l} c_i\Big(\int_0^t|\partial_t^i H_m|^r_{rp_i,\Omega}|\partial_t^{l+1-i}u_m|^r_{rq_i,\Omega}\,dt\Big)^{1/r}\equiv A_2\,,\end{aligned}$$

where $1/p_i+1/q_i=1$, $i=1,\dots,l$. By the imbedding (1.9), holding under the conditions $2i+3/r-3/(rp_i)\le 2l+1$, $2(l+1-i)+3/r-3/(rq_i)\le 2l+2$, $i=1,\dots,l$, which are satisfied if $2l+1\ge 3/r$, we obtain

$$A_2\le c\sup_t|\!|\!|H_m|\!|\!|_{2l+1,0,r,\Omega}\|u_m\|_{2l+2,r,\Omega^t}\,. \tag{3.73}$$

Therefore, (3.72) and (3.73) imply

$$\|H_m u_{mt}\|_{2l,r,\Omega^t}\le c\sup_t|\!|\!|H_m|\!|\!|_{2l+1,0,r,\Omega}\|u_m\|_{2l+2,r,\Omega^t}\,. \tag{3.74}$$

Now we consider the second term on the right-hand side of (3.70):

$$\begin{aligned}B\equiv\|(\nabla^2_{u_m}-\nabla^2_{u_{m-1}})u_m\|_{2l,r,\Omega^t}&\le\|(\xi_x^2(u_m)-\xi_x^2(u_{m-1}))u_{m\xi\xi}\|_{2l,r,\Omega^t}\\ &+\|(\xi_x(u_m)\partial_\xi\xi_x(u_m)-\xi_x(u_{m-1})\partial_\xi\xi_x(u_{m-1}))u_{m\xi}\|_{2l,r,\Omega^t}\equiv B_1+B_2\,.\end{aligned}$$

Since $\xi_x^2(u_m)-\xi_x^2(u_{m-1})=g_1(\delta+\int_0^t\widetilde u_\xi\,d\tau)\int_0^t U_{m\xi}\,d\tau$, where g_1 is some function and $\widetilde u\in[u_m,u_{m-1}]$, we obtain

$$B_1=\Big\|g_1\Big(\delta+\int_0^t\widetilde u_\xi\,d\tau\Big)\int_0^t U_{m\xi}\,d\tau\,u_{m\xi\xi}\Big\|_{2l,r,\Omega^t}\,.$$

Considering the ξ-derivatives only we see that B_1 is bounded by an expression which contains the factor $t^{1-1/r}$; this follows by applying the Hölder inequality to $\int_0^t U_{m\xi}\,d\tau$. Considering the t-derivatives we have bad terms when the factor $\int_0^t U_{m\xi}\,d\tau$ is differentiated. One of the worst terms is

$$\Big|g_1\Big(\delta+\int_0^t\widetilde u_\xi\,d\tau\Big)\partial_t^{l-1}U_{m\xi}u_{m\xi\xi}\Big|_{r,\Omega^t}\,.$$

The first factor in the above expression is estimated by $\varphi(t,A)$ which is an increasing polynomial. Hence we consider

$$|\partial_t^{l-1}U_{m\xi}u_{m\xi\xi}|_{r,\Omega^t}=\Big|\int\limits_0^t\partial_t^lU_{m\xi}(\tau)\,d\tau\,u_{m\xi}\Big|_{r,\Omega^t}\equiv B_3\,,$$

where we have used the fact that $\partial_t^{l-1}U_m|_{t=0}=0$. By the Hölder inequality and the imbedding (1.9),

$$\begin{aligned}B_3&\le c\Big(\int\limits_0^t\Big|\int\limits_0^t\|\partial_t^lU_m(\tau)\|_{2,r,\Omega}\,d\tau\Big|^r\|u_m(t)\|^r_{2l+2,r,\Omega}\,dt\Big)^{1/r}\\&\le ct^{1-1/r}\|U_m\|_{2l+2,r,\Omega^t}\|u_m\|_{2l+2,r,\Omega^t}\,.\end{aligned}$$

Summarizing, we have shown that

$$B_1\le ct^{1-1/r}\varphi(t,A)\|U_m\|_{2l+2,r,\Omega^t}\,. \tag{3.75}$$

Now we examine B_2. We have

$$\begin{aligned}B_2&\le\|(\xi_x(u_m)-\xi_x(u_{m-1}))\partial_\xi\xi_x(u_m)u_{m\xi}\|_{2l,r,\Omega^t}\\&\quad+\|\xi_x(u_{m-1})(\partial_\xi\xi_x(u_m)-\partial_\xi\xi_x(u_{m-1}))u_{m\xi}\|_{2l,r,\Omega^t}\\&\equiv\Big\|g_2\Big(\delta+\int\widetilde{u}\,d\tau\Big)\int U_{m\xi}\,d\tau\int u_{m\xi\xi}\,d\tau\,u_{m\xi}\Big\|_{2l,r,\Omega^t}\\&\quad+\Big\|g_3\Big(\delta+\int\widetilde{u}\,d\tau\Big)\int U_{m\xi\xi}\,d\tau\,u_{m\xi}\Big\|_{2l,r,\Omega^t}\,.\end{aligned}$$

Therefore, repeating the above considerations we obtain (3.75) for B_2. In this way we have shown

$$B\le ct^{1-1/r}\varphi(t,A)\|U_m\|_{2l+2,r,\Omega^t}\,. \tag{3.76}$$

Next we examine the third term on the right-hand side of (3.70):

$$\begin{aligned}C\equiv\|\nabla_{u_m}q(\eta_m)-\nabla_{u_{m-1}}q(\eta_{m-1})\|_{2l,r,\Omega^t}&\le\|(\nabla_{u_m}-\nabla_{u_{m-1}})q(\eta_m)\|_{2l,r,\Omega^t}\\&\quad+\|\nabla_{u_{m-1}}(q(\eta_m)-q(\eta_{m-1}))\|_{2l,r,\Omega^t}\equiv C_1+C_2\,.\end{aligned}$$

Write C_1 in the form

$$C_1=\Big\|g_4\Big(\delta+\int\limits_0^t\widetilde{u}_\xi\,d\tau\Big)\int\limits_0^tU_{m\xi}\,d\tau\,q'(\eta_m)\eta_{m\xi}\Big\|_{2l,r,\Omega^t}\,.$$

Let us examine the $2l$th derivative with respect to ξ of the expression within the norm signs. Using the fact that $\eta\in L_\infty(0,T;\Gamma^{2l+1,l+1/2}_{0,r}(\Omega))$, the algebra properties of the space $W_r^{2l+1,l+1/2}(\Omega^T)$ and the fact that some ξ-derivatives of $\int_0^tU_{m\xi}\,d\tau$ will always appear we obtain the estimate

$$C_1\le t\varphi(t,A)\sup_t\|\eta_m\|_{2l+1,r,\Omega}\|U_m\|_{2l+2,r,\Omega^t}\,. \tag{3.77}$$

Next write C_2 in the form

$$C_2=\Big\|g_5\Big(\delta+\int u_{m-1\xi}d\tau\Big)q'(\widetilde{\eta})H_{m\xi}\Big\|_{2l,r,\Omega^t}\,,$$

where $\widetilde{\eta} \in [\eta_m, \eta_{m-1}]$. To estimate this expression it is sufficient to consider

$$|D_\xi^{\gamma_{2l}}(q(\eta)H_\xi)|_{r,\Omega^t} + |\partial_t^l(q(\eta)H_\xi)|_{r,\Omega^t} \equiv C_3 + C_4 .$$

By the Leibniz formula

$$\begin{aligned} C_3 &\le \sum_{i\le 2l} c_i |D_\xi^{\gamma_i} q(\eta) D_\xi^{\gamma_{2l-i}} H_\xi|_{r,\Omega^t} \\ &= \sum_{i\le 2l} \sum_{\{\alpha_s\}} c_{i\alpha_s} |\partial_\eta^i q(\eta)(D_\xi^{\gamma_1}\eta)^{\alpha_i} \dots (D_\xi^{\gamma_s}\eta)^{\alpha_s} D_\xi^{\gamma_{2l-i}} H_\xi|_{r,\Omega^t} , \end{aligned}$$

where $\alpha_1 + \alpha_2 + \ldots + \alpha_s = i + 1 - s$, $\alpha_1 + 2\alpha_2 + \ldots + s\alpha_s = i$, so by the Hölder inequality,

$$C_3 \le c \sum_{i\le 2l} \sum_{\{\alpha_s\}} \Big(\int_0^t |D_\xi^{\gamma_1}\eta|^{\alpha_1 r}_{\alpha_1 r r_1,\Omega} \dots |D_\xi^{\gamma_s}\eta|^{\alpha_s r}_{\alpha_s r r_s,\Omega} |D_\xi^{\gamma_{2l-i}} H_\xi|^r_{rp,\Omega}\, dt\Big)^{1/r} ,$$

where $1/r_1 + \ldots + 1/r_s + 1/p = 1$. For the imbedding (1.9) to hold the following inequalities have to be satisfied: $1 + 3/r - 3/(\alpha_1 r r_1) \le 2l + 1, \ldots, s + 3/r - 3/(\alpha_s r r_s) \le 2l + 1$, $s \le i$, $2l - i + 3/r - 3/(rp) \le 2l$, $i \le 2l$, which hold because $(2l + 1 - 3/r)\sum \alpha_s \ge 0$. Therefore

$$C_3 \le c\varphi(A)\|H_\xi\|_{2l,r,\Omega^t} .$$

For C_4 we have the estimate

$$\begin{aligned} C_4 &\le \sum_{i\le l} c_i |\partial_t^i q(\eta) \partial_t^{l-i} H_\xi|_{r,\Omega^t} \\ &= \sum_{i\le l} \sum_{\{\alpha_s\}} c_{i\alpha_s} |\partial_\eta^i q(\eta)(\partial_t\eta)^{\alpha_1} \dots (\partial_t^s\eta)^{\alpha_s} \partial_t^{l-i} H_\xi|_{r,\Omega^t} , \end{aligned}$$

where $\alpha_1 + \alpha_2 + \ldots + \alpha_s = i + 1 - s$, $\alpha_1 + 2\alpha_2 + \ldots + s\alpha_s = i$, so by the Hölder inequality

$$C_4 \le c \sum_{i\le l} \sum_{\{\alpha_s\}} \Big(\int_0^t |\partial_t\eta|^{\alpha_1}_{\alpha_1 r r_1,\Omega} \dots |\partial_t^s\eta|^{\alpha_s}_{\alpha_s r r_s,\Omega} |\partial_t^{l-i} H_\xi|_{rp,\Omega}\, dt\Big)^{1/r} ,$$

where $1/r_1 + \ldots + 1/r_s + 1/p = 1$. To use the imbedding (1.9) we have to impose the restrictions $2 + 3/r - 3/(\alpha_1 r r_1) \le 2l + 1, \ldots, 2s + 3/r - 3/(\alpha_s r r_s) \le 2l + 1$, $2(l - i) + 3/r - 3/(rp) \le 2l$, $s \le i$, $i \le l$, which are satisfied because $(2l + 1 - 3/r)\sum \alpha_s \ge 0$. Hence we have shown that

$$C_4 \le c\varphi(A)\|H_\xi\|_{2l,r,\Omega^t} .$$

Summarizing, we have obtained

$$C \le c\varphi(A)[t\|U_m\|_{2l+2,r,\Omega^t} + \|H_\xi\|_{2l,r,\Omega^t}] . \tag{3.78}$$

Finally, we consider $D = \|H_m g\|_{2l,r,\Omega^t}$. The highest derivatives yield the expression

$$\sum_{2(i_1+j_1)+i_2+j_2=2l} |\partial_t^{i_1} D_\xi^{\gamma_{i_2}} H_m \partial_t^{j_1} D_\xi^{\gamma_{j_2}} g|_{r,\Omega^t},$$

which by the Hölder inequality is less than

$$\sum_{2(i_1+j_1)+i_2+j_2=2l} \Big(\int_0^t |\partial_t^{i_1} D_\xi^{\gamma_{i_2}} H_m|_{rp,\Omega}^r |\partial_t^{j_1} D_\xi^{\gamma_{j_2}} g|_{rq,\Omega}^r\, dt\Big)^{1/r} \equiv D_1,$$

where $1/p + 1/q = 1$. By the imbedding (1.9) we have

$$D_1 \le c\Big(\int_0^t |H_m|_{2l+1,0,r,\Omega}^r \| g \|_{2l,0,r,\Omega}^r\, dt\Big)^{1/r} \equiv D_2,$$

provided that $2i_1 + i_2 + 3/r - 3/(rp) \le 2l+1$, $2j_1 + j_2 + 3/r - 3/(rq) \le 2l$, where $2(i_1+j_1) + i_2 + j_2 = 2l$. Adding these we obtain the inequality $3/r \le 2l+1$, which is valid by the assumptions of the theorem. Hence

$$D \le c \sup_t \| H_m \|_{2l+1,0,r,\Omega} \|g\|_{2l,r,\Omega^t}. \tag{3.79}$$

Summarizing, we have shown the estimate

$$\|\overline{F}\|_{2l,r,\Omega^t} \le ct^{1-1/r}\varphi(t,A)\|U_m\|_{2l+2,r,\Omega^t} + c\varphi(t,A)\sup_t \| H_m \|_{2l+1,0,r,\Omega}. \tag{3.80}$$

Finally, we estimate the terms on the right-hand side of (3.71). First we recall that $\overline{n}(u) = g_6(\int_0^t u_\xi\, d\tau)$, where g_6 is a vector-valued function. Hence by extension theorems we can estimate the right-hand side of (3.71) by

$$\begin{aligned} E &\le c\Big(\Big\|g_7\Big(\int_0^t u_\xi d\tau\Big)\int_0^t U_{m\xi}\, d\tau\, u_{m\xi}\int_0^t u_{m\xi}\, d\tau\Big\|_{2l+1,r,\Omega^t} \\ &\quad + \Big\|q'(\widetilde{\eta})H_m g_8\Big(\int_0^t u_\xi d\tau\Big)\Big\|_{2l+1,r,\Omega^t} \\ &\quad + \Big\|g_9\Big(\int_0^t u_\xi d\tau\Big)q(\eta_{m-1})\int_0^t U_{m\xi}\, d\tau\Big\|_{2l+1,r,\Omega^t} \\ &\quad + \Big\|g_{10}\Big(\int_0^t u_\xi d\tau\Big)\int_0^t U_{m\xi}\, d\tau\Big\|_{2l+1,r,\Omega^t}\Big) \\ &\equiv E_1 + \ldots + E_4. \end{aligned}$$

By the algebra properties we have

$$E_2 \le c\varphi(t,A)\|H_m\|_{2l+1,r,\Omega^t}.$$

In the other expressions, we bound the ξ-derivatives with the factor $t^{1-1/r}$ because the term $\int U_{m\xi}\, d\tau$ always appears. Considering time derivatives we obtain the factor $t^{1/2}$. Summarizing, we have

$$\|\widetilde{G}\|_{2l+1-1/r,r,S^t} \le E \le c\varphi(t,A)(\|H_m\|_{2l+1,r,\Omega^t} + t^a\|U_m\|_{2l+2,r,\Omega^t}), \quad a>0. \tag{3.81}$$

Using (3.80) and (3.81) in the right-hand side of (3.69) we obtain

$$\begin{aligned}\|U_{m+1}\|_{2l+2,r,\Omega^t} &+ |U_{m+1}|_{2l+2-2/r,0,\infty,\Omega^t} \\ &\le \varphi_{12}(t,A)t^a[\|U_m\|_{2l+2,r,\Omega^t} + |U_m|_{2l+2-2/r,0,\infty,\Omega^t}] \\ &\quad + c\varphi_{13}(t,A)\|H_m\|_{2l+1,r,\Omega^t}.\end{aligned} \tag{3.82}$$

Therefore by (3.68) and (3.82) for $t \le T_{**}$, where T_{**} is sufficiently small, the sequence $\{u_m,\eta_m\}$ converges to a limit

$$\{u,\eta\} \in W_r^{2l+2,l+1}(\Omega^T) \times W_r^{2l+1,l+1/2}(\Omega^t) \cap C(0,T;\Gamma_{0,r}^{2l+1,l+1/2}(\Omega)),$$

$t \le \min\{T_*,T_{**}\}$, which is a solution to (3.1). Uniqueness can be proved in the standard way. Hence the theorem is proved.

To consider the global existence we need

Remark 3.7. Assume that $q_\sigma = q - p_0$ and $g=0$. Then the problem (3.1) can be written in the form

$$\begin{aligned}&\eta u_t - \mu\nabla_u^2 u - \nu\nabla_u\nabla_u\cdot u + \nabla_u q_\sigma = 0,\\ &q_{\sigma t} + \eta\nabla_u\cdot u = 0,\\ &\mathbb{T}_u(u,q_\sigma)\overline{n} = 0,\\ &u|_{t=0} = v_0, \qquad q_\sigma|_{t=0} = q_{\sigma 0},\end{aligned} \tag{3.83}$$

where $q_{\sigma0} = (q(\eta)-p_0)|_{t=0}$.

Let the assumptions of Theorem 3.6 be satisfied and let u, η be the corresponding local solution of problem (3.1). Then for solutions of (3.83) the following estimate holds:

$$\begin{aligned}\|u\|_{2l+2,r,\Omega^T} &+ \|q_\sigma\|_{2l+1,r,\Omega^T} + |q_\sigma|_{2l+1,r,0,\infty,\Omega^T} \\ &\le \varphi(T,A)[\|v_0\|_{2l+2-2/r,r,\Omega} + \|q_{\sigma0}\|_{2l+1,r,\Omega}].\end{aligned} \tag{3.84}$$

Remark 3.8 (see [37]). For a sufficiently regular solution of (3.83) such that

$$(\|u\|_{4,\Omega^T} + \|q_\sigma\|_{3,\Omega^T})T \le \delta, \tag{3.85}$$

where δ is sufficiently small, the following estimate holds:

$$\|u\|_{4,\Omega^T} + \|q_\sigma\|_{3,\Omega^T} \le \varphi(T,\|v_0\|_{3,\Omega} + \|p_{\sigma0}\|_{3,\Omega})(\|v_0\|_{3,\Omega} + \|p_{\sigma0}\|_{3,\Omega}), \tag{3.86}$$

where $u(\xi,t) = v(x(\xi,t),t)$, $q_\sigma(\xi,t) = p_\sigma(x(\xi,t),t)$. Therefore the following result can be proved.

Let $v_0 \in H^3(\Omega)$, $p_{\sigma0} \in H^3(\Omega)$, $S \in H^{4-1/2}$. Then for sufficiently small δ there exists a solution of (3.83) such that $u \in W_2^{4,2}(\Omega^T)$, $q_\sigma \in W_2^{3,3/2}(\Omega^T) \cap C(0,T;\Gamma_{0,2}^{3,3/2}(\Omega))$ and (3.86) holds.

4. Global differential inequality

Assume that we have proved the existence of a sufficiently smooth local solution. First we find a special differential inequality which enables us to prove the existence of solution by energy estimates and then to prove global existence.

To show it we consider the motion near the constant state $v_e = 0$, $p_e = p_0$, ϱ_e is a solution of the equation $p(\varrho_e) = p_0$. Therefore, we examine the system

$$
\begin{aligned}
&\varrho(v^i_t + v\cdot\nabla v^i) - \partial_{x^j}T_{ij}(v,p_\sigma) = \varrho f^i &&\text{in } \Omega_t,\ t\in[0,T]\,,\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 &&\text{in } \Omega_t,\ t\in[0,T]\,,\\
&\mathbb{T}(v,p_\sigma)\overline{n} = 0 &&\text{on } S_t,\ t\in[0,T]\,,
\end{aligned}
\tag{4.1}
$$

where $\mathbb{T} = \{T_{ij}\} = \{\mu(\partial_{x^j}v^i + \partial_{x^i}v^j) + (\nu-\mu)\delta_{ij}\operatorname{div} v - p_\sigma\delta_{ij}\}$ and $p_\sigma = p - p_0$.

Using the barotropic law $p = p(\varrho)$ we write $(4.1)_2$ in the form

$$
p_{\sigma t} + v\cdot\nabla p_\sigma + p\Psi(\varrho)\operatorname{div} v = 0\,, \tag{4.2}
$$

where $\Psi(\varrho) = p_\varrho\varrho/p$.

Set $\varrho_* = \min_{\widetilde{\Omega}^T}\varrho(x,t)$, $\varrho^* = \max_{\widetilde{\Omega}^T}\varrho(x,t)$.

Now we point out the following facts concerning the estimates in Lemmas 4.1–4.12 and Theorem 4.13:

1. The numbers δ_i are assumed to be small and are separately numbered in each lemma.

2. We distinguish absolute constants, denoted by c, which may depend on such parameters of the problem as μ, ν, κ, A and which are coefficients in those terms in the right-hand sides of the inequalities which contain the highest derivatives only and which are finally balanced by the left-hand side terms after appriopriate summing.

3. We distinguish the coefficients by the lower order terms, nonlinear terms and also by the force terms which depend on ϱ_*, ϱ^*, $T, b \equiv \|S\|_{4-1/2}$, $a \equiv \int_0^T \|v\|_{3,\Omega_{t'}}\,dt'$, on the parameters which guarantee the existence of the inverse transformation to $x = x(\xi,t)$, and also on the constants of imbedding theorems considered over Ω. Generally, the coefficients are increasing functions of the parameters. In the statements of the lemmas, we denote such coefficients by P_1, $P_2,\dots$ (common numbering for all lemmas) and independently in each lemma by $a_1, a_2,\dots$ Moreover, P_i, a_i are positive and increasing functions of a and b.

4. We have to underline that the estimates in this section are obtained under the assumption that there exists a local solution of (1.1) so all the quantities ϱ_*, ϱ^*, T, a, b are estimated by the data functions. Moreover, the existence of the inverse transformation to $x = x(\xi,t)$ is guaranteed by the estimates for the local solution. Generally, the quantities are large.

LEMMA 4.1. *Let v, p_σ be a sufficiently smooth solution of* (4.1). *Then*

$$\frac{1}{2}\frac{1}{dt}\int\limits_{\Omega_t}\left(\varrho v^2+\frac{1}{p\Psi(\varrho)}p_\sigma^2\right)dx+\frac{\mu}{2}\|v\|_{1,\Omega_t}^2+(\nu-\mu)\|\operatorname{div}v\|_{0,\Omega_t}^2 \le \varepsilon_1\|p_\sigma\|_{0,\Omega_t}^2+P_1(\varrho_*,\varrho^*,\varepsilon_1)\|v\|_{2,\Omega_t}^2\|p_\sigma\|_{1,\Omega_t}^2+P_2\varrho^{*2}\|f\|_{0,\Omega_t}^2\,, \tag{4.3}$$

where $P_1(\varepsilon_1)$ behaves as ε_1^{-a}, $a>0$, and $\varepsilon_1\in(0,1)$.

Proof. Multiplying $(4.1)_1$ by v, integrating over Ω_t and using $(4.1)_{2,3}$ implies

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\varrho v^2\,dx+\frac{\mu}{2}E_{\Omega_t}(v)+(\nu-\mu)\|\operatorname{div}v\|_{0,\Omega_t}^2-\int\limits_{\Omega_t}p_\sigma\operatorname{div}v\,dx = \int\limits_{\Omega_t}\varrho f\cdot v\,dx\,. \tag{4.4}$$

Equation (4.2) yields

$$-\int\limits_{\Omega_t}p_\sigma\operatorname{div}v\,dx=\int\limits_{\Omega_t}\frac{1}{p\Psi(\varrho)}(\partial_t+v\cdot\nabla)\frac{p_\sigma^2}{2}\,dx$$

and

$$[F_t+\operatorname{div}(Fv)+(F_\varrho\varrho-F)\operatorname{div}v]\frac{p_\sigma^2}{2}=0\,, \tag{4.5}$$

where $F=1/(p(\varrho)\Psi(\varrho))$, so

$$\begin{aligned}\int\limits_{\Omega_t}p_\sigma\operatorname{div}v\,dx&=-\frac{d}{dt}\int\limits_{\Omega_t}\frac{1}{p\Psi(\varrho)}\frac{p_\sigma^2}{2}\,dx+\int\limits_{\Omega_t}(F-F_\varrho\varrho)\operatorname{div}v\frac{p_\sigma^2}{2}\,dx\\&\le-\frac{d}{dt}\int\limits_{\Omega_t}\frac{1}{p\Psi(\varrho)}\frac{p_\sigma^2}{2}\,dx+\varepsilon_1\|p_\sigma\|_{0,\Omega_t}^2+a_1(\varrho_*,\varrho^*,\varepsilon_1)\|p_\sigma\|_{1,\Omega_t}^2\|v\|_{2,\Omega_t}^2\,.\end{aligned} \tag{4.6}$$

From Lemma 5.2 and the relation $p(\varrho)-p_0=p-p(\varrho_e)=p(\widetilde{\varrho})(\varrho-\varrho_e)$, $\widetilde{\varrho}\in[\varrho,\varrho_e]$, we have

$$\|v\|_{1,\Omega_t}^2\le c_2(\varrho^*)(E_{\Omega_t}(v)+\|p_\sigma\|_{0,\Omega_t}^2\|v\|_{2,\Omega_t}^2)\,. \tag{4.7}$$

By the Hölder and Young inequalities the right-hand side of (4.4) is estimated by

$$\delta_1\|v\|_{0,\Omega_t}^2+c(\delta_1)\varrho_*^2\|f\|_{0,\Omega_t}^2\,.$$

Hence taking δ_1 sufficiently small and using (4.6) and (4.7) in (4.4) we obtain (4.3). This concludes the proof.

LEMMA 4.2. *For a sufficiently smooth solution v, p of* (4.1) *we have*

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left(\varrho v_t^2+\frac{1}{p\Psi(\varrho)}p_{\sigma t}^2\right)dx+\frac{\mu}{4}\|v_t\|_{1,\Omega_t}^2+(\nu-\mu)\|\operatorname{div}v_t\|_{0,\Omega_t}^2 \le\varepsilon_2\|p_{\sigma t}\|_{0,\Omega_t}^2+P_3(\varrho_*,\varrho^*,\varepsilon_2)X_1^2+P_4|f|_{1,0,\Omega_t}^2\,, \tag{4.8}$$

where $P_3(\varepsilon_2)$ *behaves like* ε_2^{-a}, $a>0$, $\varepsilon_2\in(0,1)$, *and*

$$X_1 = |p_\sigma|^2_{2,1,\Omega_t} + |v|^2_{2,1,\Omega_t}\,. \tag{4.9}$$

Proof. Differentiating $(4.1)_1$ with respect to t, multiplying by v_t and integrating over Ω_t yields

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\varrho v_t^2\,dx + \frac{\mu}{2}E_{\Omega_t}(v_t) + (\nu-\mu)\|\operatorname{div} v_t\|^2_{0,\Omega_t} - \int\limits_{\Omega_t} p_{\sigma t}\operatorname{div} v_t\,dx = N_1\,, \tag{4.10}$$

where

$$N_1 \le \delta_1\|v_t\|^2_{0,\Omega_t} + c(\delta_1)[\|f\|^2_{1,\Omega_t} + X_1^2] + c(\delta_1,\varrho^*)\|f_t\|^2_{0,\Omega_t}\,.$$

From (4.2) and (4.5) with $p_{\sigma t}$ in place of p_σ we obtain

$$\int\limits_{\Omega_t} p_{\sigma t}\operatorname{div} v_t\,dx \le -\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\frac{1}{p\Psi(\varrho)}p^2_{\sigma t}\,dx + \delta_2\|p_{\sigma t}\|^2_{0,\Omega_t} + c(\varrho_*,\varrho^*,\delta_2)X_1^2\,. \tag{4.11}$$

Finally, from (4.10), (4.11) and Lemma 5.3 we obtain (4.8) for sufficiently small δ_1. This concludes the proof.

From Lemmas 4.1 and 4.2 we have

Lemma 4.3.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left[\varrho(v^2+v_t^2) + \frac{1}{p\Psi(\varrho)}(p_\sigma^2+p_{\sigma t}^2)\right]dx + \frac{\mu}{2}(\|v\|^2_{1,\Omega_t} + \|v_t\|^2_{1,\Omega_t})\\ + (\nu-\mu)(\|\operatorname{div} v\|^2_{0,\Omega_t} + \|\operatorname{div} v_t\|^2_{0,\Omega_t})\\ \le \varepsilon_3(\|p_{\sigma t}\|^2_{0,\Omega_t} + \|p_\sigma\|^2_{0,\Omega_t}) + P_5(\varrho_*,\varrho^*,\varepsilon_3)X_1^2 + P_6|f|^2_{1,0,\Omega_t}\,,\end{aligned} \tag{4.12}$$

where $\varepsilon_3\in(0,1)$.

To obtain an inequality for x-derivatives we write (4.1) in Lagrangian coordinates, so we can introduce a partition of unity in the fixed domain Ω. Therefore, we have

$$\begin{aligned} &\eta u_t^i - \nabla_{u^j}T_u^{ij}(u,q_\sigma) = \eta g^i\,,\\ &q_{\sigma t} + q\Psi(\eta)\nabla_u\cdot u = 0\,,\\ &\mathbb{T}_u(u,q_\sigma)\overline{n} = 0\,,\end{aligned} \tag{4.13}$$

where $\eta(\xi,t)=\varrho(x(\xi,t),t)$, $u(\xi,t)=v(x(\xi,t),t)$, $g(\xi,t)=f(x(\xi,t),t)$, $q(\xi,t)=p(x(\xi,t),t)$, $q_\sigma=q-p_0$, $\nabla_{u^j}=\xi^k_{x^j}\partial_{\xi^k}$, $\Psi(\eta)=q_{,\eta}\eta/q$ and

$$\mathbb{T}_u(u,q_\sigma) = \{T_u^{ij}(u,q_\sigma)\} = \{-q_\sigma\delta_{ij} + \mu(\nabla_{u^i}u^j + \nabla_{u^j}u^i) + (\nu-\mu)\delta_{ij}\nabla_u\cdot u\}\,. \tag{4.14}$$

Next we introduce a partition of unity $(\{\widetilde{\Omega}_i\},\{\zeta_i\})$, $\Omega=\bigcup_i\widetilde{\Omega}_i$. Let $\widetilde{\Omega}$ be one of the $\widetilde{\Omega}_i$'s and $\zeta(\xi)=\zeta_i(\xi)$ the corresponding function. If $\widetilde{\Omega}$ is an interior subdomain, then let $\widetilde{\omega}$ be such that $\overline{\widetilde{\omega}}\subset\widetilde{\Omega}$ and $\zeta(\xi)=1$ for $\xi\in\widetilde{\omega}$. Otherwise we assume that $\overline{\widetilde{\Omega}}\cap S\neq\emptyset$, $\overline{\widetilde{\omega}}\cap S\neq\emptyset$, $\overline{\widetilde{\omega}}\subset\overline{\widetilde{\Omega}}$. Let $\beta\in\overline{\widetilde{\omega}}\cap S\subset\overline{\widetilde{\Omega}}\cap S$, $\widetilde{S}\equiv\widetilde{\Omega}\cap S$. Introduce local coordinates $\{y\}$ connected with $\{\xi\}$ by

$$y^k=\alpha^{kl}(\xi^l-\beta^l)\,,\qquad \alpha^{3k}=n^k(\beta)\,,\ k=1,2,3\,, \tag{4.15}$$

where α^{kl} is a constant orthogonal matrix such that $\widetilde{S}$ is determined by $y^3=F(y^1,y^2)$, $F\in H^{4-1/2}$ and

$$\widetilde{\Omega}=\{y:|y^i|<d\,,\ i=1,2\,,\ F(y')<y^3<F(y')+d\,,\ y'=(y^1,y^2)\}\,.$$

Next we introduce functions u' and q' by

$$u'^i(y)=\alpha^{ij}u^j(\xi)|_{\xi=\xi(y)}\,,\qquad q'(y)=q(\xi)|_{\xi=\xi(y)}\,, \tag{4.16}$$

where $\xi=\xi(y)$ is the inverse transformation to (4.15). Further, we introduce new variables by

$$z^i=y^i\,,\quad i=1,2\,,\quad z^3=y^3-\widetilde{F}(y)\,,\quad y\in\widetilde{\Omega}\,, \tag{4.17}$$

which will be denoted by $z=\Phi(y)$, where $\widetilde{F}$ is an extension of F to $\widetilde{\Omega}$ with $\widetilde{F}\in H^4(\widetilde{\Omega})$. Let $\widehat{\Omega}=\Phi(\widetilde{\Omega})=\{z:|z^i|<d,\ i=1,2,\ 0<z^3<d\}$ and $\widehat{S}=\Phi(\widetilde{S})$. Define

$$\widehat{u}(z)=u'(y)|_{y=\Phi^{-1}(z)}\,,\qquad \widehat{q}(z)=q'(y)|_{y=\Phi^{-1}(z)}\,. \tag{4.18}$$

Introduce $\widehat{\nabla}_k=\xi^l_{x^k}(\xi)z^i_{\xi^l}\nabla_{z^i}|_{\xi=\chi^{-1}(z)}$, where $\chi(\xi)=\Phi(\psi(\xi))$ and $y=\psi(\xi)$ is described by (4.15). Introduce also the following notation:

$$\widetilde{u}(\xi)=u(\xi)\zeta(\xi)\,,\quad \widetilde{q}_\sigma(\xi)=q_\sigma(\xi)\zeta(\xi)\,,\quad \xi\in\widetilde{\Omega}\,,\ \widetilde{\Omega}\cap S=\emptyset\,, \tag{4.19}$$

$$\widetilde{u}(z)=\widehat{u}(z)\widehat{\zeta}(z)\,,\quad \widetilde{q}_\sigma(z)=\widehat{q}_\sigma(z)\widehat{\zeta}(z)\,,\quad z\in\widehat{\Omega}=\Phi(\widetilde{\Omega})\,,\ \overline{\widetilde{\Omega}}\cap S\neq\emptyset\,, \tag{4.20}$$

where $\widehat{\zeta}(z)=\zeta(\xi)|_{\xi=\chi^{-1}(z)}$.

Under the above notation problem (4.13) has the following form in an interior subdomain:

$$\begin{aligned}&\eta\widetilde{u}^i_t-\nabla_{u^j}T^{ij}_u(\widetilde{u},\widetilde{q}_\sigma)=\eta\widetilde{g}^i-\nabla_{u^j}B^{ij}_u(u,\zeta)-T^{ij}_u(u,q_\sigma)\nabla_{u^j}\zeta\equiv\eta\widetilde{g}^i+k_1\,,\\&\widetilde{q}_{\sigma t}+q\Psi(\eta)\nabla_u\cdot\widetilde{u}=q\Psi(\eta)u\cdot\nabla_u\zeta\equiv k_2\,,\end{aligned} \tag{4.21}$$

and in a boundary subdomain:

$$\begin{aligned}&\widehat{\eta}\widetilde{u}^i_t-\widehat{\nabla}_j\widehat{T}^{ij}(\widetilde{u},\widetilde{q}_\sigma)=\widehat{\eta}\widetilde{g}^i-\widehat{\nabla}_j\widehat{B}^{ij}(\widehat{u},\widehat{\zeta})-\widehat{T}^{ij}(\widehat{u},\widehat{q}_\sigma)\widehat{\nabla}_j\widehat{\zeta}\equiv\widehat{\eta}\widetilde{g}^i+k_3\,,\\&\widetilde{q}_{\sigma t}+\widehat{q}\Psi(\widehat{\eta})\widehat{\nabla}\cdot\widetilde{u}=\widehat{q}\Psi(\widehat{\eta})\widehat{u}\cdot\widehat{\nabla}\widehat{\zeta}\equiv k_4\,,\\&\widehat{\mathbb{T}}(\widetilde{u},\widetilde{q}_\sigma)\widehat{n}=k_5\,,\end{aligned} \tag{4.22}$$

where

$$\begin{aligned}&k^i_5=\widehat{B}^{ij}(\widehat{u},\widehat{\zeta})\widehat{n}_j\,,\\&B^{ij}_u(u,\zeta)=\mu(u^i\nabla_{u^j}\zeta+u^j\nabla_{u^i}\zeta)+(\nu-\mu)\delta_{ij}u\cdot\nabla_u\zeta\,,\end{aligned} \tag{4.23}$$

and $\widehat{\mathbb{T}}$, $\widehat{B}$ indicate that the operator ∇_u is replaced by $\widehat{\nabla}$.

In the next considerations we denote z^1, z^2 by τ and z^3 by n.

LEMMA 4.4. *Let the assumptions of Lemmas* 4.1 *and* 4.2 *be satisfied. Then*

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\varrho|v|_{1,0}^2+\frac{1}{p\Psi(\varrho)}|p_\sigma|_{1,0}^2\right)dx+\frac{\mu}{2}|v|_{2,1,\Omega_t}^2+|p_\sigma|_{1,0,\Omega_t}^2 \le P_7(\|f\|_{0,\Omega_t}^2+\|v\|_{0,\Omega_t}^2+\|p_\sigma\|_{0,\Omega_t}^2)+P_8X_2Y_2\,, \tag{4.24}$$

where P_7 is a positive increasing function, $P_7=P_7(a,b)$ *and*

$$\begin{aligned} X_2&=X_2(t)=|v|_{2,1,\Omega_t}^2+|p_\sigma|_{2,1,\Omega_t}^2\,,\\ Y_2&=Y_2(t)=X_2(t)+\|v\|_{3,\Omega_t}^2+\int_0^t\|v\|_{3,\Omega_\tau}^2\,d\tau\,. \end{aligned} \tag{4.25}$$

Proof. First we consider interior subdomains. Differentiating $(4.21)_1$ with respect to ξ, multiplying the result by $\widetilde{u}_\xi A$ (A is the Jacobian of the transformation $x=x(\xi)$) and integrating over $\widetilde{\Omega}$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\widetilde{\Omega}}\eta\widetilde{u}_\xi^2A\,d\xi&+\frac{\mu}{2}\int_{\widetilde{\Omega}}(\nabla_{u^i}\widetilde{u}_\xi^j+\nabla_{u^j}\widetilde{u}_\xi^i)^2A\,d\xi\\ &+(\nu-\mu)\|\nabla_u\cdot\widetilde{u}_\xi\|_{0,\widetilde{\Omega}}^2-\int_{\widetilde{\Omega}}\widetilde{q}_{\sigma\xi}\cdot\nabla_u\widetilde{u}_\xi A\,d\xi\\ &\le\delta_1(\|u_{\xi\xi}\|_{0,\widetilde{\Omega}}^2+\|q_{\sigma\xi}\|_{0,\widetilde{\Omega}}^2)+a_1(\|u\|_{1,\widetilde{\Omega}}^2+\|q_\sigma\|_{0,\widetilde{\Omega}}^2+\|g\|_{0,\widetilde{\Omega}}^2)\\ &+a_2\left(\|u\|_{2,\widetilde{\Omega}}^2\left\|\int_0^t u\,d\tau\right\|_{3,\widetilde{\Omega}}^2+\|q_\sigma\|_{1,\widetilde{\Omega}}^2|u|_{2,1,\widetilde{\Omega}}^2\right), \end{aligned} \tag{4.26}$$

where $\|h\|_{0,\widetilde{\Omega}}=(\int_{\widetilde{\Omega}}|h|^2A\,d\xi)^{1/2}$.

By the continuity equation $(4.21)_2$ we have

$$-\int_{\widetilde{\Omega}}\widetilde{q}_{\sigma\xi}\nabla_u\cdot\widetilde{u}_\xi A\,d\xi=\frac{1}{2}\frac{d}{dt}\int_{\widetilde{\Omega}}\frac{1}{q\Psi(\eta)}\widetilde{q}_{\sigma\xi}^2A\,d\xi+N_1\,, \tag{4.27}$$

where

$$\begin{aligned} |N_1|\le\delta_2\|\widetilde{q}_{\sigma\xi}\|_{0,\widetilde{\Omega}}^2&+a_4\|u\|_{1,\widetilde{\Omega}}^2\\ &+a_5\left[|\widetilde{q}_\sigma|_{2,1,\widetilde{\Omega}}^2(\|u\|_{2,\widetilde{\Omega}}^2+|q_\sigma|_{2,1,\widetilde{\Omega}}^2)+\|u\|_{2,\widetilde{\Omega}}^2\left\|\int_0^t u\,d\tau\right\|_{3,\widetilde{\Omega}}^2\right]. \end{aligned}$$

Consider the Stokes problem in $\widetilde{\Omega}$:

$$\begin{aligned} &\mu\nabla_u^2\widetilde{u}-\nu\nabla_u\nabla_u\cdot\widetilde{u}+\nabla_u\widetilde{q}_\sigma=\eta\widetilde{g}-\eta\widetilde{u}_t+k_1\,,\\ &\nabla_u\cdot\widetilde{u}=\nabla_u\cdot\widetilde{u}\,,\\ &\widetilde{u}|_{\partial\widetilde{\Omega}}=0\,. \end{aligned} \tag{4.28}$$

Hence, we have

$$\|\widetilde{u}\|^2_{2,\widetilde{\Omega}}+\|\widetilde{q}_\sigma\|^2_{1,\widetilde{\Omega}}\le a_6(\|\widetilde{g}\|^2_{0,\widetilde{\Omega}}+|u|^2_{1,0,\widetilde{\Omega}}+\|q_\sigma\|^2_{0,\widetilde{\Omega}})$$
$$+a_7(\|u\|^2_{2,\widetilde{\Omega}}+\|q_\sigma\|^2_{1,\widetilde{\Omega}})\Big\|\int_0^t u\,d\tau\Big\|^2_{3,\widetilde{\Omega}}+c\|\nabla_u\cdot\widetilde{u}\|^2_{1,\widetilde{\Omega}}\,. \tag{4.29}$$

Using Lemma 5.1 in the case $G=\widetilde{\Omega}$, $v=\widetilde{u}_\xi$, from (4.26), (4.27), (4.29) for sufficiently small δ_1 and δ_2 we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\widetilde{\Omega}}\left(\eta\widetilde{u}_\xi^2+\frac{1}{q\Psi(\eta)}\widetilde{q}^2_{\sigma\xi}\right)A\,d\xi+\frac{\mu}{2}\|\widetilde{u}_\xi\|^2_{1,\widetilde{\Omega}}$$
$$+(\nu-\mu)\|\nabla_u\cdot\widetilde{u}_\xi\|^2_{0,\widetilde{\Omega}}+\|\widetilde{q}_{\sigma\xi}\|^2_{0,\widetilde{\Omega}}$$
$$\le\delta_1(\|u_{\xi\xi}\|^2_{0,\widetilde{\Omega}}+\|q_{\sigma\xi}\|^2_{0,\widetilde{\Omega}})$$
$$+a_8(\|\widetilde{g}\|^2_{0,\widetilde{\Omega}}+|u|^2_{1,0,\widetilde{\Omega}}+\|q_\sigma\|^2_{0,\widetilde{\Omega}})+a_9X_2(\widetilde{\Omega})Y_2(\widetilde{\Omega})\,, \tag{4.30}$$

where $X_2(\widetilde{\Omega})=|u|^2_{2,1,\widetilde{\Omega}}+|q_\sigma|^2_{2,1,\widetilde{\Omega}}$, $Y_2(\widetilde{\Omega})=X_2(\widetilde{\Omega})+\|u\|^2_{3,\widetilde{\Omega}}+\int_0^t\|u\|^2_{3,\widetilde{\Omega}}\,d\tau$.

Now we consider subdomains near the boundary. Differentiating $(4.22)_1$ with respect to τ, multiplying the result by $\widetilde{u}_\tau J$ and integrating over $\widehat{\Omega}$ yields (J is the Jacobian of the transformation $x=x(z)$)

$$\frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}}\widehat{\eta}\widetilde{u}_\tau^2J\,dz+\frac{\mu}{2}\int_{\widehat{\Omega}}(\widehat{\nabla}_i\widetilde{u}^j_\tau+\widehat{\nabla}_j\widetilde{u}^i_\tau)^2J\,dz+(\nu-\mu)\|\widehat{\nabla}\cdot\widetilde{u}_\tau\|^2_{0,\widehat{\Omega}}$$
$$-\int_{\widehat{\Omega}}\widetilde{q}_{\sigma\tau}\widehat{\nabla}\cdot\widetilde{u}_\tau J\,dz-\int_{\widehat{S}}(\widehat{n}\widehat{\mathbb{T}}(\widetilde{u},\widetilde{q}_\sigma))_{,\tau}\widetilde{u}_\tau J\,dz'$$
$$\le\delta_3(\|\widetilde{u}_{zz}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma z}\|^2_{0,\widehat{\Omega}})+a_{10}(\|\widetilde{g}\|^2_{0,\widehat{\Omega}}+\|\widehat{u}\|^2_{1,\widehat{\Omega}}+\|\widehat{q}_\sigma\|^2_{0,\widehat{\Omega}})$$
$$+a_{11}\|\widehat{u}\|^2_{2,\widehat{\Omega}}\Big(\|\widehat{u}\|^2_{2,\widehat{\Omega}}+|\widehat{q}_\sigma|^2_{2,1,\widehat{\Omega}}+\Big\|\int_0^t\widehat{u}\,d\tau\Big\|^2_{3,\widehat{\Omega}}\Big)\,, \tag{4.31}$$

where we have used the inequalities

$$\int_{\widehat{\Omega}}[(\widehat{\nabla}_j\widehat{T}^{ij}(\widetilde{u},\widetilde{q}_\sigma))_{,\tau}-\widehat{\nabla}_j\widehat{T}^{ij}(\widetilde{u}_\tau,\widetilde{q}_{\sigma\tau})]\widetilde{u}_\tau J\,dz'$$
$$\le\delta_3'(\|\widetilde{u}_{zz}\|^2_{0,\widehat{\Omega}}$$
$$+\|\widetilde{q}_{\sigma z}\|^2_{0,\widehat{\Omega}})+a_{12}\Big(\|\widetilde{u}\|^2_{2,\widehat{\Omega}}\Big\|\int_0^t\widehat{u}\,d\tau\Big\|^2_{3,\widehat{\Omega}}+\|\widehat{u}\|^2_{1,\widehat{\Omega}}+\|\widehat{q}_\sigma\|^2_{0,\widehat{\Omega}}\Big)\,,$$

and

$$\int\limits_{\widehat{S}} [(\widehat{n}\widehat{\mathbb{T}}(\widetilde{u},\widetilde{q}_\sigma))_{,\tau} - \widehat{n}\widehat{\mathbb{T}}(\widetilde{u}_\tau,\widetilde{q}_{\sigma\tau})]\widetilde{u}_\tau J\,dz' \le \delta_3''(\|\widetilde{u}_{zz}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma z}\|^2_{0,\widehat{\Omega}}) + a_{13}\Big(\|\widetilde{u}\|^2_{1,\widehat{\Omega}} + \|\widetilde{q}_\sigma\|^2_{0,\widehat{\Omega}} + \|\widehat{u}\|^2_{2,\widehat{\Omega}}\Big\|\int\limits_0^t \widehat{u}\,d\tau\Big\|^2_{3,\widehat{\Omega}}\Big),$$

and the fact that ∇F can be expressed in terms of $\int_0^t u_\xi\,d\tau$.

Consider the boundary term in (4.31). Using the boundary condition $(4.22)_3$ we obtain

$$(4.32)\qquad -\int\limits_{\widehat{S}} (\widehat{n}\widehat{\mathbb{T}}(\widetilde{u},\widetilde{q}_\sigma))_{,\tau}\widetilde{u}_\tau J\,dz' \le \delta_4\|\widetilde{u}_\tau\|^2_{1,\widehat{\Omega}} + a_{14}\|\widehat{u}_\tau\|^2_{0,\widehat{S}} + a_{15}\|\widehat{u}\|^2_{2,\widehat{\Omega}}\Big\|\int\limits_0^t \widehat{u}\,d\tau\Big\|^2_{3,\widehat{\Omega}} \le \delta_5\|\widehat{u}_{zz}\|^2_{0,\widehat{\Omega}} + a_{16}\Big(\|\widehat{u}\|^2_{1,\widehat{\Omega}} + \|\widehat{u}\|^2_{2,\widehat{\Omega}}\Big\|\int\limits_0^t \widehat{u}\,d\tau\Big\|^2_{3,\widehat{\Omega}}\Big),$$

where in the last inequality we have used the interpolation inequality

$$\|\widehat{u}_\tau\|^2_{0,\widehat{S}} \le \delta\|\widehat{u}_{\tau z}\|^2_{0,\widehat{\Omega}} + c(\delta)\|\widehat{u}_\tau\|^2_{0,\widehat{\Omega}}.$$

From the continuity equation $(4.22)_2$ we get

$$(4.33)\qquad -\int\limits_{\widehat{\Omega}} \widetilde{q}_{\sigma\tau}\nabla_u\cdot\widetilde{u}_\tau J\,dz = \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}} \frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma\tau}J\,dz + N_2,$$

where

$$|N_2| \le \delta_6\|\widetilde{q}_{\sigma\tau}\|^2_{0,\widehat{\Omega}} + c\|\widehat{u}\|^2_{1,\widehat{\Omega}} + a_{17}\Big[|\widehat{q}_\sigma|^2_{2,1,\widehat{\Omega}}(\|\widehat{u}\|^2_{2,\widehat{\Omega}} + |\widehat{q}_\sigma|^2_{2,1,\widehat{\Omega}}) + P\|\widehat{u}\|^2_{2,\widehat{\Omega}}\Big\|\int\limits_0^t u\,dt'\Big\|^2_{2,\widehat{\Omega}}\Big],$$

and $P = P(|\int_0^t \widehat{u}_z\,dt'|_{\infty,\widehat{\Omega}})$, $\|h\|_{0,\widehat{\Omega}} = (\int_{\widehat{\Omega}}|h|^2J\,dz)^{1/2}$.

From (4.31)–(4.33) and Lemma 5.1 in the case $G = \widehat{\Omega}$, $v = \widetilde{u}_\tau$, we have

$$(4.34)\qquad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\Big(\widehat{\eta}\widetilde{u}_\tau^2 + \frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma\tau}\Big)J\,dz + \frac{\mu}{2}\|\widetilde{u}_\tau\|^2_{1,\widehat{\Omega}} + (\nu-\mu)\|\widehat{\nabla}\cdot\widetilde{u}_\tau\|^2_{0,\widehat{\Omega}} \le \delta_7(\|\widehat{u}_{zz}\|^2_{0,\widehat{\Omega}} + \|\widehat{q}_{\sigma z}\|^2_{0,\widehat{\Omega}}) + a_{18}(\|\widehat{u}\|^2_{1,\widehat{\Omega}} + \|\widehat{q}_\sigma\|^2_{0,\widehat{\Omega}} + \|\widetilde{g}\|^2_{0,\widehat{\Omega}}) + a_{19}X_2(\widehat{\Omega})Y_2(\widehat{\Omega}),$$

where $X_2(\widehat{\Omega}) = |\widehat{u}|^2_{2,1,\widehat{\Omega}} + |\widehat{q}_\sigma|^2_{2,1,\widehat{\Omega}}$, $Y_2(\widehat{\Omega}) = X_2(\widehat{\Omega}) + \|\widehat{u}\|^2_{3,\widehat{\Omega}} + \int_0^t \|\widehat{u}\|^2_{3,\widehat{\Omega}}\,dt'$.

Applying the operator $(\mu+\nu)\widehat{\nabla}$ to $(4.22)_2$, dividing the result by $\widehat{q}\Psi(\widehat{\eta})$ and adding to $(4.22)_1$ gives

$$(4.35)\qquad \frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widehat{\nabla}_i\widetilde{q}_{\sigma t}+\widehat{\nabla}_i\widetilde{q}_\sigma=\mu(\widehat{\nabla}^2\widetilde{u}^i-\widehat{\nabla}_i\widehat{\nabla}\cdot\widetilde{u})-\widehat{\eta}\widetilde{u}^i_t+\widehat{\eta}\widetilde{g}^i$$
$$-\frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widehat{\nabla}_i(\widehat{q}\Psi(\widehat{\eta}))\widehat{\nabla}\cdot\widetilde{u}+\frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widehat{\nabla}_i(\widehat{q}\Psi(\widehat{\eta})\widehat{u}\cdot\widehat{\nabla}\widehat{\zeta})+k^i_3\,.$$

Multiplying the normal component of (4.35) by $\widetilde{q}_{\sigma n}J$ and integrating over $\widehat{\Omega}$ implies

$$(4.36)\qquad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma n}J\,dz+\frac{1}{2}\|\widetilde{q}_{\sigma n}\|^2_{0,\widehat{\Omega}}$$
$$\le c\|\widetilde{u}_{z\tau}\|^2_{0,\widehat{\Omega}}+a_{20}(\|\widetilde{u}_t\|^2_{0,\widehat{\Omega}}+\|\widehat{u}\|^2_{1,\widehat{\Omega}}+\|\widehat{q}_\sigma\|^2_{0,\widehat{\Omega}}+\|\widetilde{g}\|^2_{0,\widehat{\Omega}})$$
$$+(\delta'_7+cd)(\|\widetilde{u}_{zz}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma z}\|^2_{0,\widehat{\Omega}})+a_{21}\Big(\|\widetilde{u}\|^2_{2,\widehat{\Omega}}\Big\|\int\limits_0^t\widehat{u}\,dt'\Big\|^2_{3,\widehat{\Omega}}$$
$$+|\widehat{q}_\sigma|^2_{2,1,\widehat{\Omega}}\Big(\|\widehat{q}_\sigma\|^2_{2,\widehat{\Omega}}+\|\widehat{u}\|^2_{2,\widehat{\Omega}}+\Big\|\int\limits_0^t\widehat{u}\,dt'\Big\|^2_{2,\widehat{\Omega}}\Big)\Big)\,.$$

We write $(4.22)_1$ in the form

$$(4.37)\qquad \widehat{\eta}\widetilde{u}^i_t-\mu\Delta\widetilde{u}^i-\nu\nabla_i\nabla\cdot\widetilde{u}=\widehat{\nabla}_i\widetilde{q}_\sigma+\widehat{\eta}\,\widetilde{g}^i+k^i_3-k^i_7\,,$$

where $k^i_7=(\mu\Delta\widetilde{u}^i+\nu\nabla_i\nabla\cdot\widetilde{u})-(\mu\widehat{\nabla}^2\widetilde{u}^i+\nu\widehat{\nabla}_i\widehat{\nabla}\cdot\widetilde{u})$.

Multiplying the third component of (4.37) by $\widetilde{u}^3_{nn}J$ and integrating over $\widehat{\Omega}$ yields

$$(4.38)\qquad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\widehat{\eta}|\widetilde{u}^3_n|^2J\,dz+\frac{\mu+\nu}{2}\|\widetilde{u}^3_{nn}\|^2_{0,\widehat{\Omega}}\le c(\|\widetilde{u}_{z\tau}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma n}\|^2_{0,\widehat{\Omega}})$$
$$+a_{22}(\|\widehat{u}\|^2_{1,\widehat{\Omega}}+\|\widehat{q}_\sigma\|^2_{0,\widehat{\Omega}}+\|\widetilde{g}\|^2_{0,\widehat{\Omega}}+\|\widetilde{u}_t\|^2_{0,\widehat{\Omega}})+\delta_8(\|\widetilde{u}^3_{nt}\|^2_{0,\widehat{\Omega}}+\|\widetilde{u}_{zz}\|^2_{0,\widehat{\Omega}})$$
$$+a_{23}\Big(|\widehat{q}_\sigma|^2_{2,1,\widehat{\Omega}}\|\widetilde{u}\|^2_{2,\widehat{\Omega}}+\|\widehat{u}\|^4_{2,\widehat{\Omega}}+\|\widetilde{u}\|^2_{2,\widehat{\Omega}}\Big\|\int\limits_0^t\widehat{u}\,dt'\Big\|^2_{3,\widehat{\Omega}}\Big)\,.$$

To estimate $\widetilde{u}^i_{nn}$, $i=1,2$, and $\widetilde{q}_{\sigma\tau}$ we write (4.37) in the form

$$(4.39)\qquad -\mu\Delta\widetilde{u}^i+\nabla_{z^i}\widetilde{q}_\sigma$$
$$=\widehat{\eta}\,\widetilde{g}^i-\widehat{\eta}\widetilde{u}^i_t+k^i_3-k^i_7+\nabla_{z^i}\widetilde{q}_\sigma-\widehat{\nabla}_i\widetilde{q}_\sigma+\nu\nabla_{z^i}\operatorname{div}\widetilde{u}$$
$$\equiv f^i+\nu\nabla_{z^i}\operatorname{div}\widetilde{u}\,,$$

and the boundary condition $(4.22)_3$ as

$$(4.40)\qquad \frac{\partial\widetilde{u}^i}{\partial z^3}=-\frac{\partial\widetilde{u}^3}{\partial z^i}+\Big(\frac{\partial\widetilde{u}^i}{\partial z^3}+\frac{\partial\widetilde{u}^3}{\partial z^i}-\frac{1}{\mu}\widehat{\tau}_i\widehat{\mathbb{T}}\widehat{n}\Big)$$
$$+\frac{1}{\mu}k_5\cdot\widehat{\tau}_i\equiv\widetilde{h}^i\,,\qquad i=1,2,\ z^3=0\,,$$

where we have also used the fact that $\widehat{\tau}_i \cdot \widehat{n} = 0$, $i = 1,2$. Considering the problem (4.39), (4.40) in $\widehat{\Omega}$ we have to add the boundary conditions

$$\widetilde{u}^i|_{|z'|=d} = 0\,, \quad \widetilde{u}^i|_{z^3=d} = 0\,, \quad i = 1,2\,, \quad \widetilde{q}_\sigma|_{|z'|=d} = 0\,, \quad \widetilde{q}_\sigma|_{z^3=d} = 0\,. \tag{4.41}$$

Multiplying (4.39) by $\widetilde{u}^i$, summing over $i = 1,2$, integrating over $\widehat{\Omega}$ and using boundary conditions (4.40) and (4.41) yields

$$\|\nabla \widetilde{u}'\|^2_{0,\widehat{\Omega}} \le \delta_9 \|\widetilde{q}_\sigma\|^2_{0,\widehat{\Omega}} + c(\|\widetilde{f}'\|^2_{0,\widehat{\Omega}} + \|\widetilde{h}'\|^2_{0,\widehat{S}} + \|\operatorname{div} \widetilde{u}\|^2_{0,\widehat{\Omega}})\,, \tag{4.42}$$

where the prime denotes that only two components ($i = 1,2$) are taken into account.

We look for a function $w \in H^1(\widehat{\Omega})$ such that

$$\begin{aligned} &\operatorname{div} w = \widetilde{q}_\sigma\,, \quad w^3|_{z^3=0} = \chi(z') \int_{\widehat{\Omega}} \widetilde{q}_\sigma dz\,, \\ &w|_{\partial\widehat{\Omega}\setminus\widehat{S}} = 0\,, \quad w^i|_{z^3=0} = 0\,, \quad i = 1,2\,, \end{aligned} \tag{4.43}$$

where $\chi(z')$ is a smooth function such that $\int_{\widehat{S}} \chi(z')\,dz' = 1$, $\chi(z') \ge 0$, $\chi|_{|z'|=d} = 0$. Moreover, $1 \le 4d^2 |\chi|_{\infty,\widehat{S}}$, so $|\chi|_{\infty,\widehat{S}} \ge 1/(4d^2)$. Finally, assuming that χ vanishes only in a neighbourhood of the boundary of $\widehat{S}$, we require that $\min_{|z'|\le d/2} \chi(z') > 0$. Hence

$$1 = \int_{\widehat{S}} \chi(z')\,dz' \ge \int_{|z'|\le d/2} \chi(z')\,dz' \ge d^2 \min_{|z'|\le d/2} \chi(z')\,,$$

so $\min_{|z'|\le d/2} \chi(z') \le 1/d^2$. Therefore, we can assume that $\chi(z') \le c/d^2$.

We look for solutions of (4.43) in the form $w = \nabla\varphi + \alpha$, where φ is a solution to the Neumann problem

$$\begin{aligned} &\Delta\varphi = \widetilde{q}_\sigma\,, \quad \partial_{z^3}\varphi|_{z^3=0} = \chi(z') \int_{\widehat{\Omega}} \widetilde{q}_\sigma\,dz \equiv \varphi_0\,, \quad \partial_{z^3}\varphi|_{z^3=d} = 0\,, \\ &\partial_{z^i}\varphi|_{|z^i|=d} = 0\,, \quad i = 1,2\,, \quad \int_{\widehat{\Omega}} \varphi\,dz = 0\,, \end{aligned} \tag{4.44}$$

and

$$\begin{aligned} &\operatorname{div}\alpha = 0\,, \quad \alpha|_{\partial\widehat{\Omega}\setminus\widehat{S}} = -\nabla\varphi|_{\partial\widehat{\Omega}\setminus\widehat{S}}\,, \quad \alpha\cdot\overline{n}|_{\widehat{S}} = 0\,, \\ &\alpha\cdot\overline{\tau}_i|_{\widehat{S}} = -\overline{\tau}_i\cdot\nabla\varphi|_{\widehat{S}}\,, \quad i = 1,2\,, \end{aligned} \tag{4.45}$$

where $\overline{n}$, $\overline{\tau}_i$, $i = 1,2$, are normal and tangent vectors to $\widehat{S}$.

Since the compatibility condition for (4.44) is satisfied there exists a unique solution to (4.44) such that $\varphi \in H^2(\widehat{\Omega})$ and

$$\|\varphi\|_{2,\widehat{\Omega}} \le c(\|\widetilde{q}_\sigma\|_{0,\widehat{\Omega}} + \|\varphi_0\|_{0,\widehat{S}} + \|\varphi_0\|_{1/2,\widehat{S}}) \le c(1 + d^{1/2})\|\widetilde{q}_\sigma\|_{0,\widehat{\Omega}}\,, \tag{4.46}$$

because

$$\|\varphi_0\|_{0,\widehat{S}} \le \Big|\int\limits_{\widehat{\Omega}} \widehat{q}_\sigma\,dz\Big|\Big(\int\limits_{\widehat{S}} |\chi(z')|^2\,dz'\Big)^{1/2} \le cd^{1/2}\|\widetilde{q}_\sigma\|_{0,\widehat{\Omega}}\,,$$

$$\|\varphi_0\|_{1/2,\widehat{S}} \le \Big|\int\limits_{\widehat{\Omega}} \widetilde{q}_\sigma\,dz\Big|\Big(\int\limits_{\widehat{S}}\int\limits_{\widehat{S}} \frac{|\chi(x')-\chi(y')|^2}{|x'-y'|^3}\,dx'dy'\Big)^{1/2}$$

$$\le \frac{c}{d^3}\Big|\int\limits_{\widehat{\Omega}} \widetilde{q}_\sigma\,dz\Big|\Big(\int\limits_{\widehat{S}}\int\limits_{\widehat{S}} |x'-y'|^{-1}\,dx'dy'\Big)^{1/2}$$

$$\le cd^{-3/2}\Big|\int\limits_{\widehat{\Omega}} \widetilde{q}_\sigma\,dz\Big| \le c\|\widetilde{q}_\sigma\|_{0,\widetilde{\Omega}}\,,$$

where we have used the fact that $|\nabla\chi| \le c/d^3$.

Similarly, the compatibility condition for (4.45) is satisfied because $\overline{n}\cdot\nabla\varphi|_{\partial\widehat{\Omega}\setminus\widehat{S}} = 0$. Hence, there exists a solution to (4.45) such that $\alpha \in H^1(\widehat{\Omega})$ and

$$\|\alpha\|_{1,\widehat{\Omega}} \le c\|\nabla\varphi\|_{1/2,\partial\widehat{\Omega}} \le c\|\varphi\|_{2,\widehat{\Omega}} \le c\|\widetilde{q}_\sigma\|_{0,\widehat{\Omega}}\,. \tag{4.47}$$

Summarizing, there exists a solution of (4.43) such that $w \in H^1(\widehat{\Omega})$ and

$$\|w\|_{1,\widehat{\Omega}} \le c\|\widetilde{q}_\sigma\|_{0,\widehat{\Omega}}\,. \tag{4.48}$$

Now we estimate $\|\widetilde{q}_\sigma\|_{0,\widehat{\Omega}}$. Multiplying (4.39) by w and integrating over $\widehat{\Omega}$ yields

$$-\mu\int\limits_{\widehat{\Omega}} \Delta\widetilde{u}\cdot w\,dz + \int\limits_{\widehat{\Omega}} \nabla\widetilde{q}_\sigma\cdot w\,dz = \int\limits_{\widehat{\Omega}} \widetilde{f}\cdot w\,dz + \nu\int\limits_{\widehat{\Omega}} \nabla\operatorname{div}\widetilde{u}\cdot w\,dz\,. \tag{4.49}$$

The boundary term which follows from integration by parts in the first term of (4.49) is estimated in the following way:

$$\Big|\mu\int\limits_{\widehat{S}} \overline{n}\cdot\nabla\widetilde{u}\cdot w\,dz'\Big| \le c\Big|\int\limits_{\widehat{S}} \widetilde{u}^3_{z^3} w^3\,dz'\Big| \le c\|\widetilde{u}^3_{z^3}\|_{-1/2,\widehat{S}}\|w^3\|_{1/2,\widehat{S}}$$

$$\le c\|\widetilde{u}^3_{z^3}\|_{0,\widehat{\Omega}}\|w\|_{1,\widehat{\Omega}}\,.$$

The second term on the left-hand side of (4.49) is equal to

$$\int\limits_{\widehat{S}} \widetilde{q}_\sigma\cdot w^3\,dz' - \int\limits_{\widehat{\Omega}} \widetilde{q}_\sigma\operatorname{div} w\,dz$$

where

$$\Big|\int\limits_{\widehat{S}} \widetilde{q}_\sigma w^3\,dz'\Big| = \Big|\int\limits_{\widehat{\Omega}} \widetilde{q}_\sigma\,dz\int\limits_{\widehat{S}} \widetilde{q}_\sigma\chi(z')\,dz'\Big| \le \frac{c}{d^2}\Big|\int\limits_{\widehat{\Omega}} \widetilde{q}_\sigma\,dz\int\limits_{\widehat{S}} \widetilde{q}_\sigma(z')\,dz'\Big|$$

$$\le cd^{-1/2}\|\widetilde{q}_\sigma\|_{0,\widehat{\Omega}}\Big|\int\limits_{\widehat{S}}\widetilde{q}_\sigma(z')\,dz'\Big|$$
$$\le \tfrac14 d^{1/2}\|\widetilde{q}_\sigma\|_{0,\widehat{\Omega}}\|\widetilde{q}_\sigma\|_{0,\widehat{S}}\le \delta_{10}^2\|\widetilde{q}_{\sigma z}\|^2_{0,\widehat{\Omega}}+c(\delta_{10})d\|\widetilde{q}_\sigma\|^2_{0,\widehat{\Omega}}\,,$$

and

$$\int\limits_{\widehat{\Omega}}\widetilde{q}_\sigma\operatorname{div}w\,dz=\|\widetilde{q}_\sigma\|^2_{0,\widehat{\Omega}}\,.$$

Finally, the last term in (4.49) can be expressed in the form

$$\int\limits_{\widehat{\Omega}}\nabla\operatorname{div}\widetilde{u}\cdot w\,dz=\int\limits_{\widehat{S}}\operatorname{div}\widetilde{u}\,w^3\,dz'-\int\limits_{\widehat{\Omega}}\operatorname{div}\widetilde{u}\operatorname{div}w\,dz\,,$$

where

$$\Big|\int\limits_{\widehat{S}}\operatorname{div}\widetilde{u}\,w^3\,dz'\Big|=\Big|\int\limits_{\widehat{\Omega}}\widetilde{q}_\sigma\,dz\int\limits_{\widehat{S}}\operatorname{div}\widetilde{u}\,\chi(z')\,dz'\Big|\le cd^{-1/2}\|\widetilde{q}_\sigma\|_{0,\widehat{\Omega}}\int\limits_{\widehat{S}}|\operatorname{div}\widetilde{u}|\,dz'$$
$$\le cd^{1/2}\|\widetilde{q}_\sigma\|_{0,\widehat{\Omega}}\|\operatorname{div}\widetilde{u}\|_{0,\widehat{S}}\le d\|\widetilde{q}_\sigma\|^2_{0,\widehat{\Omega}}+c\|\operatorname{div}\widetilde{u}\|^2_{1,\widehat{\Omega}}\,.$$

Summarizing, we obtain for sufficiently small d the estimate

$$\|\widetilde{q}_\sigma\|^2_{0,\widehat{\Omega}}\le\delta_{10}\|\widetilde{q}_{\sigma z}\|^2_{0,\widehat{\Omega}}+c(\|\widetilde{f}\|^2_{0,\widehat{\Omega}}+\|\widetilde{h}'\|^2_{0,\widehat{\Omega}}+\|\widetilde{u}^3_{z^3}\|^2_{0,\widehat{\Omega}}+\|\operatorname{div}\widetilde{u}\|^2_{1,\widehat{\Omega}})\,.\tag{4.50}$$

Now instead of problem (4.39), (4.40) we consider the problem

$$\begin{aligned}&-\mu\Delta\widetilde{u}^i_{z'}+\nabla_{z^i}\widetilde{q}_{\sigma z'}=\widetilde{f}^i_{z'}+\nabla_{z^i}\operatorname{div}\widetilde{u}_{z'}\,,\quad i=1,2,3\,,\\&\partial_{z^3}\widetilde{u}^i_{z'}=\widetilde{h}^i_{z'}\,,\quad i=1,2\,.\end{aligned}\tag{4.51}$$

Multiplying $(4.51)_1$ by $\widetilde{u}^i_{z'}$, summing over $i=1,2$, and integrating over $\widehat{\Omega}$ yields

$$\|\widetilde{u}'_{zz'}\|^2_{0,\widehat{\Omega}}\le\delta_{10}\|\widetilde{q}_{\sigma z'}\|^2_{0,\widehat{\Omega}}+c(\|\widetilde{f}'\|^2_{0,\widehat{\Omega}}+\|\widetilde{h}'_{z'}\|^2_{0,\widehat{\Omega}}+\|\operatorname{div}\widetilde{u}\|^2_{1,\widehat{\Omega}})\,.\tag{4.52}$$

Finally, let us introduce a function $w_1\in\overset{\circ}{W}{}^1_2(\widehat{\Omega})$ such that

$$\operatorname{div}w_1=\widetilde{q}_{\sigma z'}\,,\quad w_1|_{\partial\widehat{\Omega}}=0\,.\tag{4.53}$$

By $\int_{\widehat{\Omega}}\widetilde{q}_{\sigma z'}\,dz=0$ there exists a solution of (4.53) such that $w_1\in H^1(\widehat{\Omega})$ and

$$\|w_1\|_{1,\widehat{\Omega}}\le c\|\widetilde{q}_{\sigma z'}\|_{0,\widehat{\Omega}}\,.\tag{4.54}$$

Multiplying $(4.51)_1$ by w_1 and integrating over $\widehat{\Omega}$ gives

$$\|\widetilde{q}_{\sigma z'}\|^2_{0,\widehat{\Omega}}\le c(\|\widetilde{f}\|^2_{0,\widehat{\Omega}}+\|\operatorname{div}\widetilde{u}\|^2_{1,\widehat{\Omega}}+\|\widetilde{u}_{zz'}\|^2_{0,\widehat{\Omega}})\,.\tag{4.55}$$

From (4.42), (4.50), (4.52) and (4.55) we have

$$\begin{aligned}&\|\widetilde{u}'_z\|^2_{0,\widehat{\Omega}}+\|\widetilde{u}'_{zz'}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_\sigma\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma z'}\|^2_{0,\widehat{\Omega}}\\&\quad\le c(\|\widetilde{f}'\|^2_{0,\widehat{\Omega}}+\|\widetilde{h}'\|^2_{1,\widehat{\Omega}}+\|\operatorname{div}\widetilde{u}\|^2_{1,\widehat{\Omega}}+\|\widetilde{u}\|^2_{1,\widehat{\Omega}})+\delta_{10}\|\widetilde{q}_{\sigma z^3}\|^2_{0,\widehat{\Omega}}\,.\end{aligned}\tag{4.56}$$

From the form of $\widetilde{f}'$ and $\widetilde{h}'$ we have

$$
\begin{aligned}
&\|\widetilde{f}\|^2_{0,\widehat{\Omega}} \le c(\|\widetilde{g}\|^2_{0,\widehat{\Omega}} + \|\widetilde{u}_t\|^2_{0,\widehat{\Omega}} + \|\widehat{u}\|^2_{1,\widehat{\Omega}} + \|\widehat{q}_\sigma\|^2_{0,\widehat{\Omega}}) \\
&\qquad + c\Big(\Big\|\int_0^t \widehat{u}\,dt'\Big\|^2_{3,\widehat{\Omega}} + d\Big)(\|\widetilde{q}_\sigma\|^2_{2,\widehat{\Omega}} + \|\widetilde{u}\|^2_{2,\widehat{\Omega}}), \\
&\|\widetilde{h}\|^2_{1,\widehat{\Omega}} \le c\Big(\|\widetilde{u}^3_{z\tau}\|^2_{0,\widehat{\Omega}} + \Big(\Big\|\int_0^t \widehat{u}\,dt'\Big\|^2_{3,\widehat{\Omega}} + d\Big)\|\widetilde{u}\|^2_{2,\widehat{\Omega}} + \|\widehat{u}\|^2_{1,\widehat{\Omega}}\Big).
\end{aligned}
\tag{4.57}
$$

Finally, from (4.39) we obtain

$$
\|\widetilde{u}'_{nn}\|^2_{0,\widehat{\Omega}} \le c(\|\widetilde{q}_{\sigma\tau}\|^2_{0,\widehat{\Omega}} + \|\widetilde{f}\|^2_{0,\widehat{\Omega}} + \|\operatorname{div}\widetilde{u}\|^2_{1,\widehat{\Omega}}). \tag{4.58}
$$

Now, (4.56)–(4.58) imply

$$
\begin{aligned}
\|\widetilde{u}'_{nn}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma\tau}\|^2_{0,\widehat{\Omega}} &\le c(\|\widetilde{u}^3_{z\tau}\|^2_{0,\widehat{\Omega}} + \|\operatorname{div}\widetilde{u}\|^2_{1,\widehat{\Omega}}) \\
&\quad + a_{24}(\|\widetilde{g}\|^2_{0,\widehat{\Omega}} + |\widetilde{u}|^2_{1,0,\widehat{\Omega}} + \|\widetilde{q}_\sigma\|^2_{0,\widehat{\Omega}}) \\
&\quad + a_{25}\Big\|\int_0^t \widehat{u}\,dt'\Big\|^2_{3,\widehat{\Omega}}(\|\widetilde{q}_\sigma\|^2_{2,\widehat{\Omega}} + \|\widetilde{u}\|^2_{2,\widehat{\Omega}}) + \delta_{11}\|\widetilde{q}_{\sigma n}\|^2_{0,\widehat{\Omega}}.
\end{aligned}
\tag{4.59}
$$

Now (4.34), (4.36), (4.38) and (4.59) yield

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}} \Big[\widehat{\eta}(\widetilde{u}^2_\tau + |\widetilde{u}^3_n|^2) + \frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma z}\Big]\,dz + \frac{\mu}{2}\|\widetilde{u}\|^2_{2,\widehat{\Omega}} + \|\widetilde{q}_{\sigma z}\|^2_{0,\widehat{\Omega}} \\
&\qquad \le a_{26}(\|\widetilde{g}\|^2_{0,\widehat{\Omega}} + |\widehat{u}|^2_{1,0,\widehat{\Omega}} + \|\widehat{q}_\sigma\|^2_{0,\widehat{\Omega}}) + a_{27}X_2(\widehat{\Omega})Y_2(\widehat{\Omega}).
\end{aligned}
\tag{4.60}
$$

We also have

$$
\frac{d}{dt}\int_{\widehat{\Omega}} \widehat{\eta}\widetilde{u}^2_n J\,dz \le \delta_{12}\|\widetilde{u}_{nt}\|^2_{0,\widehat{\Omega}} + c\|\widetilde{u}\|^2_{1,\widehat{\Omega}} + a_{28}X_2(\widehat{\Omega})Y_2(\widehat{\Omega}). \tag{4.61}
$$

We use (4.61) in (4.60), and next we go back to the variables ξ. From the resulting estimate and (4.30), after summing over all neighbourhoods of the partition of unity and finally going back to the variables x and using (4.12) we obtain (4.24). This concludes the proof.

Now we obtain an inequality for the third derivatives.

Lemma 4.5. *For a sufficiently smooth solution v, p of* (4.1),

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} \Big(\varrho v^2_{xx} + \frac{1}{p\Psi(\varrho)}p^2_{\sigma xx}\Big)\,dx + \|v\|^2_{3,\Omega_t} + \|p_\sigma\|^2_{2,\Omega_t} \\
&\qquad \le \varepsilon_4\|v_{xxt}\|^2_{0,\Omega_t} + P_9(\|f\|^2_{1,\Omega_t} + |v|^2_{2,1,\Omega_t} + |p_\sigma|^2_{1,0,\Omega_t}) \\
&\qquad\quad + P_{10}X_3(1 + X_3)Y_3,
\end{aligned}
\tag{4.62}
$$

where

$$
\tag{4.63}
\begin{aligned}
X_3 &= \|v\|^2_{3,\Omega_t} + |p_\sigma|^2_{2,1,\Omega_t} + \|v_t\|^2_{1,\Omega_t} + \int_0^t \|v\|^2_{3,\Omega_\tau}\,d\tau\,,\\
Y_3 &= \|v\|^2_{4,\Omega_t} + \|p_\sigma\|^2_{3,\Omega_t} + |p_\sigma|^2_{2,1,\Omega_t} + \|v_t\|^2_{1,\Omega_t} + \int_0^t \|v\|^2_{4,\Omega_\tau}\,d\tau\,.
\end{aligned}
$$

P r o o f. We use the introduced partition of unity. First we consider interior subdomains. We differentiate (4.21) twice with respect to ξ, multiply the result by $\widetilde{u}_{\xi\xi}A$ and integrate over $\widetilde{\Omega}$ to get

$$
\tag{4.64}
\begin{aligned}
&\frac12\frac{d}{dt}\int_{\widetilde{\Omega}} \eta\widetilde{u}^2_{\xi\xi}A\,d\xi + \frac{\mu}{2}\int_{\widetilde{\Omega}} (\nabla_{u^i}\widetilde{u}^j_{\xi\xi} + \nabla_{u^j}\widetilde{u}^i_{\xi\xi})^2A\,d\xi\\
&\quad + (\nu-\mu)\|\nabla_u\cdot\widetilde{u}_{\xi\xi}\|^2_{0,\widetilde{\Omega}} - \int_{\widetilde{\Omega}} \widetilde{q}_{\sigma\xi\xi}\nabla_u\cdot\widetilde{u}_{\xi\xi}A\,d\xi\\
&\le \delta_1(\|\partial^3_\xi u\|^2_{0,\widetilde{\Omega}} + \|\partial^2_\xi q_\sigma\|^2_{0,\widetilde{\Omega}}) + a_1(\|u\|^2_{2,\widetilde{\Omega}} + \|q_\sigma\|^2_{1,\widetilde{\Omega}} + \|\widetilde{g}\|^2_{1,\widetilde{\Omega}})\\
&\quad + a_2X_3(\widetilde{\Omega})(1+X_3(\widetilde{\Omega}))Y_3(\widetilde{\Omega})\,,
\end{aligned}
$$

where

$$
\tag{4.65}
\begin{aligned}
X_3(\widetilde{\Omega}) &= \|u\|^2_{3,\widetilde{\Omega}} + |q_\sigma|^2_{2,1,\widetilde{\Omega}} + \|u_t\|^2_{1,\widetilde{\Omega}} + \int_0^t \|u\|^2_{3,\widetilde{\Omega}}\,dt'\,,\\
Y_3(\widetilde{\Omega}) &= \|u\|^2_{4,\widetilde{\Omega}} + \|q_\sigma\|^2_{3,\widetilde{\Omega}} + |q_\sigma|^2_{2,1,\widetilde{\Omega}} + \|u_t\|^2_{1,\widetilde{\Omega}} + \int_0^t \|u\|^2_{4,\widetilde{\Omega}}\,dt'\,.
\end{aligned}
$$

From the continuity equation $(4.21)_2$ we obtain

$$
\tag{4.66}
-\int_{\widetilde{\Omega}} \widetilde{q}_{\sigma\xi\xi}\nabla_u\cdot\widetilde{u}_{\xi\xi}A\,d\xi = \frac12\frac{d}{dt}\int_{\widetilde{\Omega}}\frac{1}{q\Psi(\eta)}\widetilde{q}^2_{\sigma\xi\xi}A\,d\xi + N_1\,,
$$

where

$$
|N_1| \le \delta_2\|\widetilde{q}_{\sigma\xi\xi}\|^2_{0,\widetilde{\Omega}} + c\|u\|^2_{2,\widetilde{\Omega}} + a_3X_3(\widetilde{\Omega})(1+X_3(\widetilde{\Omega}))Y_3(\widetilde{\Omega})\,.
$$

Using the form of k_1 (see $(4.21)_1$) from (4.28) we obtain

$$
\tag{4.67}
\begin{aligned}
\|\widetilde{u}\|^2_{3,\widetilde{\Omega}} + \|\widetilde{q}_\sigma\|^2_{2,\widetilde{\Omega}} &\le a_4(\|\widetilde{g}\|^2_{1,\widetilde{\Omega}} + \|u\|^2_{2,\widetilde{\Omega}} + \|q_\sigma\|^2_{1,\widetilde{\Omega}} + \|\widetilde{u}_t\|^2_{1,\widetilde{\Omega}})\\
&\quad + a_5\Big(\|q_\sigma\|^4_{2,\widetilde{\Omega}} + \|q_\sigma\|^2_{2,\widetilde{\Omega}}\|\widetilde{u}_t\|^2_{1,\widetilde{\Omega}}\\
&\quad + \Big\|\int_0^t u\,dt'\Big\|^2_{3,\widetilde{\Omega}}\Big(1+\Big\|\int_0^t u\,dt'\Big\|^2_{3,\widetilde{\Omega}}\Big)\|u\|^2_{3,\widetilde{\Omega}}\Big)\\
&\quad + c\|\nabla_u\cdot\widetilde{u}\|^2_{2,\widetilde{\Omega}}\,.
\end{aligned}
$$

Applying Lemma 5.1 in the case $G = \widetilde{\Omega}$, $v = \widetilde{u}_{\xi\xi}$ and (4.64), (4.66), (4.67) we get

$$\begin{aligned}(4.68)\quad &\frac{1}{2}\frac{d}{dt}\int\limits_{\widetilde{\Omega}}\left(\eta\widetilde{u}_{\xi\xi}^2 + \frac{1}{q\Psi(\eta)}\widetilde{q}_{\sigma\xi\xi}^2\right)A\,d\xi + \frac{\mu}{2}\|\widetilde{u}\|_{3,\widetilde{\Omega}}^2 + \|\widetilde{q}_\sigma\|_{2,\widetilde{\Omega}}^2\\ &\le \delta_3(\|\partial_\xi^3 u\|_{0,\widetilde{\Omega}}^2 + \|\partial_\xi^2 q_\sigma\|_{0,\widetilde{\Omega}}^2) + a_6(\|\widetilde{g}\|_{1,\widetilde{\Omega}}^2 + |u|_{2,1,\widetilde{\Omega}}^2 + \|q_\sigma\|_{1,\widetilde{\Omega}}^2)\\ &\quad + a_7 X_3(\widetilde{\Omega})(1 + X_3(\widetilde{\Omega}))Y_3(\widetilde{\Omega})\,.\end{aligned}$$

Now we consider a subdomain near the boundary. Differentiating $(4.22)_1$ twice with respect to τ, multiplying the result by $\widetilde{u}_{\tau\tau}J$ and integrating over $\widetilde{\Omega}$ yields

$$\begin{aligned}(4.69)\quad &\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\left(\widehat{\eta}\,\widetilde{u}_{\tau\tau} + \frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}_{\sigma\tau\tau}^2\right)J\,dz + \frac{\mu}{2}\|\widetilde{u}_{\tau\tau}\|_{1,\widehat{\Omega}}^2\\ &\quad - \int\limits_{\widehat{S}}(\widehat{n}\widehat{\mathbb{T}}(\widetilde{u},\widetilde{q}_\sigma))_{,\tau\tau}\cdot\widetilde{u}_{\tau\tau}J\,dz'\\ &\le \delta_4(\|\widehat{q}_{\sigma zz}\|_{0,\widehat{\Omega}}^2 + \|\widehat{u}_{zzz}\|_{0,\widehat{\Omega}}^2) + a_8(\|\widehat{u}\|_{2,\widehat{\Omega}}^2 + \|\widehat{q}_\sigma\|_{1,\widehat{\Omega}}^2 + \|\widetilde{g}\|_{1,\widehat{\Omega}}^2)\\ &\quad + a_9 X_3(\widehat{\Omega})(1 + X_3(\widehat{\Omega}))Y_3(\widehat{\Omega})\,,\end{aligned}$$

where $X_3(\widehat{\Omega})$, $Y_3(\widehat{\Omega})$ are defined by (4.65) with $\widehat{\Omega}$ in place of $\widetilde{\Omega}$, and $\widehat{u},\widehat{q}_\sigma$ in place of u, q_σ, and where we have used Lemma 5.1 in the case $G = \widehat{\Omega}$, $v = \widetilde{u}_{\tau\tau}$; moreover,

$$-\int\limits_{\widehat{\Omega}}\widetilde{q}_{\sigma\tau\tau}\widehat{\nabla}\cdot\widetilde{u}_{\tau\tau}J\,dz = \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}_{\sigma\tau\tau}^2 J\,dz + N_2\,,$$

where

$$|N_2| \le \delta_5\|\widetilde{q}_{\sigma\tau\tau}\|_{0,\widehat{\Omega}}^2 + a_{10}\|\widehat{u}\|_{2,\widehat{\Omega}}^2 + a_{11}X_3(\widehat{\Omega})(1 + X_3(\widehat{\Omega}))Y_3(\widehat{\Omega})\,.$$

Considering the boundary term in (4.69) we obtain the estimate

$$\begin{aligned}(4.70)\quad &\left|\int\limits_{\widehat{S}}(\widehat{n}\widehat{\mathbb{T}}(\widetilde{u},\widetilde{q}_\sigma))_{,\tau\tau}\cdot\widetilde{u}_{\tau\tau}J\,dz'\right| \le \delta_6\|\widehat{u}_{z\tau\tau}\|_{0,\widehat{\Omega}}^2 + a_{12}\|\widehat{u}\|_{2,\widehat{\Omega}}^2\\ &\qquad + a_{13}\|\widehat{u}\|_{3,\widehat{\Omega}}^2\left\|\int_0^t\widehat{u}\,dt'\right\|_{4,\widehat{\Omega}}^2.\end{aligned}$$

Summarizing, we have

$$\begin{aligned}(4.71)\quad &\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\left(\widehat{\eta}\,\widetilde{u}_{\tau\tau}^2 + \frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}_{\sigma\tau\tau}^2\right)J\,dz + \frac{\mu}{2}\|\widetilde{u}_{\tau\tau}\|_{1,\widehat{\Omega}}^2\\ &\le \delta_7(\|\widehat{u}_{zzz}\|_{0,\widehat{\Omega}}^2 + \|\widehat{q}_{\sigma zz}\|_{0,\widehat{\Omega}}^2) + a_{14}(\|\widehat{u}\|_{2,\widehat{\Omega}}^2 + \|\widehat{q}_\sigma\|_{1,\widehat{\Omega}}^2 + \|\widetilde{g}\|_{1,\widehat{\Omega}}^2)\\ &\quad + a_{15}X_3(\widehat{\Omega})(1 + X_3(\widehat{\Omega}))Y_3(\widehat{\Omega})\,.\end{aligned}$$

Differentiating the third component of (4.35) with respect to τ, multiplying the result by $\widetilde{q}_{\sigma n\tau}J$ and integrating over $\widehat{\Omega}$ yields

$$\begin{aligned}(4.72)\quad & \frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}}\frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}_{\sigma n\tau}^2 J\,dz+\|\widetilde{q}_{\sigma n\tau}\|_{0,\widehat{\Omega}}^2\\ &\le c\|\widetilde{u}_{z\tau\tau}\|_{0,\widehat{\Omega}}^2+(\delta_8+cd)(\|\widehat{u}_{zzz}\|_{0,\widehat{\Omega}}^2+\|\widehat{q}_{\sigma zz}\|_{0,\widehat{\Omega}}^2)\|\widehat{F}\|_{4-1/2,\widehat{S}}^2\\ &\quad+a_{16}(|\widehat{u}|_{2,1,\widehat{\Omega}}^2+\|\widehat{q}_\sigma\|_{1,\widehat{\Omega}}^2+\|\widetilde{g}\|_{1,\widehat{\Omega}}^2)+a_{17}X_3(\widehat{\Omega})(1+X_3(\widehat{\Omega}))Y_3(\widehat{\Omega})\,.\end{aligned}$$

Differentiating the third component of (4.37) with respect to τ, multiplying the result by $\widetilde{u}_{nn\tau}^3J$ and integrating over $\widehat{\Omega}$ gives

$$\begin{aligned}(4.73)\quad & \frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}}\widehat{\eta}|\widetilde{u}_{n\tau}^3|^2J\,dz+\frac{\mu+\nu}{2}\|\widetilde{u}_{nn\tau}^3\|_{0,\widehat{\Omega}}^2\\ &\le c(\|\widetilde{u}_{z\tau\tau}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma n\tau}\|_{0,\widehat{\Omega}}^2)\\ &\quad+(\delta_9+cd)(\|\widetilde{u}_{zzz}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma zz}\|_{0,\widehat{\Omega}}^2)\|\widehat{F}\|_{4-1/2,\widehat{S}}^2\\ &\quad+a_{18}(\|\widetilde{u}_t\|_{1,\widehat{\Omega}}^2+\|\widehat{u}\|_{2,\widehat{\Omega}}^2+\|\widehat{q}_\sigma\|_{1,\widehat{\Omega}}^2+\|\widetilde{g}\|_{1,\widehat{\Omega}}^2)\\ &\quad+a_{19}X_3(\widehat{\Omega})(1+X_3(\widehat{\Omega}))Y_3(\widehat{\Omega})\,.\end{aligned}$$

Differentiating (4.39) twice with respect to τ, multiplying by $\widetilde{u}'_{\tau\tau}J$, integrating over $\widehat{\Omega}$ and using the boundary condition (4.40) we get

$$\begin{aligned}(4.74)\quad & \|\widetilde{u}'_{z\tau\tau}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma\tau\tau}\|_{0,\widehat{\Omega}}^2\\ &\le(\delta_{10}+cd)(\|\widehat{u}_{zzz}\|_{0,\widehat{\Omega}}^2+\|\widehat{q}_{\sigma zz}\|_{0,\widehat{\Omega}}^2)\|\widehat{F}\|_{4-1/2,\widehat{S}}^2\\ &\quad+a_{20}(\|\operatorname{div}\widetilde{u}_\tau\|_{1,\widehat{\Omega}}^2+\|\widetilde{u}_t\|_{1,\widehat{\Omega}}^2+\|\widehat{u}\|_{2,\widehat{\Omega}}^2+\|\widehat{q}_\sigma\|_{1,\widehat{\Omega}}^2+\|\widetilde{g}\|_{1,\widehat{\Omega}}^2)\\ &\quad+a_{21}X_3(\widehat{\Omega})Y_3(\widehat{\Omega})\,.\end{aligned}$$

Moreover, from (4.39) we obtain

$$\begin{aligned}(4.75)\quad \|\widetilde{u}'_{nn\tau}\|_{0,\widehat{\Omega}}^2 &\le c(\|\widetilde{u}'_{\tau\tau\tau}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma\tau\tau}\|_{0,\widehat{\Omega}}^2+\|\operatorname{div}\widetilde{u}_{\tau\tau}\|_{0,\widehat{\Omega}}^2)\\ &\quad+(\delta_{11}+cd)(\|\widehat{u}_{zzz}\|_{0,\widehat{\Omega}}^2+\|\widehat{q}_{\sigma zz}\|_{0,\widehat{\Omega}}^2)\|\widehat{F}\|_{4-1/2,\widehat{S}}^2\\ &\quad+a_{22}(\|\widetilde{u}_t\|_{1,\widehat{\Omega}}^2+\|\widehat{u}\|_{2,\widehat{\Omega}}^2+\|\widehat{q}_\sigma\|_{1,\widehat{\Omega}}^2+\|\widetilde{g}\|_{1,\widehat{\Omega}}^2)\\ &\quad+a_{23}X_3(\widehat{\Omega})(1+X_3(\widehat{\Omega}))Y_3(\widehat{\Omega})\,.\end{aligned}$$

Summarizing, inequalities (4.71)–(4.75) yield

$$(4.76)\quad \frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}}\bigg[\widehat{\eta}(\widetilde{u}_{\tau\tau}^2+|\widetilde{u}_{n\tau}^3|^2)+\frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}_{\sigma z\tau}^2\bigg]J\,dz+\|\widetilde{u}_\tau\|_{2,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma\tau}\|_{1,\widehat{\Omega}}^2$$

$$\le (\delta_{12}+cd)(\|\widehat{u}_{zzz}\|^2_{0,\widehat{\Omega}}+\|\widehat{q}_{\sigma zz}\|^2_{0,\widehat{\Omega}})\|\widehat{F}\|^2_{4-1/2,\widehat{S}}$$
$$+a_{24}(|\widehat{u}|^2_{2,1,\widehat{\Omega}}+\|\widehat{q}_\sigma\|^2_{1,\widehat{\Omega}}+\|\widetilde{g}\|^2_{1,\widehat{\Omega}})+a_{25}X_3(\widehat{\Omega})(1+X_3(\widehat{\Omega}))Y_3(\widehat{\Omega})\,.$$

Differentiating the third component of (4.35) with respect to n, multiplying the result by $\widetilde{q}_{\sigma nn}J$ and next integrating over $\widehat{\Omega}$ implies

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma nn}J\,dz+\|\widetilde{q}_{\sigma nn}\|^2_{0,\widehat{\Omega}} \tag{4.77}$$
$$\le c\|\widetilde{u}_\tau\|^2_{2,\widehat{\Omega}}+(\delta_{13}+cd)(\|\widetilde{u}_{zzz}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma zz}\|^2_{0,\widehat{\Omega}})\|\widehat{F}\|^2_{4-1/2,\widehat{S}}$$
$$+a_{26}(\|\widetilde{u}_t\|^2_{1,\widehat{\Omega}}+\|\widehat{u}\|^2_{2,\widehat{\Omega}}+\|\widehat{q}_\sigma\|^2_{1,\widehat{\Omega}}+\|\widetilde{g}\|^2_{1,\widehat{\Omega}})$$
$$+a_{27}X_3(\widehat{\Omega})(1+X_3(\widehat{\Omega}))Y_3(\widehat{\Omega})\,.$$

We write $(4.22)_1$ in the form

$$(\mu+\nu)\nabla_{z^i}\operatorname{div}\widetilde{u}=-\mu(\Delta\widetilde{u}^i-\nabla_{z^i}\operatorname{div}\widetilde{u})+\widehat{\eta}\widetilde{u}^i_t-\widehat{\eta}\,\widetilde{g}^i-k^i_3 \tag{4.78}$$
$$-[\mu\nabla^2\widetilde{u}^i+\nu\nabla_{z^i}\operatorname{div}\widetilde{u}-\mu\widehat{\nabla}^2\widetilde{u}^i-\nu\widehat{\nabla}_i\widehat{\operatorname{div}}\widetilde{u}]-\widehat{\nabla}_i\widetilde{q}_\sigma\,.$$

Differentiating the third component of (4.78) with respect to n gives

$$\|(\operatorname{div}\widetilde{u})_{nn}\|^2_{0,\widehat{\Omega}}\le c(\|\widetilde{u}_\tau\|^2_{2,\widehat{\Omega}}+\|\widetilde{q}_{\sigma nn}\|^2_{0,\widehat{\Omega}}) \tag{4.79}$$
$$+(\delta_{14}+cd)\|\widetilde{u}\|^2_{3,\widehat{\Omega}}\|\widehat{F}\|^2_{4-1/2,\widehat{S}}$$
$$+a_{28}(|\widehat{u}|^2_{2,1,\widehat{\Omega}}+\|\widehat{q}_\sigma\|^2_{1,\widehat{\Omega}}+\|\widetilde{g}\|^2_{1,\widehat{\Omega}})$$
$$+a_{29}X_3(\widehat{\Omega})(1+X_3(\widehat{\Omega}))Y_3(\widehat{\Omega})\,.$$

Finally, differentiating (4.39) with respect to n yields

$$\|\widetilde{u}_{nnn}\|^2_{0,\widehat{\Omega}}\le c(\|\widetilde{u}_{\tau\tau}\|^2_{1,\widehat{\Omega}}+\|\widetilde{q}_{\sigma n}\|^2_{1,\widehat{\Omega}}+\|(\operatorname{div}\widetilde{u})_n\|^2_{1,\widehat{\Omega}}) \tag{4.80}$$
$$+(\delta_{15}+cd)\|\widetilde{u}\|^2_{3,\widehat{\Omega}}\|\widehat{F}\|^2_{4-1/2,\widehat{S}}$$
$$+a_{30}(|\widehat{u}|^2_{2,1,\widehat{\Omega}}+\|\widehat{q}_\sigma\|^2_{1,\widehat{\Omega}}+\|\widetilde{g}\|^2_{1,\widehat{\Omega}})$$
$$+a_{31}X_3(\widehat{\Omega})(1+X_3(\widehat{\Omega}))Y_3(\widehat{\Omega})\,.$$

From (4.76), (4.77), (4.79) and (4.80) we obtain

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\left[\widehat{\eta}(\widetilde{u}^2_{\tau\tau}+|\widetilde{u}^3_{n\tau}|^2)+\frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma zz}\right]J\,dz+\|\widetilde{u}\|^2_{3,\widehat{\Omega}}+\|\widetilde{q}_\sigma\|^2_{2,\widehat{\Omega}} \tag{4.81}$$
$$\le c(\delta_{16}+cd)(\|\widehat{u}_{zzz}\|^2_{0,\widehat{\Omega}}+\|\widehat{q}_{\sigma zz}\|^2_{0,\widehat{\Omega}})$$
$$+a_{32}(|\widehat{u}|^2_{2,1,\widehat{\Omega}}+\|\widehat{q}_\sigma\|^2_{1,\widehat{\Omega}}+\|\widetilde{g}\|^2_{1,\widehat{\Omega}})+a_{33}X_3(\widehat{\Omega})(1+X_3(\widehat{\Omega}))Y_3(\widehat{\Omega})\,.$$

To obtain the full second derivative of u under the time derivative we examine the expression

$$(4.82)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}} \widehat{\eta}\,\widetilde{u}_{zz}^2 J\,dz = \int\limits_{\widehat{\Omega}} \left(\widehat{\eta}\,\widetilde{u}_{zz}\cdot\widetilde{u}_{zzt}J + \frac{1}{2}\widehat{\eta}_t\widetilde{u}_{zz}^2 J + \frac{1}{2}\widehat{\eta}\,\widetilde{u}_{zz}^2 J_t\right) dz$$
$$\le \delta_{17}(\|\widetilde{u}_{zzt}\|_{0,\widehat{\Omega}}^2 + \|\widetilde{u}_{zz}\|_{1,\widehat{\Omega}}^2) + a_{34}\|\widehat{u}\|_{2,\widehat{\Omega}}^2\|\widetilde{u}\|_{2,\widehat{\Omega}}^2,$$

where we have used the relations

$$(4.83)\quad \widehat{\eta}_t + \widehat{\eta}\widehat{\nabla}\cdot\widehat{u} = 0 \quad\text{and}\quad J_t = J\widehat{\nabla}\cdot\widehat{u}\,.$$

Applying (4.82) in (4.81) and using the fact that δ_{17} is sufficiently small we obtain

$$(4.84)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}} \left(\widehat{\eta}\,\widetilde{u}_{zz}^2 + \frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}_{\sigma zz}^2\right) J\,dz + \|\widetilde{u}\|_{3,\widehat{\Omega}}^2 + \|\widetilde{q}_\sigma\|_{2,\widehat{\Omega}}^2$$
$$\le c(\delta_{18} + cd)(\|\widehat{u}_{zzt}\|_{0,\widehat{\Omega}}^2 + \|\widehat{u}_{zzz}\|_{0,\widehat{\Omega}}^2 + \|\widehat{q}_{\sigma zz}\|_{0,\widehat{\Omega}}^2)$$
$$+ a_{35}(|\widehat{u}|_{2,1,\widehat{\Omega}}^2 + \|\widehat{q}_\sigma\|_{1,\widehat{\Omega}}^2 + \|\widetilde{g}\|_{1,\widehat{\Omega}}^2)$$
$$+ a_{36}X_3(\widehat{\Omega})(1 + X_3(\widehat{\Omega}))Y_3(\widehat{\Omega})\,.$$

Now going back to the variables ξ in (4.84), summing over all neighbourhoods of unity (for the interior neighbourhoods we use (4.68)) and then going back to the variables x we obtain (4.62) for sufficiently small δ_i and d. This concludes the proof.

To estimate the first term on the right-hand side of (4.62) we need

LEMMA 4.6. *For a sufficiently smooth solution v, p of* (4.1),

$$(4.85)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t} \left(\varrho v_{xt}^2 + \frac{1}{p\Psi(\varrho)}p_{\sigma xt}^2\right) dx + \|v_t\|_{2,\Omega_t}^2 + \|p_{\sigma t}\|_{1,\Omega_t}^2$$
$$\le \varepsilon_5\|v_{xtt}\|_{0,\Omega_t}^2 + P_{11}(|v|_{2,0,\Omega_t}^2 + |p_\sigma|_{1,0,\Omega_t}^2 + |f|_{1,0,\Omega_t}^2)$$
$$+ P_{12}X_4(1 + X_4)Y_4\,,$$

where

$$(4.86)\quad \begin{aligned} X_4 &= |v|_{3,1,\Omega_t}^2 + |p_\sigma|_{2,0,\Omega_t}^2 + \int_0^t \|v\|_{3,\Omega_\tau}^2\,d\tau\,,\\ Y_4 &= |v|_{4,2,\Omega_t}^2 + |p_\sigma|_{3,1,\Omega_t}^2 + \int_0^t \|v\|_{4,\Omega_\tau}^2\,d\tau\,. \end{aligned}$$

Proof. We use our partition of unity. First we consider interior subdomains. Differentiating $(4.21)_1$ with respect to t and ξ, multiplying the result by $\widetilde{u}_{t\xi}$ and integrating over $\widetilde{\Omega}$ yields

$$(4.87)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\widetilde{\Omega}} \eta\widetilde{u}_{t\xi}^2 A\,d\xi + \frac{\mu}{2}\int\limits_{\widetilde{\Omega}} (\nabla_{u^i}\widetilde{u}_{t\xi}^j + \nabla_{u^j}\widetilde{u}_{t\xi}^i)^2 A\,d\xi$$

$$+ (\nu-\mu)\|\nabla_u\cdot\widetilde{u}_{t\xi}\|^2_{0,\widetilde{\Omega}} - \int\limits_{\widetilde{\Omega}} \widetilde{q}_{\sigma t\xi}\nabla_u\cdot\widetilde{u}_{t\xi}A\,d\xi$$

$$\le \delta_1\|\widetilde{u}_{t\xi}\|^2_{1,\widetilde{\Omega}} + a_1(\|u_t\|^2_{1,\widetilde{\Omega}} + \|q_{\sigma t}\|^2_{0,\widetilde{\Omega}} + |\widetilde{g}|^2_{1,0,\widetilde{\Omega}}) + a_2X_4(\widetilde{\Omega})Y_4(\widetilde{\Omega}),$$

where

$$X_4(\widetilde{\Omega}) = |u|^2_{3,1,\widetilde{\Omega}} + |q_\sigma|^2_{2,0,\widetilde{\Omega}} + \Big\|\int\limits_0^t u\,dt'\Big\|^2_{3,\widetilde{\Omega}},$$

$$Y_4(\widetilde{\Omega}) = |u|^2_{4,2,\widetilde{\Omega}} + |q_\sigma|^2_{3,1,\widetilde{\Omega}} + \Big\|\int\limits_0^t u\,dt'\Big\|^2_{4,\widetilde{\Omega}}.$$

By the continuity equation $(4.21)_2$ we have

$$-\int\limits_{\widetilde{\Omega}} \widetilde{q}_{\sigma t\xi}\nabla_u\cdot\widetilde{u}_{t\xi}A\,d\xi = \frac{1}{2}\frac{d}{dt}\int\limits_{\widetilde{\Omega}} \frac{1}{q\Psi(\eta)}\widetilde{q}^2_{\sigma t\xi}A\,d\xi + N_1, \tag{4.88}$$

where

$$|N_1| \le \delta_1\|\widetilde{q}_{\sigma t}\|^2_{1,\widetilde{\Omega}} + a_3|u|^2_{2,1,\widetilde{\Omega}} + a_4X_4(\widetilde{\Omega})(1+X_4(\widetilde{\Omega}))Y_4(\widetilde{\Omega}).$$

From (4.28) we obtain

$$\begin{aligned}\|\widetilde{u}_t\|^2_{2,\widetilde{\Omega}} + \|\widetilde{q}_{\sigma t}\|^2_{1,\widetilde{\Omega}} &\le c\|(\nabla_u\cdot\widetilde{u})_t\|^2_{1,\widetilde{\Omega}}\\ &\quad + a_5(\|\widetilde{u}_{tt}\|^2_{0,\widetilde{\Omega}} + |u|^2_{2,1,\widetilde{\Omega}} + |q_\sigma|^2_{1,0,\widetilde{\Omega}} + |\widetilde{g}|^2_{1,0,\widetilde{\Omega}})\\ &\quad + a_6X_4(\widetilde{\Omega})Y_4(\widetilde{\Omega}).\end{aligned} \tag{4.89}$$

Applying Lemma 5.1 in the case $G=\widetilde{\Omega}$, $v=\widetilde{u}_{\xi\xi}$ and (4.87)–(4.89) we obtain for sufficiently small δ_1 and δ_2

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\limits_{\widetilde{\Omega}}\Big(\eta\widetilde{u}^2_{t\xi} + \frac{1}{q\Psi(\eta)}\widetilde{q}^2_{\sigma t\xi}\Big)A\,d\xi + \|\widetilde{u}_t\|^2_{2,\widetilde{\Omega}} + \|\widetilde{q}_{\sigma t}\|^2_{1,\widetilde{\Omega}}&\\ \le a_7(\|\widetilde{u}_{tt}\|^2_{0,\widetilde{\Omega}} + |u|^2_{2,1,\widetilde{\Omega}} + |q_\sigma|^2_{1,0,\widetilde{\Omega}} + |\widetilde{g}|^2_{1,0,\widetilde{\Omega}})&\\ + a_8X_4(\widetilde{\Omega})(1+X_4(\widetilde{\Omega}))Y_4(\widetilde{\Omega}).&\end{aligned} \tag{4.90}$$

Now we obtain an estimate in a subdomain near the boundary. Differentiating $(4.22)_1$ with respect to t and τ, multiplying the result by $\widetilde{u}_{t\tau}J$ and integrating over $\widehat{\Omega}$ yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\Big(\widehat{\eta}\widetilde{u}^2_{t\tau} + \frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma t\tau}\Big)J\,dz + \frac{\mu}{2}\|\widetilde{u}_{t\tau}\|^2_{1,\widehat{\Omega}}\\ &\quad - \int\limits_{\widehat{S}} (\widehat{n}\widehat{\mathbb{T}}(\widetilde{u},\widetilde{q}_\sigma))_{,t\tau}\cdot\widetilde{u}_{t\tau}J\,dz'\end{aligned} \tag{4.91}$$

$$\le \delta_3(\|\widetilde{u}_{zzt}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_\sigma\|^2_{1,\widehat{\Omega}})+a_9(|\widehat{u}|^2_{2,0,\widehat{\Omega}}+|\widehat{q}_\sigma|^2_{1,0,\widehat{\Omega}}+|\widetilde{g}|^2_{1,0,\widehat{\Omega}})$$
$$+a_{10}X_4(\widehat{\Omega})(1+X_4(\widehat{\Omega}))Y_4(\widehat{\Omega}),$$

where $X_4(\widehat{\Omega})$, $Y_4(\widehat{\Omega})$ are equal to $X_4(\widetilde{\Omega})$, $Y_4(\widetilde{\Omega})$ with $u,q_\sigma,\widetilde{\Omega}$ replaced by $\widehat{u},\widehat{q}_\sigma,\widehat{\Omega}$, respectively. Moreover, to obtain (4.91) we have used Lemma 5.1 in the case $G=\widehat{\Omega}$, $v=\widetilde{u}_{t\tau}$ and

$$\int\limits_{\widehat{\Omega}}\widetilde{q}_{\sigma t\tau}\widehat{\nabla}\cdot\widetilde{u}_{t\tau}J\,dz=\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma t\tau}J\,dz+N_2$$

with

$$|N_2|\le\delta_4\|\widetilde{q}_{\sigma t}\|^2_{1,\widehat{\Omega}}+a_{11}|\widehat{u}|^2_{2,1,\widehat{\Omega}}+a_{12}X_4(\widehat{\Omega})(1+X_4(\widehat{\Omega}))Y_4(\widehat{\Omega}).$$

Consider the boundary term in (4.91). Using the boundary condition $(4.22)_3$ we obtain

$$\Big|-\int\limits_{\widehat{S}}(\widehat{n}\widehat{\mathbb{T}}(\widetilde{u},\widetilde{q}_\sigma))_{,t\tau}\cdot\widetilde{u}_{t\tau}J\,dz'\Big|=\Big|-\int\limits_{\widehat{S}}(\widehat{B}(\widehat{u},\widehat{\zeta})\widehat{n})_{,t\tau}\cdot\widetilde{u}_{t\tau}J\,dz'\Big| \tag{4.92}$$
$$\le\delta_5(\|\widetilde{u}_{t\tau}\|^2_{1,\widehat{\Omega}}+\|\widehat{u}_{tzz}\|^2_{0,\widehat{\Omega}})+a_{13}\Big[|\widehat{u}|^2_{2,1,\widehat{\Omega}}(1+\|\widehat{F}\|^2_{4-1/2,\widehat{S}})$$
$$+\|\widehat{u}\|^4_{3,\widehat{\Omega}}(1+\|\widehat{F}\|^2_{4-1/2,\widehat{S}})+|\widehat{u}|^2_{3,2,\widehat{\Omega}}\Big\|\int\limits_0^t\widehat{u}\,dt'\Big\|^2_{3,\widehat{\Omega}}(1+\|\widehat{u}\|^2_{3,\widehat{\Omega}})\Big].$$

Applying (4.92) in (4.91) yields

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\bigg(\widehat{\eta}\,\widetilde{u}^2_{t\tau}+\frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma t\tau}\bigg)J\,dz+\frac{\mu}{4}\|\widetilde{u}_{t\tau}\|^2_{1,\widehat{\Omega}} \tag{4.93}$$
$$\le\delta_6(\|\widehat{u}_{tzz}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma tz}\|^2_{0,\widehat{\Omega}})+a_{14}(|\widehat{u}|^2_{2,0,\widehat{\Omega}}+|\widehat{q}_\sigma|^2_{1,0,\widehat{\Omega}}+|\widetilde{g}|^2_{1,0,\widehat{\Omega}})$$
$$+a_{15}X_4(\widehat{\Omega})(1+X_4(\widehat{\Omega}))Y_4(\widehat{\Omega}).$$

Differentiating the third component of (4.35) with respect to t, multiplying the result by $\widetilde{q}_{\sigma nt}J$ and integrating over $\widehat{\Omega}$ yields

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma nt}J\,dz+\|\widetilde{q}_{\sigma nt}\|^2_{0,\widehat{\Omega}} \tag{4.94}$$
$$\le(\delta_7+cd)\|\widehat{F}\|^2_{4-1/2,\widehat{S}}(\|\widetilde{u}_{zzt}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma zt}\|^2_{0,\widehat{\Omega}})+c\|\widetilde{u}_{t\tau}\|^2_{1,\widehat{\Omega}}$$
$$+a_{16}(\|\widehat{u}\|^2_{2,0,\widehat{\Omega}}+\|\widehat{q}_{\sigma t}\|^2_{0,\widehat{\Omega}}+|\widetilde{g}|^2_{1,0,\widehat{\Omega}})$$
$$+a_{17}X_4(\widehat{\Omega})(1+X_4(\widehat{\Omega}))Y_4(\widehat{\Omega}).$$

Differentiating the third component of (4.37) with respect to t, multiplying the result by $\widetilde{u}^3_{nnt}J$ and integrating over $\widehat{\Omega}$ implies

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\widehat{\eta}|\widetilde{u}^3_{nt}|^2 J\,dz+\frac{\mu+\nu}{2}\|\widetilde{u}^3_{nnt}\|^2_{0,\widehat{\Omega}} \tag{4.95}$$
$$\le(\delta_8+cd)\|\widehat{F}\|^2_{4-1/2,\widehat{S}}(\|\widetilde{u}_{zzt}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma zt}\|^2_{0,\widehat{\Omega}})+\delta_9\|\widetilde{u}_{ztt}\|^2_{0,\widehat{\Omega}}$$
$$+c(\|\widetilde{u}_{z\tau t}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma nt}\|^2_{0,\widehat{\Omega}})+a_{18}(|\widehat{u}|^2_{2,0,\widehat{\Omega}}+|\widehat{q}_\sigma|^2_{1,0,\widehat{\Omega}}+|\widetilde{g}|^2_{1,0,\widehat{\Omega}})$$
$$+a_{19}X_4(\widehat{\Omega})(1+X_4(\widehat{\Omega}))Y_4(\widehat{\Omega})\,.$$

Differentiating (4.39) with respect to t and τ, multiplying by $\widetilde{u}'_{t\tau}J$, integrating over $\widehat{\Omega}$ and using (4.40) gives

$$\|\widetilde{u}'_{zt\tau}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma t\tau}\|^2_{0,\widehat{\Omega}} \tag{4.96}$$
$$\le(\delta_{10}+cd)\|\widehat{F}\|^2_{4-1/2,\widehat{S}}(\|\widetilde{u}_{zzt}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma zt}\|^2_{0,\widehat{\Omega}})$$
$$+c\|(\operatorname{div}\widetilde{u}')_{,\tau t}\|^2_{0,\widehat{\Omega}}+a_{20}(|\widehat{u}|^2_{2,0,\widehat{\Omega}}+|\widehat{q}_\sigma|^2_{1,0,\widehat{\Omega}}+|\widetilde{g}|^2_{1,0,\widehat{\Omega}})$$
$$+a_{20}X_4(\widehat{\Omega})(1+X_4(\widehat{\Omega}))Y_4(\widehat{\Omega})\,.$$

Moreover, from (4.39) we obtain

$$\|\widetilde{u}'_{nnt}\|^2_{0,\widehat{\Omega}}\le(\delta_{11}+cd)\|\widehat{F}\|^2_{4-1/2,\widehat{S}}(\|\widetilde{u}_{zzt}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma zt}\|^2_{0,\widehat{\Omega}}) \tag{4.97}$$
$$+c(\|\widetilde{u}_{zt\tau}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma t\tau}\|^2_{0,\widehat{\Omega}})$$
$$+a_{21}(|\widehat{u}|^2_{2,0,\widehat{\Omega}}+|\widehat{q}_\sigma|^2_{1,0,\widehat{\Omega}}+|\widetilde{g}|^2_{1,0,\widehat{\Omega}})$$
$$+a_{22}X_4(\widehat{\Omega})(1+X_4(\widehat{\Omega}))Y_4(\widehat{\Omega})\,.$$

Finally, we have

$$\frac{d}{dt}\int\limits_{\widehat{\Omega}}\widehat{\eta}\widetilde{u}^2_{zt}J\,dz\le\delta_{12}\|\widetilde{u}_{ztt}\|^2_{0,\widehat{\Omega}}+c(\|\widetilde{u}_t\|^2_{1,\widehat{\Omega}}+\|\widetilde{u}_{zt}\|^2_{0,\widehat{\Omega}}\|\widehat{u}\|^2_{3,\widehat{\Omega}})\,. \tag{4.98}$$

From (4.92), (4.94)–(4.98) we obtain for sufficiently small δ's

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\left(\widehat{\eta}\,\widetilde{u}^2_{zt}+\frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma zt}\right)J\,dz+\|\widetilde{u}_{zzt}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma zt}\|^2_{0,\widehat{\Omega}} \tag{4.99}$$
$$\le\delta_{13}\|\widetilde{u}_{ztt}\|^2_{0,\widehat{\Omega}}+(\delta_{14}+cd)(\|\widehat{u}_{zzt}\|^2_{0,\widehat{\Omega}}+\|\widehat{q}_{\sigma zt}\|^2_{0,\widehat{\Omega}})$$
$$+a_{23}(|\widehat{u}|^2_{2,0,\widehat{\Omega}}+|\widehat{q}_\sigma|^2_{1,0,\widehat{\Omega}}+|\widetilde{g}|^2_{1,0,\widehat{\Omega}})$$
$$+a_{24}X_4(\widehat{\Omega})(1+X_4(\widehat{\Omega}))Y_4(\widehat{\Omega})\,.$$

Going back to the variables ξ in (4.99), summing over all neigbourhoods of the partition of unity (where we use (4.90) for the interior subdomains) we obtain (4.85) for sufficiently small δ's and d. This concludes the proof.

To estimate the first term on the right-hand side of (4.85) we need

LEMMA 4.7. *For a sufficiently smooth solution* v, p *of* (4.1),

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\varrho v_{tt}^2+\frac{1}{p\Psi(\varrho)}p_{\sigma tt}^2\right)dx+\|v_{tt}\|_{1,\Omega_t}^2+\|p_{\sigma tt}\|_{0,\Omega_t}^2 \le c\|v_t\|_{2,\Omega_t}^2+P_{13}(\|f_{tt}\|_{0,\Omega_t}^2+|f|_{1,0,\Omega_t}^2)+P_{14}X_5(1+X_5)Y_5\,, \tag{4.100}$$

where

$$X_5=|v|_{3,1,\Omega_t}^2+|p_\sigma|_{2,0,\Omega_t}^2\,,\qquad Y_5=|v|_{4,2,\Omega_t}^2+|p_\sigma|_{3,1,\Omega_t}^2\,. \tag{4.101}$$

P r o o f. Differentiating $(4.1)_1$ twice with respect to t, multiplying by v_{tt} and integrating over Ω_t yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\varrho v_{tt}^2+\frac{1}{p\Psi(\varrho)}p_{\sigma tt}^2\right)dx+\frac{\mu}{2}\|v_{tt}\|_{1,\Omega_t}^2&\\ -\int_{S_t}(n_iT^{ij}(v,p_\sigma))_{,tt}\cdot v_{tt}^i\,ds&\\ \le \delta_1(\|v_{tt}\|_{1,\Omega_t}^2+\|p_{\sigma tt}\|_{0,\Omega_t}^2)&\\ +a_1(|f|_{1,0,\Omega_t}^2+\|f_{tt}\|_{0,\Omega_t}^2)+a_2X_5(1+X_5)Y_5\,,&\end{aligned} \tag{4.102}$$

where we have used

$$-\int_{\Omega_t}p_{\sigma tt}\operatorname{div}v_{tt}\,dx=\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\frac{1}{p\Psi(\varrho)}p_{tt}^2\,dx+N_1\,,$$

Lemma 5.4 and

$$|N_1|\le\delta_2\|p_{\sigma tt}\|_{0,\Omega_t}^2+a_3X_5(1+X_5)Y_5\,.$$

Using the boundary condition $(4.1)_3$ we see that the boundary term in (4.102) vanishes. Moreover, by the continuity equation (4.2) we have

$$\|p_{\sigma tt}\|_{0,\Omega_t}^2\le c\|v_t\|_{1,\Omega_t}^2+a_4X_5(1+X_5)Y_5\,. \tag{4.103}$$

Hence, (4.102) and (4.103) imply (4.100). This concludes the proof.

Summarizing, from Lemmas 4.5–4.7 we obtain

LEMMA 4.8.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\varrho|D_{x,t}^2u|^2+\frac{1}{p\Psi(\varrho)}|D_{x,t}^2p_\sigma|^2\right)dx+|v|_{3,1,\Omega_t}^2+|p_\sigma|_{2,0,\Omega_t}^2&\\ \le P_{15}(|v|_{2,0,\Omega_t}^2+|p_\sigma|_{1,0,\Omega_t}^2)&\\ +c(|f|_{1,0,\Omega_t}^2+\|f_{tt}\|_{0,\Omega_t}^2)+P_{16}X_6(1+X_6)Y_6\,,&\end{aligned} \tag{4.104}$$

where

$$X_6 = |v|^2_{3,1,\Omega_t} + |p_\sigma|^2_{2,0,\Omega_t} + \int_0^t \|v\|^2_{3,\Omega_\tau}\,d\tau\,, \tag{4.105}$$
$$Y_6 = |v|^2_{4,2,\Omega_t} + |p_\sigma|^2_{3,1,\Omega_t} + \int_0^t \|v\|^2_{4,\Omega_\tau}\,d\tau\,.$$

Finally, we obtain inequalities for the fourth derivatives.

LEMMA 4.9. *For a sufficiently smooth solution* v, p *of* (4.1),

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\varrho v^2_{xxx} + \frac{1}{p\Psi(\varrho)}p^2_{\sigma xxx}\right)dx + \|v\|^2_{4,\Omega_t} + \|p_\sigma\|^2_{3,\Omega_t} \tag{4.106}$$
$$\le \varepsilon_6\|v_{xxxt}\|^2_{0,\Omega_t} + P_{17}(|v|^2_{3,2,\Omega_t} + \|p_\sigma\|^2_{2,\Omega_t} + \|f\|^2_{2,\Omega_t})$$
$$+ P_{18}X_7(1+X_7^2)Y_7\,,$$

where

$$X_7 = |v|^2_{3,2,\Omega_t} + |p_\sigma|^2_{3,2,\Omega_t} + \int_0^t \|v\|^2_{3,\Omega_\tau}\,d\tau\,, \tag{4.107}$$
$$Y_7 = |v|^2_{4,3,\Omega_t} + |p_\sigma|^2_{3,2,\Omega_t} + \int_0^t \|v\|^2_{4,\Omega_\tau}\,d\tau\,.$$

Proof. We use the partition of unity. First we consider interior subdomains. Differentiate $(4.21)_1$ three times with respect to ξ, multiply by $\widetilde{u}_{\xi\xi\xi}A$ and integrate over $\widehat{\Omega}$ to get

$$\frac{1}{2}\frac{d}{dt}\int_{\widetilde{\Omega}}\left(\eta\widetilde{u}^2_{\xi\xi\xi} + \frac{1}{q\Psi(\eta)}\widetilde{q}^2_{\sigma\xi\xi\xi}\right)A\,d\xi \tag{4.108}$$
$$+ \frac{\mu}{2}\int_{\widetilde{\Omega}}(\nabla_{u^i}\widetilde{u}^j_{\xi\xi\xi} + \nabla_{u^j}\widetilde{u}^i_{\xi\xi\xi})^2A\,d\xi + (\nu-\mu)\|\nabla_u\cdot\widetilde{u}_{\xi\xi\xi}\|^2_{0,\widetilde{\Omega}}$$
$$\le \delta_1(\|\widetilde{u}_{\xi\xi\xi}\|^2_{1,\widetilde{\Omega}} + \|\widetilde{q}_{\sigma\xi\xi\xi}\|^2_{0,\widetilde{\Omega}}) + a_1(\|u\|^2_{3,\widetilde{\Omega}} + \|q_\sigma\|^2_{2,\widetilde{\Omega}} + \|\widetilde{g}\|^2_{2,\widetilde{\Omega}})$$
$$+ a_2X_7(\widetilde{\Omega})(1+X_7^2(\widetilde{\Omega}))Y_7(\widetilde{\Omega})\,,$$

where

$$X_7(\widetilde{\Omega}) = |u|^2_{3,2,\widetilde{\Omega}} + |q_\sigma|^2_{3,2,\widetilde{\Omega}} + \int_0^t \|u\|^2_{3,\widetilde{\Omega}}\,dt'\,,$$
$$Y_7(\widetilde{\Omega}) = |u|^2_{4,3,\widetilde{\Omega}} + |q_\sigma|^2_{3,2,\widetilde{\Omega}} + \int_0^t \|u\|^2_{4,\widetilde{\Omega}}\,dt'\,.$$

We have also used

$$-\int\limits_{\widetilde{\Omega}} \widetilde{q}_{\sigma\xi\xi\xi}\nabla_u\cdot\widetilde{u}_{\xi\xi\xi}A\,d\xi=\frac{1}{2}\frac{d}{dt}\int\limits_{\widetilde{\Omega}}\frac{1}{q\Psi(\eta)}\widetilde{q}^2_{\sigma\xi\xi\xi}A\,d\xi+N_1\,,$$

where

$$|N_1|\le\delta_2\|\widetilde{q}_{\sigma\xi\xi\xi}\|^2_{0,\widetilde{\Omega}}+a_3\|u\|^2_{3,\widehat{\Omega}}+a_4X_7(\widetilde{\Omega})(1+X_7^2(\widetilde{\Omega}))Y_7(\widetilde{\Omega})\,.$$

Moreover, the following relation has been employed:

$$\begin{aligned}\Big|\int\limits_{\widetilde{\Omega}}[(\nabla_u\nabla_u\widetilde{u})_{\xi\xi\xi}-\nabla_u\nabla_u\widetilde{u}_{\xi\xi\xi}]\cdot\widetilde{u}_{\xi\xi\xi}A\,d\xi+\int\limits_{\widetilde{\Omega}}[(\nabla_u\widetilde{q}_\sigma)_{\xi\xi\xi}-\nabla_u\widetilde{q}_{\sigma\xi\xi\xi}]\cdot\widetilde{u}_{\xi\xi\xi}A\,d\xi\Big|&\\ \le\delta_3\|\widetilde{u}_{\xi\xi\xi}\|^2_{1,\widetilde{\Omega}}+a_5X_7(\widetilde{\Omega})(1+X_7^2(\widetilde{\Omega}))Y_7(\widetilde{\Omega})\,.&\end{aligned}$$

From (4.28) we obtain

$$\begin{aligned}\|\widetilde{u}\|^2_{4,\widetilde{\Omega}}+\|\widetilde{q}_\sigma\|^2_{3,\widetilde{\Omega}}&\le c\|\nabla_u\cdot\widetilde{u}\|^2_{3,\widetilde{\Omega}}+a_6(|u|^2_{3,2,\widetilde{\Omega}}+\|q_\sigma\|^2_{2,\widetilde{\Omega}}+\|\widetilde{g}\|^2_{2,\widetilde{\Omega}})\\&\quad+a_7X_7(\widetilde{\Omega})(1+X_7^2(\widetilde{\Omega}))Y_7(\widetilde{\Omega})\,.\end{aligned}\tag{4.109}$$

From (4.108) and (4.109) for sufficiently small δ_1 we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\limits_{\widetilde{\Omega}}\left(\eta\widetilde{u}^2_{\xi\xi\xi}+\frac{1}{q\Psi(\eta)}\widetilde{q}^2_{\sigma\xi\xi\xi}\right)A\,d\xi+\|\widetilde{u}\|^2_{4,\widetilde{\Omega}}+\|\widetilde{q}_\sigma\|^2_{3,\widetilde{\Omega}}\\&\quad\le a_8(|u|^2_{3,2,\widetilde{\Omega}}+\|q_\sigma\|^2_{2,\widetilde{\Omega}}+\|\widetilde{g}\|^2_{2,\widetilde{\Omega}})+a_9X_7(\widetilde{\Omega})(1+X_7(\widetilde{\Omega}))Y_7(\widetilde{\Omega})\,,\end{aligned}\tag{4.110}$$

where Lemma 5.1 in the case $G=\widetilde{\Omega}$ and $v=\widetilde{u}_{\xi\xi\xi}$ has been used.

Now we consider a subdomain near the boundary. Differentiating $(4.22)_1$ three times with respect to τ, multiplying by $\widetilde{u}_{\tau\tau\tau}J$ and integrating over $\widehat{\Omega}$ yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\left[\widehat{\eta}\,\widetilde{u}^2_{\tau\tau\tau}+\frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma\tau\tau\tau}\right]J\,dz+\frac{\mu}{2}\|\widetilde{u}_{\tau\tau\tau}\|^2_{1,\widehat{\Omega}}\\&\quad+(\nu-\mu)\|\widehat{\nabla}\cdot\widetilde{u}_{\tau\tau\tau}\|^2_{0,\widehat{\Omega}}-\int\limits_{\widehat{S}}(\widehat{n}\widehat{\mathbb{T}}(\widetilde{u},\widetilde{q}_\sigma))_{,\tau\tau\tau}\cdot\widetilde{u}_{\tau\tau\tau}J\,dz'\\&\quad\le\delta_4(\|\widetilde{u}_{z\tau\tau\tau}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma z\tau\tau}\|^2_{0,\widehat{\Omega}})+a_{10}(|\widehat{u}|^2_{3,2,\widehat{\Omega}}+\|\widehat{q}_\sigma\|^2_{2,\widehat{\Omega}}+\|\widetilde{g}\|^2_{2,\widehat{\Omega}})\\&\quad+a_{11}X_7(\widehat{\Omega})(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,,\end{aligned}\tag{4.111}$$

where $X_7(\widehat{\Omega})$, $Y_7(\widehat{\Omega})$ have the form of $X_7(\widetilde{\Omega})$, $Y_7(\widetilde{\Omega})$ with $\widehat{u}$, $\widehat{q}_\sigma$, $\widehat{\Omega}$ in place of u, q_σ, $\widetilde{\Omega}$, respectively. Moreover, we have used

$$-\int\limits_{\widehat{\Omega}}\widetilde{q}_{\sigma\tau\tau\tau}\widehat{\nabla}\cdot\widetilde{u}_{\tau\tau\tau}J\,dz=\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma\tau\tau\tau}J\,dz+N_2\,,$$

where

$$|N_2| \le \delta_5 \|\widetilde{q}_{\sigma\tau\tau\tau}\|^2_{0,\widehat{\Omega}} + a_{12}\|\widehat{u}\|^2_{3,\widehat{\Omega}} + a_{13}X_7(\widehat{\Omega})(1 + X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,.$$

We have also employed the following estimates:

$$\begin{aligned}\Big|\int\limits_{\widehat{\Omega}} [(\widehat{\nabla}\widehat{\nabla}\widetilde{u})_{,\tau\tau\tau} - \widehat{\nabla}\widehat{\nabla}\widetilde{u}_{\tau\tau\tau}] \cdot \widetilde{u}_{\tau\tau\tau} J\,dz + \int\limits_{\widehat{\Omega}} [(\widehat{\nabla}\widetilde{q}_\sigma)_{,\tau\tau\tau} - \widehat{\nabla}\widetilde{q}_{\sigma\tau\tau\tau}] \cdot \widetilde{u}_{\tau\tau\tau} J\,dz\Big| \\ \le \delta_6(\|\widetilde{u}_{zz\tau\tau}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma z\tau\tau}\|^2_{0,\widehat{\Omega}} + a_{12}(\|\widehat{u}\|^2_{3,\widehat{\Omega}} + \|\widehat{q}_\sigma\|^2_{2,\widehat{\Omega}}) \\ + a_{13}X_7(\widehat{\Omega})(1 + X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,,\end{aligned}$$

and

$$\begin{aligned}\Big|\int\limits_{\widehat{S}} [(\widehat{n}\widehat{\mathbb{T}}(\widetilde{u}, \widetilde{q}_\sigma))_{,\tau\tau\tau} - \widehat{n}\widehat{\mathbb{T}}(\widetilde{u}_{\tau\tau\tau}, \widetilde{q}_{\sigma\tau\tau\tau})] \cdot \widetilde{u}_{\tau\tau\tau} J\,dz'\Big| \\ \le \delta_7(\|\widetilde{u}_{z\tau\tau\tau}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma z\tau\tau}\|^2_{0,\widehat{\Omega}}) \\ + a_{14}(\|\widehat{u}\|^2_{3,\widehat{\Omega}} + \|\widehat{q}_\sigma\|^2_{2,\widehat{\Omega}}) \\ + a_{15}X_7(\widehat{\Omega})(1 + X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,,\end{aligned}$$

where δ_6 and δ_7 have been assumed to be sufficiently small.

Finally, Lemma 5.1 in the case $G = \widehat{\Omega}$ and $v = \widetilde{u}_{\tau\tau\tau}$ has been used.

Using the boundary condition $(4.22)_3$ we estimate the boundary term in (4.111) as follows:

$$\begin{aligned}\Big| - \int\limits_{\widehat{S}} (\widehat{n}\widehat{\mathbb{T}}(\widetilde{u}, \widetilde{q}_\sigma))_{,\tau\tau\tau} \cdot \widetilde{u}_{\tau\tau\tau} J\,dz'\Big| \\ \le \delta_8\|\widehat{u}_{zzzz}\|^2_{0,\widehat{\Omega}} + a_{16}\|\widehat{u}\|^2_{3,\widehat{\Omega}} + a_{17}X_7(\widehat{\Omega})(1 + X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,.\end{aligned} \tag{4.112}$$

Assuming δ_8 to be sufficiently small, from (4.111) and (4.112) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}} \left(\widehat{\eta}\,\widetilde{u}^2_{\tau\tau\tau} + \frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma\tau\tau\tau}\right) J\,dz + \frac{\mu}{4}\|\widetilde{u}_{\tau\tau\tau}\|^2_{1,\widehat{\Omega}} \\ + (\nu - \mu)\|\widehat{\nabla} \cdot \widetilde{u}_{\tau\tau\tau}\|^2_{0,\widehat{\Omega}} \\ \le \delta_9(\|\widehat{u}_{z\tau\tau\tau}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma z\tau\tau}\|^2_{0,\widehat{\Omega}}) \\ + a_{18}(|\widehat{u}|^2_{3,2,\widehat{\Omega}} + \|\widehat{q}_\sigma\|^2_{2,\widehat{\Omega}} + \|\widetilde{g}\|^2_{2,\widehat{\Omega}}) \\ + a_{19}X_7(\widehat{\Omega})(1 + X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,.\end{aligned} \tag{4.113}$$

Differentiating the third component of (4.35) twice with respect to τ, multiplying the result by $\widetilde{q}_{\sigma n\tau\tau}J$ and integrating over $\widehat{\Omega}$ gives

$$(4.114)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}} \frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma n\tau\tau}J\,dz+\frac{1}{2}\|\widetilde{q}_{\sigma n\tau\tau}\|^2_{0,\widehat{\Omega}}$$
$$\le(\delta_{10}+cd)(\|\widetilde{u}_{zz\tau\tau}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma n\tau\tau}\|^2_{0,\widehat{\Omega}})$$
$$+c\|\widetilde{u}_{\tau\tau\tau}\|^2_{1,\widehat{\Omega}}+a_{20}(|\widehat{u}|^2_{3,2,\widehat{\Omega}}+\|q_\sigma\|^2_{2,\widehat{\Omega}}+\|\widetilde{g}\|^2_{2,\widehat{\Omega}})$$
$$+a_{21}X_7(\widehat{\Omega})(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,.$$

Differentiating the third component of (4.37) twice with respect to τ, multiplying the result by $\widetilde{u}^3_{nn\tau\tau}J$ and integrating over $\widehat{\Omega}$ implies

$$(4.115)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\widehat{\eta}|\widetilde{u}^3_{n\tau\tau}|^2J\,dz+\frac{\mu+\nu}{2}\|u^3_{nn\tau\tau}\|^2_{0,\widehat{\Omega}}$$
$$\le(\delta_{11}+cd)(\|\widetilde{u}_{zz\tau\tau}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma z\tau\tau}\|^2_{0,\widehat{\Omega}})+\delta_{12}\|\widetilde{u}_{n\tau\tau t}\|^2_{0,\widehat{\Omega}}$$
$$+c(\|\widetilde{u}_{z\tau\tau\tau}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma n\tau\tau}\|^2_{0,\widehat{\Omega}})$$
$$+a_{22}(|\widehat{u}|^2_{3,2,\widehat{\Omega}}+\|\widehat{q}_\sigma\|^2_{2,\widehat{\Omega}}+\|\widetilde{g}\|^2_{2,\widehat{\Omega}})$$
$$+a_{23}X_7(\widehat{\Omega})(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,.$$

Differentiating (4.39) three times with respect to τ, multiplying by $\widetilde{u}'_{\tau\tau\tau}J$, integrating over $\widehat{\Omega}$ and using the boundary condition (4.40) we obtain

$$(4.116)\quad \|\widetilde{u}'_{z\tau\tau\tau}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma\tau\tau\tau}\|^2_{0,\widehat{\Omega}}\le(\delta_{12}+cd)(\|\widehat{u}_{zzzz}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma zzz}\|^2_{0,\widehat{\Omega}})$$
$$+c\|\operatorname{div}\widetilde{u}_{\tau\tau}\|^2_{1,\widehat{\Omega}}+a_{24}(|\widehat{u}|^2_{3,2,\widehat{\Omega}}+\|\widehat{q}_\sigma\|^2_{2,\widehat{\Omega}}+\|\widetilde{g}\|^2_{2,\widehat{\Omega}})$$
$$+a_{25}X_7(\widehat{\Omega})(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,.$$

Moreover, from (4.39) we find

$$(4.117)\quad \|\widetilde{u}'_{nn\tau\tau}\|^2_{0,\widehat{\Omega}}\le(\delta_{13}+cd)(\|\widetilde{u}_{zzzz}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma zzz}\|^2_{0,\widehat{\Omega}})$$
$$+c(\|\widetilde{u}'_{\tau\tau\tau\tau}\|^2_{0,\widehat{\Omega}}+\|(\operatorname{div}\widetilde{u})_{\tau\tau\tau}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma\tau\tau\tau}\|^2_{0,\widehat{\Omega}})$$
$$+a_{26}(|\widehat{u}|^2_{3,2,\widehat{\Omega}}+\|\widehat{q}_\sigma\|^2_{2,\widehat{\Omega}}+\|\widetilde{g}\|^2_{2,\widehat{\Omega}})+a_{27}X_7(\widehat{\Omega})(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,.$$

Summarizing, from (4.114)–(4.117) we obtain

$$(4.118)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\left(\widehat{\eta}|\widetilde{u}^3_{n\tau\tau}|^2+\frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma n\tau\tau}\right)J\,dz+\|\widetilde{q}_{\sigma z\tau\tau}\|^2_{0,\widehat{\Omega}}+\|\widetilde{u}_{zz\tau\tau}\|^2_{0,\widehat{\Omega}}$$
$$\le(\delta_{14}+cd)(\|\widetilde{u}_{zzzz}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma zzz}\|^2_{0,\widehat{\Omega}})+\delta_{15}\|\widetilde{u}_{n\tau\tau t}\|^2_{0,\widehat{\Omega}}$$
$$+c\|\widetilde{u}_{z\tau\tau\tau}\|^2_{0,\widehat{\Omega}}+a_{28}(|\widehat{u}|^2_{3,2,\widehat{\Omega}}+\|\widehat{q}_\sigma\|^2_{2,\widehat{\Omega}}+\|\widetilde{g}\|^2_{2,\widehat{\Omega}})$$
$$+a_{29}X_7(\widehat{\Omega})(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,.$$

Differentiating the third component of (4.35) with respect to n and τ, multiplying by $\widetilde{q}_{\sigma nn\tau}J$ and integrating over $\widehat{\Omega}$ yields

$$
\begin{aligned}
(4.119)\qquad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}} \frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma nn\tau}J\,dz + \|\widetilde{q}_{\sigma nn\tau}\|^2_{0,\widehat{\Omega}}&\\
\le (\delta_{16}+cd)(\|\widetilde{u}_{zzz\tau}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma zz\tau}\|^2_{0,\widehat{\Omega}})&\\
+ c\|\widetilde{u}_{zz\tau\tau}\|^2_{0,\widehat{\Omega}} + a_{30}(|\widehat{u}|^2_{3,2,\widehat{\Omega}} + |\widehat{q}_\sigma|^2_{2,1,\widehat{\Omega}} + \|\widetilde{g}\|^2_{2,\widehat{\Omega}})&\\
+ a_{31}X_7(\widehat{\Omega})(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,.&
\end{aligned}
$$

Differentiating the third component of (4.78) with respect to n and τ gives

$$
\begin{aligned}
(4.120)\qquad \|(\operatorname{div}\widetilde{u})_{nn\tau}\|^2_{0,\widehat{\Omega}} \le (\delta_{17}+cd)(\|\widetilde{u}_{zzzz}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma zzz}\|^2_{0,\widehat{\Omega}})&\\
+ c(\|\widetilde{u}_{zz\tau\tau}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma nn\tau}\|^2_{0,\widehat{\Omega}})&\\
+ a_{32}(|\widehat{u}|^2_{3,2,\widehat{\Omega}} + \|\widehat{q}_\sigma\|^2_{2,\widehat{\Omega}} + \|\widetilde{g}\|^2_{2,\widehat{\Omega}})&\\
+ a_{33}X_7(\widehat{\Omega})(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,.&
\end{aligned}
$$

Next we differentiate (4.39) with respect to n and τ. Hence we get

$$
\begin{aligned}
(4.121)\qquad \|\widetilde{u}_{nnn\tau}\|^2_{0,\widehat{\Omega}} \le (\delta_{18}+cd)(\|\widetilde{u}_{zzzz}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma zzz}\|^2_{0,\widehat{\Omega}})&\\
+ c(\|\widetilde{u}_{zz\tau\tau}\|^2_{0,\widehat{\Omega}} + \|(\operatorname{div}\widetilde{u})_{,zn\tau}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma zn\tau}\|^2_{0,\widehat{\Omega}})&\\
+ a_{34}(|\widehat{u}|^2_{3,2,\widehat{\Omega}} + \|\widehat{q}_\sigma\|^2_{2,\widehat{\Omega}} + \|\widetilde{g}\|^2_{2,\widehat{\Omega}})&\\
+ a_{35}X_7(\widehat{\Omega})(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,.&
\end{aligned}
$$

From (4.118)–(4.121) we obtain

$$
\begin{aligned}
(4.122)\qquad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}} \left(\widehat{\eta}|\widetilde{u}^3_{n\tau\tau}|^2 + \frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma zn\tau}\right)J\,dz&\\
+ \|\widetilde{u}_{zzz\tau}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma zz\tau}\|^2_{0,\widehat{\Omega}}&\\
\le (\delta_{19}+cd)(\|\widetilde{u}_{zzzz}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma zzz}\|^2_{0,\widehat{\Omega}}) + \delta_{20}\|\widetilde{u}_{n\tau\tau t}\|^2_{0,\widehat{\Omega}}&\\
+ c\|\widetilde{u}_{z\tau\tau\tau}\|^2_{0,\widehat{\Omega}} + a_{36}(|\widehat{u}|^2_{3,2,\widehat{\Omega}} + \|\widehat{q}_\sigma\|^2_{2,\widehat{\Omega}} + \|\widetilde{g}\|^2_{2,\widehat{\Omega}})&\\
+ a_{37}X_7(\widehat{\Omega})(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,.&
\end{aligned}
$$

Differentiating the third component of (4.35) twice with respect to n, multiplying the result by $\widetilde{q}_{\sigma nnn}J$ and integrating over $\widehat{\Omega}$ yields

$$(4.123)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}_{\sigma nnn}^2 J\,dz+\|\widetilde{q}_{\sigma nnn}\|_{0,\widehat{\Omega}}^2 \le(\delta_{21}+cd)(\|\widetilde{u}_{zzzz}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma zzz}\|_{0,\widehat{\Omega}}^2)+c\|\widetilde{u}_{zzz\tau}\|_{0,\widehat{\Omega}}^2 + a_{38}(|\widehat{u}|_{3,2,\widehat{\Omega}}^2+|\widehat{q}_\sigma|_{2,1,\widehat{\Omega}}^2+\|\widetilde{g}\|_{2,\widehat{\Omega}}^2) + a_{39}X_7(\widehat{\Omega})(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,.$$

Differentiating the third component of (4.78) twice with respect to n implies

$$(4.124)\quad \|(\operatorname{div}\widetilde{u})_{,nnn}\|_{0,\widehat{\Omega}}^2\le(\delta_{22}+cd)(\|\widetilde{u}_{zzzz}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma zzz}\|_{0,\widehat{\Omega}}^2) + c(\|\widetilde{u}_{zzz\tau}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma nnn}\|_{0,\widehat{\Omega}}^2)+a_{40}(|\widehat{u}|_{3,2,\widehat{\Omega}}^2+\|\widehat{q}_\sigma\|_{2,\widehat{\Omega}}^2+\|\widetilde{g}\|_{2,\widehat{\Omega}}^2) + a_{41}X_7(\widehat{\Omega})(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,.$$

We differentiate (4.39) twice with respect to n. Hence after integrating over $\widehat{\Omega}$ we obtain

$$(4.125)\quad \|\widetilde{u}_{nnnn}\|_{0,\widehat{\Omega}}^2\le(\delta_{23}+cd)(\|\widetilde{u}_{zzzz}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma zzz}\|_{0,\widehat{\Omega}}^2) + c(\|\widetilde{u}_{zzn\tau}\|_{0,\widehat{\Omega}}^2+\|(\operatorname{div}\widetilde{u})_{,znn}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma znn}\|_{0,\widehat{\Omega}}^2) + a_{42}(|\widehat{u}|_{3,2,\widehat{\Omega}}^2+\|\widehat{q}_\sigma\|_{2,\widehat{\Omega}}^2+\|\widetilde{g}\|_{2,\widehat{\Omega}}^2) + a_{43}X_7(\widehat{\Omega})(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,.$$

Finally, note that

$$(4.126)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\widehat{\eta}\,\widetilde{u}_{zzz}^2 J\,dz\le\delta_{24}(\|\widetilde{u}_{zzzt}\|_{0,\widehat{\Omega}}^2+\|\widetilde{u}_{zzzz}\|_{0,\widehat{\Omega}}^2) + c\|\widetilde{u}\|_{3,\widehat{\Omega}}^2+c(|\widehat{q}_\sigma|_{2,1,\widehat{\Omega}}^2+\|\widehat{u}\|_{3,\widehat{\Omega}}^2)\|\widetilde{u}\|_{3,\widehat{\Omega}}^2\,.$$

From (4.122)–(4.126) we obtain

$$(4.127)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\bigg(\widehat{\eta}\,\widetilde{u}_{zzz}^2+\frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}_{\sigma zzz}^2\bigg)J\,dz+\|\widetilde{u}_{zzzz}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma zzz}\|_{0,\widehat{\Omega}}^2 \le\delta_{25}\|\widetilde{u}_{zzzt}\|_{0,\widehat{\Omega}}^2+(\delta_{26}+cd)(\|\widehat{u}_{zzzz}\|_{0,\widehat{\Omega}}^2+\|\widehat{q}_{\sigma zzz}\|_{0,\widehat{\Omega}}^2) + c\|\widetilde{u}_{z\tau\tau\tau}\|_{0,\widehat{\Omega}}^2+a_{44}(|\widehat{u}|_{3,2,\widehat{\Omega}}^2+|\widehat{q}_\sigma|_{2,1,\widehat{\Omega}}^2+\|\widetilde{g}\|_{2,\widehat{\Omega}}^2) + a_{45}X_7(\widehat{\Omega})(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,.$$

From (4.113) and (4.127) it follows that

$$(4.128)\qquad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\bigg(\widehat{\eta}\,\widetilde{u}_{zzz}^2+\frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}_{\sigma zzz}^2\bigg)J\,dz+\|\widetilde{u}_{zzzz}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma zzz}\|_{0,\widehat{\Omega}}^2$$
$$\le c\delta_{27}\|\widetilde{u}_{zzzt}\|_{0,\widehat{\Omega}}^2+(\delta_{28}+cd)(\|\widehat{u}_{zzzz}\|_{0,\widehat{\Omega}}^2+\|\widehat{q}_{\sigma zzz}\|_{0,\widehat{\Omega}}^2$$
$$+a_{46}(|\widehat{u}|_{3,2,\widehat{\Omega}}^2+|\widehat{q}_\sigma|_{2,1,\widehat{\Omega}}^2+\|\widetilde{g}\|_{2,\widehat{\Omega}}^2)$$
$$+a_{47}X_7(\widehat{\Omega})(1+X_7^2(\widehat{\Omega}))Y_7(\widehat{\Omega})\,.$$

Going back to the variables ξ in (4.128), summing the result and (4.110) over all neighbourhoods of the partition of unity, using the fact that δ_{28} and d are sufficiently small and finally going back to the variables x we obtain (4.106). This concludes the proof.

To estimate the first term on the right-hand side of (4.106) we need

LEMMA 4.10. *For a sufficiently smooth solution v, p of* (4.1),

$$(4.129)\qquad \frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\bigg(\varrho v_{xxt}^2+\frac{\mu+\nu}{p\Psi(\varrho)}p_{\sigma xxt}^2\bigg)dx+\|v_{xxxt}\|_{0,\Omega_t}^2+\|p_{\sigma xxt}\|_{0,\Omega_t}^2$$
$$\le\varepsilon_7(\|v_{xxxx}\|_{0,\Omega_t}^2+\|v_{xxtt}\|_{0,\Omega_t}^2)$$
$$+P_{19}(|v|_{3,1,\Omega}^2+|p_\sigma|_{2,1,\Omega_t}^2+|f|_{2,1,\Omega_t}^2)$$
$$+P_{20}X_8(1+X_8^2)Y_8\,,$$

where ε_7 may be assumed arbitrarily small and

$$(4.130)\qquad \begin{aligned} X_8&=|v|_{3,2,\Omega_t}^2+|p_\sigma|_{3,1,\Omega_t}^2+\int\limits_0^t\|v\|_{3,\Omega_\tau}^2\,d\tau\,,\\ Y_8&=|v|_{4,3,\Omega_t}^2+|p_\sigma|_{3,1,\Omega_t}^2+\int\limits_0^t\|v\|_{4,\Omega_\tau}^2\,d\tau\,.\end{aligned}$$

Proof. We use the partition of unity. First we consider interior subdomains. Differentiating $(4.21)_1$ twice with respect to ξ and once with respect to time, multiplying by $\widetilde{u}_{t\xi\xi}A$ and integrating over $\widetilde{\Omega}$ yields

$$(4.131)\qquad \frac{1}{2}\frac{d}{dt}\int\limits_{\widetilde{\Omega}}\bigg(\eta\widetilde{u}_{t\xi\xi}^2+\frac{1}{q\Psi(\eta)}\widetilde{q}_{\sigma t\xi\xi}^2\bigg)A\,d\xi$$
$$+\frac{\mu}{2}\int\limits_{\widetilde{\Omega}}(\nabla_{u^i}\widetilde{u}_{t\xi\xi}^j+\nabla_{u^j}\widetilde{u}_{t\xi\xi}^i)^2A\,d\xi+(\nu-\mu)\|\nabla_u\cdot\widetilde{u}_{t\xi\xi}\|_{0,\widetilde{\Omega}}^2$$
$$\le\delta_1(\|\widetilde{u}_{t\xi\xi}\|_{1,\widetilde{\Omega}}^2+\|\widetilde{q}_{\sigma t\xi\xi}\|_{0,\widetilde{\Omega}}^2)+a_1(|u|_{3,2,\widetilde{\Omega}}^2+|\widetilde{q}_\sigma|_{2,1,\widetilde{\Omega}}^2+|\widetilde{g}|_{2,1,\widetilde{\Omega}}^2)$$
$$+a_2X_8(\widetilde{\Omega})(1+X_8^2(\widetilde{\Omega}))Y_8(\widetilde{\Omega})\,,$$

where

$$X_8(\widetilde{\Omega}) = |u|^2_{3,2,\widetilde{\Omega}} + |q_\sigma|^2_{3,1,\widetilde{\Omega}} + \int_0^t \|u\|^2_{3,\widetilde{\Omega}}\,dt\,,$$

$$Y_8(\widetilde{\Omega}) = |u|^2_{4,3,\widetilde{\Omega}} + |q_\sigma|^2_{3,1,\widetilde{\Omega}} + \int_0^t \|u\|^2_{4,\widetilde{\Omega}}\,dt\,.$$

Moreover, the following has been used:

$$-\int_{\widetilde{\Omega}} \widetilde{q}_{\sigma t\xi\xi}\nabla_u \cdot \widetilde{u}_{t\xi\xi} A\,d\xi = \frac{1}{2}\frac{d}{dt}\int_{\widetilde{\Omega}} \frac{1}{q\Psi(\eta)}\widetilde{q}^2_{\sigma t\xi\xi} A\,d\xi + N_1\,,$$

$$\begin{aligned}\bigg|\int_{\widetilde{\Omega}} [(\nabla_u\nabla_u\widetilde{u})_{,t\xi\xi} - \nabla_u\nabla_u\widetilde{u}_{t\xi\xi}]\cdot\widetilde{u}_{t\xi\xi}A\,d\xi \\ + \int_{\widetilde{\Omega}} [(\nabla_u\widetilde{q}_\sigma)_{,t\xi\xi} - \nabla_u\widetilde{q}_{\sigma t\xi\xi}]\cdot\widetilde{u}_{t\xi\xi}A\,d\xi\bigg| \\ \le \delta_2\|\widetilde{u}_{t\xi\xi}\|^2_{1,\widetilde{\Omega}} + a_3X_8(\widetilde{\Omega})(1+X_8^2(\widetilde{\Omega}))Y_8(\widetilde{\Omega})\,,\end{aligned}$$

and

$$|N_1| \le \delta_3\|\widetilde{q}_{\sigma t\xi\xi}\|^2_{0,\widetilde{\Omega}} + a_4|u|^2_{3,2,\widetilde{\Omega}} + a_5X_8(\widetilde{\Omega})(1+X_8^2(\widetilde{\Omega}))Y_8(\widetilde{\Omega})\,.$$

From (4.28) we obtain

$$\begin{aligned}\|\widetilde{u}_t\|^2_{3,\widetilde{\Omega}} + \|\widetilde{q}_{\sigma t}\|^2_{2,\widetilde{\Omega}} \le c\|\nabla_u\cdot\widetilde{u}_t\|^2_{2,\widetilde{\Omega}} + a_6(|u|^2_{3,1,\widetilde{\Omega}} + |q_\sigma|^2_{2,1,\widetilde{\Omega}} + |\widetilde{g}|^2_{2,1,\widetilde{\Omega}}) \\ + a_7X_8(\widetilde{\Omega})(1+X_8^2(\widetilde{\Omega}))Y_8(\widetilde{\Omega})\,.\end{aligned} \tag{4.132}$$

Now by applying Lemma 5.1 for $G = \widetilde{\Omega}$ and $v = \widetilde{u}_{t\xi\xi}$, from (4.131) and (4.132) for sufficiently small δ_1 we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\widetilde{\Omega}} \bigg(\eta\widetilde{u}^2_{t\xi\xi} + \frac{1}{q\Psi(\eta)}\widetilde{q}^2_{\sigma t\xi\xi}\bigg)A\,d\xi + \|\widetilde{u}_t\|^2_{3,\widetilde{\Omega}} + \|\widetilde{q}_{\sigma t}\|^2_{2,\widetilde{\Omega}} \\ \le a_8(|u|^2_{3,1,\widetilde{\Omega}} + |q_\sigma|^2_{2,1,\widetilde{\Omega}} + |\widetilde{g}|^2_{2,1,\widetilde{\Omega}}) + a_9X_8(\widetilde{\Omega})(1+X_8^2(\widetilde{\Omega}))Y_8(\widetilde{\Omega})\,.\end{aligned} \tag{4.133}$$

Now we consider boundary subdomains. Differentiating $(4.22)_1$ with respect to t and twice with respect to τ, multiplying by $\widetilde{u}_{t\tau\tau}J$, integrating over $\widetilde{\Omega}$ and applying Lemma 5.1 for $G = \widehat{\Omega}$ and $v = \widetilde{u}_{t\tau\tau}$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}} \bigg[\widehat{\eta}\widetilde{u}^2_{t\tau\tau} + \frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma t\tau\tau}\bigg]J\,dz + \|\widetilde{u}_{t\tau\tau}\|^2_{1,\widehat{\Omega}} \\ +(\nu-\mu)\|(\operatorname{div}\widetilde{u})_{,t\tau\tau}\|^2_{0,\widehat{\Omega}} - \int_{\widehat{S}} (\widehat{n}\widehat{\mathbb{T}}(\widetilde{u},\widetilde{q}_\sigma))_{,t\tau\tau}\cdot\widetilde{u}_{t\tau\tau}J\,dz'\end{aligned} \tag{4.134}$$

$$\le \delta_4(\|\widetilde u_{zz\tau t}\|^2_{0,\widehat\Omega} + \|\widetilde q_{\sigma z\tau t}\|^2_{0,\widehat\Omega} + a_{10}(|\widehat u|^2_{3,1,\widehat\Omega} + |\widehat q_\sigma|^2_{2,1,\widehat\Omega} + |\widehat g|^2_{2,1,\widehat\Omega})$$
$$+ a_{11}X_8(\widehat\Omega)(1 + X_8^2(\widehat\Omega))Y_8(\widehat\Omega)\,,$$

where we have used the following relations:

$$-\int\limits_{\widehat\Omega} \widetilde q_{\sigma\tau\tau t}\widehat\nabla\cdot\widetilde u_{\tau\tau t}J\,dz = \frac12\frac{d}{dt}\int\limits_{\widehat\Omega}\frac{1}{\widehat q\Psi(\widehat\eta)}\widetilde q^2_{\sigma\tau\tau t}J\,dz + N_2\,,$$

$$\Big|\int\limits_{\widehat\Omega}[(\widehat\nabla\widehat{\mathbb T}(\widetilde u,\widetilde q_\sigma))_{,\tau\tau t} - \widehat\nabla\widehat{\mathbb T}(\widetilde u_{\tau\tau t},\widetilde q_{\sigma\tau\tau t})]\widetilde u_{\tau\tau t}J\,dz\Big|$$
$$\le \delta_4'(\|\widetilde u_{zz\tau t}\|^2_{0,\widehat\Omega} + \|\widetilde q_{\sigma z\tau t}\|^2_{0,\widehat\Omega}) + a_{12}(|\widehat u|^2_{3,1,\widehat\Omega} + |\widehat q_\sigma|^2_{2,1,\widehat\Omega})$$
$$+ a_{13}X_8(\widehat\Omega)(1 + X_8^2(\widehat\Omega))Y_8(\widehat\Omega)$$

and

$$|N_2| \le \delta_4''\|\widetilde q_{\sigma\tau\tau t}\|^2_{0,\widehat\Omega} + a_{14}|\widehat u|^2_{3,2,\widehat\Omega} + a_{15}X_8(\widehat\Omega)(1 + X_8(\widehat\Omega))Y_8(\widehat\Omega)\,.$$

Moreover, $X_8(\widehat\Omega)$, $Y_8(\widehat\Omega)$ are obtained from $X_8(\widetilde\Omega)$, $Y_8(\widetilde\Omega)$ upon replacing u, q_σ, $\widetilde\Omega$ by $\widehat u$, $\widehat q_\sigma$, $\widehat\Omega$, respectively.

In view of the boundary condition $(4.22)_3$ the boundary term in (4.134) can be estimated in the following way:

$$\Big|\int\limits_{\widehat S}(\widehat n\widehat{\mathbb T}(\widetilde u,\widetilde q_\sigma))_{,t\tau\tau}\cdot\widetilde u_{t\tau\tau}J\,dz'\Big| = \Big|\int\limits_{\widehat S}(\widehat B(\widehat u,\widehat\zeta)\widehat n)_{,t\tau\tau}\cdot\widetilde u_{t\tau\tau}J\,dz'\Big| \tag{4.135}$$
$$\le \delta_5\|\widehat u_{t\tau\tau z}\|^2_{0,\widehat\Omega} + a_{16}\|\widehat u_t\|^2_{2,\widehat\Omega} + a_{17}X_8(\widehat\Omega)(1 + X_8^2(\widehat\Omega))Y_8(\widehat\Omega)\,.$$

From (4.134) and (4.135) for sufficiently small δ_5 we obtain

$$\frac12\frac{d}{dt}\int\limits_{\widehat\Omega}\Big[\widehat\eta\,\widetilde u^2_{t\tau\tau} + \frac{1}{\widehat q\Psi(\widehat\eta)}\widetilde q^2_{\sigma t\tau\tau}\Big]J\,dz \tag{4.136}$$
$$+ \|\widetilde u_{t\tau\tau}\|^2_{1,\widehat\Omega} + (\nu-\mu)\|(\operatorname{div}\widetilde u)_{,t\tau\tau}\|^2_{0,\widehat\Omega}$$
$$\le \delta_6(\|\widehat u_{t\tau\tau z}\|^2_{0,\widehat\Omega} + \|\widehat q_{\sigma t\tau\tau}\|^2_{0,\widehat\Omega}) + a_{18}(|\widehat u|^2_{3,1,\widehat\Omega} + |\widehat q_\sigma|^2_{2,1,\widehat\Omega} + |\widetilde g|^2_{2,1,\widehat\Omega})$$
$$+ a_{19}X_8(\widehat\Omega)(1 + X_8^2(\widehat\Omega))Y_8(\widehat\Omega)\,.$$

Differentiating the third component of (4.35) with respect to τ and t, multiplying the result by $\widetilde q_{\sigma n\tau t}J$ and integrating over $\widehat\Omega$ implies

$$\frac12\frac{d}{dt}\int\limits_{\widehat\Omega}\frac{\mu+\nu}{\widehat q\Psi(\widehat\eta)}\widetilde q^2_{\sigma n\tau t}J\,dz + \|\widetilde q_{\sigma n\tau t}\|^2_{0,\widehat\Omega} \tag{4.137}$$

$$\le (\delta_7 + cd)(\|\widetilde{u}_{zz\tau t}\|^2_{0,\widehat{\Omega}} + \|\widehat{q}_{\sigma z\tau t}\|^2_{0,\widehat{\Omega}}) + c\|\widetilde{u}_{z\tau\tau t}\|^2_{0,\widehat{\Omega}}$$
$$+ a_{20}(|\widehat{u}|^2_{3,2,\widehat{\Omega}} + |\widehat{q}_\sigma|^2_{2,1,\widehat{\Omega}} + |\widetilde{g}|^2_{2,1,\widehat{\Omega}})$$
$$+ a_{21}X_8(\widehat{\Omega})(1 + X_8^2(\widehat{\Omega}))Y_8(\widehat{\Omega}) .$$

Differentiating the third component of (4.37) with respect to τ and t, multiplying the result by $\widetilde{u}^3_{nn\tau t}J$ and integrating over $\widehat{\Omega}$ gives

$$\begin{aligned}(4.138)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}} \widehat{\eta}|\widetilde{u}^3_{n\tau t}|^2 J\,dz + \frac{\mu+\nu}{2}\|\widetilde{u}^3_{nn\tau t}\|^2_{0,\widehat{\Omega}} &\le \delta_8\|\widetilde{u}_{zztt}\|^2_{0,\widehat{\Omega}} + (\delta_9 + cd)(\|\widetilde{u}_{zzzt}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma z\tau t}\|^2_{0,\widehat{\Omega}})\\
&\quad + c(\|\widetilde{u}_{z\tau\tau t}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma n\tau t}\|^2_{0,\widehat{\Omega}})\\
&\quad + a_{22}(|\widehat{u}|^2_{3,1,\widehat{\Omega}} + |\widehat{q}_\sigma|^2_{2,1,\widehat{\Omega}} + |\widetilde{g}|^2_{2,1,\widehat{\Omega}})\\
&\quad + a_{23}X_8(\widehat{\Omega})(1 + X_8^2(\widehat{\Omega}))Y_8(\widehat{\Omega}) .\end{aligned}$$

From (4.39)–(4.41) we have

$$\begin{aligned}(4.139)\quad \|\widetilde{u}'_{n\tau\tau t}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma\tau\tau t}\|^2_{0,\widehat{\Omega}} &\le (\delta_{10} + cd)(\|\widetilde{u}_{zzzt}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma zzt}\|^2_{0,\widehat{\Omega}})\\
&\quad + c(\|\widetilde{u}^3_{nn\tau t}\|^2_{0,\widehat{\Omega}} + \|\widetilde{u}_{z\tau\tau t}\|^2_{0,\widehat{\Omega}})\\
&\quad + a_{24}(|\widehat{u}|^2_{3,1,\widehat{\Omega}} + |\widehat{q}_\sigma|^2_{2,1,\widehat{\Omega}} + |\widetilde{g}|^2_{2,1,\widehat{\Omega}})\\
&\quad + a_{25}X_8(\widehat{\Omega})(1 + X_8^2(\widehat{\Omega}))Y_8(\widehat{\Omega}) ,\end{aligned}$$

where the prime denotes that only components u^1, u^2 are taken into consideration. Moreover, from (4.39) we get

$$\begin{aligned}(4.140)\quad \|\widetilde{u}'_{nn\tau t}\|^2_{0,\widehat{\Omega}} &\le c(\|(\operatorname{div}\widetilde{u})_{,\tau\tau t}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma\tau\tau t}\|^2_{0,\widehat{\Omega}})\\
&\quad + (\delta_{11} + cd)(\|\widetilde{u}_{zzzt}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma zzt}\|^2_{0,\widehat{\Omega}})\\
&\quad + a_{26}(|\widehat{u}|^2_{3,1,\widehat{\Omega}} + |\widehat{q}_\sigma|^2_{2,1,\widehat{\Omega}} + |\widetilde{g}|^2_{2,1,\widehat{\Omega}})\\
&\quad + a_{27}X_8(\widehat{\Omega})(1 + X_8^2(\widehat{\Omega}))Y_8(\widehat{\Omega}) .\end{aligned}$$

Summarizing, from (4.137)–(4.140) we obtain

$$\begin{aligned}(4.141)\quad &\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}} \bigg(\widehat{\eta}|\widetilde{u}^3_{n\tau t}|^2 + \frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma n\tau t}\bigg) J\,dz + \|\widetilde{u}_{zz\tau t}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma z\tau t}\|^2_{0,\widehat{\Omega}}\\
&\le \delta_{12}\|\widetilde{u}_{tt}\|^2_{2,\widehat{\Omega}} + c\|\widetilde{u}_{\tau\tau t}\|^2_{1,\widehat{\Omega}} + (\delta_{13} + cd)(\|\widehat{u}_{zzzt}\|^2_{0,\widehat{\Omega}} + \|\widehat{q}_{\sigma zzt}\|^2_{0,\widehat{\Omega}})\\
&\quad + a_{28}(|\widehat{u}|^2_{3,1,\widehat{\Omega}} + |\widehat{q}_\sigma|^2_{2,1,\widehat{\Omega}} + |\widetilde{g}|^2_{2,1,\widehat{\Omega}}) + a_{29}X_8(\widehat{\Omega})(1 + X_8^2(\widehat{\Omega}))Y_8(\widehat{\Omega}) .\end{aligned}$$

Differentiating the third component of (4.35) with respect to t and n, multiplying the result by $\widetilde{q}_{\sigma nnt}J$ and integrating over $\widehat{\Omega}$ yields

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma nnt}J\,dz+\frac{1}{2}\|\widetilde{q}_{\sigma nnt}\|^2_{0,\widehat{\Omega}} \tag{4.142}$$
$$\le c\|\widetilde{u}_{zz\tau t}\|^2_{0,\widehat{\Omega}}+(\delta_{14}+cd)(\|\widetilde{u}_{zzzt}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma zzt}\|^2_{0,\widehat{\Omega}})$$
$$+a_{30}(|\widehat{u}|^2_{3,1,\widehat{\Omega}}+|\widehat{q}_\sigma|^2_{2,1,\widehat{\Omega}}+|\widetilde{g}|^2_{2,1,\widehat{\Omega}})$$
$$+a_{31}X_8(\widehat{\Omega})(1+X_8^2(\widehat{\Omega}))Y_8(\widehat{\Omega}).$$

Differentiating the third component of (4.37) with respect to n and t, multiplying the result by $\widetilde{u}^3_{nnnt}J$ and integrating over $\widehat{\Omega}$ implies

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\widehat{\eta}|\widetilde{u}^3_{nnt}|^2J\,dz+\frac{\mu+\nu}{2}\|\widetilde{u}^3_{nnnt}\|^2_{0,\widehat{\Omega}} \tag{4.143}$$
$$\le c(\|\widetilde{u}_{zz\tau t}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma nnt}\|^2_{0,\widehat{\Omega}})+\delta_{15}\|\widetilde{u}_{tt}\|^2_{2,\widehat{\Omega}}$$
$$+(\delta_{16}+cd)(\|\widetilde{u}_{zzzt}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma zzt}\|^2_{0,\widehat{\Omega}})$$
$$+a_{32}(|\widehat{u}|^2_{3,1,\widehat{\Omega}}+|\widehat{q}_\sigma|^2_{2,1,\widehat{\Omega}}+|\widetilde{g}|^2_{2,1,\widehat{\Omega}})$$
$$+a_{33}X_8(\widehat{\Omega})(1+X_8^2(\widehat{\Omega}))Y_8(\widehat{\Omega}).$$

Finally, from (4.39) we get

$$\|\widetilde{u}'_{nnnt}\|^2_{0,\widehat{\Omega}}\le c(\|(\operatorname{div}\widetilde{u})_{,n\tau t}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma n\tau t}\|^2_{0,\widehat{\Omega}}) \tag{4.144}$$
$$+c(\delta_{17}+cd)(\|\widetilde{u}_{zzzt}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma zzt}\|^2_{0,\widehat{\Omega}})$$
$$+a_{34}(|\widehat{u}|^2_{3,1,\widehat{\Omega}}+|\widehat{q}_\sigma|^2_{2,1,\widehat{\Omega}}+|\widetilde{g}|^2_{2,1,\widehat{\Omega}})$$
$$+a_{35}X_8(\widehat{\Omega})(1+X_8^2(\widehat{\Omega}))Y_8(\widehat{\Omega}).$$

Hence, from (4.141)–(4.144) it follows that

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\left(\widehat{\eta}|\widetilde{u}^3_{znt}|^2+\frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma znt}\right)J\,dz+\|\widetilde{u}_{nzt}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma zzt}\|^2_{0,\widehat{\Omega}} \tag{4.145}$$
$$\le\delta_{18}\|\widetilde{u}_{zztt}\|^2_{0,\widehat{\Omega}}+c\|\widetilde{u}_{z\tau\tau t}\|^2_{0,\widehat{\Omega}}+(\delta_{19}+cd)(\|\widetilde{u}_{zzzt}\|^2_{0,\widehat{\Omega}}+\|\widetilde{q}_{\sigma zzt}\|^2_{0,\widehat{\Omega}})$$
$$+a_{36}(|\widehat{u}|^2_{3,1,\widehat{\Omega}}+|\widehat{q}_\sigma|^2_{2,1,\widehat{\Omega}}+|\widetilde{g}|^2_{2,1,\widehat{\Omega}})+a_{37}X_8(\widehat{\Omega})(1+X_8^2(\widehat{\Omega}))Y_8(\widehat{\Omega}).$$

To obtain the full derivative $\widetilde{u}_{zzt}$ under the integral over $\widehat{\Omega}$ and under the time derivative we need the following:

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\widehat{\eta}\,\widetilde{u}_{zzt}^2 J\,dz \le \delta_{20}\|\widetilde{u}_{zztt}\|_{0,\widehat{\Omega}}^2 + c\|\widetilde{u}_{zzt}\|_{2,\widehat{\Omega}}^2 + c(\|\widehat{u}\|_{3,\widehat{\Omega}}^2 + |\widehat{q}_\sigma|_{2,1,\widehat{\Omega}}^2)|\widehat{u}|_{3,2,\widehat{\Omega}}^2\,. \tag{4.146}$$

From (4.136), (4.145) and (4.146) we obtain for sufficiently small δ and d

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}&\left(\widehat{\eta}\,\widetilde{u}_{zzt}^2 + \frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}_{\sigma zzt}^2\right)J\,dz + \|\widetilde{u}_{zzzt}\|_{0,\widehat{\Omega}}^2 + \|\widetilde{q}_{\sigma zzt}\|_{0,\widehat{\Omega}}^2\\ &\le \delta_{21}(\|\widetilde{u}_{zzzz}\|_{0,\widehat{\Omega}}^2 + \|\widetilde{u}_{zztt}\|_{0,\widehat{\Omega}}^2 + \|\widehat{u}_{zzzt}\|_{0,\widehat{\Omega}}^2 + \|\widehat{q}_{\sigma zzt}\|_{0,\widehat{\Omega}}^2)\\ &\quad + a_{38}(|\widehat{u}|_{3,1,\widehat{\Omega}}^2 + |\widehat{q}_\sigma|_{2,1,\widehat{\Omega}}^2 + |\widetilde{g}|_{2,1,\widehat{\Omega}}^2)\\ &\quad + a_{39}X_8(\widehat{\Omega})(1+X_8^2(\widehat{\Omega}))Y_8(\widehat{\Omega})\,.\end{aligned}\tag{4.147}$$

Going back to the variables ξ in (4.147), next summing over all neighbourhoods of the partition of unity and using (4.133) and the smallness of δ_{21}, we finally obtain (4.129) after going back to the variables x. This concludes the proof.

To estimate the first term on the right-hand side of (4.129) we need the following result.

LEMMA 4.11. *For a sufficiently smooth solution v, p of* (4.1),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}&\left(\varrho v_{xtt}^2 + \frac{\mu+\nu}{p\Psi(\varrho)}q_{\sigma xtt}^2\right)dx + \|v_{tt}\|_{2,\Omega_t}^2 + \|p_{\sigma tt}\|_{1,\Omega_t}^2\\ &\le \varepsilon_8(\|v_{xxxt}\|_{0,\Omega_t}^2 + \|v_{xttt}\|_{0,\Omega_t}^2)\\ &\quad + P_{21}(|v|_{3,0,\Omega_t}^2 + |p_\sigma|_{2,0,\Omega_t}^2 + |f|_{2,0,\Omega_t}^2) + P_{22}X_9(1+X_9^2)Y_9\,,\end{aligned}\tag{4.148}$$

where

$$\begin{aligned}X_9 &= |v|_{3,0,\Omega_t}^2 + |p_\sigma|_{3,0,\Omega_t}^2 + \int\limits_0^t \|v\|_{3,\Omega_\tau}^2\,d\tau\,,\\ Y_9 &= |v|_{4,1,\Omega_t}^2 + |p_\sigma|_{3,0,\Omega_t}^2 + \int\limits_0^t \|v\|_{4,\Omega_\tau}^2\,d\tau\,.\end{aligned}\tag{4.149}$$

Proof. We use the partition of unity. First we consider interior subdomains. Differentiating $(4.21)_1$ twice with respect to t and once with respect to ξ, multiplying the result by $\widetilde{u}_{tt\xi}A$ and integrating over $\widehat{\Omega}$ yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\limits_{\widetilde{\Omega}}&\left(\eta\widetilde{u}_{tt\xi}^2 + \frac{1}{q\Psi(\eta)}\widetilde{q}_{\sigma tt\xi}^2\right)A\,d\xi\\ &+\frac{\mu}{2}\int\limits_{\widetilde{\Omega}}(\nabla_{u^i}\widetilde{u}_{tt\xi}^j + \nabla_{u^j}\widetilde{u}_{tt\xi}^i)^2 A\,d\xi + (\nu-\mu)\|\nabla_u\cdot\widetilde{u}_{tt\xi}\|_{0,\widetilde{\Omega}}^2\end{aligned}\tag{4.150}$$

$$\le \delta_1(\|\widetilde{u}_{tt\xi\xi}\|^2_{0,\widetilde{\Omega}} + \|\widetilde{q}_{\sigma tt\xi}\|^2_{0,\widetilde{\Omega}}) + a_1(|u|^2_{3,1,\widetilde{\Omega}} + |q_\sigma|^2_{2,0,\widetilde{\Omega}} + |\widetilde{g}|^2_{2,0,\widetilde{\Omega}})$$
$$+ a_2 X_9(\widetilde{\Omega})(1 + X_9^2(\widetilde{\Omega}))Y_9(\widetilde{\Omega}),$$

where

$$X_9(\widetilde{\Omega}) = |u|^2_{3,0,\widetilde{\Omega}} + |q_\sigma|^2_{3,0,\widetilde{\Omega}} + \int_{\widetilde{\Omega}} \|u\|^2_{3,\widetilde{\Omega}}\,dt,$$
$$Y_9(\widetilde{\Omega}) = |u|^2_{4,1,\widetilde{\Omega}} + |q_\sigma|^2_{3,0,\widetilde{\Omega}} + \int_{\widetilde{\Omega}} \|u\|^2_{4,\widetilde{\Omega}}\,dt.$$

We have used the facts that

$$-\int_{\widetilde{\Omega}} \widetilde{q}_{\sigma tt\xi}\nabla_u \cdot \widetilde{u}_{tt\xi} A\,d\xi = \frac{1}{2}\frac{d}{dt}\int_{\widetilde{\Omega}} \frac{1}{q\Psi(\eta)}\widetilde{q}^2_{\sigma tt\xi} A\,d\xi + N_1,$$

$$\bigg|\int_{\widetilde{\Omega}} [(\nabla_u\nabla_u\widetilde{u})_{,tt\xi} - \nabla_u\nabla_u\widetilde{u}_{tt\xi}]\cdot\widetilde{u}_{tt\xi}A\,d\xi$$
$$+ \int_{\widetilde{\Omega}} [(\nabla_u\widetilde{q}_\sigma)_{,tt\xi} - \nabla_u\widetilde{q}_{\sigma tt\xi}]\cdot\widetilde{u}_{tt\xi}A\,d\xi\bigg|$$
$$\le \delta_2\|\widetilde{u}_{tt\xi\xi}\|^2_{0,\widetilde{\Omega}} + a_3 X_9(\widetilde{\Omega})(1 + X_9^2(\widetilde{\Omega}))Y_9(\widetilde{\Omega}),$$

and

$$|N_1| \le \delta_3\|\widetilde{q}_{\sigma tt\xi}\|^2_{0,\widetilde{\Omega}} + a_4|\widehat{u}|^2_{3,1,\widehat{\Omega}} + a_5 X_9(\widetilde{\Omega})(1 + X_9^2(\widetilde{\Omega}))Y_9(\widetilde{\Omega}).$$

From (4.28) we obtain

$$\|\widetilde{u}_{tt}\|^2_{2,\widetilde{\Omega}} + \|\widetilde{q}_{\sigma tt}\|^2_{1,\widetilde{\Omega}} \le c\|u_{zztt}\|^2_{0,\widetilde{\Omega}} + a_6(|u|^2_{3,0,\widetilde{\Omega}} + |q_\sigma|^2_{2,0,\widetilde{\Omega}} + |\widetilde{g}|^2_{2,0,\widetilde{\Omega}})$$
$$+ a_7 X_9(\widetilde{\Omega})(1 + X_9^2(\widetilde{\Omega}))Y_9(\widetilde{\Omega}). \tag{4.151}$$

Now from (4.150) and (4.151) for sufficiently small δ_1 and from Lemma 5.1 for $G = \widetilde{\Omega}$ and $v = \widetilde{u}_{tt\xi}$ we get

$$\frac{1}{2}\frac{d}{dt}\int_{\widetilde{\Omega}} \left(\eta\widetilde{u}^2_{tt\xi} + \frac{1}{q\Psi(\eta)}\widetilde{q}^2_{\sigma tt\xi}\right)A\,d\xi + \|\widetilde{u}_{tt}\|^2_{2,\widetilde{\Omega}} + \|\widetilde{q}_{\sigma tt}\|^2_{1,\widetilde{\Omega}}$$
$$\le a_8(|u|^2_{3,0,\widetilde{\Omega}} + |q_\sigma|^2_{2,0,\widetilde{\Omega}} + |\widetilde{g}|^2_{2,0,\widetilde{\Omega}}) + a_9 X_9(\widetilde{\Omega})(1 + X_9^2(\widetilde{\Omega}))Y_9(\widetilde{\Omega}). \tag{4.152}$$

Consider now boundary subdomains. Differentiating $(4.22)_1$ twice with respect to t and once with respect to τ, multiplying the result by $\widetilde{u}_{tt\tau}J$ and integrating over $\widehat{\Omega}$ gives

$$\text{(4.153)}\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\bigg[\widehat{\eta}\,\widetilde{u}_{tt\tau}^2+\frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}_{\sigma tt\tau}^2\bigg]J\,dz+\|\widetilde{u}_{tt\tau}\|_{1,\widehat{\Omega}}^2$$
$$+(\nu-\mu)\|(\operatorname{div}\widetilde{u})_{,tt\tau}\|_{0,\widehat{\Omega}}^2-\int\limits_{\widehat{S}}(\widehat{n}\widehat{\mathbb{T}}(\widetilde{u},\widetilde{q}_\sigma))_{,tt\tau}\cdot\widetilde{u}_{tt\tau}J\,dz'$$
$$\le\delta_4(\|\widetilde{q}_{\sigma tt\tau}\|_{0,\widehat{\Omega}}^2+\|\widetilde{u}_{zztt}\|_{0,\widehat{\Omega}}^2)+a_{10}(|\widehat{u}|_{3,0,\widehat{\Omega}}^2+|\widehat{q}_\sigma|_{2,0,\widehat{\Omega}}^2+|\widetilde{g}|_{2,0,\widehat{\Omega}}^2)$$
$$+a_{11}X_9(\widehat{\Omega})(1+X_9^2(\widehat{\Omega}))Y_9(\widehat{\Omega}),$$

where we have used Lemma 5.1 in the case $G=\widehat{\Omega}$, $v=\widetilde{u}_{tt\tau}$, and

$$\Big|\int\limits_{\widehat{\Omega}}[(\widehat{\nabla}\widehat{\mathbb{T}}(\widetilde{u},\widetilde{q}_\sigma))_{,tt\tau}-\widehat{\nabla}\widehat{\mathbb{T}}(\widetilde{u}_{tt\tau},\widetilde{q}_{\sigma tt\tau})]\widetilde{u}_{tt\tau}J\,dz\Big|$$
$$\le\delta_4'(\|\widetilde{u}_{zztt}\|_{0,\widehat{\Omega}}^2+\|\widehat{q}_{\sigma ztt}\|_{0,\widehat{\Omega}}^2)$$
$$+a_{12}|\widetilde{u}|_{3,1,\widehat{\Omega}}^2+a_{13}X_9(\widehat{\Omega})(1+X_9^2(\widehat{\Omega}))Y_9(\widehat{\Omega}),$$
$$\int\limits_{\widehat{\Omega}}\widehat{\nabla}\cdot\widetilde{u}_{tt\tau}\widetilde{q}_{\sigma tt\tau}J\,dz=-\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}_{\sigma tt\tau}^2J\,dz+N_2,$$

where

$$|N_2|\le\delta_4''\|\widetilde{q}_{\sigma tt\tau}\|_{0,\widehat{\Omega}}^2+a_{14}|\widehat{u}|_{3,1,\widehat{\Omega}}^2+a_{15}X_9(\widehat{\Omega})(1+X_9^2(\widehat{\Omega}))Y_9(\widehat{\Omega}).$$

Moreover, $X_9(\widehat{\Omega})$, $Y_9(\widehat{\Omega})$ are obtained from $X_9(\widetilde{\Omega})$, $Y_9(\widetilde{\Omega})$ upon replacing u, q_σ, $\widetilde{\Omega}$ by $\widehat{u}$, $\widehat{q}_\sigma$, $\widehat{\Omega}$, respectively.

By using the boundary condition $(4.22)_3$ the boundary term in (4.153) can be estimated in the following way:

$$\text{(4.154)}\quad \Big|\int\limits_{\widehat{S}}(\widehat{n}\widehat{\mathbb{T}}(\widetilde{u},\widetilde{q}_\sigma))_{,tt\tau}\cdot\widetilde{u}_{tt\tau}J\,dz'\Big|$$
$$\le\delta_5\|\widehat{u}_{ztt\tau}\|_{0,\widehat{\Omega}}^2+a_{16}\|\widehat{u}_{tt}\|_{1,\widehat{\Omega}}^2+a_{17}X_9(\widehat{\Omega})(1+X_9^2(\widehat{\Omega}))Y_9(\widehat{\Omega}).$$

From (4.153) and (4.154) we obtain for sufficiently small δ_5

$$\text{(4.155)}\quad \frac{d}{dt}\int\limits_{\widehat{\Omega}}\bigg(\widehat{\eta}\,\widetilde{u}_{tt\tau}^2+\frac{1}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}_{\sigma tt\tau}^2\bigg)J\,dz'+\|\widetilde{u}_{tt\tau}\|_{1,\widehat{\Omega}}^2+(\nu-\mu)\|(\operatorname{div}\widetilde{u})_{,tt\tau}\|_{0,\widehat{\Omega}}^2$$
$$\le\delta_6(\|\widehat{u}_{ztt\tau}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma tt\tau}\|_{0,\widehat{\Omega}}^2)$$
$$+a_{18}(|\widehat{u}|_{3,0,\widehat{\Omega}}^2+|\widehat{q}_\sigma|_{2,0,\widehat{\Omega}}^2+|\widetilde{g}|_{2,0,\widehat{\Omega}}^2)$$
$$+a_{19}X_9(\widehat{\Omega})(1+X_9^2(\widehat{\Omega}))Y_9(\widehat{\Omega})).$$

Differentiating the third component of (4.35) twice with respect to t, multiplying the result by $\widetilde{q}_{\sigma ntt}J$ and integrating over $\widehat{\Omega}$ implies

$$(4.156)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}_{\sigma ntt}^2 J\,dz+\frac{1}{2}\|\widetilde{q}_{\sigma ntt}\|_{0,\widehat{\Omega}}^2$$
$$\le c\|\widetilde{u}_{z\tau tt}\|_{0,\widehat{\Omega}}^2+(\delta_7+cd)(\|\widetilde{u}_{zztt}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma ztt}\|_{0,\widehat{\Omega}}^2)$$
$$+a_{20}(|\widehat{u}|_{3,0,\widehat{\Omega}}^2+|\widehat{q}_\sigma|_{2,0,\widehat{\Omega}}^2+|\widetilde{g}|_{2,0,\widehat{\Omega}}^2)$$
$$+a_{21}X_9(\widehat{\Omega})(1+X_9^2(\widehat{\Omega}))Y_9(\widehat{\Omega})\,.$$

Differentiating the third component of (4.37) twice with respect to t, multiplying the result by $\widetilde{u}_{nntt}^3 J$ and integrating over $\widehat{\Omega}$ one has

$$(4.157)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\widehat{\eta}|\widetilde{u}_{ntt}^3|^2 J\,dz+\frac{\mu+\nu}{2}\|\widetilde{u}_{nntt}^3\|_{0,\widehat{\Omega}}^2$$
$$\le c(\|\widetilde{u}_{z\tau tt}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma ntt}\|_{0,\widehat{\Omega}}^2)+\delta_8\|\widetilde{u}_{zttt}\|_{0,\widehat{\Omega}}^2$$
$$+(\delta_9+cd)(\|\widetilde{u}_{zztt}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma ztt}\|_{0,\widehat{\Omega}}^2)$$
$$+a_{22}(|\widehat{u}|_{3,0,\widehat{\Omega}}^2+|\widehat{q}_\sigma|_{2,0,\widehat{\Omega}}^2+|\widetilde{g}|_{2,0,\widehat{\Omega}}^2)$$
$$+a_{23}X_9(\widehat{\Omega})(1+X_9^2(\widehat{\Omega}))Y_9(\widehat{\Omega})\,.$$

From (4.39)–(4.41) we have

$$(4.158)\quad \|\widetilde{u}'_{n\tau tt}\|_{0,\widehat{\Omega}}^2+\|\widehat{q}_{\sigma\tau tt}\|_{0,\widehat{\Omega}}^2$$
$$\le(\delta_{10}+cd)(\|\widetilde{u}_{zztt}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma ztt}\|_{0,\widehat{\Omega}}^2)+c(\|\widetilde{u}_{nntt}\|_{0,\widehat{\Omega}}^2+\|\widetilde{u}_{z\tau tt}\|_{0,\widehat{\Omega}}^2)$$
$$+a_{24}(|\widehat{u}|_{3,0,\widehat{\Omega}}^2+|\widehat{q}_\sigma|_{2,0,\widehat{\Omega}}^2+|\widetilde{g}|_{2,0,\widehat{\Omega}}^2)+a_{25}X_9(\widehat{\Omega})(1+X_9^2(\widehat{\Omega}))Y_9(\widehat{\Omega})\,.$$

Moreover, from (4.39) it follows that

$$(4.159)\quad \|\widetilde{u}'_{nntt}\|_{0,\widehat{\Omega}}^2\le c(\|(\operatorname{div}\widetilde{u})_{,\tau tt}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma\tau tt}\|_{0,\widehat{\Omega}}^2)$$
$$+(\delta_{11}+cd)(\|\widetilde{u}_{zztt}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma ztt}\|_{0,\widehat{\Omega}}^2)$$
$$+a_{26}(|\widehat{u}|_{3,0,\widehat{\Omega}}^2+|\widehat{q}_\sigma|_{2,0,\widehat{\Omega}}^2+|\widetilde{g}|_{2,0,\widehat{\Omega}}^2)$$
$$+a_{27}X_9(\widehat{\Omega})(1+X_9^2(\widehat{\Omega}))Y_9(\widehat{\Omega})\,.$$

Summarizing, from (4.156)–(4.159) we have

$$(4.160)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\left(\widehat{\eta}|\widetilde{u}_{ntt}^3|^2+\frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}_{\sigma ntt}^2\right)J\,dz+\|\widetilde{u}_{zztt}\|_{0,\widehat{\Omega}}^2+\|\widetilde{q}_{\sigma ztt}\|_{0,\widehat{\Omega}}^2$$

$$\le \delta_{12}\|\widetilde{u}_{ttt}\|^2_{1,\widehat{\Omega}} + (\delta_{13}+cd)(\|\widetilde{u}_{zztt}\|^2_{0,\widehat{\Omega}} + \|\widetilde{q}_{\sigma ztt}\|^2_{0,\widehat{\Omega}})$$
$$+ c\|\widetilde{u}_{z\tau tt}\|^2_{0,\widehat{\Omega}} + a_{28}(|\widehat{u}|^2_{3,0,\widehat{\Omega}} + |\widehat{q}_\sigma|^2_{2,0,\widehat{\Omega}} + |\widehat{g}|^2_{2,0,\widehat{\Omega}})$$
$$+ a_{29}X_9(\widehat{\Omega})(1+X_9^2(\widehat{\Omega}))Y_9(\widehat{\Omega})\,.$$

Finally, using the inequality

$$(4.161)\qquad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}} \widehat{\eta}\,\widetilde{u}^2_{ztt}J\,dz \le \delta_{14}\|\widetilde{u}_{ttt}\|^2_{1,\widehat{\Omega}} + c\|\widetilde{u}_{tt}\|^2_{1,\widehat{\Omega}} + c(\|\widehat{u}\|^2_{3,\widehat{\Omega}} + |\widehat{q}_\sigma|^2_{3,2,\widehat{\Omega}})\|\widetilde{u}_{ztt}\|^2_{0,\widehat{\Omega}}\,,$$

from (4.155) and (4.160) we obtain

$$(4.162)\qquad \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\left(\widehat{\eta}\,\widetilde{u}^2_{ztt} + \frac{\mu+\nu}{\widehat{q}\Psi(\widehat{\eta})}\widetilde{q}^2_{\sigma ztt}\right)J\,dz + \|\widetilde{u}_{tt}\|^2_{2,\widehat{\Omega}} + \|\widetilde{q}_{\sigma tt}\|^2_{1,\widehat{\Omega}}$$
$$\le \delta_{15}(\|\widetilde{u}_{zzzt}\|^2_{0,\widehat{\Omega}} + \|\widetilde{u}_{zttt}\|^2_{0,\widehat{\Omega}} + \|\widehat{u}_{zztt}\|^2_{0,\widehat{\Omega}})$$
$$+ a_{30}(|\widehat{u}|^2_{3,0,\widehat{\Omega}} + |\widehat{q}_\sigma|^2_{2,0,\widehat{\Omega}} + |\widehat{g}|^2_{2,0,\widehat{\Omega}})$$
$$+ a_{31}X_9(\widehat{\Omega})(1+X_9^2(\widehat{\Omega}))Y_9(\widehat{\Omega})\,.$$

Going back to the variables ξ in (4.162), next summing the result and (4.153) over all neighbourhoods of the partition of unity we finally obtain (4.148) after going back to the variables x and upon assuming that δ_{15} is sufficiently small. This concludes the proof.

Finally, to obtain an estimate for the second term on the right-hand side of (4.148) we have to show

LEMMA 4.12. *For a sufficiently smooth solution v, p of* (4.1),

$$(4.163)\qquad \frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left(\varrho v^2_{ttt} + \frac{1}{p\Psi(\varrho)}p^2_{\sigma ttt}\right)dx + \|v_{ttt}\|^2_{1,\Omega_t} + \|p_{\sigma ttt}\|^2_{0,\Omega_t}$$
$$\le c\|v_{xtt}\|^2_{0,\Omega_t} + P_{23}(\|f_{ttt}\|^2_{0,\Omega_t} + |f|^2_{2,0,\Omega_t}) + P_{24}X_{10}(1+X_{10}^3)Y_{10}\,,$$

where $X_{10} = |v|^2_{3,0,\Omega_t} + |p_\sigma|^2_{3,0,\Omega_t}$, $Y_{10} = |v|^2_{4,1,\Omega_t} + |p_\sigma|^2_{3,0,\Omega_t}$.

Proof. Differentiating $(4.1)_1$ three times with respect to t, multiplying the result by v_{ttt}, integrating over Ω_t and using Lemma 5.5 one obtains

$$(4.164)\qquad \frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left(\varrho v^2_{ttt} + \frac{1}{p\Psi(\varrho)}p^2_{\sigma ttt}\right)dx + \|v_{ttt}\|^2_{1,\Omega_t}$$
$$- \int\limits_{S_t}(n_iT^{ij}(v,p_\sigma))_{,ttt}\cdot v^j_{ttt}\,ds$$
$$\le \delta_1\|v_{ttt}\|^2_{0,\Omega_t} + c(\|f_{ttt}\|^2_{0,\Omega_t} + |f|^2_{2,0,\Omega_t}) + cX_{10}(1+X_{10}^3)Y_{10}\,,$$

where by the boundary condition $(4.1)_3$ the boundary term vanishes.

By (4.2) we have

$$\|p_{\sigma ttt}\|^2_{0,\Omega_t} \le c\|v_{xtt}\|^2_{0,\Omega_t} + cX_{10}(1+X_{10}^2)Y_{10}\,. \tag{4.165}$$

Therefore, from (4.164) and (4.165) we obtain (4.163). This concludes the proof.

From the above lemmas for sufficiently small ε's we obtain

THEOREM 4.13. *For a sufficiently smooth solution v, p of* (4.1),

$$\frac{d}{dt}\varphi + \Phi \le c_1 P(X)X(1+X^3)Y + c_2F + c_3\Psi \tag{4.166}$$

where c_i, $i=1,2,3$, depend on ϱ_*, ϱ^*, T, $\|S\|_{4-1/2}$, $\int_0^T \|v\|_{3,\Omega_{t'}}\,dt'$, and

$$\begin{aligned} &X = |v|^2_{3,0,\Omega_t} + |p_\sigma|^2_{3,0,\Omega_t} + \int_0^t \|v\|^2_{3,\Omega_{t'}}\,dt'\,,\\ &Y = |v|^2_{4,1,\Omega_t} + |p_\sigma|^2_{3,0,\Omega_t} + \int_0^t \|v\|^2_{4,\Omega_{t'}}\,dt'\,,\\ &\varphi(t) = |v|^2_{3,0,\Omega_t} + |p_\sigma|^2_{3,0,\Omega_t}\,, \qquad \Psi(t) = \|v\|^2_{0,\Omega_t} + \|p_\sigma\|^2_{0,\Omega_t}\,,\\ &\Phi(t) = |v|^2_{4,1,\Omega_t} + |p_\sigma|^2_{3,0,\Omega_t}\,, \qquad F(t) = \|f_{ttt}\|^2_{0,\Omega_t} + |f|^2_{2,0,\Omega_t}\,. \end{aligned} \tag{4.167}$$

5. Korn inequality

In this section we show Korn type inequalities which are necessary to prove global existence of solutions. We follow the ideas from [26]. First we show

LEMMA 5.1. *Let $G \subset \mathbb{R}^3$ be a given bounded domain. Let $v \in L_2(G)$ be such that*

$$E_G(v) = \int_G (\partial_{x^i} v^j + \partial_{x^j} v^i)^2\,dx < \infty\,. \tag{5.1}$$

Then there exists a constant c such that

$$\|v\|^2_{1,G} \le c(E_G(v) + \|v\|^2_{0,G})\,. \tag{5.2}$$

Proof. Introduce a function u by

$$u = \sum_{i=1}^3 b_i\varphi_i(x) + v\,, \tag{5.3}$$

where

$$\varphi_i = (x-\overline{x}) \times e_i\,, \tag{5.4}$$

with $x=(x^1,x^2,x^3)$, $\overline{x}=\frac{1}{|G|}(\int_G x^1\,dx,\ \int_G x^2\,dx,\ \int_G x^3\,dx)$ and $e_i=(\delta_{i1},\delta_{i2},\delta_{i3})$, $i=1,2,3$. Define $b=(b_1,b_2,b_3)$ by

$$b=\frac{1}{2|G|}\int_G \operatorname{rot} v\,dx\,. \tag{5.5}$$

Since $\operatorname{rot}\varphi_i=-2e_i$, $i=1,2,3$, equations (5.3) and (5.5) imply

$$\int_G \operatorname{rot} u\,dx=0\,. \tag{5.6}$$

From (5.4) we have $\int_G\varphi_i\,dx=0$, $i=1,2,3$, so

$$\int_G u\,dx=\int_G v\,dx\,, \tag{5.7}$$

and also $E_G(\varphi_i)=0$, $i=1,2,3$, so

$$E_G(u)=E_G(v)\,. \tag{5.8}$$

By Theorem 1 of [26] we have

$$\partial_{x^j}w_i=\varepsilon_{ikl}\partial_{x^k}S_{jl}\,,\quad i=1,2,3,\quad w=\operatorname{rot}u,\quad S_{ij}=\partial_{x^i}u^j+\partial_{x^j}u^i\,, \tag{5.9}$$

so by (5.6) and Lemma 2.4 of [6] it follows that

$$\|\operatorname{rot}u\|_{0,G}^2\le c\sum_{i,j=1}^3\|S_{ij}\|_{0,G}^2=cE_G(u)=cE_G(v)\,. \tag{5.10}$$

Employing the identity

$$\partial_{x^j}u^i=\tfrac12(\partial_{x^j}u^i+\partial_{x^i}u^j)+\tfrac12(\partial_{x^j}u^i-\partial_{x^i}u^j)$$

and (5.10) we have

$$\|\nabla u\|_{0,G}^2\le c(E_G(u)+\|\operatorname{rot}u\|_{0,G}^2)\le cE_G(u)=cE_G(v)\,. \tag{5.11}$$

Using (5.3) we obtain

$$\|\nabla v\|_{0,G}^2\le cE_G(v)+c|b|^2\,. \tag{5.12}$$

To estimate the last term we consider the system of equations

$$\sum_{i=1}^3 b_i\int_G\varphi_i(x)\cdot\varphi_j(x)\,dx=\int_G(u-v)\varphi_j(x)\,dx\,,\qquad j=1,2,3\,, \tag{5.13}$$

which follows from (5.3). Since $\det\mathit{\Gamma}\ne0$, where $\mathit{\Gamma}=\{\mathit{\Gamma}_{ij}\}$, $\mathit{\Gamma}_{ij}=\int_G\varphi_i(x)\cdot\varphi_j(x)\,dx$, we can calculate b from (5.13), so

$$|b|^2\le c(\|u\|_{0,G}^2+\|v\|_{0,G}^2)\,. \tag{5.14}$$

Now by the Poincaré inequality and (5.7), (5.8) we obtain

$$\|u\|_{0,G}^2\le 2\Big\|u-\frac{1}{|G|}\int_G u\,dx\Big\|_{0,G}^2+2\Big\|\frac{1}{|G|}\int_G u\,dx\Big\|_{0,G}^2 \tag{5.15}$$

$$\le c\Big(\|\nabla u\|_{0,G}^2 + \Big\|\frac{1}{|G|}\int_G v\,dx\Big\|_{0,G}^2\Big)$$
$$\le c(E_G(v) + \|v\|_{0,G}^2)\,.$$

Using (5.14) and (5.15) in (5.12) we obtain (5.2). This concludes the proof.

Assuming the relations (which hold by Remark 2.4)

$$\int_{\Omega_t} \varrho v\,dx = 0\,, \tag{5.16}$$

$$\int_{\Omega_t} \varrho v\cdot\varphi_i\,dx = 0\,, \qquad i = 1,2,3\,, \tag{5.17}$$

where φ_i is described by (5.4), we have

LEMMA 5.2. *Let $\Omega_t \subset \mathbb{R}^3$ be a bounded domain. Let $v \in L_2(\Omega_t)$ satisfy* (5.16), (5.17) *and*

$$E_{\Omega_t}(v) \equiv \int_{\Omega_t} (\partial_{x^i} v^j + \partial_{x^j} v^i)^2\,dx < \infty\,. \tag{5.18}$$

Then there exists a constant c such that

$$\|v\|_{1,\Omega_t}^2 \le c\Big[E_{\Omega_t}(v) + \Big(\int_{\Omega_t} |\varrho - \varrho_e|\,|v|\,dx\Big)^2\Big]\,, \tag{5.19}$$

where ϱ_e is a constant density of the equilibrium state.

Proof. We repeat the proof of Lemma 5.1 up to (5.12) with $G = \Omega_t$. Next, to calculate b we write (5.16) and (5.17) in the form

$$\int_{\Omega_t} (\varrho - \varrho_e)v\,dx + \varrho_e \int_{\Omega_t} v\,dx = 0\,, \tag{5.20}$$

$$\int_{\Omega_t} (\varrho - \varrho_e)v\cdot\varphi_i\,dx + \varrho_e \int_{\Omega_t} v\cdot\varphi_i\,dx = 0\,, \qquad i = 1,2,3\,. \tag{5.21}$$

Calculating v from (5.3) and inserting it in the second term in (5.21) we obtain

$$\sum_{k=1}^3 b_k \int_{\Omega_t} \varphi_k\cdot\varphi_i\,dx = \int_{\Omega_t} u\cdot\varphi_i\,dx + \frac{1}{\varrho_e}\int_{\Omega_t} (\varrho - \varrho_e)v\cdot\varphi_i\,dx \tag{5.22}$$

so

$$|b| \le c\Big(\|u\|_{0,\Omega_t} + \int_{\Omega_t} |\varrho - \varrho_e|\,|v|\,dx\Big) \tag{5.23}$$

and (5.12) yields

$$\|\nabla v\|_{0,\Omega_t}^2 \le cE_{\Omega_t}(v) + c\|u\|_{0,\Omega_t}^2 + c\Big(\int_{\Omega_t} |\varrho - \varrho_e|\,|v|\,dx\Big)^2\,. \tag{5.24}$$

From (5.3) and (5.23) we have

$$\|v\|_{0,\Omega_t}^2 \le c\|u\|_{0,\Omega_t}^2 + c\Big(\int\limits_{\Omega_t} |\varrho-\varrho_e|\,|v|\,dx\Big)^2, \tag{5.25}$$

so (5.24) and (5.25) imply

$$\|v\|_{1,\Omega_t}^2 \le cE_{\Omega_t}(v) + c\|u\|_{0,\Omega_t}^2 + c\Big(\int\limits_{\Omega_t} |\varrho-\varrho_e|\,|v|\,dx\Big)^2. \tag{5.26}$$

Employing (5.7) in (5.20) we get

$$\int\limits_{\Omega_t} u\,dx = -\frac{1}{\varrho_e}\int\limits_{\Omega_t} (\varrho-\varrho_e)v\,dx \tag{5.27}$$

so

$$\begin{aligned}\|u\|_{0,\Omega_t}^2 &\le 2\Big\|u - \frac{1}{|\Omega_t|}\int\limits_{\Omega_t} u\,dx\Big\|_{0,\Omega_t}^2 + \frac{2}{|\Omega_t|}\Big|\int\limits_{\Omega_t} u\,dx\Big|^2 \\ &\le c\Big(\|\nabla u\|_{0,\Omega_t}^2 + \Big(\int\limits_{\Omega_t} |\varrho-\varrho_e|\,|v|\,dx\Big)^2\Big) \\ &\le c\Big(E_{\Omega_t}(v) + \Big(\int\limits_{\Omega_t} |\varrho-\varrho_e|\,|v|\,dx\Big)^2\Big).\end{aligned} \tag{5.28}$$

Hence, (5.26) and (5.28) imply (5.19). This concludes the proof.

Lemma 5.3. *Let $\Omega_t \subset \mathbb{R}^3$ be a bounded domain. Let* (5.16) *and* (5.17) *be satisfied. Let $v \in \Gamma_1^2(\Omega_t)$, $p_\sigma \in \Gamma_1^2(\Omega_t)$ and*

$$E_{\Omega_t}(v_t) \equiv \int\limits_{\Omega_t} (\partial_{x^i} v_t^j + \partial_{x^j} v_t^i)^2\,dx < \infty. \tag{5.29}$$

Then there exists a constant c such that

$$\|v_t\|_{1,\Omega_t}^2 \le c(E_{\Omega_t}(v_t) + Z_1^2(1+Z_1)), \tag{5.30}$$

where $Z_1 = |v|_{2,1,\Omega_t}^2 + |p_\sigma|_{2,1,\Omega_t}^2$.

Proof. We use the proofs of Lemmas 5.1 and 5.2 with v, G replaced by v_t, Ω_t, respectively. Moreover, $u = \sum_{i=1}^3 b_i\varphi_i + v_t$. Differentiating (5.20) and (5.21) with respect to time gives

$$\begin{aligned}\int\limits_{\Omega_t} v_t\,dx = -\Big[&\int\limits_{\Omega_t} (v\cdot\nabla v + v\operatorname{div} v)\,dx \\ &+ \frac{1}{\varrho_e}\int\limits_{\Omega_t} (\varrho-\varrho_e)(v_t + v\cdot\nabla v + v\operatorname{div} v)\,dx \\ &+ \frac{1}{\varrho_e}\int\limits_{\Omega_t} (\varrho_t + v\cdot\nabla\varrho)v\,dx\Big] \equiv N_1,\end{aligned} \tag{5.31}$$

$$
(5.32)\quad \int\limits_{\Omega_t} v_t\cdot\varphi_i\,dx = -\bigg[\int\limits_{\Omega_t}(v\cdot\nabla v + v\operatorname{div} v)\cdot\varphi_i\,dx + \int\limits_{\Omega_t} v\cdot\varphi_{it}\,dx + \frac{1}{\varrho_e}\int\limits_{\Omega_t}[(\varrho_t + v\cdot\nabla\varrho)v\cdot\varphi_i + (\varrho-\varrho_e)(v_t + v\cdot\nabla v)\cdot\varphi_i + (\varrho-\varrho_e)v\cdot\varphi_i\operatorname{div} v]\,dx\bigg] \equiv N_{2i}\,.
$$

Using the Hölder inequality and Sobolev imbedding theorems we have

$$
(5.33)\qquad N_1^2 + \sum_{i=1}^3 N_{2i}^2 \le cZ_1^2(1+Z_1)\,.
$$

Next following the proofs of Lemmas 5.1 and 5.2 we prove the lemma. This concludes the proof.

LEMMA 5.4. *Let $\Omega_t \subset \mathbb{R}^3$ be a bounded domain. Let* (5.16) *and* (5.17) *be satisfied. Let $v, p_\sigma \in \Gamma_0^2(\Omega_t)$, and*

$$
(5.34)\qquad E_{\Omega_t}(v_{tt}) \equiv \int\limits_{\Omega_t}(\partial_{x^i}v_{tt}^j + \partial_{x^j}v_{tt}^i)\,dx < \infty\,.
$$

Then there exists a constant c such that

$$
(5.35)\qquad \|v_{tt}\|_{1,\Omega_t}^2 \le c(E_{\Omega_t}(v_{tt}) + Z_2^2(1+Z_2^2))\,,
$$

where $Z_2 \equiv |v|_{2,0,\Omega_t}^2 + |p_\sigma|_{2,0,\Omega_t}^2$.

Proof. Let $w = v_{tt}$. Introduce the function

$$
(5.36)\qquad u = \sum_{i=1}^3 b_i\varphi_i + w\,,
$$

where the φ_i are described by (5.4). The rest of the argument is as in Lemmas 5.1 and 5.2, with (5.20) and (5.21) replaced by

$$
(5.37)\qquad \int\limits_{\Omega_t} v_{tt}\,dx = N_3\,,\qquad \int\limits_{\Omega_t} v_{tt}\cdot\varphi_i\,dx = N_{4i}\,,\qquad i=1,2,3\,,
$$

and $|N_3| + \sum_{i=1}^3 |N_{4i}| \le cZ_2(1+Z_2)$.

LEMMA 5.5. *Let $\Omega_t \subset \mathbb{R}^3$ be a bounded domain. Let* (5.16) *and* (5.17) *be satisfied. Let $v, p_\sigma \in \Gamma_0^3(\Omega_t)$ and*

$$
(5.38)\qquad E_{\Omega_t}(v_{ttt}) \equiv \int\limits_{\Omega_t}(\partial_{x^i}v_{ttt}^j + \partial_{x^j}v_{ttt}^i)^2\,dx < \infty\,.
$$

Then there exists a constant c such that

$$
(5.39)\qquad \|v_{ttt}\|_{1,\Omega_t}^2 \le c(E_{\Omega_t}(v_{ttt}) + Z_3^2(1+Z_3^2))\,,
$$

where $Z_3 = |v|_{3,0,\Omega_t}^2 + |p_\sigma|_{3,0,\Omega_t}^2$.

P r o o f. Let $w = v_{ttt}$, $G = \Omega_t$ and

$$u = \sum_{i=1}^{3} b_i \varphi_i + w\,,$$

and proceed as before with (5.20) and (5.21) replaced by

$$\int_{\Omega_t} v_{ttt}\,dx = N_5\,, \qquad \int_{\Omega_t} v_{ttt}\cdot\varphi_i = N_{6i}\,, \qquad i = 1,2,3\,, \tag{5.40}$$

and $|N_5| + |\sum_{i=1}^3 N_{6i}| \le cZ_3(1+Z_3^2)$.

R e m a r k 5.6. The expressions (5.37) and (5.40) are obtained from (5.20) and (5.21) by going to the variables ξ, differentiating with respect to time and going back to the variables x. Similar considerations have been used in (5.31) and (5.32).

In Lemmas 5.1–5.5 we have proved Korn type inequalities (5.2), (5.19), (5.30), (5.35), (5.39) for a given moment of time. Hence, in general, the constants in these inequalities depend on time. To prove the global existence (see Theorem 6.5) we have to show that the constants in (5.2), (5.19), (5.30), (5.35), (5.39) do not depend on time. Therefore, we show

LEMMA 5.7. *Assume that* $v \in H^3(\Omega_t)$, $t \in \mathbb{R}^1_+$, *and*

$$\sup_{\xi,t}\Big|\int_0^t v(x(\xi,\tau))\,d\tau\Big| \le \int_0^t |v(x,\tau)|_{\infty,\Omega_\tau}\,d\tau \le \varepsilon\,, \tag{5.41}$$

where ε *is sufficiently small. Then the constants in* (5.2), (5.19), (5.30), (5.35) *and* (5.39) *do not depend on* t.

P r o o f. For sufficiently small ε the invertible transformation (1.4) exists for all t, because

$$\Big|\int_0^t u_\xi(\xi,\tau)\,d\tau\Big|_{\infty,\Omega} \le |x_\xi|_{\infty,\Omega^t}\int_0^t |v_x(x,\tau)|_{\infty,\Omega_\tau}\,d\tau \le c\varepsilon\,,$$

and

$$|1 - |x_\xi|_{\infty,\Omega}| \le \frac{1}{|\Omega|}|\,|\Omega| - |\Omega_t|\,| \le c\frac{\varepsilon_0}{|\Omega|}$$

(see Remark 2.3). Consider first Lemma 5.1. Assume that it holds for $G = \Omega$ and $v = u$. Therefore,

$$\|u\|^2_{1,\Omega} \le c(E_\Omega(u) + \|u\|^2_{0,\Omega})\,, \tag{5.42}$$

where $v(\xi,0) = u(\xi)$.

Now we want to prove (5.42) for Ω_t and v, where $x = x(\xi,t)$ and $t \in \mathbb{R}^1_+$. We have

$$\|v\|^2_{1,\Omega_t} = \int_{\Omega_t} (|v|^2 + |v_x|^2)\,dx = \int_\Omega (|u|^2 + |u_\xi \xi_x|^2)|x_\xi|\,d\xi \tag{5.43}$$

$$\le \max(1, |\xi_x|^2_{\infty,\Omega})|x_\xi|_{\infty,\Omega}\|u\|^2_{1,\Omega} \le c_1\|u\|^2_{1,\Omega}\,,$$

where in view of (5.41) and the properties of the transformation $x = x(\xi, t)$ (see (1.4)) the constant c_1 does not depend on t.

Next we consider

$$E_\Omega(u) = \int_\Omega (u^i_{\xi^j} + u^j_{\xi^i})^2\,d\xi = \int_{\Omega_t} (v^i_{x^k}x^k_{\xi^j} + v^j_{x^k}x^k_{\xi^i})^2|\xi_x|\,dx$$
$$\le c|\xi_x|_{\infty,\Omega_t}\int_{\Omega_t}(v^i_{x^j} + v^j_{x^i})^2\,dx + c\Big|\int_0^t u_\xi\,d\tau\Big|^2_{\infty,\Omega_t}\int_{\Omega_t}|v_x|^2\,dx\,, \tag{5.44}$$

where c is an absolute constant and $|\xi_x|_{\infty,\Omega_t} \le c_2$, which does not depend on t for sufficiently small ε.

Finally,

$$\|u\|^2_{0,\Omega} \le \int_{\Omega_t} v^2|\xi_x|\,dx \le c_3\|v\|^2_{0,\Omega_t}\,, \tag{5.45}$$

where c_3 is a constant also independent of t.

Hence, from (5.41)–(5.45) for sufficiently small ε we obtain

$$\|v\|^2_{1,\Omega_t} \le c_4(E_{\Omega_t}(v) + \|v\|^2_{0,\Omega_t})\,, \tag{5.46}$$

where c_4 does not depend on t.

Now we consider (5.19). Assume that the following inequality holds (it can be proved in the same way as (5.19)):

$$\|u\|^2_{1,\Omega} \le c\Big(E_\Omega(u) + \Big(\int_\Omega |\eta - \varrho_e|\,|u|\,d\xi\Big)^2\Big)\,, \tag{5.47}$$

where c does not depend on t and $\eta(\xi) = \varrho(x(\xi,0),0)$. In view of the previous considerations it is sufficient to examine the last integral. We have

$$\int_\Omega |\eta - \varrho_e|\,|u|\,d\xi = \int_{\Omega_t} |\varrho - \varrho_e|\,|v|\,|\xi_x|\,dx \le |\xi_x|_{\infty,\Omega_t}\int_{\Omega_t}|\varrho - \varrho_e|\,|v|\,dx$$
$$\le c_5\int_{\Omega_t}|\varrho - \varrho_e|\,|v|\,dx\,.$$

Therefore, (5.19) holds with a constant independent of t.

For (5.30), we repeat the proofs of Lemmas 5.1 and 5.2 in the case $G = \Omega_t$ and $v = v_t$, where we control all constants by employing the assumption (5.41). Introduce the function

$$u = \sum_{i=1}^3 b_i\varphi_i(x) + v_t\,, \tag{5.48}$$

where

$$\varphi_i = (x - \overline{x}) \times e_i\,, \tag{5.49}$$

with $e_i,\ i=1,2,3$, defined in (5.4), and

$$\overline{x}=\frac{1}{|\Omega_t|}\Big(\int\limits_{\Omega_t}x^1\,dx,\ \int\limits_{\Omega_t}x^2\,dx,\ \int\limits_{\Omega_t}x^3\,dx\Big).$$

Define $b=(b_1,b_2,b_3)$ by

$$b=\frac{1}{2|\Omega_t|}\int\limits_{\Omega_t}\operatorname{rot}v_t\,dx\,. \tag{5.50}$$

Since $\operatorname{rot}\varphi_i=-2e_i,\ i=1,2,3$, equations (5.49) and (5.50) imply

$$\int\limits_{\Omega_t}\operatorname{rot}u\,dx=0\,. \tag{5.51}$$

By (5.49) we have $\int_{\Omega_t}\varphi_i\,dx=0$, so

$$\int\limits_{\Omega_t}u\,dx=\int\limits_{\Omega_t}v_t\,dx \tag{5.52}$$

and also $E_{\Omega_t}(\varphi_i)=0,\ i=1,2,3$, so

$$E_{\Omega_t}(u)=E_{\Omega_t}(v_t)\,. \tag{5.53}$$

Since (5.9) holds, by (5.51) and Lemma 2.4 of [6] we have

$$\|\operatorname{rot}u\|^2_{0,\Omega_t}\le c\sum_{i,j=1}^{2}\|S_{ij}\|^2_{0,\Omega_t}=cE_{\Omega_t}(u)=cE_{\Omega_t}(v_t)\,, \tag{5.54}$$

where c does not depend on t because the volume $|\Omega_t|$ and in view of (5.41) also the shape of Ω_t change very little for all t.

Using (5.54) we have (see also the decomposition before (5.11))

$$\|\nabla u\|^2_{0,\Omega_t}\le\tfrac{1}{2}(E_{\Omega_t}(u)+\|\operatorname{rot}u\|^2_{0,\Omega_t})\le cE_{\Omega_t}(u)=cE_{\Omega_t}(v_t)\,, \tag{5.55}$$

where c also does not depend on t.

From (5.48) and (5.55) we obtain

$$\|\nabla v_t\|^2_{0,\Omega_t}\le c(E_{\Omega_t}(v_t)+|b|^2)\,, \tag{5.56}$$

where c is independent of t. The coordinates of the vector b are calculated from

$$\sum_{k=1}^{3}b_k\int\limits_{\Omega_t}\varphi_k\cdot\varphi_i\,dx=\int\limits_{\Omega_t}u\cdot\varphi_i\,dx-N_{2i}(\varphi)\,, \tag{5.57}$$

where $N_{2i}(\varphi)$ is determined by (5.32). Hence

$$b_k=\sum_{i=1}^{3}G^{-1}_{ki}\Big[\int\limits_{\Omega_t}u\cdot\varphi_i\,dx-N_{2i}(\varphi)\Big]\,, \tag{5.58}$$

and $G_{ki} = \int_{\Omega_t} \varphi_k \cdot \varphi_i \, dx$. We have $\det G \neq 0$ and $|\varphi_i| \le cR$, where $R = \max\{R_t\}$ and $|\Omega_t| = 4\pi R_t^3//3$. By (5.41) the maximum is attained. Therefore

$$|b|^2 \le c\Big(\|u\|^2_{0,\Omega_t} + \sum_{i=1}^3 N_{2i}^2\Big)\,, \tag{5.59}$$

with c independent of t. Hence

$$\|\nabla v_t\|^2_{0,\Omega_t} \le c\Big(E_{\Omega_t}(v_t) + \|u\|^2_{0,\Omega_t} + \sum_{i=1}^3 N_{2i}^2\Big)\,. \tag{5.60}$$

From (5.48) and (5.59) we obtain

$$\|v_t\|^2_{0,\Omega_t} \le c\Big(\|u\|^2_{0,\Omega_t} + \sum_{i=1}^3 N_{2i}^2\Big)\,, \tag{5.61}$$

with c independent of t. Moreover, (5.52) and (5.31) imply

$$\int_{\Omega_t} u \, dx = N_1\,, \tag{5.62}$$

therefore (see (5.28))

$$\|u\|^2_{0,\Omega_t} \le c(E_{\Omega_t}(v_t) + N_1^2)\,, \tag{5.63}$$

where the constant c does not depend on t, because it depends on the constant from the Poincaré inequality and $|\Omega_t|$, but under our assumptions these quantities can be bounded by constants independent of t. From (5.60), (5.61) and (5.63) we obtain the conclusion in the case of the inequality (5.30). The proofs for (5.35) and (5.39) are similar. This concludes the proof.

6. Global existence

To prove global existence we assume that the external force vanishes, so

$$f = 0\,. \tag{6.1}$$

Let $\varphi(t)$ and $\Phi(t)$ be defined by (4.167). Then we introduce the spaces

$$\mathfrak{N}(t) = \{(v, p_\sigma) : \varphi(t) < \infty\}\,, \qquad \mathfrak{M}(t) = \Big\{(v, p_\sigma) : \varphi(t) + \int_0^t \Phi(\tau)\, d\tau < \infty\Big\}\,.$$

LEMMA 6.1. *Let the initial data v_0, $p_{\sigma 0}$, S of the problem* (1.1) *be such that* $(v(0), p_\sigma(0)) \in \mathfrak{N}(0)$ *and $S \in W_2^{4-1/2}$. Let*

$$\int_\Omega \varrho_0 v_0 \, dx = 0\,, \qquad \int_\Omega \varrho_0 x \, dx = 0\,.$$

Let the initial data v_0, $p_{\sigma 0}$ satisfy

$$\varphi(0) \le \varepsilon_1 \,, \tag{6.2}$$

where ε_1 is sufficiently small. Then there exists a local solution v, p of (1.1) *such that $(v(t), p_\sigma(t)) \in \mathfrak{M}(t)$, $t \le T$, where T is the time of local existence and*

$$\varphi(t) + \int_0^t \Phi(\tau)\, d\tau \le c_1 \varepsilon_1 \,. \tag{6.3}$$

Proof. Take $(v(0), p_\sigma(0)) \in \mathfrak{N}(0)$, $S \in W_2^{4-1/2}$. Then $(v_0, p_{\sigma 0}) \in H^3(\Omega)$, so by Theorem 3.6 and Remark 3.8 there exists a solution of (1.1) such that

$$u \in W_2^{4,2}(\Omega^T) \,, \qquad q_\sigma \in W_2^{3,3/2}(\Omega^T) \cap C(0,T; \Gamma_{0,2}^{3,3/2}(\Omega)) \,,$$

and

$$\begin{aligned} \|u\|^2_{4,\Omega^T} + \|q_\sigma\|^2_{3,\Omega^T} + {}& [\![q_\sigma]\!]^2_{3,2,0,\infty,\Omega^T} \\ &\le c(\|v_0\|^2_{3,\Omega} + \|p_\sigma\|^2_{3,\Omega}) \le c\varphi(0) \le c\varepsilon_1 \,, \end{aligned} \tag{6.4}$$

where $u = v(x(\xi,t),t)$, $q_\sigma = p_\sigma(x(\xi,t),t)$.

Writing (4.2) in Lagrangian coordinates we have

$$q_{\sigma t} + q\Psi(\eta) \operatorname{div}_u u = 0$$

so

$$q_\sigma = q_\sigma(0) - \int_0^t q\Psi(\eta) \operatorname{div}_u u \, d\tau \,. \tag{6.5}$$

Using the estimate (6.4) for the local solution we obtain the following estimates for the solution (3.28) of the continuity equation:

$$\begin{aligned} \sup_t(\|\eta_{tt}\|^2_{0,\Omega} + \|\eta_t\|^2_{2,\Omega} + \|\eta\|^2_{3,\Omega}) + \|\eta_{tt}\|^2_{1,2,2,\Omega^T} &+ \|\eta_t\|^2_{3,2,2,\Omega^T} \\ &\le \varphi_1(T, \varphi(0))\varphi(0) \le c\varepsilon_1 \,, \end{aligned} \tag{6.6}$$

where we have used the imbedding

$$\begin{aligned} N_1 \equiv {}& \sup_t(\|u\|^2_{3,\Omega} + \|u_t\|^2_{1,\Omega}) \\ &\le c(\|u\|^2_{4,\Omega^T} + \|u(0)\|_{3,\Omega} + |u(0)|^2_{1,0,\Omega}) \,. \end{aligned} \tag{6.7}$$

Similar considerations can be applied to q_σ, so we have (the inequality (6.6) must be used)

$$\begin{aligned} N_2 \equiv \sup_t(\|q_{\sigma tt}\|^2_{0,\Omega} + \|q_{\sigma t}\|^2_{2,\Omega} + \|q_\sigma\|^2_{3,\Omega}) + \|q_{\sigma tt}\|^2_{1,2,2,\Omega^T} + \|q_{\sigma t}\|^2_{3,2,2,\Omega^T} \\ \le c\varphi_2(T, \varphi(0))\varphi(0) \le c\varepsilon_1 \,. \end{aligned} \tag{6.8}$$

In the above considerations we have also used the fact that

$$\int_0^t |u_\xi|_{\infty,\Omega} \, d\tau \le cT^{1/2} \|u\|_{4,\Omega^T} \le cT^{1/2} \varphi(0) \,.$$

Repeating the proof of Lemma 4.10 we obtain

$$\frac{d}{dt}(\|v_{xxt}\|^2_{0,\Omega_t}+\|p_{\sigma xxt}\|^2_{0,\Omega_t})+\|v_{xxt}\|^2_{1,\Omega_t}+\|p_{\sigma xxt}\|^2_{0,\Omega_t} \le (\varepsilon_1'+cN)(\|v_{xttt}\|^2_{0,\Omega_t}+\|v_{xxxt}\|^2_{0,\Omega_t})+cM(1+N)^2\,, \tag{6.9}$$

where $N=N_1+N_2$ and M is such that

$$\int\limits_0^T M\,d\tau \le c\varphi(0)$$

in view of the estimates for the local solution.

Similarly, Lemma 4.11 yields

$$\frac{d}{dt}(\|v_{xtt}\|^2_{0,\Omega_t}+\|p_{\sigma xtt}\|^2_{0,\Omega_t})+\|v_{xtt}\|^2_{1,\Omega_t}+\|p_{\sigma xtt}\|^2_{0,\Omega_t} \le (\varepsilon_2'+cN)(\|v_{xttt}\|^2_{0,\Omega_t}+\|v_{xxtt}\|^2_{0,\Omega_t}+\|v_{xxxt}\|^2_{0,\Omega_t})+cM(1+N)^2\,. \tag{6.10}$$

Finally, Lemma 4.12 implies

$$\frac{d}{dt}\|v_{ttt}\|^2_{0,\Omega_t}+\|v_{ttt}\|^2_{1,\Omega_t}\le c(N+M)\|v_{ttt}\|^2_{0,\Omega_t} + cN(\|v_{xtt}\|^2_{0,\Omega_t}+\|v_{xttt}\|^2_{0,\Omega_t}+\|p_{\sigma ttt}\|^2_{0,\Omega_t})+cM(1+N)^2\,, \tag{6.11}$$

where by virtue of the continuity equation (4.2) we have

$$\|p_{\sigma ttt}\|^2_{0,\Omega_t}\le c\|v_{xtt}\|^2_{0,\Omega_t}+cM(1+N)^2\,. \tag{6.12}$$

From (6.9)–(6.12) for sufficiently small $\varepsilon_1',\varepsilon_2'$, N and $\int_0^T M\,d\tau$ we deduce that v, $p_\sigma\in\mathfrak{M}(T)$. Of course to prove the last statement the standard technique of mollifiers or differences should be used. This concludes the proof.

LEMMA 6.2. *Assume that there exists a local solution to problem* (1.1) *which belongs to* $\mathfrak{M}(T)$. *Let the assumptions of Lemma* 2.2 *be satisfied. Then there exists* $\delta=\delta(\delta',\varepsilon)\in(0,1)$ *such that*

$$\|p_\sigma\|^2_{0,\Omega_t}\le c_2\delta\,, \tag{6.13}$$

where $\delta'\in(0,1)$ *and* $\delta=\delta'+c(\delta')\varepsilon_0(\varepsilon)$, $c(\delta')$ *is a decreasing function of* δ' *and* c_2 *depends on* $c_1\varepsilon_1$ (see (6.3)).

P r o o f. Let

$$\overline{p}_{\Omega_t}=\frac{1}{|\Omega_t|}\int\limits_{\Omega_t}p\,dx \quad\text{and}\quad p_{\Omega_t}=p-\overline{p}_{\Omega_t}\,.$$

Then

$$\|p_\sigma\|_{0,\Omega_t}\le\|p_{\Omega_t}\|_{0,\Omega_t}+\|\overline{p}_{\Omega_t}-p_0\|_{0,\Omega_t}\,. \tag{6.14}$$

Introduce a function ϑ as a solution of the problem

$$\begin{aligned}\operatorname{div}\vartheta&=p_{\Omega_t}\quad&&\text{in }\Omega_t\,,\\ \vartheta&=0&&\text{on }S_t\,.\end{aligned}\tag{6.15}$$

In view of Lemma 2.2 of [6] the solution exists, with $\vartheta \in \mathring{W}^1_2(\Omega_t) = \{u \in W^1_2(\Omega_t) : u|_{S_t} = 0\}$ and

$$\|\vartheta\|_{1,\Omega_t} \le c\|p_{\Omega_t}\|_{0,\Omega_t} . \tag{6.16}$$

Multiplying $(1.1)_1$ in the form $\varrho v_t + \varrho v\cdot\nabla v + \nabla p_{\Omega_t} - \operatorname{div}\mathbb{D}(v) = 0$ by ϑ, integrating the result over Ω_t and performing integration by parts we have

$$\int\limits_{\Omega_t} p_{\Omega_t} \operatorname{div}\vartheta\, dx = \int\limits_{\Omega_t} \mathbb{D}(v)\cdot\nabla\vartheta\, dx + \int\limits_{\Omega_t} \varrho(v_t + v\cdot\nabla v)\cdot\vartheta\, dx .$$

Taking into account (6.16) and the estimates

$$\begin{aligned}
|\varrho|_{\infty,\Omega_t} &\le |\varrho_0|_{\infty,\Omega} \exp\Big|\int\limits_0^t \operatorname{div}_u u\, d\tau\Big|_{\infty,\Omega} \le |\varrho_0|_{\infty,\Omega} f\Big(\int\limits_0^t |u_\xi|_{\infty,\Omega}\, d\tau\Big)\\
&\le |\varrho_0|_{\infty,\Omega} f\Big(t^{1/2}\Big(\int\limits_0^t \varphi(\tau)\, d\tau\Big)^{1/2}\Big),\\
|v|_{\infty,\Omega_t} &\le c\|v\|_{2,\Omega_t} \le c\varphi^{1/2},
\end{aligned}$$

we obtain

$$\|p_{\Omega_t}\|^2_{0,\Omega_t} \le c(\|v_x\|^2_{0,\Omega_t} + \|v_t\|^2_{0,\Omega_t}) . \tag{6.17}$$

Finally, by the interpolation inequalities we have

$$\|p_{\Omega_t}\|^2_{0,\Omega_t} \le \varepsilon'_1\|u\|^2_{4,\Omega^T} + c(\varepsilon'_1)\sup_t \|v\|^2_{0,\Omega_t} , \tag{6.18}$$

where $t \le T$ and $\varepsilon'_1 \in (0,1)$.

To estimate the second term on the right-hand side of (6.14) we use

$$\int\limits_{\Omega_t} |\overline{p}_{\Omega_t} - p_0|^2\, dx = |\Omega_t|\, |\overline{p}_{\Omega_t} - p_0|^2 , \tag{6.19}$$

so the boundary conditions $(1.1)_4$ imply

$$\begin{aligned}
|\overline{p}_{\Omega_t} - p_0|^2 &\le \frac{c}{|S_t|}(\|p_{\Omega_t}\|^2_{0,S_t} + \|v_x\|^2_{0,S_t})\\
&\le \varepsilon'_2(\|p_x\|^2_{0,\Omega_t} + \|v_{xx}\|^2_{0,\Omega_t}) + c(\varepsilon'_2)(\|p_{\Omega_t}\|^2_{0,\Omega_t} + \|v\|^2_{0,\Omega_t}) .
\end{aligned} \tag{6.20}$$

Now from (6.14) and (6.18)–(6.20) it follows that

$$\begin{aligned}
\sup_t \|p_\sigma\|^2_{0,\Omega_t} &\le \varepsilon'_3(\|v\|^2_{3,\Omega^T} + \sup_t(\|p_{\sigma x}\|^2_{0,\Omega_t} + \|v_{xx}\|^2_{0,\Omega_t}))\\
&\quad + c(\varepsilon'_3)\sup_t \|v\|^2_{0,\Omega_t}\\
&\le \varepsilon'_3\Big(\int\limits_0^T \varphi(\tau)\, d\tau + \sup_t \varphi(t)\Big) + c(\varepsilon'_3)\sup_t \|v\|^2_{0,\Omega_t} .
\end{aligned} \tag{6.21}$$

Finally, from the existence of local solutions it follows that the minimum (ϱ_*) and maximum (ϱ^*) of the density ϱ are attained. Hence (2.24) yields

$$\|v\|^2_{0,\Omega_t} \le (1/\varrho_*) \int\limits_{\Omega_t} \varrho v^2\,dx \le 2\kappa_0\varepsilon_0/\varrho_* \,. \tag{6.22}$$

Now from the assumption that the local solution is in $\mathfrak{M}$ and for sufficiently small ε_3' we get (6.13). This concludes the proof.

Lemma 6.3. *Assume that there exists a local solution of* (1.1) *in* $\mathfrak{M}(t)$, $0 \le t \le T$. *Assume that the initial data are in* $\mathfrak{N}(0)$ *and*

$$\varphi(0) \le \gamma\,, \qquad \gamma \in (0,1/2]\,, \tag{6.23}$$

where γ *is sufficiently small. Assume also that* $\psi(t)$, $t \in [0,T]$, *is sufficiently small* (see Remark 2.3 and Lemma 6.2). *Then the solution at* $t \in (0,T]$ *belongs to* $\mathfrak{N}(t)$ *and*

$$\varphi(t) \le \gamma\,. \tag{6.24}$$

Proof. First we find a differential inequality which enables us to prove (6.24). From the notation in (4.167), differential inequality (4.166) and (6.3) with sufficiently small ε_1 it follows that

$$\begin{aligned} X(t) &\le \varphi(t) + \int\limits_0^t \Phi(\tau)\,d\tau \le cT \sup_{t'\le t} \psi(t') + \varphi(0)\,, \\ Y(t) &\le \Phi(t) + \int\limits_0^t \Phi(\tau)\,d\tau \le \Phi(t) + cT \sup_{t'\le t} \psi(t') + \varphi(0)\,. \end{aligned} \tag{6.25}$$

Using (6.25) in (4.166) gives

$$\begin{aligned} &\frac{d}{dt}\varphi + \Phi \\ &\le c_1'\Big(\varphi + \int\limits_0^t \Phi(\tau)\,d\tau\Big)\Big[1 + \Big(\varphi + \int\limits_0^t \Phi(\tau)\,d\tau\Big)^3\Big]\Big(\Phi + \int\limits_0^t \Phi(\tau)\,d\tau\Big) + c_2'\psi(t) \\ &\le c_1'\varphi(1+\varphi^3)\Phi + c_1'\varphi\Big[\Big(\varphi + \int\limits_0^t \Phi(\tau)\,d\tau\Big)^3 - \varphi^3\Big]\Phi \\ &\quad + c_1'\Big[1 + \Big(\varphi + \int\limits_0^t \Phi(\tau)\,d\tau\Big)^3\Big] \int\limits_0^t \Phi(\tau)\,d\tau \Big(\Phi + \int\limits_0^t \Phi(\tau)\,d\tau\Big) + c_2'\psi(t)\,. \end{aligned} \tag{6.26}$$

Using the fact that

$$\Big(\int\limits_0^t \Phi(\tau)\,d\tau\Big)^2 \le c_3'((\sup_t \psi(t))^2 + \varphi(0)^2)$$

and that the right-hand sides of (6.25) are so small that

$$c_1'\varphi\Big[\Big(\varphi+\int\limits_0^t\Phi(\tau)\,d\tau\Big)^3-\varphi^3\Big]\Phi+c_1'\Big[1+\Big(\varphi+\int\limits_0^t\Phi(\tau)\,d\tau\Big)^3\Big]\Phi\int\limits_0^t\Phi(\tau)\,d\tau\le\Phi/2\,,$$

from (6.26) we obtain

$$\frac{d}{dt}\varphi+\frac{1}{2}\Phi\le c_1[\varphi(1+\varphi^3)\Phi+\varphi^2(0)+\sup_{\tau\le t}\psi(\tau)]\,. \tag{6.27}$$

We have $\Phi\ge c_2\varphi$ and $c_1\ge 1$. Let $\varphi(0)\le\gamma/(2c_1)$, $\gamma\in(0,1/2]$. Assume that $t_*=\inf\{t\in[0,T]:\varphi(t)>\gamma/(2c_1)\}$. Let $\psi\le\varepsilon_0$. Consider (6.27) in the interval $[0,t_*]$. From the definition of t_* we have $\varphi(t_*)=\gamma/(2c_1)$. Then for $t\le t_*$ we have $\varphi^2(0)+\psi\le\gamma^2/(4c_1^2)+\varepsilon_0$. Assume that γ and ε_0 are so small that $\gamma^2/(4c_1)+c_1\varepsilon_0<(c_2/(16c_1))\gamma$. Then from (6.27) we obtain

$$\varphi_t(t_*)\le-\Phi[1/2-c_1(\gamma/(2c_1)+\gamma^3/(8c_1^3))]+(c_2/(16c_1))\gamma$$

so since $\Phi\ge c_2\varphi$ and γ is sufficiently small we have

$$\varphi_t(t_*)\le-\frac{c_2\gamma}{2c_1}[1/2-(\gamma/2+\gamma^3/(8c_1^2))]+\frac{c_2}{16c_1}\gamma$$

and hence because $c_1\ge 1$ and $\gamma\le 1/2$ we get

$$\varphi_t(t_*)\le-\frac{c_2\gamma}{2c_1}[1/8+7/64]+\frac{c_2\gamma}{16c_1}<0\,.$$

Hence $\varphi_t(t_*)<0$, a contradiction. Therefore, (6.24) holds. This concludes the proof.

Lemma 6.3 suggests that the solution can be continued to the interval $[T,2T]$, but to do this we need the following facts:

(6.28) (a) The existence of the transformation $x=x(\xi,t)$ and its inverse for $t\in[T,2T]$.

(b) The validity of the Korn inequality with the same constant for the whole interval $[0,2T]$.

(c) The variations of the shape of Ω_t for $t\in[0,2T]$ are so small that the constants in the imbedding theorem (1.9) can be chosen independently of t.

Generally to prove global existence we need these facts for all t. Lemma 2.2 implies that the volume of Ω_t does not change much but we have not shown yet any restriction on the variations of its shape. In the case of surface tension such a restriction follows from the global conservation laws (see [35]).

It is sufficient to show (c), because then (a) and (b) follow.

Lemma 6.4. *Assume that there exists a local solution of* (1.1) *in* $\mathfrak{M}(t)$, $0\le t\le T$, *with initial data in* $\mathfrak{N}(0)$ *sufficiently small* (see (6.2)). *Then there exists a constant* $\mu_0>0$ *such that*

$$\varphi(t) \le e^{-\mu_0 t}\varphi(0)\,, \qquad t \le T\,, \tag{6.29}$$

and T is the time of local existence.

Proof. Multiplying $(1.1)_1$ by v, integrating over Ω_t and using the continuity equation $(1.1)_2$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t} \varrho v^2\,dx + \frac{\mu}{2}E_{\Omega_t}(v) + (\nu-\mu)\|\operatorname{div} v\|^2_{0,\Omega_t} - \int\limits_{\Omega_t} p_\sigma \operatorname{div} v\,dx = 0\,. \tag{6.30}$$

In the case of the barotropic fluid $\Psi(\varrho) = p_\varrho \varrho/p = \kappa$, so (4.2) takes the form

$$\frac{1}{\kappa p}(p_{\sigma t} + v\cdot\nabla p_\sigma) = -\operatorname{div} v\,. \tag{6.31}$$

Multiplying (6.31) by p_σ, integrating over Ω_t and using (4.5) with $F = 1/(\kappa A \varrho^\kappa)$ one obtains

$$-\int\limits_{\Omega_t} p_\sigma \operatorname{div} v\,dx = \frac{d}{dt}\int\limits_{\Omega_t}\frac{1}{\kappa A\varrho^\kappa}\frac{p_\sigma^2}{2}\,dx - \frac{\kappa+1}{2\kappa A}\int\limits_{\Omega_t}\frac{1}{\varrho^\kappa}\operatorname{div} v\, p_\sigma^2\,dx\,. \tag{6.32}$$

Using (6.32) in (6.30) and applying the Hölder and Young inequalities in the last term of (6.32) one gets

$$\begin{aligned}\frac{d}{dt}\int\limits_{\Omega_t}\left(\frac{1}{2}\varrho v^2 + \frac{1}{2\kappa p}p_\sigma^2\right)dx + \frac{\mu}{2}E_{\Omega_t}(v) + \frac{\nu-\mu}{2}\|\operatorname{div} v\|^2_{0,\Omega_t}\\ \le \frac{(\kappa+1)^2}{8\kappa^2(\nu-\mu)A^2\varrho_*^{2\kappa}}\|p_\sigma\|^4_{1,\Omega_t} \le c_1'\varphi^2(t)\,.\end{aligned} \tag{6.33}$$

Differentiating $(1.1)_1$ with respect to t, multiplying by v_t, integrating over Ω_t and using the continuity equation $(1.1)_1$ implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\varrho v_t^2\,dx + \frac{\mu}{2}E_{\Omega_t}(v_t) + (\nu-\mu)\|\operatorname{div} v_t\|^2_{0,\Omega_t}\\ \le \int\limits_{\Omega_t} p_{\sigma t}\operatorname{div} v_t\,dx + \varepsilon_1'\|v_t\|^2_{1,\Omega_t} + c_2'\varphi^2(t)(1+\varphi(t))\,.\end{aligned} \tag{6.34}$$

Differentiating (6.31) with respect to t, multiplying the result by $p_{\sigma t}$ and integrating over Ω_t yields

$$\begin{aligned}\int\limits_{\Omega_t}\operatorname{div} v_t\, p_{\sigma t}\,dx \le -\int\limits_{\Omega_t}\frac{1}{\kappa p}(\partial_t + v\cdot\nabla)\frac{p_{\sigma t}^2}{2}\,dx + \varepsilon_2'\|p_{\sigma t}\|^2_{0,\Omega_t}\\ + c_3'\varphi^2(t)(1+\varphi(t))\,.\end{aligned} \tag{6.35}$$

Using (4.5) with $F = 1/(p\kappa)$ and with p_σ replaced by $p_{\sigma t}$ in (6.35) implies

$$\begin{aligned}\int\limits_{\Omega_t}\operatorname{div} v_t\, p_{\sigma t}\,dx \le -\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\frac{1}{\kappa p}p_{\sigma t}^2\,dx + \varepsilon_3'\|p_{\sigma t}\|^2_{0,\Omega_t}\\ + c_4'\varphi^2(t)(1+\varphi(t))\,.\end{aligned} \tag{6.36}$$

Lemma 5.3 gives

$$\|v_t\|^2_{1,\Omega_t} \le c_5'(E_{\Omega_t}(v_t) + \varphi^2(t)(1+\varphi(t)))\,. \tag{6.37}$$

From (6.34), (6.36) and (6.37) for ε_1' sufficiently small we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left(\varrho v_t^2 + \frac{1}{\kappa p}p_{\sigma t}^2\right)dx + \frac{\mu}{4}\|v_t\|^2_{1,\Omega_t} + (\nu-\mu)\|\operatorname{div} v_t\|^2_{0,\Omega_t}&\\ \le \varepsilon_2'\|p_{\sigma t}\|^2_{0,\Omega_t} + c_6'\varphi^2(t)(1+\varphi(t))\,.&\end{aligned} \tag{6.38}$$

The continuity equation (4.2) yields

$$\|p_{\sigma t}\|^2_{0,\Omega_t} \le c_7'(\|\operatorname{div} v\|^2_{0,\Omega_t} + \varphi^2(t))\,. \tag{6.39}$$

Now from (6.33), (6.38), (6.39) and Lemma 5.2 for sufficiently small ε_2' we obtain

$$\begin{aligned}\frac{d}{dt}\int\limits_{\Omega_t}\left[\varrho(v^2+v_t^2) + \frac{1}{\kappa p}(p_\sigma^2 + p_{\sigma t}^2)\right]dx + \frac{\mu}{2}(\|v\|^2_{1,\Omega_t} + \|v_t\|^2_{1,\Omega_t})&\\ + \frac{\nu-\mu}{2}(\|\operatorname{div} v\|^2_{0,\Omega_t} + \|\operatorname{div} v_t\|^2_{0,\Omega_t}) \le c_8'\varphi^2(t)(1+\varphi(t))\,.&\end{aligned} \tag{6.40}$$

Repeating the proof of inequality (4.166) we see that it can be written in the form

$$\frac{d}{dt}\varphi + \Phi \le c_9' P(X)\Big(\varphi + \int\limits_0^t \|v\|^2_{4,\Omega_\tau}\,d\tau\Big)(1+X^3)\Phi + c_{10}'\psi\,, \tag{6.41}$$

because the factor $\int_0^t \|v\|^2_{4,\Omega_\tau}\,d\tau$ appears only as a coefficient of the derivatives of v and p_σ which determine Φ. From the assumption that the data are sufficiently small we deduce that $\varphi + \int_0^t \|v\|^2_{4,\Omega_\tau}\,d\tau$ is also small (see (6.3) and the proof of Lemma 6.1). Therefore, for sufficiently small data, from (6.41) we get

$$\frac{d}{dt}\varphi + \Phi \le c_{11}'(\|v\|^2_{0,\Omega_t} + \|p_\sigma\|^2_{0,\Omega_t})\,. \tag{6.42}$$

Similarly to the proof of Lemma 6.2 we obtain

$$\|p_\sigma\|^2_{0,\Omega_t} \le \varepsilon_3'(\|p_{\sigma x}\|^2_{0,\Omega_t} + \|v_{xx}\|^2_{0,\Omega_t}) + c(\varepsilon_3')(\|v\|^2_{0,\Omega_t} + \|v_t\|^2_{0,\Omega_t})\,. \tag{6.43}$$

From (6.42) and (6.43) we have

$$\frac{d}{dt}\varphi + \Phi \le c_{12}'(\|v\|^2_{0,\Omega_t} + \|v_t\|^2_{0,\Omega_t})\,. \tag{6.44}$$

Multiplying (6.40) by a sufficiently large constant c_{13}', adding the result to (6.44) and using the fact that $\varphi(0)$ is sufficiently small we obtain

$$\frac{d}{dt}\widetilde{\varphi} + \widetilde{\Phi} \le 0\,, \tag{6.45}$$

where

$$\widetilde{\varphi} = \varphi + c_{13}'\int\limits_{\Omega_t}\left[\varrho(v^2+v_t^2) + \frac{1}{\kappa p}(p_\sigma^2 + p_{\sigma t}^2)\right]dx\,,$$

$$\widetilde{\Phi} = \Phi + c'_{13}\,\frac{\mu}{2}(\|v\|^2_{1,\Omega_t} + \|v_t\|^2_{1,\Omega_t}) + \frac{\nu-\mu}{2}(\|\operatorname{div} v\|^2_{0,\Omega_t} + \|\operatorname{div} v_t\|^2_{0,\Omega_t})\,.$$

There exist constants c', c'' such that $c'\varphi \le \widetilde{\varphi} \le c''\varphi$, $c'\Phi \le \widetilde{\Phi} \le c''\Phi$. Moreover, $\widetilde{\varphi} \le \widetilde{c}\widetilde{\Phi}$. Hence we obtain the inequality

$$\frac{d}{dt}\widetilde{\varphi} + c'_{14}\widetilde{\varphi} \le 0\,, \tag{6.46}$$

which implies (6.29). This concludes the proof.

Finally, we prove the main result of this paper.

Theorem 6.5. *Assume that* $f = 0$, $(v(0), p_\sigma(0)) \in \mathfrak{N}(0)$ *and*

$$\varphi(0) \le \delta_1\,, \tag{6.47}$$

where $\delta_1 \in (0,1)$ *is sufficiently small and* $p_\sigma(0) = p(0) - p_0 = A\varrho_0^\kappa - p_0$. *Assume also that the initial data are chosen in such a way that*

$$0 < A_1 \equiv \int_\Omega (A\varrho_0^\kappa - p_0)\,dx + \frac{\kappa-1}{2}\int_\Omega \varrho_0 v_0^2\,dx + \kappa|\Omega|^\kappa(p_0 - A\overline{\varrho}_0^\kappa) \le \delta_2\,, \tag{6.48}$$

where $\delta_2 \in (0,1)$ *is sufficiently small*, *and* $\overline{\varrho}_0 = (1/|\Omega|)\int_\Omega \varrho_0\,dx$. *Assume moreover that there exist positive numbers* ψ_*, $|\Omega_*|$ *such that*

$$0 < A_2 \equiv \tfrac{1}{2}\int_\Omega \varrho_0 v_0^2\,dx + \psi - \psi_* + p_0(|\Omega| - |\Omega_*|) \le \delta_2\,, \tag{6.49}$$

where $\psi = \frac{1}{\kappa-1}\int_\Omega A\varrho_0^\kappa\,dx$. *Assume finally that* $S \in W_2^{4-1/2}$ *and*

$$\int_\Omega \varrho_0 v_0\cdot\eta\,dx = 0\,,\qquad \int_\Omega \varrho_0 x\,dx = 0\,,\qquad \int_\Omega \varrho_0\,dx = M\,, \tag{6.50}$$

where $\eta = a + b\times x$, a, b *are arbitrary constant vectors. Then there exists a global solution of* (1.1) *such that* $(v(t), p_\sigma(t)) \in \mathfrak{M}(t)$, $t \in \mathbb{R}_+$, *and*

$$\begin{aligned} 0 \le \tfrac{1}{2}|\Omega_t|^{\kappa-1}\int_\Omega \varrho_0 v_0^2\,dx + p_0|\Omega_t|^{\kappa-1}(|\Omega| - |\Omega_t|)& \\ + \frac{A}{\kappa-1}\Big[|\Omega_t|^{\kappa-1}\int_\Omega \varrho_0^\kappa\,dx - \Big(\int_{\Omega_t}\varrho\,dx\Big)^\kappa\Big] &\le A_1 \end{aligned}$$

and

$$\begin{aligned} 0 \le \tfrac{1}{2}\int_{\Omega_t}\varrho v^2\,dx + \psi_t - \psi_* + p_0(|\Omega_t| - |\Omega_*|)& \\ + \int_0^t \Big[\frac{\mu}{2}E_{\Omega_\tau}(v) + (\nu-\mu)\|\operatorname{div} v\|^2_{0,\Omega_\tau}\Big]\,d\tau &\le A_2\,, \end{aligned}$$

where $\psi_t = \frac{1}{\kappa-1}\int_{\Omega_t} A\varrho^\kappa\,dx$, *and*

$$\int_{\Omega_t} \varrho v\cdot\eta\,dx = 0\,,\qquad \int_{\Omega_t} \varrho x\,dx = 0\,,\qquad \int_{\Omega_t} \varrho\,dx = M\,.$$

P r o o f. The theorem is proved step by step using local existence in a fixed time interval. Under the assumption that

$$(v(0), p_\sigma(0)) \in \mathfrak{N}(0)\,, \tag{6.51}$$

Theorem 3.6 and Remark 3.7 yield local existence of solutions of (1.1) such that

$$u \in W_2^{4,2}(\Omega^T)\,,\qquad q_\sigma \in W_2^{3,3/2}(\Omega^T) \cap C(0,T;\Gamma_{0,2}^{3,3/2}(\Omega))\,, \tag{6.52}$$

where T is the time of existence. To show this we needed Lagrangian coordinates. By (6.51) and (6.52) Lemma 6.1 implies that the local solution belongs to $\mathfrak{M}(t)$, $t \le T$. For small δ_1 the existence time T is correspondingly large, so we can assume it is a fixed positive number.

To prove the last result we needed the Korn inequalities (see Section 5) and imbedding theorems (see (1.9), (1.10)). The constants in those theorems depend on Ω_t and the shape of S_t, so generally they are functions of t.

But in view of (6.3) with sufficiently small δ_1 (δ_1 replaces ε_1), we obtain

$$\Big|\int_0^t v\,d\tau\Big| \le c\delta_1\,,\qquad t\in[0,T]\,. \tag{6.53}$$

Hence from the relation

$$x = \xi + \int_0^t u(\xi,\tau)\,d\tau\,,\qquad \xi\in S\,,\ t\le T\,, \tag{6.54}$$

for sufficiently small δ_1 and fixed T, the shape of Ω_t, $t\le T$, does not change too much, so the constants from the imbedding theorems can be chosen independent of time.

By taking δ_2 from (6.48) and (6.49) sufficiently small, Remark 2.3 (see (2.24)) yields that $\sup_t \|v\|_{0,\Omega_t}^2 \le c\varepsilon_0$ and next Lemma 6.2 implies that $\sup_t \|p_\sigma\|_{0,\Omega_t}^2 \le c\delta$, with ε_0 and δ as small as we need (ε_0 and δ depend on the choice of the parameters of problem (1.1) (see Lemma 2.2 and Remark 2.3)). Then from the definition of $\psi(t)$ (see (4.167)) we have

$$\sup_t \psi(t) \le c(\varepsilon_0+\delta)\,,\qquad t\le T\,. \tag{6.55}$$

Hence (6.47) and Lemma 6.3 imply

$$\varphi(T) \le \delta_1\,, \tag{6.56}$$

for sufficiently small δ_1 and δ_2.

Now we wish to extend the solution to the interval $[T, 2T]$. Using (6.56) we can prove the existence of local solutions in $\mathfrak{M}(t)$, $T \le t \le 2T$. To prove

$$\varphi(2T) \le \delta_1 \tag{6.57}$$

we need inequality (4.166) where the constants depend on the constants from the imbedding theorems and Korn inequalities for $t \in [T, 2T]$. Therefore we have to show that the shape of S_t, $t \le 2T$, does not change more than for $t \le T$.

For this we need the following (see the condition (6.28)). Assume that there exists a local solution in the interval $[0, kT]$. Then in view of Lemma 6.4 we have, for $t \in [0, kT]$,

$$\begin{aligned}
\Big|\int_0^t v\,dx\Big| + \Big|\int_0^t v_x\,dx\Big| &\le c_1' \int_0^t \|v\|_{3,\Omega_\tau}\,d\tau \le c_1' \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} \|v\|_{3,\Omega_\tau}\,d\tau \\
&\le c_1' T^{1/2} \sum_{i=0}^{k-1} \Big(\int_{iT}^{(i+1)T} \|v\|_{3,\Omega_\tau}^2\,d\tau\Big)^{1/2} \le c_1' T^{1/2} \sum_{i=0}^{k-1} \Big(\int_{iT}^{(i+1)T} \varphi(\tau)\,d\tau\Big)^{1/2} \\
&\le c_1' T^{1/2} \sum_{i=0}^{k-1} \Big(\varphi(iT) \int_{iT}^{(i+1)T} e^{-\mu_0(t-iT)}\,dt\Big)^{1/2} \\
&\le c_1' [T(1-e^{-\mu_0 T})/\mu_0]^{1/2} \sum_{i=0}^{k-1} (\varphi(iT))^{1/2} \\
&\le c_1' \{T[(1-e^{-\mu_0 T})/\mu_0]\varphi(0)(1 + e^{-\mu_0 T/2} + e^{-2\mu_0 T/2} + \ldots)\}^{1/2} \\
&= c_1' [T(1/\mu_0)\varphi(0)(1-e^{-\mu_0 T})(1-e^{-\mu_0 T/2})^{-1}]^{1/2} \\
&= c_1' [T(1/\mu_0)\varphi(0)(1+e^{-\mu_0 T/2})]^{1/2} = c_2'(T\varphi(0))^{1/2} \le c_3' T^{1/2} \delta_1^{1/2}\,.
\end{aligned} \tag{6.58}$$

Taking $k = 2$ and δ_1 sufficiently small we see that $|\int_0^t u(x,\tau)\,d\tau|$ is small for any $t \in [T, 2T]$, so (6.54) implies that the shape of S_t changes no more than in $[0, T]$, and then the differential inequality (4.166) can also be shown for this interval with the same constants. Hence in view of Lemma 6.1 the solution of (1.1) belongs to $\mathfrak{M}(t)$, $t \in [T, 2T]$. Next Lemmas 6.1–6.3 imply (6.57).

Repeating the above considerations for the intervals $[kT, (k+1)T], k \ge 2$, we prove the existence for all $t \in \mathbb{R}_+$. This concludes the proof.

Remark 6.6. We proved the existence of a global solution such that its boundary shape does not change much for all $t \in \mathbb{R}_+$. Moreover, we have some freedom in the choice of an initial domain (see $(6.50)_{1,2}$). However, the freedom is restricted in that the initial domain must be chosen in such a way that the boundary of the drop does not intersect in time after small variations permitted in Theorem 6.5

Remark 6.7. Lemma 6.4 implies that $\varphi(t) \to 0$ as $t \to \infty$. Hence the considered motion converges to the constant state.

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