POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CCCXXII

BARBARA OPOZDA

Some contributions to the differential geometry of submanifolds

WARSZAWA 1992

Barbara Opozda
Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków
Poland

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TEX at the Institute
Printed and bound by
drukarnia
herman & herman
02-240 Warszawa, ul. Jakobinów 23, tel: 846-79-66, tel/fax: 49-89-95

PRINTED IN POLAND

ISBN 83-85116-63-X ISSN 0012-3862

CONTENTS

1991 *Mathematics Subject Classification*: Primary 53C20; Secondary 53A10.
Received 24.12.1987; revised version 1.4.1992.

I

1. Introduction. Pinching problems are classical but also nowadays intensively studied problems in the geometry of minimal submanifolds. The aim of the second section of this paper is to collect some theorems dealing with a pinching problem (see Problem B below) for totally real minimal submanifolds of the unit six-sphere $S^6(1)$. The results of Chapter II have been obtained together with F. Dillen, L. Verstraelen and L. Vrancken of the University of Louvain. It is known that, by using the Cayley algebra, the sphere $S^6(1)$ can be endowed with a non-integrable almost complex structure which together with the standard metric tensor field gives $S^6(1)$ a nearly Kähler structure (see $[C]_2$ or [Eh]). We will briefly recall this structure in §1 of Chapter II. Related to this structure there are two special kinds of submanifolds in $S^6(1)$, namely almost complex and totally real submanifolds. We will consider such submanifolds with respect to the following Problems A and B.

PROBLEM A. *Which real numbers can be realized as constant sectional curvatures of some minimal totally real or almost complex submanifolds of* $S^6(1)$?

It appears that the set of such numbers is finite. Namely, if M is an almost complex submanifold of $S^6(1)$ (and it is not an open part of $S^6(1)$), then it is minimal and by a theorem of A. Gray (see [G]), M is 2-dimensional. The set of all possible constant Gaussian curvatures of M is $\{0, 1/6, 1\}$. This was proved by K. Sekigawa in [S]. If M is a totally real submanifold of $S^6(1)$, then it is 3- or 2-dimensional. N. Ejiri proved in $[E]_1$ that if M is a 3-dimensional totally real submanifold of $S^6(1)$ then it is minimal, and if it has constant sectional curvature K then $K = 1$ or $1/16$. In the case where M is a minimal totally real surface, the set of numbers satisfying the condition of Problem A is $\{0, 1\}$. A proof of this fact is given in §3 of Chapter II.

Since the set of numbers satisfying the condition of Problem A is finite, we can state the following pinching problem.

PROBLEM B. *Let* K_1, K_2 *be two neighbouring numbers which can be realized as constant sectional curvatures of some* 3*-dimensional totally real* (*or* 2*-dimensional minimal totally real or* 2*-dimensional almost complex*) *submanifolds of* $S^6(1)$. *Does there exist a compact* 3*-dimensional totally real* (*resp.* 2*-dimensional minimal totally real*, *or* 2*-dimensional almost complex*) *submanifold of* $S^6(1)$ *whose* *sectional curvature K is not constant and satisfies the inequalities $K_1 \le K \le K_2$ on the whole of M?*

In all the cases mentioned the answer is negative. In the case of almost complex submanifolds the first partial result in this respect was given by K. Sekigawa in [S]. The result dealt with the interval $(1/6, 1]$. The main tool in the study were formulas of Simons' type. Namely, J. Simons obtained in 1968 (see [Si]) a formula for the Laplacian of the square of the length of the second fundamental form for minimal submanifolds of real space forms. This formula can be roughly written in the following way: $\Delta\|h\|^2 = \{\ldots\}\|h\|^2$. Under suitable conditions the portion $\{\ldots\}$ is non-negative on the whole of M. Hence, by Hopf's lemma, $\Delta\|h\|^2$ vanishes on M and this gives some new information about the curvature.

A complete answer to Problem B can be obtained by using the following lemma of A. Ros.

LEMMA 1.1 (integral formulas of Ros, $[\mathrm{R}]_2$, [MRU]). *Let T be a k-covariant tensor field on a compact Riemannian manifold M. If ∇ is the Levi-Civita connection on M, then*

$$\int_{UM} \nabla T(V, \ldots, V)\, dV = 0\,, \tag{1.1}$$

$$\int_{UM} \sum_{i=1}^{n} \nabla T(E_i, E_i, V, \ldots, V)\, dV = 0\,, \tag{1.2}$$

where dV is the volume element of the canonical measure on the unit tangent bundle UM, and $E_1, \ldots, E_n$ denotes an orthonormal basis of the tangent space T_xM for $x \in M$.

As regards the interval $[1/6, 1]$ the proof using Lemma 1.1 is much shorter than that obtained by using Simons' formula (see $[\mathrm{DOVV}]_1$). Namely, it is sufficient to consider the tensor field T_1 defined by

$$T_1(X_1, X_2, X_3, X_4) = \langle h(X_1, X_2), h(X_3, X_4)\rangle\,, \tag{1.3}$$

where $\langle\, ,\rangle$ is the Riemannian tensor field on $S^6(1)$ and h is the second fundamental form of a submanifold. If we apply (1.1) to ∇T_1 and use the equations of Gauss, Ricci, Codazzi and some properties of the nearly Kähler structure of $S^6(1)$, then we get

$$\int_{UM} (1-K)(3K-1/2)\, dV + \int_{UM} s(V)\, dV = 0$$

where s is some non-negative function on UM. It is now clear that if $1/6 \le K \le 1$, then $K \equiv 1$ or $K \equiv 1/6$. Lemma 1.1 is also the main tool in proving that

$$\{0 \le K \le 1/6\} \;\Rightarrow\; \{K = 1/6 \text{ or } K = 0\}$$

(see [DVV]). Summing up, in the case of almost complex surfaces we have the following complete answer to Problems A and B:

Let M be a compact complex surface in $S^6(1)$. If the Gaussian curvature K of M is constant, then $K = 0$ or $1/6$ or 1. If $0 \le K \le 1/6$ or $1/6 \le K \le 1$, then K is constant.

We can also use Lemma 1.1 in the study of 3-dimensional totally real submanifolds of $S^6(1)$. The lemma applied to several tensor fields and combined with Green's theorem applied to some functions defined on fibres of UM gives the following result:

$$\{1/16 < K \le 1\} \;\Rightarrow\; \{K = 1\}$$

(see §2 of Chapter II). Although it is possible to use the integral formulas of Ros for settling Problem B for totally real minimal surfaces in $S^6(1)$, we give another proof which yields the following more general assertion (see §3 of Chapter II and [DOVV_3]):

If M is a compact surface of genus 0 minimally and totally real immersed in $S^6(1)$, then M is a 2-dimensional great sphere of $S^6(1)$.

The proof is based on some properties of minimal immersions of topological 2-spheres into real space forms. The properties were proved by S. S. Chern in $[\mathrm{Che}]_1$. We prove them in §2 of Chapter III, where we also prove several other properties of minimal (and more generally, with parallel mean curvature vector) surfaces in Kähler manifolds with constant holomorphic sectional curvature and in Riemannian manifolds with constant sectional curvature (see Theorem 2.11 and Remark 2.12 of Chapter III).

In the more general situation where M is a minimal submanifold in a Euclidean unit sphere $S^n(1)$ of arbitrary dimension n, we have a complete answer to Problem A (where, of course, M is just minimal in $S^n(1)$) in the case where M is a surface, and some partial answers to Problem B also for minimal surfaces. Namely, O. Borůvka [Bo] constructed full (i.e. such that the codimension cannot be reduced) minimal immersions of the 2-sphere $S^2(2/(m(m+1)))$ of Gaussian curvature $2/(m(m+1))$ into $S^{2m}(1)$ for every $m \in \mathbb{N}$. Later, E. Calabi showed in $[\mathrm{C}]_3$ that, up to rigid motions, the Borůvka spheres were the only compact minimal surfaces with constant positive Gaussian curvature in unit spheres. Moreover, N. Wallach and R. Bryant proved the following results.

THEOREM 1.2 [W]. *Let U be an open subset of $S^2(K)$ minimally immersed in $S^n(1)$. Then the immersion can be extended to a minimal immersion from the whole of $S^2(K)$ into $S^n(1)$.*

Roughly speaking, if M is a surface of constant positive Gaussian curvature and M is minimally immersed in $S^n(1)$, then M is at least locally an open part of a Borůvka sphere.

THEOREM 1.3 [Br]. *There are no minimal surfaces of constant negative Gaussian curvature in any $S^n(1)$.*

As regards Problem B, U. Simon conjectured the following:

Let M be a compact surface with Gaussian curvature K minimally immersed in $S^n(1)$. If $2/(m(m+1)) \le K \le 2/(m(m-1))$, where $m \in \mathbb{N} \setminus \{1\}$, then $K \equiv 2/(m(m+1))$ or $K \equiv 2/(m(m-1))$ on M.

The conjecture is proved in the cases $m = 2$ and $m = 3$ (see $[\mathrm{L}]_1$, [BKSS], [K–S], [O]). The main tool were formulas of Simons' type and some properties of eigenfunctions of the Laplacian for the 2-sphere. In §3 of Chapter I we give another, in our opinion easier proof of these results. The proof is based on Ros' integral formulas. Another approach is given in [D], [J–R].

If the ambient space is a complex projective space, then there are complete answers to the corresponding Problems A and B in the case where M is a complex submanifold or a totally real submanifold of the highest possible dimension. If $\mathbb{C}P^n(H)$ denotes the n-dimensional complex projective space endowed with the Fubini–Study metric of holomorphic sectional curvature H, then for any positive integer k, and using the homogeneous coordinates, we have the following full (i.e. the complex codimension cannot be reduced) complex imbedding:

$$\psi_k : \mathbb{C}P^m(1/k) \to \mathbb{C}P^{m(k)}(1)\,, \qquad m(k) = \binom{m+k}{k} - 1\,,$$

where

$$\psi_k : (z_i)_{0\le i\le m} \to \left(\sqrt{\frac{k!}{k_0!\ldots k_m!}}\, z_0^{k_0}\ldots z_m^{k_m}\right)_{k_0+\ldots+k_m=k}.$$

This imbedding is called the *k-th standard imbedding.* Standard imbeddings provide all Kähler immersions of the complex projective space into itself, even locally. More precisely, we have the following theorem due to E. Calabi $[\mathrm{C}]_1$:

Theorem 1.4. *Let M be an m-dimensional Kähler submanifold immersed in $\mathbb{C}P^n(1)$. Suppose that M has a constant holomorphic sectional curvature H. Then there exists a positive integer k such that $H = 1/k$. If the immersion is full, then $n = \binom{m+k}{k} - 1$ and M is locally congruent to the k-th standard imbedding of $\mathbb{C}P^m(1/k)$ into $\mathbb{C}P^n(1)$. Moreover, if M is complete, then M is an imbedded submanifold congruent to the k-th standard imbedding of $\mathbb{C}P^m(1/k)$.*

Let H and K be the holomorphic sectional and the sectional curvatures of a compact Kähler submanifold M immersed in $\mathbb{C}P^n(1)$. K. Ogiue [Og] conjectured the following facts: If

(a) $H > 1/2$, or
(b) $n \ge 2$ and $K > 1/8$,

then M is totally geodesic in $\mathbb{C}P^n(1)$, i.e. it is a linear subvariety. This was proved by A. Ros $[\mathrm{R}]_1$ and A. Ros and L. Verstraelen [R–V]. They used natural arguments on the minimum of the holomorphic sectional curvature in the unit tangent bundle of M. Combining the positive solution of the Frankel conjecture (see [S–Y]) with ideas used by Lawson in $[\mathrm{L}]_2$, A. Ros obtained the following complete answer to Problem B in the case of complex submanifolds of $\mathbb{C}P^n(1)$. (In the case where $n = 1$ the theorem was proved by Lawson in $[L]_2$.)

THEOREM 1.5. *Let M be an m-dimensional compact Kähler submanifold immersed in the complex projective space $\mathbb{C}P^n(1)$. If*

(1) $1/k \geq H \geq 1/(k+1)$ *for some* $k = 2, 3, \ldots,$ *or*
(2) $m \geq 2$ *and* $1/k \geq K \geq 1/(4(k+1))$ *for some* $k = 1, 2, \ldots,$

then M is congruent to the standard imbedding of $\mathbb{C}P^m(1/k)$ or of $\mathbb{C}P^m(1/(k+1))$ into $\mathbb{C}P^n(1)$.

If $k = 1$, then we have moreover the following

THEOREM 1.6 $[R]_2$. *Let M be a compact Kähler submanifold immersed in $\mathbb{C}P^n(1)$. Then $H \geq 1/2$ if and only if M has parallel second fundamental form.*

Notice here that Kähler submanifolds with parallel second fundamental form were classified by H. Nakagawa and R. Takagi [N–Ta].

The proof of Theorem 1.6 is based on (1.1).

Assume now that M is an n-dimensional totally real minimal submanifold of $\mathbb{C}P^n(4)$. It is known (see [Ch–O]) that if M has constant sectional curvature K, then $K = 0$ or 1. If $K = 1$, then M is totally geodesic, and if M is moreover complete, then it is congruent to the standard imbedding of a real projective space into a complex projective space (see [A]). Concerning Problem B, K. Ogiue [Og] also conjectured the following fact:

If M is an n-dimensional compact minimal totally real submanifold of $\mathbb{C}P^n$ and the sectional curvatures K of M are everywhere positive, then M is totally geodesic.

S.-T. Yau [Y] proved the conjecture in the case where $n = 2$. The method he used was Simons' formula. However, in higher dimensions this formula only allows one to obtain partial results. Using the same method as in $[R]_1$, F. Urbano positively solved the Ogiue conjecture (see $[U]_1$). Moreover, by using the integral formulas of Ros he proved in $[U]_2$ the following more general

THEOREM 1.7. *Let M be an n-dimensional compact totally real submanifold immersed in an n-dimensional complex space form with parallel mean curvature vector. If M has non-negative sectional curvature, then M has parallel second fundamental form.*

Submanifolds with parallel second fundamental form in complex space forms were completely classified in [F], [N] and [N–T].

If the dimension of M is not the highest possible the problem seems to be more difficult. Consider, for instance, the opposite case, i.e. assume that M is a surface. We have the following example.

EXAMPLE 1.1. Let M be a surface minimally immersed in a unit sphere $S^n(1)$. Since $S^n(1)$ is the Riemannian universal covering space of the real projective space

$\mathbb{R}P^n(1)$, $S^n(1)$ is isometrically and totally geodesically immersed in $\mathbb{R}P^n(1)$. By composing these immersions with the standard imbedding of $\mathbb{R}P^n(1)$ into $\mathbb{C}P^n(4)$, we obtain a minimal totally real immersion of M into $\mathbb{C}P^n(4)$:

$$M \to S^n(1) \to \mathbb{R}P^n(1) \to \mathbb{C}P^n(4)\,.$$

This composition of immersions provides the easiest examples of minimal totally real immersions into a complex projective space. Moreover, it inspires the following analogue of Simon's conjecture:

If M is a compact minimal totally real surface in $\mathbb{C}P^n(1)$ and the Gaussian curvature K of M satisfies $2/(m(m+1)) \le K \le 2/(m(m-1))$ for some $m \in \mathbb{N}\setminus\{1\}$, then constantly $K = 2/(m(m+1))$ or $K = 2/(m(m-1))$.

Of course, this problem contains Simon's conjecture, but in fact it turns out to coincide with it. Namely, in §3 of Chapter III we prove that if M is a surface of genus 0, then the immersions constructed in Example 1.1 are the only examples of minimal totally real immersions of M into $\mathbb{C}P^n(4)$. If in Example 1.1 the immersion from M into $S^n(1)$ is assumed to have parallel mean curvature vector, then we obtain a totally real immersion with parallel mean curvature vector from M into $\mathbb{C}P^n(4)$. Also in this case such immersions provide all examples of totally real immersions with parallel mean curvature vector from surfaces of genus 0 into $\mathbb{C}P^n(4)$. A similar result is true for surfaces in any complex space form. We discuss this problem in §3 of Chapter III.

The class of submanifolds with parallel mean curvature vector contains minimal submanifolds and submanifolds with parallel second fundamental form. This class has been investigated, for instance, by H. Hopf, S.-T. Yau, S. S. Chern, B.-Y. Chen. In particular, S.-T. Yau in [Y] and B.-Y. Chen in $[\mathrm{Ch}]_{1,2}$ proved that every surface with parallel mean curvature vector in a real space form is actually contained in a totally geodesic 3-space or is a minimal surface of an umbilical hypersurface. This reduces the theory of surfaces with parallel mean curvature vector in real space forms to the theory of minimal surfaces. The main purpose of Chapter III of the present paper is to prove that a surface M of genus 0 totally real and with parallel mean curvature vector immersed in any complex space form $\widetilde{M}$ is actually a submanifold of a real space form and the real codimension of M in $\widetilde{M}$ can be reduced at least twice. More precisely, we shall prove the following theorem (see also $[\mathrm{O}]_2$). In the theorem, by a complex space form we mean a complete simply connected Kähler manifold with constant holomorphic sectional curvature.

THEOREM 1.8. *If M is a compact surface of genus 0 totally real and with parallel mean curvature vector immersed in a complex space form $\widetilde{M}$, then there is a totally real totally geodesic submanifold M' of $\widetilde{M}$ such that the image of M is contained in M'.*

With the aim of proving this theorem we consider normal bundles of all orders. They are built from covariant derivatives of the second fundamental form. An important tool in the construction is a theorem of Chern on isolated zeros of a solution of a system of differential equations (see Theorem 2.3 in Chapter III). By using isothermal coordinates on M we also define a sequence of holomorphic forms on M which, by the Riemann–Roch theorem, vanish on M. From this we derive some important properties of normal bundles. These results are given in §2 of Chapter III. In §3 of Chapter III we prove "reduction theorems" for totally real submanifolds in complex space forms (Propositions 3.1 and 3.2). By applying these theorems to the sum of normal bundles of all orders we obtain a reduction of the real codimension which is required in Theorem 1.8. The proof of Theorem 1.8 also yields several other assertions. In particular, we get the following generalization of the well-known theorem of Hopf. It was also proved by S. S. Chern in $[\mathrm{Che}]_2$.

THEOREM 1.9. *Let M be a surface of genus zero with constant mean curvature immersed in a* 3*-dimensional Riemannian manifold $\widetilde{M}$ of constant sectional curvature. Then M is totally umbilical.*

Recall that in Hopf's original theorem $\widetilde{M}$ was a Euclidean vector space. Recall also that the famous conjecture of Hopf dealt with the possibility of dropping the assumption that M is of genus 0. The first counterexample was given by Wente in [We].

Among other assertions which can be derived from the proof of Theorem 1.8 we have a theorem of Calabi (Remark 2.12) and a generalization of Yau's theorem (Theorem 2.10).

The last part of this paper is devoted to the study of $\mathcal{C}$-totally real surfaces in Sasakian space forms. By using a similar argument to that given in §§2 and 3 of Chapter III we obtain corresponding results for $\mathcal{C}$-totally real surfaces of genus 0 in Sasakian space forms (Theorems 4.1, 4.5, Corollary 4.7). In particular, it is known (by Calabi's theorem) that if M is a surface of genus 0 minimally immersed in a Euclidean odd-dimensional sphere, then the codimension of M can be reduced. We prove that if the tangent space T_pM is perpendicular to ip for every $p \in M$, then the codimension of M can be reduced at least twice. We also have a pinching theorem for $\mathcal{C}$-totally real surfaces in 7-dimensional Sasakian space forms. The last theorem is related to Problem A for $\mathcal{C}$-totally real surfaces in odd-dimensional spheres.

2. Preliminaries. All the objects defined and used in this work are assumed to be smooth, i.e. of class C^∞. Manifolds are connected and by a submanifold we shall mean an immersed submanifold.

If M is a submanifold of a Riemannian manifold $\widetilde{M}$, then N will denote its normal bundle, $\langle\, ,\rangle$ a metric tensor on $\widetilde{M}$ as well as the induced one on M, and $\|\cdot\|$ the induced norm. If $\widetilde{\nabla}$ and ∇ are the Levi-Civita connections on $\widetilde{M}$ and M respectively, then the Gauss and Weingarten formulas are

$$\widetilde{\nabla}_W Z = \nabla_W Z + h(W, Z) \tag{2.1}$$

and

$$\widetilde{\nabla}_W \xi = -A_\xi W + D_W \xi \,, \tag{2.2}$$

where ξ is a normal vector field on M; W, Z are tangent vector fields on M (i.e. $W, Z \in \mathfrak{X}(M)$); h is the second fundamental form; A_ξ is the Weingarten endomorphism associated with ξ; and D is the normal connection of M. The covariant derivative of h is defined by

$$\nabla h(X, Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{2.3}$$

where $X, Y, Z \in \mathfrak{X}(M)$. In a similar way we define the covariant derivative of any 2-covariant tensor field on M with values in the normal bundle. A submanifold is said to have *parallel second fundamental form* if $\nabla h = 0$. By induction we define higher derivatives of h in the following way:

$$\begin{aligned} \nabla^l h(W_1, \ldots, W_{l+2}) &= D_{W_1}(\nabla^{l-1} h(W_2, \ldots, W_{l+2})) \\ &\quad - \nabla^{l-1} h(\nabla_{W_1} W_2, \ldots, W_{l+2}) - \ldots - \nabla^{l-1} h(W_2, \ldots, \nabla_{W_1} W_{l+2}) \end{aligned} \tag{2.4}$$

for $W_1, \ldots, W_{l+2} \in \mathfrak{X}(M)$. For technical reasons we set $\nabla^0 h = h$. If $l \geq 2$, then we have the following Ricci identity:

$$\begin{aligned} \nabla^l h(W_1, W_2, W_3, \ldots, W_{l+2}) &- \nabla^l h(W_2, W_1, W_3, \ldots, W_{l+2}) \\ &= R^\perp(W_1, W_2)\nabla^{l-2} h(W_3, \ldots, W_{l+2}) \\ &\quad - \nabla^{l-2} h(R(W_1, W_2)W_3, \ldots, W_{l+2}) \\ &\quad - \ldots - \nabla^{l-2} h(W_3, \ldots, R(W_1, W_2)W_{l+2}) \,, \end{aligned} \tag{2.5}$$

where R, $R^\perp$ are the curvature tensors of ∇ and D respectively. We will also use the Gauss and Ricci equations:

$$\begin{aligned} \langle \widetilde{R}(X, Y)Z, W \rangle &= \langle R(X, Y)Z, W \rangle + \langle h(X, Z), h(Y, W) \rangle \\ &\quad - \langle h(Y, Z), h(X, W) \rangle \,, \end{aligned} \tag{2.6}$$

$$\langle \widetilde{R}(X, Y)\xi, \eta \rangle = \langle R^\perp(X, Y)\xi, \eta \rangle - \langle [A_\xi, A_\eta]X, Y \rangle \,, \tag{2.7}$$

where $\widetilde{R}$ is the curvature tensor of $\widetilde{\nabla}$; $W, X, Y, Z \in \mathfrak{X}(M)$; and ξ, η are normal vector fields on M.

In all cases considered in this paper the Codazzi equation is

$$\nabla h(X, Y, Z) = \nabla h(Y, X, Z) \tag{2.8}$$

for $X, Y, Z \in \mathfrak{X}(M)$; it plays an important role in our investigations. It follows, in particular, that $\nabla^l h(X_1, \ldots, X_{l+2})$ is symmetric in the last three arguments for any $l \geq 1$.

A submanifold M is *minimal* if its mean curvature vector field (usually called the *mean curvature vector*)

$$\varkappa = \frac{1}{\dim M} \operatorname{tr} h$$

vanishes on M.

Let M be a compact Riemannian manifold, UM its unit tangent bundle and UM_p the fibre of UM at a point p of M. Recall that for any continuous function $f : UM \to \mathbb{R}$ the integral of f is defined by

$$\int_{UM} f\, dV = \int_{M} \Big(\int_{UM_p} f\, dV_p \Big)\, dp \,,$$

where dp, dV and dV_p denote the volume elements of the canonical measures on M, UM and UM_p respectively. In what follows the volume elements in integrals will be dropped.

Throughout the paper $S^n(1)$ will denote the unit hypersphere in $\mathbb{R}^{n+1}$ centred at the origin.

3. On Simon's conjecture. As mentioned in the Introduction the following partial answer to Simon's conjecture was proved in [K–S], [O], [BKSS]. We give another proof of this result.

THEOREM 3.1. *Let M be a compact surface with Gaussian curvature K minimally immersed in $S^n(1)$.*

(1) *If $1/3 \le K$, then constantly $K = 1$ or $K = 1/3$.*
(2) *If $1/6 \le K \le 1/3$, then constantly $K = 1/3$ or $K = 1/6$.*

Proof. We may assume that M is oriented. Indeed, if M is not orientable, then instead of M we can consider its double covering space. Throughout the proof V will mean a unit vector or a unit (local) vector field on M and U will denote the unit vector or the unit vector field on M such that the pair V, U is positively oriented.

By the assumptions of the theorem and by the Gauss–Bonnet theorem M is homeomorphic to the sphere $S^2(1)$. Therefore M is isotropic, i.e.

$$\|h(V,V)\| = \|h(V,U)\| \tag{3.1}$$

and

$$\langle h(V,V), h(V,U)\rangle = 0 \tag{3.2}$$

(see $[\mathrm{Che}]_1$, $[\mathrm{E}]_2$, [G–R] or §2 of Chapter III of this paper). Hence the Gauss equation is

$$K = 1 - 2\|h(V,V)\|^2 \,. \tag{3.3}$$

Formulas (3.1) and (3.2) also imply

$$\langle \nabla h(V,V,V), h(V,U)\rangle = -\langle \nabla h(V,V,U), h(V,V)\rangle \tag{3.4}$$

and

$$\langle \nabla h(V,V,V), h(V,V)\rangle = \langle \nabla h(V,V,U), h(V,U)\rangle \,. \tag{3.5}$$

The Ricci equation is

$$R^{\perp}(U,V)\xi = 2\langle h(V,V),\xi\rangle h(V,U) - 2\langle h(V,U),\xi\rangle h(V,V) \tag{3.6}$$

for any normal vector ξ.

By (2.5), (3.1)–(3.3) and (3.6) we obtain

$$\nabla^2 h(U,V,V,V) - \nabla^2 h(V,U,V,V) = (1-3K)h(U,V)\,, \tag{3.7}$$

$$\nabla^2 h(V,V,V,V) + \nabla^2 h(U,U,V,V) = (3K-1)h(V,V)\,, \tag{3.8}$$

$$\nabla^2 h(U,V,V,U) - \nabla^2 h(V,U,V,U) = (3K-1)h(V,V)\,. \tag{3.9}$$

Consider the tensor field T_1 defined by (1.3), i.e.

$$T_1(X_1,X_2,X_3,X_4) = \langle h(X_1,X_2), h(X_3,X_4)\rangle\,.$$

We have

$$\nabla^2 T_1(V,V,V,V,V,V) = 2\langle \nabla^2 h(V,V,V,V), h(V,V)\rangle + 2\|\nabla h(V,V,V)\|^2\,, \tag{3.10}$$

$$\nabla^2 T_1(U,U,V,V,V,V) = 2\langle \nabla^2 h(U,U,V,V), h(V,V)\rangle + 2\|\nabla h(U,V,V)\|^2\,. \tag{3.11}$$

Therefore the integral formula (1.2) applied to ∇T_1 and formulas (3.8), (3.3) yield

$$0 = \int\limits_{UM} (3K-1)(1-K) + \int\limits_{UM} \{\|\nabla h(V,V,V)\|^2 + \|\nabla h(V,V,U)\|^2\}\,, \tag{3.12}$$

which proves the first assertion.

Now we prove (2). We have

$$\begin{aligned}\nabla^3 h(V,V,V,V,V) &+ \nabla^3 h(U,U,V,V,V)\\ &= \{\nabla^3 h(V,V,V,V,V) + \nabla^3 h(V,U,U,V,V)\}\\ &\quad + \{\nabla^3 h(U,U,V,V,V) - \nabla^3 h(U,V,U,V,V)\}\\ &\quad + \{\nabla^3 h(U,V,U,V,V) - \nabla^3 h(V,U,U,V,V)\}\,.\end{aligned} \tag{3.13}$$

Let $p \in M$ and let the vector fields V, U be such that $\nabla_V V = \nabla_V U = \nabla_U U = \nabla_U V = 0$ at p. By (3.7), (3.3) and (3.4) we get at p

$$\begin{aligned}\nabla^3 h(U,U,V,V,V) - \nabla^3 h(U,V,U,V,V) &= D_U((1-3K)h(U,V))\\ &= (3K-1)\nabla h(V,V,V) - 12\langle \nabla h(V,V,V), h(V,U)\rangle h(U,V)\,.\end{aligned} \tag{3.14}$$

Similarly, (3.8), (3.3) and (3.4) imply

$$\begin{aligned}\nabla^3 h(V,V,V,V,V) &+ \nabla^3 h(V,U,U,V,V)\\ &= (3K-1)\nabla h(V,V,V) - 12\langle \nabla h(V,V,V), h(V,V)\rangle h(V,V)\,.\end{aligned} \tag{3.15}$$

By (2.5), (3.6), (3.4) and (3.5) we obtain

$$\begin{aligned}(3.16)\qquad \nabla^3 h(U,V,U,V,V) - \nabla^3 h(V,U,U,V,V) &= 3K\nabla h(V,V,V) - 2\langle \nabla h(V,V,V), h(V,V)\rangle h(V,V) \\ &\quad - 2\langle \nabla h(V,V,V), h(V,U)\rangle h(V,U)\,.\end{aligned}$$

By virtue of (3.13)–(3.16) we have

$$\begin{aligned}(3.17)\qquad \nabla^3 h(V,V,V,V,V) + \nabla^3 h(U,U,V,V,V) &= (9K-2)\nabla h(V,V,V) - 14\langle \nabla h(V,V,V), h(V,V)\rangle h(V,V) \\ &\quad - 14\langle \nabla h(V,V,V), h(U,V)\rangle h(U,V)\,.\end{aligned}$$

Define T_2 by

$$(3.18)\qquad T_2(X_1,X_2,X_3,X_4,X_5,X_6,X_7) = \langle \nabla^2 h(X_1,X_2,X_3,X_4), \nabla h(X_5,X_6,X_7)\rangle\,.$$

Because of (3.17) we have

$$\begin{aligned}(3.19)\qquad \nabla T_2(V,\dots,V) + \nabla T_2(U,U,V,\dots,V) &= (9K-2)\|\nabla h(V,V,V)\|^2 - 14\langle \nabla h(V,V,V), h(V,V)\rangle^2 \\ &\quad - 14\langle \nabla h(V,V,V), h(U,V)\rangle^2 \\ &\quad + \|\nabla^2 h(V,V,V,V)\|^2 + \|\nabla^2 h(U,V,V,V)\|^2\,.\end{aligned}$$

Since rotation through $+\pi/2$ in T_pM is measure preserving, we get

$$(3.20)\qquad \int_{UM_p} \|\nabla h(V,V,U)\|^2 = \int_{UM_p} \|\nabla h(V,V,V)\|^2\,,$$

$$(3.21)\qquad \int_{UM_p} \|\nabla^2 h(V,V,V,V)\|^2 = \int_{UM_p} \|\nabla^2 h(U,U,V,V)\|^2\,,$$

$$(3.22)\qquad \int_{UM_p} \|\nabla^2 h(U,V,V,V)\|^2 = \int_{UM_p} \|\nabla^2 h(V,U,V,V)\|^2\,.$$

By applying Green's theorem to the 1-form α defined on UM_p by

$$\alpha_V(U) = \langle \nabla^2 h(V,V,V,U), \nabla^2 h(V,V,V,V)\rangle\,,$$

and by using (3.7)–(3.9), we get

$$(3.23)\qquad \int_{UM_p} \|\nabla^2 h(V,V,V,V)\|^2 = \int_{UM_p} \|\nabla^2 h(V,V,V,U)\|^2\,.$$

Combining this with (3.21) and (3.23), we obtain

$$(3.24)\qquad \int_{UM_p} \|\nabla^2 h(U,V,V,V)\|^2 = \int_{UM_p} \|\nabla^2 h(U,U,V,V)\|^2\,.$$

Consequently, by applying (1.2) to T_2, we get

$$(3.25)\qquad 0=\int_{UM}\{(9K-2)\|\nabla h(V,V,V)\|^2-14t + \tfrac12\|\nabla^2h(V,V,V,V)+\nabla^2h(U,U,V,V)\|^2 + \tfrac12\|\nabla^2h(V,V,V,V)-\nabla^2h(U,U,V,V)\|^2\}\,,$$

where

$$t=\langle\nabla h(V,V,V),h(V,V)\rangle^2+\langle\nabla h(V,V,V),h(U,V)\rangle^2\,.$$

Denote by Δ the Laplacian on M. It is easy to verify that

$$(3.26)\qquad \Delta K=-4\{\|\nabla h(V,V,V)\|^2+\|\nabla h(V,V,U)\|^2\}-2(3K-1)(1-K)$$

and

$$(3.27)\qquad \Delta K^r=r(r-1)K^{r-2}((UK)^2+(VK)^2)+\Delta KrK^{r-1}$$

for any integer $r\ge 2$. In particular,

$$(3.28)\qquad \Delta K^2=2\{16t-8Ks+4K(3K-1)(1-K)\}\,,$$

where $s=\|\nabla h(V,V,V)\|^2+\|\nabla h(V,V,U)\|^2$.

Formulas (3.26) and (3.28) imply that s and t are functions on M. We have

$$(3.29)\qquad 2\int_{UM}s=\int_{UM}(1-3K)(1-K)\,,$$

$$(3.30)\qquad \int_{UM}Ks=\int_{UM}\{8t+(1-3K)(1-K)K\}\,.$$

Using (3.20) and substituting (3.29) and (3.30) into (3.25) we get

$$(3.31)\qquad 0=\int_{UM}\{4t+\tfrac14(1-K)(1-3K)(6K-1) + \tfrac12\|\nabla^2h(V,V,V,V)-\nabla^2h(U,U,V,V)\|^2\}\,.$$

If $1/6\le K\le 1/3$, then the above formula implies that $K=1/3$ or $1/6$ on M. The proof is complete.

Some new approaches to Simon's problem are given in [BJRW], [B–W] and [I].

II. Pinching theorems for submanifolds of the nearly Kähler 6-sphere

1. The nearly Kähler structure on $S^6(1)$. Let $e_0,e_1,\ldots,e_7$ be the canonical basis of $\mathbb{R}^8$. Then each point α of $\mathbb{R}^8$ can be written in a unique way as

$$\alpha=Ae_0+x\,,$$

where $A\in\mathbb{R}$ and x is a linear combination of $e_1,\ldots,e_7$. α can be viewed as a Cayley number, and it is called *purely imaginary* when $A=0$. For any pair of purely imaginary x and y, we consider the multiplication $\cdot$ given by

$$x \cdot y = -\langle x, y\rangle e_0 + x \times y\,,$$

where $\langle\,,\rangle$ is the standard scalar product on $\mathbb{R}^8$ and $x \times y$ is defined by the following multiplication table for $e_j \times e_k$:

j \ k	1	2	3	4	5	6	7
1	0	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
2	$-e_3$	0	e_1	e_6	$-e_7$	$-e_4$	e_5
3	e_2	$-e_1$	0	$-e_7$	$-e_6$	e_5	e_4
4	$-e_5$	$-e_6$	e_7	0	e_1	e_2	$-e_3$
5	e_4	e_7	e_6	$-e_1$	0	$-e_3$	$-e_2$
6	$-e_7$	e_4	$-e_5$	$-e_2$	e_3	0	e_1
7	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	0

For two Cayley numbers $\alpha = Ae_0 + x$ and $\beta = Be_0 + y$, the Cayley multiplication $\cdot$, which makes $\mathbb{R}^8$ the Cayley algebra $\mathcal{C}$, is defined by

$$\alpha \cdot \beta = ABe_0 + Ay + Bx + x \cdot y\,.$$

We recall that the multiplication $\cdot$ of $\mathcal{C}$ is neither commutative nor associative. The set $\mathcal{C}_+$ of all purely imaginary Cayley numbers can clearly be viewed as the 7-dimensional linear subspace $\mathbb{R}^7$ of $\mathbb{R}^8$. In $\mathcal{C}_+$ we consider the unit hypersphere $S^6(1)$. The tangent space T_xS^6 of $S^6(1)$ at a point x can be identified with the vector subspace of $\mathcal{C}_+$ orthogonal to x. Now we define on $S^6(1)$ a $(1,1)$-tensor J by putting

$$J_xU = x \times U\,,$$

where $x \in S^6(1)$ and $U \in T_xS^6$. This tensor field is well defined (i.e. $J_xU \in T_xS^6$) and determines an almost complex structure on $S^6(1)$, i.e.

$$J^2 = -\,\mathrm{id}\,,$$

where id is the identity $(1,1)$-tensor field on $S^6(1)$.

The compact simple Lie group G_2 is the group of automorphisms of $\mathcal{C}$ and acts transitively on $S^6(1)$. Moreover, it preserves both J and the standard metric on $S^6(1)$ (see [F–I]). Further, let G be the $(2,1)$-tensor field on $S^6(1)$ defined by

$$G(X,Y) = (\widetilde{\nabla}_X J)Y\,, \tag{1.1}$$

where $X, Y \in \mathfrak{X}(S^6(1))$ and $\widetilde{\nabla}$ is the Levi-Civita connection on $S^6(1)$. This tensor field has the following properties:

$$G(X,X) = 0\,, \tag{1.2}$$

$$G(X,Y) + G(Y,X) = 0\,, \tag{1.3}$$

$$(1.4)\qquad G(X,JY)+JG(X,Y)=0\,,$$

$$(1.5)\qquad (\widetilde{\nabla}_X G)(Y,Z)=\langle Y,JZ\rangle X+\langle X,Z\rangle JY-\langle X,Y\rangle JZ\,,$$

$$(1.6)\qquad \langle G(X,Y),Z\rangle+\langle G(X,Z),Y\rangle=0\,,$$

$$(1.7)\qquad \langle G(X,Y),G(Z,W)\rangle=\langle X,Z\rangle\langle Y,W\rangle-\langle X,W\rangle\langle Z,Y\rangle + \langle JX,Z\rangle\langle Y,JW\rangle-\langle JX,W\rangle\langle Y,JZ\rangle\,,$$

$$(1.8)\qquad G(X,G(Y,Z))=\langle X,Z\rangle Y-\langle X,Y\rangle Z+\langle JX,Z\rangle JY-\langle JX,Y\rangle JZ\,,$$

where $X,Y,Z,W\in\mathfrak{X}(S^6(1))$ (see [S], [G], $[\mathrm{E}]_1$). Recall that (1.2) means that the structure $(J,\langle\,,\rangle)$ is nearly Kähler. For further properties of this structure we refer to [Eh], [Fr], $[\mathrm{C}]_2$, [F–I], [M].

Recall also that if $(\widetilde{M},J,\langle\,,\rangle)$ is an almost Hermitian manifold, i.e. J is an almost complex structure on $\widetilde{M}$ and $\langle\,,\rangle$ is a Hermitian metric tensor field associated with J, then a submanifold M of $\widetilde{M}$ is called *almost complex* (resp. *totally real*) if $J(TM)=TM$ (resp. $J(TM)$ is orthogonal to TM). Of course, if M is totally real, then $\dim M\le\frac{1}{2}\dim\widetilde{M}$. Hence a totally real submanifold of $S^6(1)$ is either 3- or 2-dimensional. Recall also that an almost complex submanifold of a nearly Kähler manifold is minimal.

Throughout this chapter S^6 will mean $S^6(1)$ endowed with the nearly Kähler structure described above.

2. 3-dimensional totally real submanifolds of S^6. N. Ejiri proved in $[\mathrm{E}]_1$ the following

THEOREM 2.1. *A 3-dimensional totally real submanifold M of S^6 is minimal and orientable. If M has constant sectional curvature K, then $K=1/16$ or $K=1$.*

Examples of 3-dimensional totally real 1/16-curved submanifolds of S^6 are given in [M]. We also have the following example of a 3-dimensional totally real geodesic submanifold of S^6:

$$M=\{x\in S^6: x=x_1e_1+x_3e_3+x_5e_5+x_7e_7\}\,,$$

where as in the previous section $e_1,\dots,e_7$ is the canonical basis of $\mathbb{R}^7$.

Let N be a 3-dimensional totally real submanifold of S^6. It is known that G restricted to M has values in N and the values of $G_{|M}$ span N (see $[\mathrm{E}]_1$). By (1.4) it follows that $G(X,\xi)\in TM$ for any $X\in T_pM$ and $\xi\in N_p$, $p\in M$. By the Gauss and Weingarten formulas we have

$$(2.1)\qquad D_X(JY)=G(X,Y)+J\nabla_XY\,,$$

$$(2.2)\qquad A_{JX}Y=-Jh(X,Y)$$

for $X,Y\in\mathfrak{X}(M)$. The last formula implies

$$(2.3)\qquad \langle h(X,Y),JZ\rangle=\langle h(X,Z),JY\rangle\,,$$

for any $X, Y, Z \in \mathfrak{X}(M)$. The Gauss and Ricci equations have the form

$$\begin{aligned}(2.4)\quad \langle R(X,Y)W,Z\rangle &= \langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle \\ &\quad + \langle h(X,Z),h(Y,W)\rangle - \langle h(X,W),h(Y,Z)\rangle\,,\end{aligned}$$

$$(2.5)\qquad \langle R^{\perp}(X,Y)\xi,\eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle$$

where $X, Y, Z, W \in \mathfrak{X}(M)$, and ξ and η are normal vector fields on M. The Codazzi equation is given by (2.8). Formula (1.8) implies

$$(2.6)\qquad G(X, G(Z,W)) = \langle X,W\rangle Z - \langle X,Z\rangle W$$

for $X, Y, Z \in \mathfrak{X}(M)$. It is also known (see $[\mathrm{E}]_1$) that

$$(2.7)\qquad (\nabla JG)(X,Y,Z) = \langle X,Y\rangle Z - \langle X,Z\rangle Y - G(X,W,G(Y,Z))$$

for $X, Y, Z \in \mathfrak{X}(M)$. Combining this with (2.6) we get on M

$$(2.8)\qquad \nabla(JG) = 0\,.$$

We have

$$\begin{aligned}(2.9)\quad (\widetilde{\nabla}_X G)(Y,Z) &= -A_{G(Y,Z)}X + D_X G(Y,Z) - G(\widetilde{\nabla}_X Y, Z) - G(Y, \widetilde{\nabla}_X Z) \\ &= A_{J(JG(Y,Z))}X - D_X J(JG(Y,Z)) - G(\widetilde{\nabla}_X Y, Z) - G(Y, \widetilde{\nabla}_X Z) \\ &= -\,Jh(JG(Y,Z),X) - G(X, JG(Y,Z)) - J\nabla_X JG(Y,Z) \\ &\quad - G(\widetilde{\nabla}_X Y, Z) - G(Y, \widetilde{\nabla}_X Z) \\ &= -\,Jh(JG(Y,Z),X) - JG(X, G(Y,Z)) - J(\nabla_X JG)(Y,Z) \\ &\quad - J\{JG(\nabla_X Y, Z) + JG(Y, \nabla_X Z)\} - G(\widetilde{\nabla}_Y Y, Z) - G(Y, \widetilde{\nabla}_X Z) \\ &= -\,Jh(JG(Y,Z),X) + JG(X, G(Y,Z)) - G(h(X,Y),Z) \\ &\quad - G(Y, h(X,Z))\,.\end{aligned}$$

By comparing this with (1.5), we get

$$(2.10)\qquad h(X, JG(Y,Z)) = JG(h(X,Y),Z) + JG(h(Z,X),Y)$$

for any $X, Y, Z \in \mathfrak{X}(M)$. Let $X, Y, Z \in T_pM$ and let $\gamma(t)$ be the geodesic determined by X. Denote by the same letters vector fields defined along $\gamma(t)$ and such that $\nabla_X Y = \nabla_X X = \nabla_X Z = 0$. Then, by (2.2), (2.8), (1.4) and (2.6), we obtain at p

$$\begin{aligned}(2.11)\quad \nabla G(X,Y,Z) &= D_X(G(Y,Z)) = -D_X(J(JG(Y,Z))) \\ &= -\,G(X, JG(Y,Z)) = \langle X,Z\rangle JY - \langle X,Y\rangle JZ\,.\end{aligned}$$

Now observe that

$$(2.12)\qquad \langle \nabla h(W,X,Z), JY\rangle = \langle \nabla h(W,X,Y), JZ\rangle + \langle G(Y,Z), h(X,W)\rangle\,.$$

Indeed, by differentiating (2.3), we get

$$(2.13)\qquad \langle \nabla h(W,X,Y),JZ\rangle + \langle h(X,Y),G(W,Z)\rangle = \langle \nabla h(W,X,Z),JY\rangle + \langle h(X,Z),G(W,Y)\rangle\,.$$

By (2.3), (2.10) and (1.6) we obtain

$$\begin{aligned}\langle h(X,Y),G(W,Z)\rangle &= -\langle h(X,Y),J(JG(W,Z))\rangle = -\langle h(X,JG(W,Z)),JY\rangle\\ &= -\langle JG(h(X,W),Z),JY\rangle - \langle JG(W,h(Z,X)),JY\rangle\\ &= \langle h(X,W),G(Y,Z)\rangle + \langle G(W,Y),h(X,Z)\rangle\,,\end{aligned}$$

i.e.

$$\langle h(X,Y),G(W,Z)\rangle = \langle h(X,W),G(Y,Z)\rangle + \langle G(W,Y),h(X,Z)\rangle\,.$$

Combining the last formula with (2.13), we get (2.12).

As an immediate consequence of (2.12) and the fact that the values of $G_{|M}$ span the normal bundle, we obtain

PROPOSITION 2.2 [Op]$_1$. *Let M be a 3-dimensional totally real submanifold of S^6. If the second fundamental form of M is parallel, then M is totally geodesic.*

By differentiating (2.12) we get

$$\begin{aligned}(2.14)\qquad \langle \nabla^2 h(U,W,X,Z),JY\rangle &+ \langle \nabla h(W,X,Z),G(U,Y)\rangle\\ &= \langle \nabla^2 h(U,W,X,Y),JZ\rangle + \langle \nabla h(W,X,Y),G(U,Z)\rangle\\ &\quad + \langle \nabla G(U,Y,Z),h(X,W)\rangle + \langle G(Y,Z),\nabla h(U,X,W)\rangle\,.\end{aligned}$$

Concerning Problem B stated in the Introduction we have the following theorem:

THEOREM 2.3 [DOVV]$_3$. *If all sectional curvatures K of a compact 3-dimensional totally real submanifold of S^6 satisfy $1/16 < K \le 1$, then constantly $K = 1$.*

Proof. Let $V \in UM_p$ and let E_2, E_3 be orthonormal vectors in T_pM orthogonal to V. Then $\{E_2,E_3\}$ can be considered as an orthonormal basis of $T_V(UM_p)$. The vectors $V = E_1, E_2, E_3$ form an orthonormal basis of T_pM. It is easy to observe that the vectors $G(E_1,E_2)$, $G(E_3,E_1)$ and $G(E_2,E_3)$ are parallel to JE_3, JE_2 and JE_1 respectively. For instance, by (1.4), (1.6) and (1.2), we have

$$\begin{aligned}\langle G(E_1,E_2),JE_1\rangle &= -\langle JG(E_1,E_2),E_1\rangle = \langle G(E_1,JE_2),E_1\rangle\\ &= -\langle G(E_1,E_1),JE_2\rangle = 0\,.\end{aligned}$$

By (1.7), $G(E_1,E_2)$, $G(E_2,E_3)$ and $G(E_3,E_1)$ are unit vectors. Hence we can order E_1, E_2, E_3 in such a way that $JE_1 = G(E_2,E_3)$, $JE_2 = G(E_3,E_1)$ and $JE_3 = G(E_1,E_2)$.

LEMMA 1. *For every $p \in M$*

$$3\int_{UM_p} \|h(V,V)\|^2 = 7\int_{UM_p} \langle h(V,V),JV\rangle^2\,.$$

P r o o f. Consider the following 1-form α on UM_p:

$$\alpha_V(E) = \langle h(V,V), JV\rangle\langle h(V,V), JE\rangle\,.$$

If we denote by δ the codifferential operator on UM_p, then we get

$$(\delta\alpha)(V) = 3\|h(V,V)\|^2 - 7\langle h(V,V), JV\rangle^2\,.$$

Integrating this and using Green's theorem completes the proof.

Lemma 2.

$$\int_{UM}\sum_{i=1}^{3}\langle\nabla h(E_i,V,V), JV\rangle\langle G(V,E_i), h(V,V)\rangle = \tfrac{1}{3}\int_{UM}\langle h(V,V), JV\rangle^2\,.$$

P r o o f. Define the covariant tensor field T_3 by

$$T_3(X_1,\ldots,X_7) = \langle G(X_1,X_2), h(X_3,X_4)\rangle\langle h(X_5,X_6), JX_7\rangle\,.$$

Using (2.11) we get

$$\begin{aligned}\sum_{i=1}^{3} T_3(E_i,E_i,V,\ldots,V) = &-2\langle h(V,V), JV\rangle^2\\ &+\sum_{i=1}^{3}\langle G(E_i,V), \nabla h(E_i,V,V)\rangle\langle h(V,V), JV\rangle\\ &+\sum_{i=1}^{3}\langle G(E_i,V), h(V,V)\rangle\langle\nabla h(E_i,V,V), JV\rangle\\ &+\|h(V,V)\|^2 - \langle h(V,V), JV\rangle^2\,.\end{aligned}$$

Moreover, by (2.13), we have

$$\begin{aligned}(2.15)\qquad &\sum_{i=1}^{3}\langle G(E_i,V), \nabla h(E_i,V,V)\rangle\\ &= \langle G(E_2,V), \nabla h(E_2,V,V)\rangle + \langle G(E_2,V), \nabla h(E_3,V,V)\rangle\\ &= -\langle JE_3, \nabla h(E_2,V,V)\rangle + \langle JE_2, \nabla h(E_3,V,V)\rangle\\ &= -\langle G(E_3,E_2), h(V,V)\rangle = \langle JV, h(V,V)\rangle\,.\end{aligned}$$

Consequently,

$$\begin{aligned}\sum_{i=1}^{3}\nabla T_3(E_i,E_i,V,\ldots,V) = &-2\langle h(V,V), JV\rangle^2 + \|h(V,V)\|^2\\ &+\sum_{i=1}^{3}\langle\nabla h(E_i,V,V), JV\rangle\langle G(E_i,V), h(V,V)\rangle\,.\end{aligned}$$

Now, applying to T_3 the integral formula (1.2) of Chapter I and using Lemma 1 finishes the proof.

LEMMA 3.

$$\int\limits_{UM} \|\nabla h(V,V,V)\|^2 = \int\limits_{UM} \sum_{i=1}^{3} \langle \nabla h(E_i,V,V), JV\rangle^2 + \tfrac{2}{3} \int\limits_{UM} \langle h(V,V), JV\rangle^2 \,.$$

Proof. Because of (2.11) we have

$$\begin{aligned} \|\nabla h(V,V,V)\|^2 &= \sum_{i=1}^{3} \langle \nabla h(V,V,V), JE_i\rangle^2 \\ &= \sum_{i=1}^{3} \{\langle \nabla h(V,V,E_i), JV\rangle + \langle G(E_i,V), h(V,V)\rangle\}^2 \\ &= \sum_{i=1}^{3} \langle \nabla h(V,V,E_i), JV\rangle^2 \\ &\quad - 2\sum_{i=1}^{3} \langle \nabla h(V,V,E_i), JV\rangle \langle G(V,E_i), h(V,V)\rangle \\ &\quad + \|h(V,V)\|^2 - \langle h(V,V), JV\rangle^2 \,. \end{aligned}$$

By integrating this and using also Lemmas 1 and 2 we get the assertion.

LEMMA 4.

$$\int\limits_{UM} \sum_{i=1}^{3} \langle \nabla h(E_i,V,V), JV\rangle^2 = \tfrac{9}{4} \int\limits_{UM} \langle \nabla h(V,V,V), JV\rangle^2 + \tfrac{1}{12} \int\limits_{UM} \langle h(V,V), JV\rangle^2 \,.$$

Proof. Define the 1-form β on UM_p by

$$\beta_V(E) = \langle \nabla h(V,V,V), JV\rangle \langle \nabla h(V,V,V), JE\rangle \,.$$

We have

$$\begin{aligned} (\delta\beta)(V) = &\sum_{i=2}^{3} 3\langle \nabla h(E_i,V,V), JV\rangle \langle \nabla h(V,V,V), JE_i\rangle \\ &+ \sum_{i=2}^{3} \langle \nabla h(V,V,V), JE_i\rangle^2 \\ &+ 3\sum_{i=2}^{3} \langle \nabla h(E_i,V,V), JE_i\rangle \langle \nabla h(V,V,V), JV\rangle \\ &- 2\langle \nabla h(V,V,V), JV\rangle^2 \\ = 3&\sum_{i=2}^{3} \langle \nabla h(V,V,E_i), JV\rangle^2 \\ &+ 3\sum_{i=2}^{3} \langle \nabla h(V,V,E_i), JV\rangle \langle G(E_i,V), h(V,V)\rangle \end{aligned}$$

$$
\begin{aligned}
&\quad + \|\nabla h(V,V,V)\|^2 - 6\langle \nabla h(V,V,V), JV\rangle^2\\
&= 3\sum_{i=1}^{3}\langle \nabla h(V,V,E_i), JV\rangle^2\\
&\quad - 3\sum_{i=1}^{3}\langle \nabla h(V,V,E_i), JV\rangle\langle G(E_i,V), h(V,V)\rangle\\
&\quad + \|\nabla h(V,V,V)\|^2 - 9\langle \nabla h(V,V,V), JV\rangle^2 .
\end{aligned}
$$

If we integrate this, use Green's theorem and Lemmas 2 and 3, we get the assertion.

By combining Lemmas 3 and 4, we obtain

LEMMA 5.

$$\int_{UM} \|\nabla h(V,V,V)\|^2 = \tfrac{9}{4}\int_{UM} \langle \nabla h(V,V,V), JV\rangle^2 + \tfrac{3}{4}\int_{UM} \langle h(V,V), JV\rangle^2 .$$

LEMMA 6.

$$\int_{UM} \langle \nabla h(V,V,V), JV\rangle^2 + \int_{UM} \langle \nabla^2 h(V,V,V,V), JV\rangle\langle h(V,V), JV\rangle = 0 .$$

Proof. Define T_4 by

$$T_4(X_1,\ldots,X_6) = \langle h(X_1,X_2), JX_3\rangle\langle h(X_4,X_5), JX_6\rangle .$$

We have

$$(\nabla^2 T_4)(V,\ldots,V) = 2\langle \nabla^2 h(V,V,V,V), JV\rangle\langle h(V,V), JV\rangle + 2\langle \nabla h(V,V,V), JV\rangle^2 .$$

Applying the first integral formula (1.1) of Chapter I to ∇T_4 completes the proof of Lemma 6.

LEMMA 7.

$$\int_{UM} \|\nabla h(V,V,V)\|^2 + \int_{UM} \sum_{i=1}^{3}\langle \nabla^2 h(E_i,E_i,V,V), JV\rangle\langle h(V,V), JV\rangle = 0 .$$

Proof. This is a consequence of the second integral formula (1.2) of Chapter I applied to ∇T_4. In fact, by a straightforward computation and using also (2.11) and (2.12), we obtain

$$
\begin{aligned}
(\nabla^2 T_4)(E_i,E_i,V,\ldots,V) &= 2\langle \nabla h(V,V,V), JE_i\rangle^2\\
&\quad + 2\langle \nabla^2 h(E_i,E_i,V,V), JV\rangle\langle h(V,V), JV\rangle\\
&\quad + 4\langle \nabla h(E_i,V,V), G(E_i,V)\rangle\langle h(V,V), JV\rangle\\
&\quad - 4\langle h(V,V), JV\rangle^2 .
\end{aligned}
$$

Hence, by (2.15), we have

$$\sum_{i=1}^{3}(\nabla^2 T_4)(E_i,E_i,V,\dots,V)$$
$$=\sum_{i=1}^{3}2\langle\nabla^2 h(E_i,E_i,V,V),JV)\rangle\langle h(V,V),JV\rangle+2\|\nabla h(V,V,V)\|^2\,.$$

Integrating this completes the proof.

LEMMA 8.

$$\int\limits_{UM}\|\nabla h(V,V,V)\|^2+\int\limits_{UM}\langle\nabla^2 h(V,V,V,V),h(V,V)\rangle=0\,.$$

Proof. Consider the tensor field T_1 defined by (3.10), i.e.

$$T_1(X_1,X_2,X_3,X_4)=\langle h(X_1,X_2),h(X_3,X_4)\rangle\,.$$

Lemma 8 follows from (1.1) of Chapter I and (3.11).

LEMMA 9.

$$0=\tfrac{3}{4}\int\limits_{UM}\langle\nabla h(V,V,V),JV\rangle^2$$
$$-\tfrac{1}{12}\int\limits_{UM}\langle h(V,V),JV\rangle^2+\int\limits_{UM}\langle R(V,A_{JV}V)A_{JV}V,V\rangle\,.$$

Proof. Define the function g on UM_p, $p\in M$, by

$$g(V)=\langle h(V,V),JV\rangle-\langle\nabla^2 h(V,V,V,V),JV\rangle\,.$$

We have

$$\begin{aligned}(\varDelta g)(V)=\;&-22g(V)+6\Big\{\sum_{i=2}^{3}\langle h(V,V),JE_i\rangle\langle\nabla^2 h(E_i,V,V,V),JV\rangle\\&+3\sum_{i=2}^{3}\langle h(V,V),JE_i\rangle\langle\nabla^2 h(V,E_i,V,V),JV\rangle\\&+\sum_{i=2}^{3}\langle h(V,V),JE_i\rangle\langle\nabla^2 h(V,V,V,V),JE_i\rangle\Big\}\\&+6\sum_{i=2}^{3}\langle h(V,V),JV\rangle\langle\nabla^2 h(E_i,E_i,V,V),JV\rangle\\&+2\sum_{i=2}^{3}\langle h(V,V),JV\rangle\langle\nabla^2 h(E_i,V,V,V),JE_i\rangle\end{aligned}$$

$$+ 6\Big\{ \sum_{i=2}^{3} \langle h(V,V), JV\rangle \langle \nabla^2 h(V,E_i,V,V), JE_i\rangle$$
$$+ \sum_{i=2}^{3} \langle h(V,V), JV\rangle \langle \nabla^2 h(V,V,E_i,E_i), JV\rangle \Big\}.$$

By (2.2), (2.12) and the Ricci equation we get

$$\begin{aligned}
\langle \nabla^2 h(E_i,V,V,V), JV\rangle &= \langle \nabla^2 h(V,V,V,E_i), JV\rangle \\
&\quad + \langle R^{\perp}(E_i,V)h(V,V), JV\rangle - 2\langle h(R(E_i,V)V,V), JV\rangle \\
&= \langle \nabla^2 h(V,V,V,V), JE_i\rangle + \langle \nabla h(V,V,V), G(V,E_i)\rangle \\
&\quad + \langle \nabla G(V,V,E_i), h(V,V)\rangle + \langle G(V,E_i), \nabla h(V,V,V)\rangle \\
&\quad + \langle [A_{h(V,V)}, A_{JV}]E_i, V\rangle - 2\langle A_{JV}V, R(E_i,V)V\rangle \\
&= \langle \nabla^2 h(V,V,V,V), JE_2\rangle + 2\langle G(V,E_i), \nabla h(V,V,V)\rangle \\
&\quad - \langle JE_i, h(V,V)\rangle + \langle A_{JE_i}V, A_{h(V,V)}V\rangle - \langle A_{JV}V, A_{h(V,V)}E_i\rangle \\
&\quad - 2\langle A_{JV}V, R(E_i,V)V\rangle.
\end{aligned}$$

Hence, by using also the Gauss equation, we obtain

$$\begin{aligned}
(2.16)\quad & \sum_{i=2}^{3} \langle \nabla^2 h(E_i,V,V,V), JV\rangle \langle h(V,V), JE_i\rangle \\
&= \langle \nabla^2 h(V,V,V,V), h(V,V)\rangle - g(V) \\
&\quad + 2\{\langle \nabla h(V,V,V), JE_3\rangle \langle h(V,V), G(E_3,V)\rangle \\
&\quad - \langle \nabla h(V,V,V), JE_2\rangle \langle h(V,V), G(V,E_2)\rangle\} \\
&\quad - \|h(V,V)\|^2 + \langle h(V,V), JV\rangle^2 + \langle A_{h(V,V)}V, A_{h(V,V)}V\rangle \\
&\quad - \langle h(V,V), JV\rangle \langle A_{JV}V, A_{h(V,V)}V\rangle - 2\langle A_{JV}V, R(A_{JV}V,V)V\rangle \\
&= \langle \nabla^2 h(V,V,V,V), h(V,V)\rangle - g(V) \\
&\quad + 2\sum_{i=1}^{3} \langle \nabla h(V,V,E_i), JV\rangle \langle h(V,V), G(E_i,V)\rangle + 2\|h(V,V)\|^2 \\
&\quad - 2\langle h(V,V), JV\rangle^2 - \|h(V,V)\|^2 + \langle h(V,V), JV\rangle^2 \\
&\quad - R(V,A_{JV}V,V,A_{JV}V) - \langle h(V,V), JV\rangle^2 + \|h(V,V)\|^2 \\
&\quad - 2R(V,A_{JV}V,V,A_{JV}V) \\
&= \langle \nabla^2 h(V,V,V,V), h(V,V)\rangle - g(V) \\
&\quad + \sum_{i=1}^{3} \langle \nabla h(V,V,E_i), JV\rangle \langle h(V,V), G(E_i,V)\rangle + 2\|h(V,V)\|^2 \\
&\quad - 2\langle h(V,V), JV\rangle^2 - 3R(V,A_{JV}V,V,A_{JV}V).
\end{aligned}$$

In a similar way we compute

$$\sum_{i=2}^{3}\langle h(V,V),JE_i\rangle\langle\nabla^2 h(V,E_i,V,V),JV\rangle\,,$$

and consequently we get

$$\begin{aligned}
6\Big\{&\sum_{i=2}^{3}\langle h(V,V),JE_i\rangle\langle\nabla^2 h(E_i,V,V,V),JV\rangle\\
&+3\sum_{i=2}^{3}\langle h(V,V),JE_i\rangle\langle\nabla^2 h(V,E_i,V,V),JV\rangle\\
&+\sum_{i=2}^{3}\langle h(V,V),JE_i\rangle\langle\nabla^2 h(V,V,V,V),JE_i\rangle\Big\}\\
&=30\langle h(V,V),\nabla^2 h(V,V,V,V)\rangle+30\|h(V,V)\|^2-30\langle h(V,V),JV\rangle^2\\
&\quad-30g(V)+48\sum_{i=1}^{3}\langle\nabla h(V,V,E_i),JV\rangle\langle h(V,V),G(E_i,V)\rangle\\
&\quad-18R(V,A_{JV}V,V,A_{JV}V)\,.
\end{aligned}$$

We also have

$$\begin{aligned}
\sum_{i=2}^{3}\{&6\langle h(V,V),JV\rangle\langle\nabla^2 h(E_i,E_i,V,V),JV\rangle\\
&+2\langle h(V,V),JV\rangle\langle\nabla^2 h(E_i,V,V,V),JE_i\rangle\}\\
&=8\sum_{i=1}^{3}\langle h(V,V),JV\rangle\langle\nabla^2 h(E_i,E_i,V,V),JV\rangle-8g(V)
\end{aligned}$$

and

$$\begin{aligned}
6\Big\{&\sum_{i=2}^{3}\langle h(V,V),JV\rangle\langle\nabla^2 h(V,E_i,V,V),JE_i\rangle\\
&+\langle h(V,V),JV\rangle\langle\nabla^2 h(V,V,E_i,E_i),JV\rangle\Big\}\\
&=-12g(V)+6\langle h(V,V),JV\rangle^2\,.
\end{aligned}$$

Consequently,

$$\begin{aligned}
(\Delta g)(V)=&-72g(V)+30\langle h(V,V),\nabla^2 h(V,V,V,V)\rangle\\
&+30\|h(V,V)\|^2-24\langle h(V,V),JV\rangle^2\\
&-48\sum_{i=1}^{3}\langle\nabla h(V,V,E_i),JV\rangle\langle h(V,V),G(V,E_i)\rangle
\end{aligned}$$

$$- 18R(V, A_{JV}V, V, A_{JV}V) + 8\sum_{i=1}^{3}\langle h(V,V), JV\rangle\langle\nabla^2 h(E_i, E_i, V, V), JV\rangle\,.$$

Integrating this over UM and using Lemmas 6, 8, 1, 3 and 7 gives

$$72\int\limits_{UM}\langle\nabla h(V,V,V), JV\rangle^2 + 30\int\limits_{UM}\langle h(V,V), JV\rangle^2 - 38\int\limits_{UM}\|\nabla h(V,V,V)\|^2 - 18\int\limits_{UM} R(V, A_{JV}V, V, A_{JV}V) = 0\,.$$

Lemma 5 completes the proof of Lemma 9.

Making use of Lemma 1 we can rewrite Lemma 9 as follows:

LEMMA 10.

$$\tfrac{3}{4}\int\limits_{UM}\langle h(V,V,V), JV\rangle^2 + \int\limits_{UM}\{R(V, A_{JV}V, V, A_{JV}V) - \tfrac{1}{16}(\|A_{JV}V\|^2 - \langle A_{JV}V, V\rangle^2)\} = 0\,.$$

Since $K > 1/16$,

$$R(V, A_{JV}V, V, A_{JV}V) - \tfrac{1}{16}(\|A_{JV}V\|^2 - \langle A_{JV}V, V\rangle^2) \geq 0$$

for any $V \in UM$. Therefore, by Lemma 10,

$$\langle\nabla h(V,V,V), JV\rangle = 0\,, \tag{2.17}$$

$$R(V, A_{JV}V, V, A_{JV}V) = \tfrac{1}{16}(\|A_{JV}V\|^2 - \langle A_{JV}V, V\rangle^2) \tag{2.18}$$

for any $V \in UM$. If there exists a vector $V \in UM$ such that $A_{JV}V$ is not parallel to V, then V and $A_{JV}V$ determine a plane whose sectional curvature equals $1/16$ according to (2.18). This is a contradiction. Therefore all the $A_{JV}V$ are parallel to V. Because of (2.2) this implies

$$\|h(V,V)\|^2 = \langle h(V,V), JV\rangle^2 \tag{2.19}$$

for all $V \in UM$. Formula (2.19) together with Lemma 1 implies that M is totally geodesic. This completes the proof of Theorem 2.3.

3. Totally real surfaces in S^6. Whereas every 3-dimensional totally real submanifold of S^6 is minimal, a 2-dimensional totally real submanifold of S^6 is not necessarily so. For instance, since $S^3(1)$ can be totally real immersed in S^6 (see the beginning of §2), every small hypersphere M in $S^3(1)$ can be immersed in S^6 as a totally real submanifold. The curvature of M is constant, but M is not minimal in S^6. Similarly to the case of 3-dimensional totally real submanifolds of S^6 we have the following formulas for totally real surfaces in S^6:

$$A_{JY}X = -t(Jh(X,Y))\,, \tag{3.1}$$

$$D_X JY = J\nabla_X Y + nJh(X,Y) + G(X,Y) \tag{3.2}$$

for $X, Y \in \mathfrak{X}(M)$, where t and n denote the orthogonal projections of $TS^6_{|M}$ onto TM and the normal bundle N respectively. From (3.1) we get (2.3). Formula (2.6) is also true in the case of totally real surfaces.

Throughout this section we assume that M is a surface of genus 0 minimally and totally real immersed in S^6. V, U will mean a pair of orthonormal vectors tangent to M at some point $p \in M$ or a pair of orthonormal vector fields on M defined in a neighbourhood of p and such that $\nabla_V V = \nabla_V U = \nabla_U V = \nabla_U U = 0$ at p.

Since M is minimal and of genus 0, we have (3.1), (3.2) of Chapter I and

$$\|\nabla h(V,V,V)\| = \|\nabla h(V,V,U)\|\,, \tag{3.3}$$

$$\langle \nabla h(V,V,V), \nabla h(V,V,U)\rangle = 0 \tag{3.4}$$

(see $[\mathrm{Che}]_1$, $[\mathrm{E}]_2$). Formulas (3.3) and (3.4) will also be proved in §2 of Chapter III.

The Gauss equation is given by (3.3) of Chapter I.

From (1.7) it follows that $\|G(V,U)\| = 1$. Formulas (1.2) and (1.6) imply that

$$\langle G(V,U), V\rangle = \langle G(V,JU), V\rangle = 0\,.$$

Similarly, using also (1.4), we obtain

$$\langle G(V,U), U\rangle = \langle G(V,U), JV\rangle = \langle G(V,JU), JV\rangle = \langle G(V,U), JU\rangle = 0\,.$$

Therefore $\{V, U, JV, JU, G(V,U), JG(V,U)\}$ is an orthonormal basis of T_pS^6 at any $p \in M$. Put $\xi = G(V,U)$. By (1.4) and (2.6) we have

$$\begin{aligned} G(\xi,U) &= -V\,, & G(J\xi,U) &= JV\,,\\ G(\xi,V) &= U\,, & G(J\xi,V) &= -JU\,. \end{aligned} \tag{3.5}$$

If we define

$$\begin{aligned} a_1 &= \langle h(V,V), JV\rangle\,, & a_2 &= \langle h(V,V), JU\rangle\,,\\ a_3 &= \langle h(V,V), \xi\rangle\,, & a_4 &= \langle h(V,V), J\xi\rangle\,,\\ c &= \langle h(V,U), \xi\rangle\,, & d &= \langle h(V,U), J\xi\rangle\,, \end{aligned}$$

then, by using also (2.3), we get

$$h(V,V) = a_1 JV + a_2 JU + a_3\xi + a_4 J\xi\,, \tag{3.6}$$

$$h(V,U) = a_2 JV - a_1 JU + c\xi + dJ\xi\,. \tag{3.7}$$

Formulas (3.1) and (3.2) of Chapter I yield

$$a_3^2 + a_4^2 = c^2 + d^2\,, \tag{3.8}$$

$$a_3 c + a_4 d = 0\,. \tag{3.9}$$

We also have

$$\begin{aligned} A_\xi U &= cV - a_3 U\,, & A_\xi V &= a_3 V + cU\,,\\ A_{J\xi} U &= dV - a_4 U\,, & A_{J\xi} V &= a_4 V + dU\,. \end{aligned} \tag{3.10}$$

Using (3.2) we get

$$D_U JV = cJ\xi - d\xi - \xi\,, \qquad D_V JV = a_3 J\xi - a_4\xi\,,$$
$$D_U JU = b_3 J\xi - b_4\xi\,, \qquad D_V JU = cJ\xi - d\xi + \xi\,. \tag{3.11}$$

By (1.5) we have

$$(\widetilde{\nabla}_V G)(V,U) = -JU\,. \tag{3.12}$$

On the other hand, we obtain

$$\begin{aligned}(\widetilde{\nabla}_V G)(V,U) &= \widetilde{\nabla}_V G(V,U) - G(h(V,V),U) - G(V,h(V,U))\\ &= -A_\xi V + D_V\xi - G(h(V,V),U) - G(V,h(V,U))\\ &= -a_4 JV - dJU + D_V\xi\,,\end{aligned} \tag{3.13}$$

where we have used (3.5)–(3.7) and (3.10). Hence

$$D_V\xi = a_4 JV + (d-1)JU\,. \tag{3.14}$$

It follows that $0 = \langle D_V\xi, J\xi\rangle = -\langle \xi, D_V J\xi\rangle$. Since also $\langle D_V J\xi, J\xi\rangle = 0$, we have

$$\begin{aligned}D_V J\xi &= \langle D_V J\xi, JV\rangle JV + \langle D_V J\xi, JU\rangle JU\\ &= -\langle J\xi, D_V JV\rangle JV - \langle J\xi, D_V JU\rangle JU = -a_3 JV - cJU\,.\end{aligned} \tag{3.15}$$

Similarly, we get

$$\begin{aligned}JV = (\widetilde{\nabla}_U G)(V,U) &= \widetilde{\nabla}_U G(V,U) - G(h(U,V),U) - G(V,h(U,U))\\ &= D_U\xi - dJV - b_4 JU\,,\end{aligned}$$

i.e.

$$D_U\xi = (1+d)JV - a_4 JU\,, \tag{3.16}$$

$$D_U J\xi = -cJV + a_3 JU\,. \tag{3.17}$$

LEMMA 3.1. $a_3 = a_4 = c = d = 0$.

Proof. First compute $\nabla h(U,V,V) = \nabla h(V,V,U)$ in two different ways. Namely, we have at p (where $\nabla_V V = \nabla_V U = \nabla_U U = \nabla_U V = 0$)

$$\begin{aligned}\nabla h(U,V,V) &= D_U h(V,V)\\ &= (Ua_1)JV + (Ua_2)JU + (Ua_3)\xi + (Ua_4)J\xi\\ &\quad + a_1(-\xi + cJ\xi - d\xi) + a_2(a_4\xi - a_3 J\xi)\\ &\quad + a_3((d+1)JV - a_4 JU) + a_4(-cJV + a_3 JU)\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\nabla h(V,V,U) &= D_V h(V,U)\\ &= (Va_2)JV - (Va_1)JU + (Vc)\xi + (Vd)J\xi\\ &\quad + a_2(a_3 J\xi - a_4\xi) - a_1(\xi + cJ\xi - d\xi)\\ &\quad + c(a_4 JV + (d-1)JU) + d(-a_3 JV - cJU)\,.\end{aligned} \tag{3.19}$$

By comparing the terms involving JV in (3.18) and (3.19) we obtain

$$Ua_1 + 2a_3d + a_3 - 2ca_4 - Va_2 = 0\,. \tag{3.20}$$

Next we compute $\nabla h(V,V,V) = -\nabla h(U,V,U)$ at p in two different ways:

$$\begin{aligned}\nabla h(V,V,V) &= D_V h(V,V)\\ &= (Va_1)JV + (Va_2)JU + (Va_3)\xi + (Va_4)J\xi\\ &\quad + a_1(a_3J\xi - a_4\xi) + a_2((1-d)\xi + cJ\xi)\\ &\quad + a_3(a_4JV + (d-1)JU) + a_4(-a_3JV - cJU)\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}\nabla h(U,V,U) &= D_U h(V,V)\\ &= (Ua_2)JV - (Ua_1)JU + (Uc)\xi + (Ud)J\xi\\ &\quad + a_2(-\xi + cJ\xi - d\xi) + a_1(a_3J\xi - a_4\xi)\\ &\quad + c((d+1)JV - a_4JU) + d(-cJV + a_3JU)\,.\end{aligned} \tag{3.22}$$

By comparing (3.21) and (3.22) we get

$$Va_2 - Ua_1 + 2da_3 - 2ca_4 - a_3 = 0\,. \tag{3.23}$$

Adding (3.20) and (3.23) yields

$$a_3d - ca_4 = 0\,. \tag{3.24}$$

This, together with (3.8) and (3.9), implies the conclusion of the lemma.

By using Lemma 3.1 and formulas (3.21), (3.19), we find

$$\nabla h(V,V,V) = (Va_1)JV + (Va_2)JU + a_2\xi\,, \tag{3.25}$$

$$\nabla h(V,V,U) = (Va_2)JV - (Va_1)JU - a_1\xi\,. \tag{3.26}$$

Consequently, (3.3) and (3.4) give $a_1a_2 = 0$ and $a_1^2 = a_2^2$. Therefore $h = 0$ and we have

Theorem 3.2. *If M is a surface of genus 0 minimally and totally real immersed in S^6, then M is totally geodesic.*

As a corollary we obtain the following pinching theorem:

Corollary 3.3. *If M is a compact minimal totally real surface in S^6 with Gaussian curvature $K \ge 0$, then either $K \equiv 0$ or $K \equiv 1$.*

Proof. If M is orientable and K is not constantly 0 on M, then the Gauss–Bonnet theorem implies that M has genus 0. If M is not orientable, then we apply Theorem 3.2 to the double covering space of M.

We also have the following complete answer to Problem A (stated in the Introduction) in this case:

Corollary 3.4. *If M is a minimal totally real surface in S^6 with constant Gaussian curvature K, then either $K \equiv 0$ or $K \equiv 1$.*

P r o o f. By Theorem 1.3 of Chapter I we know that $K \geq 0$. Assume $K > 0$. Then M is at least locally an open part of a Borůvka sphere (comp. Theorem 1.2 of Chapter I) and, consequently, formulas (3.1), (3.2) of Chapter I, (3.3) and (3.4) still hold for M. The same proof as that of Theorem 3.2 yields that M is totally geodesic.

As mentioned in §2 the sphere $S^3(1)$ can be imbedded in S^6 as a totally real geodesic submanifold. If M is a surface in $S^3(1)$, then M is a totally real surface in S^6. Therefore the study of totally real surfaces in S^6 extends the study of hypersurfaces in $S^3(1)$, and the results of this section extend some results from, for instance, $[\mathrm{L}]_1$.

It is known (see [K–Y]) that the imbedding

$$T_2 = S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2}) \ni ((x_1, x_2), (y_1, y_2)) \to (x_1, x_2, y_1, y_2) \in S^3(1)\,,$$

where $S^1(1/\sqrt{2}) \subset \mathbb{R}^2$ is the sphere $\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1/2\}$, is minimal. Hence the torus T_2 provides an example of a locally flat minimal real surface in S^6.

III. Surfaces in complex and Sasakian space forms with parallel mean curvature vector

1. Totally real surfaces in Kähler manifolds. Let $\widetilde{M}$ be a Kähler manifold with almost complex structure J and Hermitian metric $\langle\,,\rangle$. We assume moreover that $\widetilde{M}$ has constant holomorphic sectional curvature c. Let M be a totally real submanifold of $\widetilde{M}$. The orthogonal complement to $JTM \subset N$ in N will be denoted by $\mathcal{H}$. Since M is Kähler, we have

$$-A_{JZ}W = tJh(W, Z)\,, \tag{1.1}$$

$$D_W JZ = J\nabla_W Z + nJh(W, Z) \tag{1.2}$$

for $W, Z \in \mathfrak{X}(M)$, where t, n, as in the previous chapter, denote the projections onto TM and N respectively in $T\widetilde{M}_{|M} = TM \oplus N$. The equality (1.1) implies

$$\langle h(W, Z), JS\rangle = \langle h(W, S), JZ\rangle \tag{1.3}$$

for $W, Z, S \in \mathfrak{X}(M)$. If ξ, η are normal vector fields belonging to $\mathcal{H}$, i.e. with values in $\mathcal{H}$, then

$$\langle D_W J\xi, \eta\rangle = \langle JD_W\xi, \eta\rangle\,. \tag{1.4}$$

If $\widetilde{R}$ denotes the curvature tensor of $\widetilde{M}$, then

$$\begin{aligned} &\widetilde{R}(W, Z)S \\ &\quad = \frac{c}{4}\{\langle Z, S\rangle W - \langle W, S\rangle Z - \langle Z, JS\rangle JW + \langle W, JS\rangle JZ + 2\langle W, JZ\rangle JS\} \end{aligned} \tag{1.5}$$

for $W, Z, S \in \mathfrak{X}(M)$, where c is the holomorphic curvature of $\widetilde{M}$. This implies the Codazzi equation given by (2.8) of Chapter I. Denote by $\varkappa$ the mean curvature vector of M.

From now on we assume that M is an orientable surface. V, U will mean a pair of orthonormal vectors tangent to M at a point $p \in M$ or a pair of orthonormal vector fields tangent to M defined in a neighbourhood of a point $p \in M$ and such that $\nabla_V V = \nabla_U U = \nabla_V U = \nabla_U V = 0$ at p. Of course, V and U can be prescribed at p. In the considered case the Ricci and Gauss equations are

$$(1.6)\qquad R^{\perp}(V,U)\xi = \frac{c}{4}(\langle V, J\xi\rangle JU - \langle U, J\xi\rangle JV) + \langle h(V,U),\xi\rangle\langle h(V,V) - h(U,U)\rangle - \langle h(V,V) - h(U,U),\xi\rangle h(V,U)$$

for any normal vector ξ, and

$$(1.7)\qquad c/4 = K + \|h(V,U)\|^2 - \langle h(V,V), h(U,U)\rangle\,,$$

where K is the Gaussian curvature of M. If M is a compact surface of genus 0 minimally immersed in $\widetilde{M}$, then (1.7) and the Gauss–Bonnet theorem yield that c is positive.

It is known that we can introduce on M an atlas of isothermal coordinates which endows M with a complex structure. If $z = x + iy$ is an isothermal coordinate on an open subset $\mathcal{U}$ of M, then we put

$$(1.8)\qquad X = \partial/\partial x\,,\qquad Y = \partial/\partial y\,;$$

we keep this notation throughout the rest of the paper. The vector fields X, Y are orthogonal and $\langle X, X\rangle = \langle Y, Y\rangle = E$, where E is a positive function on $\mathcal{U}$. We have

$$(1.9)\qquad \nabla_X X = -\nabla_Y Y = \frac{1}{2E}((XE)X - (YE)Y)\,,$$

$$(1.10)\qquad \nabla_Y X = \nabla_X Y = \frac{1}{2E}((YE)X + (XE)Y)\,.$$

If $\widetilde{z} = \widetilde{x} + i\widetilde{y}$ is a second isothermal coordinate on $\mathcal{U}$ and we denote

$$\widetilde{X} = \partial/\partial\widetilde{x}\,,\qquad \widetilde{Y} = \partial/\partial\widetilde{y}\,,$$

then we have

$$(1.11)\qquad d\widetilde{z}(X) = a + ib\,,\qquad d\widetilde{z}(Y) = i(a+ib)\,,$$

$$(1.12)\qquad \widetilde{X} = \frac{a}{F}X - \frac{b}{F}Y\,,\qquad \widetilde{Y} = \frac{b}{F}X + \frac{a}{F}Y$$

at $p \in U$, where

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = d_p\widetilde{z}\circ(d_p z)^{-1}$$

and $F = a^2 + b^2$. Using the above facts we can prove the following two lemmas. If in these lemmas $k > 2$, then any argument of T not written out is X.

LEMMA 1.1. *Let M be a surface equipped with an atlas of isothermal coordinates and let X, Y be given by* (1.8). *If T is a symmetric k-covariant* ($k \geq 2$) *tensor field on M such that*

$$T(W_1,\ldots,W_{k-2},X,X)+T(W_1,\ldots,W_{k-2},Y,Y)=0 \tag{1.13}$$

for any $W_1,\ldots,W_{k-2}\in\mathfrak{X}(M)$, then the form

$$(T(X,\ldots,X)-iT(X,\ldots,X,Y))dz^k \tag{1.14}$$

is well defined on the whole of M. Moreover,

$$T(\nabla_X X,\ldots,X)=-T(\nabla_Y X,\ldots,Y)\,, \tag{1.15}$$

$$T(\nabla_Y X,\ldots,X)=T(\nabla_X X,\ldots,Y)\,. \tag{1.16}$$

Proof. Let $\widetilde{z}=\widetilde{x}+i\widetilde{y}$ be another isothermal coordinate on $\mathcal{U}$. By the symmetry of T and by (1.13), (1.12), we obtain at p

$$T(\widetilde{X},\ldots,\widetilde{X})-iT(\widetilde{X},\ldots,\widetilde{Y})=\frac{(a-bi)^k}{F^k}(T(X,\ldots,X)-iT(X,\ldots,Y))\,.$$

This formula, together with (1.11), implies that the form given by (1.14) is well defined on M. By a straightforward computation and by (1.9), (1.10) we obtain (1.15) and (1.16).

LEMMA 1.2. *Let T be a k-covariant* ($k\geq 2$) *symmetric and $\mathcal{V}$-valued tensor field on M, where $\mathcal{V}$ is a Riemannian vector bundle over M with a metric tensor $\langle\,,\rangle$ and the induced norm $\|\ \|$. If T satisfies* (1.13), *then the form*

$$(\|T(X,\ldots,X)\|^2-\|T(X,\ldots,X,Y)\|^2 \\ -2i\langle T(X,\ldots,X),T(X,\ldots,X,Y)\rangle)dz^{2k} \tag{1.17}$$

is well defined on M. Moreover,

$$\begin{aligned} &-\langle T(\nabla_Y X,\ldots,X),T(X,\ldots,Y)\rangle-\langle T(X,\ldots,X),T(\nabla_Y X,\ldots,Y)\rangle \\ &\quad=\langle T(\nabla_X X,\ldots,X),T(X,\ldots,X)\rangle-\langle T(\nabla_X X,\ldots,Y),T(X,\ldots,Y)\rangle\,, \end{aligned} \tag{1.18}$$

$$\begin{aligned} &\langle T(\nabla_Y X,\ldots,X),T(X,\ldots,X)\rangle-\langle T(\nabla_Y X,\ldots,Y),T(X,\ldots,Y)\rangle \\ &\quad=\langle T(\nabla_X X,\ldots,X),T(X,\ldots,Y)\rangle+\langle T(X,\ldots,X),T(\nabla_X X,\ldots,Y)\rangle\,. \end{aligned} \tag{1.19}$$

Proof. Let $\xi_1,\ldots,\xi_m$ be an orthonormal frame of $\mathcal{V}$ defined on $\mathcal{U}$. By Lemma 1.1, for each $\alpha\in\{1,\ldots,m\}$

$$(\langle T(X,\ldots,X),\xi_\alpha\rangle-i\langle T(X,\ldots,X,Y),\xi_\alpha\rangle)dz^k$$

is independent of the choice of isothermal coordinates on $\mathcal{U}$. It follows that the form

$$\begin{aligned} &\sum_{\alpha=1}^{m}(\langle T(X,\ldots,X),\xi_\alpha\rangle-i\langle T(X,\ldots,Y),\xi_\alpha\rangle)^2dz^{2k} \\ &\quad=(\|T(X,\ldots,X)\|^2-\|T(X,\ldots,Y)\|^2-2i\langle T(X,\ldots,X),T(X,\ldots,Y)\rangle)dz^{2k} \end{aligned}$$

is also independent of the choice of isothermal coordinates. From (1.9) and (1.10), we can easily obtain (1.18) and (1.19), which completes the proof of the lemma.

The *first normal space* at $p \in M$ is defined by

$$\widetilde{N}_{1p} = \operatorname{span}\{h(W_1, W_2) : W_1, W_2 \in T_pM\}.$$

If we set

$$M_1 = \{p \in M : \dim \widetilde{N}_{1p} = \max_{q \in M} \dim \widetilde{N}_{1q}\},$$

then M_1 is an open subset of M. If we denote by $\widetilde{N}_1$ the collection $\bigcup_{p \in M_1} \widetilde{N}_{1p}$, then $\widetilde{N}_1$ is a vector bundle over M_1. If we define $\widetilde{\sigma}_3$ to be the orthogonal projection of $\nabla h_{|M_1}$ onto the orthogonal complement to $\widetilde{N}_1$, then $\widetilde{\sigma}_3$ is a symmetric N-valued tensor field on M_1. Now we set

$$\widetilde{N}_{2p} = \operatorname{span}\{\widetilde{\sigma}_3(W_1, W_2, W_3) : W_1, W_2, W_3 \in T_pM,\ p \in M_1\},$$
$$M_2 = \{p \in M_1 : \dim \widetilde{N}_{2p} = \max_{q \in M_1} \dim \widetilde{N}_{2q}\}, \qquad \widetilde{N}_2 = \bigcup_{p \in M_2} \widetilde{N}_{2p}.$$

The set M_2 is open in M, $\widetilde{N}_2$ is a vector bundle over M_2 and if we define $\widetilde{\sigma}_4$ as the orthogonal projection of $\nabla^2 h_{|M_2}$ onto the orthogonal complement to $\widetilde{N}_1 + \widetilde{N}_2$, then $\widetilde{\sigma}_4$ is an N-valued 4-covariant tensor field on M_2. Continuing this process, we can define a sequence $M_0 = M \supset M_1 \supset \ldots$ of open subsets of M, a sequence of vector bundles: $\widetilde{N}_0 = 0$, $\widetilde{N}_1$ defined on $M_1, \ldots, \widetilde{N}_l$ on M_l, and a sequence of N-valued $(l+2)$-covariant tensor fields $\widetilde{\sigma}_{l+2}$, for $l = 1, \ldots$, where $\widetilde{\sigma}_{l+2}$ is defined on M_l as the orthogonal projection of $\nabla^l h_{|M_l}$ onto the orthogonal complement to $\widetilde{N}_0 + \ldots + \widetilde{N}_l$. If $\widetilde{\sigma}_{l+2} \equiv 0$, then $\widetilde{N}_{l+k} = 0$, $M_{k+l} = M_l$, $\nabla^{l+k-1} h$ has values in $\widetilde{N}_0 + \ldots + \widetilde{N}_l$ on M_l and $\widetilde{\sigma}_{l+k+2} \equiv 0$ for any $k \geq 1$.

2. Surfaces of genus 0 with parallel mean curvature vector. Throughout this section we use the same notations as in §1. Let M be a compact surface of genus 0 totally real immersed into $\widetilde{M}$. We assume moreover that the mean curvature vector of the immersion is parallel with respect to D. If W, Z is a pair of orthogonal vectors tangent to M and such that their lengths are equal, then for any $l \geq 0$ we define

$$\Omega^l(W, W) = \begin{cases} \frac{1}{2}(h(W,W) - h(Z,Z)) & \text{for } l = 0, \\ \nabla^l h(W, \ldots, W) & \text{for } l > 0, \end{cases} \tag{2.1}$$

$$\Omega^l(W, Z) = \nabla^l h(W, \ldots, W, Z) \quad \text{for every } l \geq 0. \tag{2.2}$$

Since $D\varkappa = 0$, we have

$$\nabla^l h(W_1, \ldots, W_l, W, W) = -\nabla^l h(W_1, \ldots, W_l, Z, Z) \tag{2.3}$$

provided $l > 0$. We shall write Ω instead of Ω^0.

Consider the following symmetric 2-form defined by using isothermal coordinates:

$$(2.4)\qquad (\langle \Omega(X,X),\varkappa\rangle - i\langle \Omega(X,Y),\varkappa\rangle)dz^2\,.$$

By making use of (1.12) and (1.11) it is easy to check that this form is well defined on the whole of M. In fact, let $\widetilde{z}=\widetilde{x}+i\widetilde{y}$ be another isothermal coordinate on $\mathcal{U}$. By (1.12), we obtain

$$(2.5)\qquad \begin{aligned} \Omega(\widetilde{X},\widetilde{X}) &= \frac{a^2-b^2}{F^2}\Omega(X,X) - \frac{2ab}{F^2}\Omega(X,Y)\,,\\ \Omega(\widetilde{X},\widetilde{Y}) &= \frac{2ab}{F^2}\Omega(X,X) + \frac{a^2-b^2}{F^2}\Omega(X,Y)\,. \end{aligned}$$

Consequently, using also (1.11), we find

$$\begin{aligned} &(\langle \Omega(\widetilde{X},\widetilde{X}),\varkappa\rangle - i\langle \Omega(\widetilde{X},\widetilde{Y}),\varkappa\rangle)d\widetilde{z}^2\\ &\qquad = \frac{(a-bi)^2}{F^2}(\langle \Omega(X,X),\varkappa\rangle - i\langle \Omega(\widetilde{X},\widetilde{Y}),\varkappa\rangle)d\widetilde{z}^2\\ &\qquad = (\langle \Omega(X,X),\varkappa\rangle - i\langle \Omega(X,Y),\varkappa\rangle)dz^2\,. \end{aligned}$$

It is known (see [Ru], [G–R]) that if $D\varkappa=0$, then the Codazzi equation gives

$$(2.6)\quad D_X(\Omega(X,X))+D_Y(\Omega(X,Y))=0\,,\qquad D_Y(\Omega(X,X))-D_X(\Omega(X,Y))=0\,.$$

It follows that the function $\langle \Omega(X,X),\varkappa\rangle - i\langle \Omega(X,Y),\varkappa\rangle$ satisfies the Cauchy–Riemann equations. Hence the form given by (2.4) is holomorphic and by the Riemann–Roch theorem it is zero. An isothermal coordinate around a point $p\in M$ can be chosen in such a way that the directions of X_p and Y_p are prescribed. Therefore we have

LEMMA 2.1. *For any orthonormal basis V, U of T_pM the vectors $\Omega(V,V)$ and $\Omega(V,U)$ are orthogonal to $\varkappa$.*

Consider now the form

$$(2.7)\qquad (\|\Omega(X,X)\|^2 - \|\Omega(X,Y)\|^2 - 2i\langle \Omega(X,X),\Omega(X,Y)\rangle)dz^4\,.$$

This form is also well defined on M and by (2.6) it is holomorphic. The computation can be found in [G–R]. Hence, by the Riemann–Roch theorem, it is zero and consequently we get

LEMMA 2.2. *For any orthonormal basis V, U of T_pM the vectors $\Omega(V,V)$, $\Omega(V,U)$ are orthogonal and have the same length.*

We denote by $\widetilde{C}_p$ the orthogonal complement to $\varkappa_p$ in $\widetilde{N}_{1p}$. The space $\widetilde{C}_p$ is spanned by $\Omega(V,V)$ and $\Omega(V,U)$. Since $\langle \Omega(V,U),\varkappa\rangle=0$, we have

$$\langle h(V,U),h(V,V)\rangle = -\langle h(V,U),h(U,U)\rangle\,.$$

But $\langle \Omega(V,V),\Omega(V,U)\rangle=0$, i.e.

$$\langle h(V,V),h(V,U)\rangle = \langle h(U,U),h(U,V)\rangle\,.$$

This means that $\langle h(V,V), h(V,U)\rangle = 0$ and consequently $\|h(V,V)\|$ is constant on UM_p.

Let $\xi_1,\dots,\xi_{2n-2}$ be an orthonormal normal frame defined in a neighbourhood $\mathcal{U}$ of p. Assume also that $z(p)=0$, where z is, as before, an isothermal coordinate on $\mathcal{U}$. We define

$$w_\alpha = \langle \Omega(X,X),\xi_\alpha\rangle - i\langle \Omega(X,Y),\xi_\alpha\rangle\,. \tag{2.8}$$

The functions w_α satisfy

$$\frac{\partial}{\partial \overline{z}} w_\alpha = \sum_{\beta=1}^{2n-2} c_{\alpha\beta}\Big(\frac{\partial}{\partial \overline{z}}\Big) w_\beta\,, \tag{2.9}$$

where

$$c_{\alpha\beta}(Z) = \langle D_Z\xi_\alpha,\xi_\beta\rangle\,. \tag{2.10}$$

Namely, we have

$$\begin{aligned}(X+iY)(\langle \Omega(X,X),\xi_\alpha\rangle - i\langle \Omega(X,Y),\xi_\alpha\rangle)\\ = \langle D_X\Omega(X,X),\xi_\alpha\rangle - i\langle D_X\Omega(X,Y),\xi_\alpha\rangle\\ + i\langle D_Y\Omega(X,X),\xi_\alpha\rangle + \langle D_Y\Omega(X,Y),\xi_\alpha\rangle\\ + \langle \Omega(X,X),D_X\xi_\alpha\rangle - i\langle \Omega(X,Y),D_X\xi_\alpha\rangle\\ + i\langle \Omega(X,X),D_Y\xi_\alpha\rangle + \langle \Omega(X,Y),D_Y\xi_\alpha\rangle\end{aligned}$$

and

$$\begin{aligned}\sum_{\beta=1}^{2n-2}\langle D_{X+iY}\xi_\alpha,\xi_\beta\rangle\{\langle \Omega(X,X),\xi_\beta\rangle - i\langle \Omega(X,Y),\xi_\beta\rangle\}\\ = \langle D_{X+iY}\xi_\alpha,\Omega(X,X)\rangle - i\langle D_{X+iY}\xi_\alpha,\Omega(X,Y)\rangle\,.\end{aligned}$$

By (2.6) we obtain the required equality.

Recall now the following theorem due to S. S. Chern [Che]$_1$.

THEOREM 2.3. *Let $w_\alpha(z)$, $\alpha=1,\dots,m$, be complex-valued functions defined in a neighbourhood $\mathcal{U}$ of $0\in\mathbb{C}$ and satisfying*

$$\frac{\partial w_\alpha}{\partial \overline{z}} = \sum_{\beta=1}^{m} a_{\alpha\beta}w_\beta\,,\qquad \alpha,\beta=1,\dots,m\,,$$

where $a_{\alpha\beta}$ are complex-valued C^1 functions. Let $w(z)=(w_1(z),\dots,w_m(z))$ and suppose $w(0)=0$. Then either w is identically zero on $\mathcal{U}$ or $z=0$ is an isolated zero of w and there exists an integer $r>0$ such that $\widetilde{w}(z)=w(z)/z^r$ is continuous and non-zero at $z=0$.

We shall apply the above theorem to the functions defined by (2.8). Suppose that $\widetilde{C}$ is not constantly zero on $\mathcal{U}$ and $\widetilde{C}_p=0$. Then p is an isolated zero of the function $M\ni p\to \dim\widetilde{C}_p$. Let r be the integer satisfying the condition of

Theorem 2.3. Then by the proof of Lemma 3.3 of [Ba], we know that $w(z)$ is smooth at 0. Let $N^{\mathbb{C}}$ be the complexification of the normal bundle N. Consider the $N^{\mathbb{C}}$-valued vector field on $\mathcal{U}$ given by

$$\xi(z) = \sum_{\alpha=1}^{2n+2} \frac{w_\alpha(z)\xi_\alpha(z)}{z^r}\,.$$

Then ξ is smooth on $\mathcal{U}$ and non-zero at $z = 0$. If we set $\operatorname{Re} z^r = \nu(z)$ and $\operatorname{Im} z^r = \mu(z)$, then we have $\xi(z) = \xi^1(z) - i\xi^2(z)$, where

$$\xi^1(z) = \frac{\nu(z)\Omega(X,X) - \mu(z)\Omega(X,Y)}{\nu(z)^2 + \mu(z)^2}\,,$$
$$\xi^2(z) = \frac{\mu(z)\Omega(X,X) - \nu(z)\Omega(X,Y)}{\nu(z)^2 + \mu(z)^2}\,.$$

If $z \neq 0$, then $\|\xi^1(z)\| = \|\xi^2(z)\|$, $\xi^1(z)$ is perpendicular to $\xi^2(z)$, the vectors $\xi^1(z)$, $\xi^2(z)$ belong to $\widetilde{C}_z$ and have limits as $z \to 0$. Since one of these limits is non-zero, so is the other one. The limits are orthogonal. Hence we can extend the bundle $\widetilde{C}$ to a smooth bundle, say C, over the whole of M. Therefore we have

LEMMA 2.4. *The set $M \setminus M_1$ consists of isolated points and $\widetilde{N}_1$ can be extended to a smooth bundle, say N_1, over the whole of M.*

Remark 2.5. If we assume that M is minimal and $\widetilde{M} = \mathbb{C}P^n$, then N_1 is real-analytic. Namely, in this case the given immersion is real-analytic and the functions w_α and $c_{\alpha\beta}(\partial/\partial\bar{z})$ are real-analytic. It is clear that except at 0

$$\frac{\partial}{\partial\bar{z}}\widetilde{w}(z) = c(z)\widetilde{w}(z)\,,$$

where $c(z)$ is the matrix function $[c_{\alpha\beta}(\partial/\partial\bar{z})]_{\alpha,\beta=1,\ldots,2n-2}$. Since $\widetilde{w}$ is smooth, we have

$$\frac{\partial}{\partial\bar{z}}\widetilde{w}(0) = \lim_{\substack{z\to 0\\ z\neq 0}} \frac{\partial}{\partial\bar{z}}\widetilde{w}(z) = \lim_{z\to 0} c(z)\widetilde{w}(z) = c(0)\widetilde{w}(0)\,.$$

Thus w is also a solution of the system $(\partial/\partial\bar{z})u = cu$. If we restrict $w(z)$ to the line $\{y = 0\}$, then $w(z) = a_0 + a_1x + a_2x^2 + \ldots$ in a neighbourhood of 0. Since $w(z)/z^s$ is continuous at 0 for every $s \le r$, we have $a_0 = \ldots = a_{r-1} = 0$. This means that w is real-analytic on $\{y = 0\} \cap \mathcal{U}$. By the Cauchy–Kovalevskaya and Holmgren theorems, w is real-analytic in $\mathcal{U}$.

The bundle C is 0- or 2-dimensional. If $\dim C = 0$, then M is totally umbilical and $\nabla h = 0$. Indeed, by differentiating the equality $h(V,V) = h(U,U)$ at a point $p \in M$, where, as in the previous section, $\nabla_V V = \nabla_V U = \nabla_U U = \nabla_U V = 0$, we get

$$\nabla h(V,V,V) - \nabla h(V,U,U) = 0\,, \qquad \nabla h(U,V,V) - \nabla h(U,U,U) = 0\,.$$

But, by (2.3), $\nabla h(V,U,U) = -\nabla h(V,V,V)$ and $\nabla h(U,U,U) = -\nabla h(U,V,V)$. Hence $\nabla h = 0$.

Assume for a moment that $\widetilde{M}$ is a Riemannian manifold of constant sectional curvature. Of course, we do not assume that M is totally real. The facts we have just obtained in §2 are also valid in this case. By combining them with the "reduction theorem" of Erbacher [Er], we obtain

THEOREM 2.6. *Let M be a surface of genus 0 immersed in a constant curved Riemannian manifold $\widetilde{M}$ with parallel mean curvature vector.*

(1) *If $\dim \widetilde{N}_{1p} \le 1$ for every $p \in M$, then M is totally umbilical. If moreover M is not totally geodesic, then it is contained in a 3-dimensional totally geodesic submanifold of $\widetilde{M}$.*

(2) *If $\dim \widetilde{M} = 4$, then M is minimal or totally umbilical.*

The assertion (2) is related to Theorem 5 of [Y].

If the codimension of M is 1 and M has constant mean curvature, then it has parallel mean curvature vector. Hence Theorem 1.9 of Chapter I is an immediate corollary of the above proposition.

Return now to the "totally real" case. By (1.3) we get

$$\langle h(V,V), JU\rangle = \langle h(V,U), JV\rangle\,, \qquad \langle h(U,V), JU\rangle = \langle h(U,U), JV\rangle\,.$$

If we differentiate these equalities and use (1.2), then we get

$$\begin{aligned} 2\langle h(V,V), Jh(V,U)\rangle &= \langle \nabla h(V,V,U), JV\rangle - \langle \nabla h(V,V,V), JU\rangle\,,\\ -2\langle h(U,U), Jh(V,U)\rangle &= -\langle \nabla h(U,V,V), JV\rangle + \langle \nabla h(V,V,V), JU\rangle\,. \end{aligned} \tag{2.11}$$

Therefore

$$\langle \Omega(V,V), J\Omega(V,U)\rangle = 0\,. \tag{2.12}$$

Hence C is a totally real subbundle of N. It is easy to verify that the following form is well defined on M:

$$(\langle JX, \varkappa\rangle - i\langle JY, \varkappa\rangle)\, dz\,. \tag{2.13}$$

Moreover, by (1.2), we find that

$$\begin{aligned} X\langle JX, \varkappa\rangle &= \langle J\nabla_X X, \varkappa\rangle + \langle Jh(X,X), \varkappa\rangle\,,\\ -Y\langle JY, \varkappa\rangle &= -\langle J\nabla_Y Y, \varkappa\rangle - \langle Jh(Y,Y), \varkappa\rangle\,. \end{aligned}$$

By (1.9) and the fact that $h(X,X) + h(Y,Y)$ is parallel to $\varkappa$, we obtain the first Cauchy–Riemann equation. The verification of the second equation is also straightforward. By the Riemann–Roch theorem the form given by (2.13) vanishes on M. Hence we have

LEMMA 2.7. *$\varkappa$ belongs to $\mathcal{H}$.*

Consequently, by differentiating the equalities $\langle JX, \varkappa\rangle = 0$ and $\langle JY, \varkappa\rangle = 0$, we get

$$\langle Jh(Y,X), \varkappa\rangle = 0\,, \qquad \langle Jh(X,X), \varkappa\rangle = 0\,, \qquad \langle Jh(Y,Y), \varkappa\rangle = 0\,.$$

By combining these equalities with (2.12), we obtain

LEMMA 2.8. *N_1 is a totally real subbundle of N.*

Since $\langle \varkappa, JW\rangle = 0$ for any $W \in \mathfrak{X}(M)$, we have

$$\begin{aligned} \langle \Omega(X,X), JY\rangle &= \langle \Omega(X,Y), JX\rangle\,, \\ \langle \Omega(X,Y), JY\rangle &= -\langle \Omega(X,X), JX\rangle\,. \end{aligned} \tag{2.14}$$

Consider now the following form:

$$(\langle \Omega(X,X), JX\rangle - i\langle \Omega(X,Y), JX\rangle)dz^3\,. \tag{2.15}$$

It is well defined on M because

$$\begin{aligned} &(\langle \Omega(\widetilde{X},\widetilde{X}), J\widetilde{X}\rangle - i\langle \Omega(\widetilde{X},\widetilde{Y}), J\widetilde{X}\rangle) \\ &\qquad = \frac{(a-ib)^3}{F^3}(\langle \Omega(X,X), JX\rangle - i\langle \Omega(X,Y), JX\rangle)\,, \end{aligned}$$

where $\widetilde{X}$, $\widetilde{Y}$ are defined by an isothermal coordinate $\widetilde{z}$. Since N_1 is totally real, we have

$$X\langle \Omega(X,X), JX\rangle = \langle D_X\Omega(X,X), JX\rangle + \langle \Omega(X,X), J\nabla_X X\rangle\,, \tag{2.16}$$

$$-Y\langle \Omega(X,Y), JX\rangle = -\langle D_Y\Omega(X,Y), JX\rangle - \langle \Omega(X,Y), J\nabla_Y X\rangle\,. \tag{2.17}$$

Moreover, by (1.9) and (2.14), we get

$$\begin{aligned} \langle \Omega(X,X), J\nabla_X X\rangle &= \frac{1}{2E}\{(XE)\langle \Omega(X,X), JX\rangle - (YE)\langle \Omega(X,X), JY\rangle\} \\ &= -\frac{1}{2E}\{(YE)\langle \Omega(X,Y), JX\rangle + (XE)\langle \Omega(X,Y), JY\rangle\}\,. \end{aligned}$$

Now, by making use of (2.6) and (1.10), we find that the functions defined by (2.16) and (2.17) are equal. The second Cauchy–Riemann equation can be verified in the same way. Applying the Riemann–Roch theorem to the form (2.15) implies that it is zero. By replacing X by Y and Y by $-X$ we obtain

$$\langle \Omega(Y,Y), JY\rangle = 0\,, \qquad \langle \Omega(Y,X), JY\rangle = 0\,.$$

Hence C is contained in $\mathcal{H}$ and this fact together with Lemma 2.7 yields

LEMMA 2.9. *N_1 is a subbundle of $\mathcal{H}$.*

As a consequence of the above considerations we obtain

THEOREM 2.10. *Let M be a surface of genus 0 totally real and with parallel mean curvature vector immersed in a Kähler manifold $\widetilde{M}$ of constant holomorphic sectional curvature c.*

(1) *If $\dim_{\mathbb{C}} \widetilde{M} = 2$, then M is totally geodesic and $c > 0$.*
(2) *If $\dim_{\mathbb{C}} \widetilde{M} = 3$, then M is totally umbilical and $\nabla h = 0$.*
(3) *If $\dim_{\mathbb{C}} \widetilde{M} = 4$, then M is minimal or totally umbilical.*

In particular, if M is a minimal totally real surface of genus 0 in a 3-dimensional Kähler manifold of constant holomorphic sectional curvature, then M is totally geodesic. This is a generalization of Theorem 7(i) of [Y]. Another generalization of this theorem is (1) of Theorem 2.10.

Observe also that

$$\langle \nabla^l h(W_1, \ldots, W_{l+2}), \varkappa\rangle = 0 \tag{2.18}$$

for $l > 1$. Indeed, by differentiating $\langle \Omega(V, V), \varkappa\rangle = 0$, we find that

$$\langle \nabla h(V, V, V), \varkappa\rangle = 0\,, \qquad \langle \nabla h(U, V, V), \varkappa\rangle = 0\,.$$

Hence, by (2.3), formula (2.18) is true for $l = 1$. Now, by induction, one can easily verify (2.18) for every l.

Let us now state the main theorem of this section.

THEOREM 2.11. *Let M be a surface of genus 0 totally real and with parallel mean curvature vector immersed in a Kähler manifold of constant holomorphic sectional curvature. For any $l \geq 0$ we have*

(1) *There is a smooth vector bundle N_{l+1} over M such that $N_{l+1} = \widetilde{N}_{l+1}$ on M_{l+1}. For every $l \geq 1$, N_{l+1} is 0- or 2-dimensional.*

Define σ_{l+3} to be the orthogonal projection of $\nabla^{l+1}h$ onto the orthogonal complement to $N_1 + \ldots + N_{l+1}$. Also, set $\sigma_2 = h$ and

$$\omega_{l+2}(W, W) = \tfrac{1}{2}(\sigma_{l+2}(W, \ldots, W) - \sigma_{l+2}(W, \ldots, W, Z, Z))\,,$$
$$\omega_{l+2}(W, Z) = \sigma_{l+2}(W, \ldots, W, Z)$$

for any orthogonal basis W, Z of T_pM such that $\|W\| = \|Z\|$. (For $l = 2$, $\sigma_l(W, \ldots, W, Z, Z)$ means $\sigma_2(Z, Z)$.)

(2) *$M \setminus M_{l+1}$ consists of isolated points.*

(3) *For every $k \leq l$*

$$\langle \Omega^l(V, V), \Omega^k(V, V)\rangle = \langle \Omega^l(V, U), \Omega^k(V, U)\rangle\,,$$
$$\langle \Omega^l(V, V), \Omega^k(V, U)\rangle = -\langle \Omega^l(V, U), \Omega^k(V, V)\rangle\,.$$

(4) *For every $k \leq l$*

$$\langle \Omega^l(V, V), \omega_{k+2}(V, V)\rangle = \langle \Omega^l(V, U), \omega_{k+2}(V, U)\rangle\,,$$
$$\langle \Omega^l(V, V), \omega_{k+2}(V, U)\rangle = -\langle \Omega^l(V, U), \omega_{k+2}(V, V)\rangle\,.$$

(5) $\langle \omega_{l+2}(V, V), \omega_{l+2}(V, U)\rangle = 0\,,\ \|\omega_{l+2}(V, V)\| = \|\omega_{l+2}(V, U)\|\,.$

(6) *The orthogonal projection of $\nabla^{l+1}h$ onto orthogonal complement to $N_0 + \ldots + N_l$ is symmetric. In particular, σ_{l+3} is symmetric.*

(7) *For any $W_1, \ldots, W_{l+3} \in \mathfrak{X}(M)$*

$$\nabla\sigma_{l+2}(W_1, W_2, W_3, \ldots, W_{l+3}) - \nabla\sigma_{l+2}(W_2, W_1, W_3, \ldots, W_{l+3}) \in N_0 + \ldots + N_l\,.$$

(8) *$\nabla^l h$ has values in $\mathcal{H}$.*

(9) *For any $W_1, \ldots, W_{l+2}, Z_1, \ldots, Z_{k+2} \in \mathfrak{X}(M)$ and $k \leq l$*

$$\langle \nabla^l h(W_1, \ldots, W_{l+2}), J\nabla^k h(Z_1, \ldots, Z_{k+2})\rangle = 0\,.$$

P r o o f. We have proved (1)–(9) for $l = 0$. If M is totally umbilical, i.e. if $C = 0$ and consequently $\nabla h = 0$, then all the assertions for $l > 0$ are trivial. Assume M is not totally umbilical and (1)–(9) are true for all natural numbers less than or equal to l. If $\sigma_{l+2} \equiv 0$ on M, then all the assertions except (3) and (4) are trivial for $l + 1$. Hence, in general, we assume $\sigma_{l+2} \not\equiv 0$ and we will admit the case $\sigma_{l+2} \equiv 0$ only in the proof of (3) and (4). Notice also that the inductive assumption on (9) is equivalent to the assumption that $N_1 + \ldots + N_{l+1}$ is orthogonal to $J(N_1 + \ldots + N_{l+1})$.

By differentiating at p the first equality of (3) we obtain

$$(2.19) \quad \langle \Omega^{l+1}(V,V), \Omega^k(V,V)\rangle + \langle \Omega^l(V,V), \Omega^{k+1}(V,V)\rangle = \langle \Omega^{l+1}(V,U), \Omega^k(V,U)\rangle + \langle \Omega^l(V,U), \Omega^{k+1}(V,U)\,.$$

If $k \le l-1$, then by using the inductive assumption on (3), we get

$$(2.20) \quad \langle \Omega^{l+1}(V,V), \Omega^k(V,V)\rangle = \langle \Omega^{l+1}(V,U), \Omega^k(V,U)\rangle\,.$$

If $k = l$, then (2.20) is an immediate consequence of (2.19). Hence (2.20) holds for any $k \le l$.

Similarly, we obtain

$$(2.21) \quad \langle \Omega^{l+1}(V,V), \Omega^k(V,U)\rangle = -\langle \Omega^{l+1}(V,U), \Omega^k(V,V)\rangle$$

for any $k \le l$, and

$$(2.22) \quad \langle \nabla^{l+1}h(U,V,\ldots,V), \Omega^l(V,U)\rangle = -\langle \nabla^{l+1}(U,V,\ldots,V,U), \Omega^l(V,V)\rangle\,,$$

$$(2.23) \quad \langle \nabla^{l+1}h(U,V,\ldots,V,U), \Omega^l(V,U)\rangle = \langle \nabla^{l+1}h(U,V,\ldots,V), \Omega^l(V,V)\rangle\,.$$

If we differentiate the equality

$$\langle \nabla^l h(W_2,\ldots,W_{l+3}), JZ\rangle = 0$$

and use the inductive assumption on (8) and (9), then we obtain

$$\langle \nabla^{l+1} h(W_1,\ldots,W_{l+3}), JZ\rangle = 0$$

for every $W_1,\ldots,W_{l+3}, Z \in \mathfrak{X}(M)$, which proves (8) for $l+1$. Similarly, by differentiating the equality in (9) and using the inductive assumption on (8) and (9), we get

$$(2.24) \quad \langle \nabla^{l+1}h(W_1,\ldots,W_{l+3}), J\nabla^k h(Z_1,\ldots,Z_{k+2})\rangle = 0$$

for any $W_1,\ldots,W_{l+3}, Z_1,\ldots,Z_{k+2} \in \mathfrak{X}(M)$ and $k \le l-1$.

Assume now $k = l$. We have

$$(2.25) \quad \langle \nabla^{l+1}h(V,\ldots,V), J\nabla^l h(V,\ldots,V,U)\rangle = \langle J\nabla^l h(V,\ldots,V), \nabla^{l+1}h(V,\ldots,V,U)\rangle$$

and

$$\langle \nabla^{l+1}h(U,V,\ldots,V), J\nabla^l h(V,\ldots,V,U)\rangle = \langle J\nabla^l h(V,\ldots,V), \nabla^{l+1}h(U,V,\ldots,V,U)\rangle\,.$$

By the last equality and the inductive assumption on (6) and (9), we get

$$(2.26)\quad \langle \nabla^{l+1}h(U,V,\ldots,V), J\nabla^l h(V,\ldots,V,U)\rangle \\ = -\langle J\nabla^l h(V,\ldots,V), \nabla^{l+1}h(V,\ldots,V)\rangle\,.$$

If $l=0$, then differentiating $\langle h(V,V), Jh(U,U)\rangle = 0$ and using (2.3) for $l=1$, we get

$$\langle \nabla h(V,V,V), Jh(U,U)\rangle = -\langle \nabla h(V,V,V), Jh(V,V)\rangle\,,$$
$$\langle \nabla h(U,V,V), Jh(U,U)\rangle = -\langle \nabla h(U,V,V), Jh(V,V)\rangle\,.$$

The above formulas and the inductive assumption on (6) and (9) imply that the tensor field

$$(2.27)\quad (\mathfrak{X}(M))^{2l+5} \ni (W_1,\ldots,W_{2l+5}) \to \\ \langle \nabla^{l+1}h(W_1,\ldots,W_{l+3}), J\nabla^l h(W_{l+4},\ldots,W_{2l+5})\rangle$$

is symmetric.

We shall prove (6) for $l+1$. It is sufficient to show that for any $1\le r\le l+3$ and at any $p\in M_{l+1}$

$$(2.28)\quad P = \nabla^{l+2}h(W_1,\ldots,W_{r-1},W_r,W_{r+1},W_{r+2},\ldots,W_{l+4}) \\ -\nabla^{l+2}h(W_1,\ldots,W_{r-1},W_{r+1},W_r,W_{r+2},\ldots,W_{l+4}) \in N_0+\ldots+N_{l+1}$$

for $W_1,\ldots,W_{l+4} = V$ or U and $W_{r+1}=V$, $W_{r+2}=U$. Assume $r>1$. Then we have at p

$$P = D_{W_1}(\nabla^{l+1}h(W_2,\ldots,W_r,W_{r+1},\ldots,W_{l+4}) \\ -\nabla^{l+1}h(W_2,\ldots,W_{r+1},W_r,\ldots,W_{l+4}))$$

and P belongs to $N_0+\ldots+N_{l+1}$ by the inductive assumption on (6). If $r=1$, then (2.5) of Chapter I, (1.6) and the inductive assumption on (8) give

$$P = 2\langle h(V,U), \nabla^l h(W_3,\ldots,W_{l+4})\rangle \Omega(V,V) \\ -2\langle \Omega(V,V), \nabla^l h(W_3,\ldots,W_{l+4})\rangle h(V,U) - \nabla^l h(R(V,U)W_3,\ldots,W_{l+4}) \\ -\ldots - \nabla^l h(W_3,\ldots,R(V,U)W_{l+4}) \in N_0+\ldots+N_{l+1}\,.$$

Since M_{l+1} is dense in M, the proof of (6) for $l+1$ is complete.

Recall now the assumption $\sigma_{l+2}\not\equiv 0$. Let $p\in M_{l+1}$. By the inductive assumption on (4), (5) and by (2.17) we know that for any $r\le l$ there exist a neighbourhood $\mathcal{U}\subset M_{l+1}$ of p and smooth functions $\beta_1^r,\ldots,\beta_{2r}^r$ defined on $\mathcal{U}$ such that

$$(2.29)\quad \Omega^r(V,V) = \beta_1^r\omega_2(V,V)+\beta_2^r\omega_2(V,U)+\ldots \\ +\beta_{2r-1}^r\omega_{r+1}(V,V)+\beta_{2r}^r\omega_{r+1}(V,U)+\omega_{r+2}(V,V)\,,$$

$$(2.30)\quad \Omega^r(V,U) = -\beta_2^r\omega_2(V,V)+\beta_1^r\omega_2(V,U)-\ldots \\ -\beta_{2r}^r\omega_{r+1}(V,V)+\beta_{2r-1}^r\omega_{r+1}(V,U)+\omega_{r+2}(V,U)\,.$$

There are also smooth functions $\beta_1^{l+1}, \dots, \beta_{2l+2}^{l+1}, \widetilde{\beta}_1^{l+1}, \dots, \widetilde{\beta}_{2l+2}^{l+1}$ on $\mathcal{U}$ such that

$$\begin{aligned} \Omega^{l+1}(V,V) = {} & \beta_1^{l+1}\omega_2(V,V) + \beta_2^{l+1}\omega_2(V,U) + \dots \\ & + \beta_{2l+1}^{l+1}\omega_{l+2}(V,V) + \beta_{2l+2}^{l+1}\omega_{l+2}(V,U) + \omega_{l+3}(V,V) \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} \Omega^{l+1}(V,U) = {} & \widetilde{\beta}_1^{l+1}\omega_2(V,V) + \widetilde{\beta}_2^{l+1}\omega_2(V,U) + \dots \\ & + \widetilde{\beta}_{2l+1}^{l+1}\omega_{l+2}(V,V) + \widetilde{\beta}_{2l+2}^{l+1}\omega_{l+2}(V,U) + \omega_{l+3}(V,U) . \end{aligned}$$

By using (2.20), (2.21) for $k = 0$ and (5) for $l = 0$, we obtain

$$\widetilde{\beta}_1^{l+1} = -\beta_2^{l+1} , \qquad \widetilde{\beta}_2^{l+1} = \beta_1^{l+1} .$$

Assume now that

$$\langle \Omega^{l+1}(V,V), \omega_r(V,V)\rangle = \langle \Omega^{l+1}(V,U), \omega_r(V,U)\rangle , \tag{2.32}$$

$$\langle \Omega^{l+1}(V,U), \omega_r(V,V)\rangle = - \langle \Omega^{l+1}(V,V), \omega_r(V,U)\rangle \tag{2.33}$$

for every $r \le k+1$ and some $k \le l$. By (2.20), (2.29) and (2.30) we obtain

$$\begin{aligned} & \langle \Omega^{l+1}(V,V), \beta_1^k\omega_2(V,V) + \beta_2^k\omega_2(V,U) + \dots \\ & \qquad + \beta_{2k-1}^k\omega_{k+1}(V,V) + \beta_{2k}^k\omega_k(V,U) + \omega_{k+2}(V,V)\rangle \\ & = \langle \Omega^{l+1}(V,U), - \beta_2^k\omega_2(V,V) + \beta_1^k\omega_2(V,U) - \dots \\ & \qquad - \beta_{2k}^k\omega_{k+1}(V,V) + \beta_{2k-1}^k\omega_{k+1}(V,U) + \omega_{k+2}(V,U)\rangle . \end{aligned}$$

Consequently, using (2.32) and (2.33) we get

$$\langle \Omega^{l+1}(V,V), \omega_{k+2}(V,V)\rangle = \langle \Omega^{l+1}(V,U), \omega_{k+2}(V,U)\rangle . \tag{2.34}$$

Similarly, formulas (2.21), (2.29), (2.30), (2.32) and (2.33) imply

$$\langle \Omega^{l+1}(V,U), \omega_{k+2}(V,V)\rangle = -\langle \Omega^{l+1}(V,V), \omega_{k+2}(V,U)\rangle . \tag{2.35}$$

Hence (2.34) and (2.35) hold for any $k \le l$.

Since $\|\omega_{k+2}(V,U)\| = \|\omega_{k+2}(V,V)\| \ne 0$ and $\langle \omega_{k+2}(V,U), \omega_{k+2}(V,V)\rangle = 0$ for $k \le l$, we have

$$\begin{aligned} \Omega^{l+1}(V,U) = {} & - \beta_2^{l+1}\omega_2(V,V) + \beta_1^{l+1}\omega_2(V,U) - \dots \\ & - \beta_{2l+2}^{l+1}\omega_{l+2}(V,V) + \beta_{2l+1}^{l+1}\omega_{l+2}(V,U) + \omega_{l+3}(V,U) . \end{aligned} \tag{2.36}$$

We have proved (4) for $l+1$ and $k \le l$ on M_{l+1}, i.e. by (2) it holds on the whole of M (under the assumption $\sigma_{l+2} \not\equiv 0$). Assume now $\sigma_{l+2} \equiv 0$, where $l > 0$. Since we have also assumed that M is not totally umbilical, there exists $0 \le r_0 \le l-1$ such that $\sigma_{r_0+3} \equiv 0$ and $\omega_{r_0+2}(V,V) \not\equiv 0$, $\omega_{r_0+2}(V,U) \not\equiv 0$. By using the same arguments as above, in a neighbourhood of any point $p \in M_{r_0+1}$ we obtain

$$\begin{aligned} \Omega^{l+1}(V,V) = {} & \beta_1^{l+1}\omega_2(V,V) + \beta_2^{l+1}\omega_2(V,U) + \dots \\ & + \beta_{2r_0+1}^{l+1}\omega_{r_0+2}(V,V) + \beta_{2r_0+2}^{l+1}\omega_{r_0+2}(V,U) , \end{aligned}$$

$$\begin{aligned}\Omega^{l+1}(V,U) = & -\beta_2^{l+1}\omega_2(V,V)+\beta_1^{l+1}\omega_2(V,U)-\dots\\ & -\beta_{2r_0+2}^{l+1}\omega_{r_0+2}(V,V)+\beta_{2r_0+1}^{l+1}\omega_{r_0+2}(V,U)\,,\end{aligned}$$

which finishes the proof of (3) and gives (4) in this case.

We know that

$$\nabla^{l+2}h(U,V,\dots,V)-\nabla^{l+2}h(V,\dots,V,U)\in N_0+\dots+N_{l+1}\,.$$

Therefore, by differentiating (2.31), (2.36) and by the fact that $\sigma_2,\dots,\sigma_{l+2}$ have values in $N_0+\dots+N_{l+1}$, we obtain at p

$$\begin{aligned}&\beta_1^{l+1}\nabla\sigma_2(U,V,V)+\beta_2^{l+1}\nabla\sigma_2(U,V,U)+\dots+\beta_{2l+1}^{l+1}\nabla\sigma_{l+2}(U,V,\dots,V)\\ &\quad+\beta_{2l+2}^{l+1}\nabla\sigma_{l+2}(U,V,\dots,V,U)+\nabla\sigma_{l+3}(U,V,\dots,V)\\ &\quad+\beta_2^{l+1}\nabla\sigma_2(V,V,V)-\beta_1^{l+1}\nabla\sigma_2(V,V,U)+\dots+\beta_{2l+2}^{l+1}\nabla\sigma_{l+2}(V,\dots,V)\\ &\quad-\beta_{2l+1}^{l+1}\nabla\sigma_{l+2}(V,\dots,V,U)-\nabla\sigma_{l+3}(V,\dots,V,U)\in N_0+\dots+N_{l+1}\,.\end{aligned}$$

Hence, the inductive assumption on (7) and the symmetry of $\sigma_2,\dots,\sigma_{l+3}$ yield

$$\nabla\sigma_{l+3}(U,V,\dots,V)-\nabla\sigma_{l+3}(V,U,V,\dots,V)\in N_0+\dots+N_{l+1}\,. \tag{2.37}$$

Similarly, we have

$$\nabla^{l+2}h(V,\dots,V)+\nabla^{l+2}h(U,V,\dots,V,U)\in N_0+\dots+N_{l+1}\,,$$

i.e.

$$\begin{aligned}&\beta_1^{l+1}\nabla\sigma_2(V,V,V)+\beta_2^{l+1}\nabla\sigma_2(V,V,U)+\dots+\beta_{2l+1}^{l+1}\nabla\sigma_{l+2}(V,\dots,V)\\ &\quad+\beta_{2l+2}^{l+1}\nabla\sigma_{l+2}(V,\dots,V,U)+\nabla\sigma_{l+3}(V,\dots,V)-\beta_2^{l+1}\nabla\sigma_2(U,V,V)\\ &\quad+\beta_1^{l+1}\nabla\sigma_2(U,V,U)-\dots-\beta_{2l+2}^{l+1}\nabla\sigma_{l+2}(U,V,\dots,V)\\ &\quad+\beta_{2l+1}^{l+1}\nabla\sigma_{l+2}(U,V,\dots,V,U)+\nabla\sigma_{l+3}(U,V,\dots,V,U)\in N_0+\dots+N_{l+1}\,.\end{aligned}$$

Consequently, $\nabla\sigma_{l+3}(V,\dots,V,V)+\nabla\sigma_{l+3}(U,V,\dots,V,U)\in N_0+\dots+N_{l+1}$, i.e.

$$\nabla\sigma_{l+3}(U,V,\dots,V,U)-\nabla\sigma_{l+3}(V,U,V,\dots,V,U)\in N_0+\dots+N_{l+1}\,. \tag{2.38}$$

Formulas (2.37) and (2.38) imply (7) for $l+1$.

By (2.31) and (2.36) we have

$$\begin{aligned}&\langle\Omega^{l+1}(V,V),\Omega^{l+1}(V,U)\rangle=\langle\omega_{l+3}(V,V),\omega_{l+3}(V,U)\rangle\\ &\qquad=\langle\Omega^{l+1}(V,V),\omega_{l+3}(V,U)\rangle=\langle\Omega^{l+1}(V,U),\omega_{l+3}(V,V)\rangle\,,\\ &\qquad\langle\Omega^{l+1}(V,U),\omega_{l+3}(V,U)\rangle=\|\omega_{l+3}(V,U)\|^2\,,\\ &\qquad\langle\Omega^{l+1}(V,V),\omega_{l+3}(V,V)\rangle=\|\omega_{l+3}(V,V)\|^2\end{aligned} \tag{2.39}$$

and

$$\|\Omega^{l+1}(V,V)\| = \|\Omega^{l+1}(V,U)\| \quad \text{iff} \quad \|\omega_{l+3}(V,V)\| = \|\omega_{l+3}(V,U)\|\,. \tag{2.40}$$

Consider now the form

$$(\|\sigma_{l+3}(X,\ldots,X)\|^2 - \|\sigma_{l+3}(X,\ldots,X,Y)\|^2 \\ -2i\langle\sigma_{l+3}(X,\ldots,X),\sigma_{l+3}(X,\ldots,X,Y)\rangle)\,dz^{2(l+3)}\,. \tag{2.41}$$

By Lemma 1.2 it is well defined on the whole of M. It is also holomorphic. Indeed, we have

$$\begin{aligned} X(\|\sigma_{l+3}(X,\ldots,X)\|^2 &- \|\sigma_{l+3}(X,\ldots,X,Y)\|^2) \\ &= 2\{\langle\nabla\sigma_{l+3}(X,\ldots,X),\sigma_{l+3}(X,\ldots,X)\rangle \\ &\quad + (l+3)\langle\sigma_{l+3}(\nabla_X X,\ldots,X),\sigma_{l+3}(X,\ldots,X)\rangle \\ &\quad - \langle\nabla\sigma_{l+3}(X,\ldots,X,Y),\sigma_{l+3}(X,\ldots,X,Y)\rangle \\ &\quad - (l+2)\langle\sigma_{l+3}(\nabla_X X,X,\ldots,X,Y),\sigma_{l+3}(X,\ldots,X,Y)\rangle \\ &\quad - \langle\sigma_{l+3}(X,\ldots,X,\nabla_X Y),\sigma_{l+3}(X,\ldots,X,Y)\rangle\} \end{aligned}$$

and

$$\begin{aligned} -2Y\langle\sigma_{l+3}(X,\ldots,X),&\sigma_{l+3}(X,\ldots,X,Y)\rangle \\ &= -2\{\langle\nabla\sigma_{l+3}(Y,X,\ldots,X),\sigma_{l+3}(X,\ldots,X,Y)\rangle \\ &\quad + (l+3)\langle\sigma_{l+3}(\nabla_Y X,\ldots,X),\sigma_{l+3}(X,\ldots,X,Y)\rangle \\ &\quad + \langle\sigma_{l+3}(X,\ldots,X),\nabla\sigma_{l+3}(Y,X,\ldots,X,Y)\rangle \\ &\quad + (l+2)\langle\sigma_{l+3}(X,\ldots,X),\sigma_{l+3}(\nabla_Y X,\ldots,X,Y)\rangle \\ &\quad + \langle\sigma_{l+3}(X,\ldots,X),\sigma_{l+3}(X,\ldots,X,\nabla_Y Y)\rangle\}\,. \end{aligned}$$

By using (7) for $l+1$ and (1.19) from Lemma 1.2 we obtain the first Cauchy–Riemann equation. In a similar way one can verify the second equation. Since the form given by (2.41) is holomorphic, it is zero and consequently by (2.39) and (2.40) the proof of (5), (3) and (4) for $l+1$ is complete.

Let $p \in M_{l+1}$ and let β^r_j, $j = 1,\ldots,2r$, $r = 1,\ldots,l+1$ be smooth functions defined in a neighbourhood $\mathcal{U} \subset M_{l+1}$ of p such that (2.29)–(2.31) and (2.36) are satisfied. There are functions $\alpha_1,\ldots,\alpha_{2l+2}$ such that

$$\Omega^{l+1}(V,V) = \alpha_1\Omega(V,V) + \alpha_2\Omega(V,U) + \ldots \\ + \alpha_{2l+1}\Omega^l(V,V) + \alpha_{2l+2}\Omega^l(V,U) + \omega_{l+3}(V,V)\,, \tag{2.42}$$

$$\Omega^{l+1}(V,U) = \widetilde{\alpha}_1\Omega(V,V) + \widetilde{\alpha}_2\Omega(V,U) + \ldots \\ + \widetilde{\alpha}_{2l+1}\Omega^l(V,V) + \widetilde{\alpha}_{2l+2}\Omega^l(V,U) + \omega_{l+3}(V,U)\,. \tag{2.43}$$

We also have

$$\begin{aligned}(2.44)\qquad \Omega^{l+1}(V,V) &= \alpha_1\omega_2(V,V) + \alpha_2\omega_2(V,U) \\ &+ \alpha_3(\beta_1^1\omega_2(V,V) + \beta_2^1\omega_2(V,U)) + \alpha_3\omega_3(V,V) \\ &+ \alpha_4(-\beta_2^1\omega_2(V,V) + \beta_1^1\omega_2(V,U)) + \alpha_4\omega_3(V,U) + \dots \\ &+ \alpha_{2l-1}(\beta_1^{l-1}\omega_2(V,V) + \beta_2^{l-1}\omega_2(V,U) + \dots \\ &+ \beta_{2l-3}^{l-1}\omega_l(V,V) + \beta_{2l-2}^{l-1}\omega_l(V,U)) + \alpha_{2l-1}\omega_{l+1}(V,V) \\ &+ \alpha_{2l}(-\beta_2^{l-1}\omega_2(V,V) + \beta_1^{l-1}\omega_2(V,U) - \dots \\ &- \beta_{2l-2}^{l-1}\omega_l(V,V) + \beta_{2l-3}^{l-1}\omega_l(V,U)) + \alpha_{2l}\omega_{l+1}(V,U) \\ &+ \alpha_{2l+1}(\beta_1^{l}\omega_2(V,V) + \beta_2^{l}\omega_2(V,U) + \dots \\ &+ \beta_{2l-1}^{l}\omega_{l+1}(V,V) + \beta_{2l}^{l}\omega_{l+1}(V,U)) \\ &+ \alpha_{2l+1}\omega_{l+2}(V,V) + \alpha_{2l+2}(-\beta_2^{l}\omega_2(V,V) + \beta_1^{l}\omega_2(V,U) \\ - \dots - \beta_{2l}^{l}\omega_{l+1}(V,V) &+ \beta_{2l-1}^{l}\omega_{l+1}(V,U)) + \alpha_{2l+2}\omega_{l+2}(V,U) + \omega_{l+3}(V,V)\end{aligned}$$

and

$$\begin{aligned}(2.45)\qquad \Omega^{l+1}(V,U) &= \widetilde{\alpha}_1\omega_2(V,V) + \widetilde{\alpha}_2\omega_2(V,U) \\ &+ \widetilde{\alpha}_3(\beta_1^1\omega_2(V,V)) + \beta_2^1\omega_2(V,U) + \widetilde{\alpha}_3\omega_3(V,V) \\ &+ \widetilde{\alpha}_4(-\beta_2^1\omega_2(V,V) + \beta_1^1\omega_2(V,U)) + \widetilde{\alpha}_4\widetilde{\omega}_3(V,U) + \dots \\ &+ \widetilde{\alpha}_{2l-1}(\beta_1^{l-1}\omega_2(V,V) + \beta_2^{l-1}\omega_2(V,U) + \dots \\ &+ \beta_{2l-3}^{l-1}\omega_l(V,V) + \beta_{2l-2}^{l-1}\omega_l(V,U)) + \widetilde{\alpha}_{2l-1}\omega_{l+1}(V,V) \\ &+ \widetilde{\alpha}_{2l}(-\beta_2^{l-1}\omega_2(V,V) + \beta_1^{l-1}\omega_2(V,U) - \dots \\ &- \beta_{2l-2}^{l-1}\omega_l(V,V) + \beta_{2l-3}^{l-1}\omega_l(V,U)) + \widetilde{\alpha}_{2l}\omega_{l+1}(V,U) \\ &+ \widetilde{\alpha}_{2l+1}(\beta_1^{l}\omega_2(V,V) + \beta_2^{l}\omega_2(V,U) + \dots \\ &+ \beta_{2l-1}^{l}\omega_{l+1}(V,V) + \beta_{2l}^{l}\omega_{l+1}(V,U)) \\ &+ \widetilde{\alpha}_{2l+1}\omega_{l+2}(V,V) + \widetilde{\alpha}_{2l+2}(-\beta_2^{l}\omega_2(V,V) + \beta_1^{l}\omega_2(V,U) \\ - \dots - \beta_{2l}^{l}\omega_{l+1}(V,V) &+ \beta_{2l-1}^{l}\omega_{l+1}(V,U)) + \widetilde{\alpha}_{2l+2}\omega_{l+2}(V,U) + \omega_{l+3}(V,U)\,.\end{aligned}$$

By comparing (2.31) with (2.44) and (2.36) with (2.45) we obtain

$$\begin{aligned}\alpha_{2l+1} &= \beta_{2l+1}^{l+1}\,, & \alpha_{2l+2} &= \beta_{2l+2}^{l+1}\,,\\ \widetilde{\alpha}_{2l+1} &= -\,\beta_{2l+2}^{l+1}\,, & \widetilde{\alpha}_{2l+2} &= \beta_{2l+1}^{l+1}\,.\end{aligned}$$

This means that $\widetilde{\alpha}_{2l+1} = -\alpha_{2l+2}$ and $\widetilde{\alpha}_{2l+2} = \alpha_{2l+1}$. Now we compare the terms involving $\omega_{l+1}(V,V)$ and $\omega_{l+1}(V,U)$ in (2.31) and (2.44) as well as in (2.36) and (2.45). We get

$$-\alpha_{2l+2}\beta^l_{2l} + \alpha_{2l+1}\beta^l_{2l-1} + \alpha_{2l-1} = \beta^{l+1}_{2l-1}\,,$$
$$\alpha_{2l+2}\beta^l_{2l-1} + \alpha_{2l+1}\beta^l_{2l} + \alpha_{2l} = \beta^{l+1}_{2l}$$

and

$$-\beta^l_{2l}\alpha_{2l+1} - \alpha_{2l+2}\beta^l_{2l-1} + \widetilde{\alpha}_{2l-1} = -\,\beta^{l+1}_{2l}\,,$$
$$\alpha_{2l+1}\beta^l_{2l-1} - \alpha_{2l+2}\beta^l_{2l} + \widetilde{\alpha}_{2l} = \beta^{l+1}_{2l-1}\,.$$

Consequently, $\widetilde{\alpha}_{2l-1} = -\alpha_{2l}$ and $\widetilde{\alpha}_{2l} = \alpha_{2l-1}$. Continuing this process, we obtain

$$\widetilde{\alpha}_1 = -\alpha_2\,, \quad \widetilde{\alpha}_2 = \alpha_1\,,\ \dots\,,\ \widetilde{\alpha}_{2l+1} = -\alpha_{2l+2}\,, \quad \widetilde{\alpha}_{2l+2} = \alpha_{2l+1}\,.$$

Therefore, we have on $\mathcal{U}$

$$\begin{aligned} \Omega^{l+1}(V,U) = &-\alpha_2\Omega(V,V) + \alpha_1\Omega(V,U) - \dots - \alpha_{2l+2}\Omega^l(V,V) \\ &+ \alpha_{2l+1}\Omega(V,U) + \omega_{l+3}(V,U)\,. \end{aligned} \tag{2.46}$$

By the inductive assumption on (8) and (8) for $l+1$ we know that $\omega_{l+3}(V,V)$ and $\omega_{l+3}(V,U)$ belong to $\mathcal{H}$. If we project these vectors orthogonally onto $J(N_1+\dots+N_{l+1})$ and use (2.24) and the symmetry of (2.27), then we get

$$\varrho_1 J\Omega^l(V,V) + \varrho_2 J\Omega^l(V,U) \quad \text{and} \quad \varrho_2 J\Omega^l(V,V) - \varrho_1 J\Omega^l(V,U)$$

respectively, for some functions ϱ_1, ϱ_2. If the orthogonal projection of $\omega_{l+3}(V,V)$ onto $J(TM+N_1+\dots+N_{l+1})$ is equal to $\gamma\xi$, where ξ is the unit vector field, then the orthogonal projection of $\omega_{l+3}(V,U)$ onto the same bundle is equal to $\gamma_1\xi + \delta J\xi + \eta$ for some function δ, where η is perpendicular to $J(TM+N_1+\dots$ $\dots+N_{l+1})+N_1+\dots+N_{l+1}+\operatorname{span}\{\xi, J\xi\}$. By making use of (5) for $l+1$ and the inductive assumption on (3), we get $\gamma^2 = \gamma_1^2 + \delta^2 + \|\eta\|^2$ and $\gamma\gamma_1 = 0$. It follows that $\gamma_1 = 0$. Consequently, we can write

$$\begin{aligned} \Omega^{l+1}(V,V) = {}&\alpha_1\Omega(V,V) + \alpha_2\Omega(V,U) + \dots + \alpha_{2l+1}\Omega^l(V,V)l \\ &+ \alpha_{2l+2}\Omega^l(V,U) + \varrho_1 J\Omega^l(V,V) + \varrho_2 J\Omega^l(V,U) + \gamma\xi\,, \end{aligned} \tag{2.47}$$

$$\begin{aligned} \Omega^{l+1}(V,U) = &-\alpha_2\Omega(V,V) + \alpha_1\Omega(V,U) - \dots - \alpha_{2l+2}\Omega^l(V,V) \\ &+ \alpha_{2l+1}\Omega^l(V,U) + \varrho_2 J\Omega^l(V,V) - \varrho_1 J\Omega^l(V,U) + \delta J\xi + \eta\,. \end{aligned} \tag{2.48}$$

By the inductive assumption on (9) and (3), we get

$$\begin{aligned} &\langle \Omega^{l+1}(V,V), J\Omega^{l+1}(V,U)\rangle \\ &= \langle \alpha_1\Omega(V,V) + \alpha_2\Omega(V,U) + \dots + \alpha_{2l+1}\Omega^l(V,V) \\ &\quad + \alpha_{2l+2}\Omega^l(V,U), -\varrho_2\Omega^l(V,V) + \varrho_1\Omega^l(V,U)\rangle \\ &\quad + \langle \varrho_1 J\Omega^l(V,V) + \varrho_2 J\Omega^l(V,U), -\alpha_2 J\Omega(V,V) + \alpha_1 J\Omega(V,U) \\ &\quad - \dots - \alpha_{2l+2}J\Omega^l(V,V) + \alpha_{2l+1}J\Omega^l(V,U)\rangle - \gamma\delta = -\gamma\delta\,. \end{aligned} \tag{2.49}$$

If we differentiate (2.47) at p and use the inductive assumption on (9), (3), (8), then we obtain

$$\begin{aligned}\langle \Omega^{l+2}(V,V), J\Omega^l(V,U)\rangle &= \langle \alpha_{2l+1}\Omega^{l+1}(V,V) + \alpha_{2l+2}\Omega^{l+1}(V,U)\\ &\quad + \varrho_1 J\Omega^{l+1}(V,V) + \varrho_2 J\Omega^{l+1}(V,U)\\ &\quad + (V\varrho_2)J\Omega^{l+1}(V,U) + \gamma D_V\xi, J\Omega^l(V,U)\rangle\\ &= (\alpha_{2l+1}\varrho_2 - \alpha_{2l+2}\varrho_1 + V\varrho_2)\|\Omega^l(V,V)\|^2 + \varrho_1\langle\Omega^{l+1}(V,V),\Omega^l(V,U)\rangle\\ &\quad + \varrho_2\langle\Omega^{l+1}(V,U),\Omega^l(V,U)\rangle + \gamma\langle D_V\xi, J\Omega^l(V,U)\rangle\,.\end{aligned}$$

Moreover,

$$\langle D_V\xi, J\Omega^l(V,U)\rangle = -\langle\xi, J\Omega^{l+1}(V,U)\rangle = \langle J\xi, \Omega^{l+1}(V,U)\rangle = \delta\,.$$

Consequently, using also (3) for $l+1$, we obtain

$$\begin{aligned}(2.50)\quad \langle \Omega^{l+2}(V,V), J\Omega^l(V,U)\rangle &= (\alpha_{2l+1}\varrho_2 - \alpha_{2l+2}\varrho_1 + V\varrho_2)\|\Omega^l(V,V)\|^2\\ &\quad + \varrho_1\langle\Omega^{l+1}(V,V),\Omega^l(V,U)\rangle + \varrho_2\langle\Omega^{l+1}(V,V),\Omega^l(V,V)\rangle + \gamma\delta\,.\end{aligned}$$

Similarly we get

$$\begin{aligned}&\langle\nabla^{l+2}h(U,V,\dots,V,U), J\Omega^l(V,U)\rangle\\ &\quad= \langle -\alpha_{2l+2}\nabla^{l+1}h(U,V,\dots,V) + \alpha_{2l+1}\nabla^{l+1}h(U,V,\dots,V,U)\\ &\qquad + \varrho_2 J\nabla^{l+1}h(U,V,\dots,V) - \varrho_1 J\nabla^{l+1}h(U,V,\dots,V,U)\\ &\qquad - (U\varrho_1)J\Omega^l(V,U) + \delta J D_U\xi + D_U\eta, J\Omega^l(V,U)\rangle\,.\end{aligned}$$

By the inductive assumption on (6) and (9) and by (3) for $l+1$, we have

$$\begin{aligned}\langle\nabla^{l+1}h(U,V,\dots,V), J\Omega^l(V,U)\rangle &= \langle\Omega^{l+1}(V,U), J\Omega^l(V,U)\rangle\\ &= -\varrho_1\|\Omega^l(V,V)\|^2\,,\\ \langle\nabla^{l+1}h(U,V,\dots,V,U), J\Omega^l(V,U)\rangle &= -\langle\Omega^{l+1}(V,V), J\Omega^l(V,U)\rangle\\ &= -\varrho_2\|\Omega^l(V,U)\|^2\,,\\ \langle D_U\xi, \Omega^l(V,U)\rangle &= -\langle\xi, \nabla^{l+1}h(U,V,\dots,V,U)\rangle\\ &= \langle\xi, \Omega^{l+1}(V,V)\rangle = \gamma\,,\\ \langle D_U\eta, J\Omega^l(V,U)\rangle &= -\langle\eta, J\nabla^{l+1}h(U,V,\dots,V,U)\rangle\\ &= -\langle J\eta, \Omega^{l+1}(V,V)\rangle = 0\,.\end{aligned}$$

Consequently, by using also (2.23), we obtain

$$\begin{aligned}(2.51)\quad &\langle\nabla^{l+2}h(U,V,\dots,V,U), J\Omega^l(V,U)\rangle\\ &\quad= (\alpha_{2l+2}\varrho_1 - \alpha_{2l+1}\varrho_2 - U\varrho_1)\|\Omega^l(V,V)\|^2\\ &\qquad + \varrho_2\langle\nabla^{l+1}h(U,V,\dots,V), \nabla^l h(V,\dots,V,U)\rangle\\ &\qquad - \varrho_1\langle\nabla^{l+1}h(U,V,\dots,V), \nabla^l h(V,\dots,V)\rangle + \delta\gamma\,.\end{aligned}$$

By (6) for $l+1$, the inductive assumption on (9) and equalities (2.50) and (2.51), we get

$$(2.52)\quad \begin{aligned} U\varrho_1\|\Omega^l(V,V)\|^2 - \varrho_2\langle\nabla^{l+1}h(U,V,\dots,V),\Omega^l(V,U)\rangle \\ + \varrho_1\langle\nabla^{l+1}h(U,V,\dots,V),\Omega^l(V,V)\rangle - \gamma\delta \\ = V\varrho_2\|\Omega^l(V,V)\|^2 + \varrho_1\langle\Omega^{l+1}(V,V),\Omega^l(V,U)\rangle \\ + \varrho_2\langle\Omega^{l+1}(V,V),\Omega^l(V,V) + \gamma\delta\,. \end{aligned}$$

In a similar way, by using also (2.22), we obtain

$$(2.53)\quad \begin{aligned} \langle\nabla^{l+2}h(U,V,\dots,V),J\Omega^l(V,V)\rangle \\ = (\alpha_{2l+1}\varrho_2 - \varrho_1\alpha_{2l+2} + U\varrho_1)\|\Omega^l(V,V)\|^2 \\ + \varrho_1\langle\nabla^{l+1}h(U,V,\dots,V),\Omega^l(V,V)\rangle \\ - \varrho_2\langle\nabla^{l+1}h(U,V,\dots,V),\Omega^l(V,U)\rangle + \gamma\delta \end{aligned}$$

and

$$(2.54)\quad \begin{aligned} \langle\Omega^{l+2}(V,U),J\Omega^l(V,V)\rangle = (-\alpha_{2l+2}\varrho_1 + \varrho_2\alpha_{2l+1} + V\varrho_2)\|\Omega^l(V,V)\|^2 \\ + \varrho_2\langle\Omega^{l+1}(V,V),\Omega^l(V,V)\rangle + \varrho_1\langle\Omega^{l+1}(V,V),\Omega^l(V,U)\rangle - \delta\gamma\,. \end{aligned}$$

Since, by (6) for $l+1$ and (9) for l,

$$\langle\nabla^{l+2}h(U,V,\dots,V),J\Omega^l(V,V)\rangle = \langle\Omega^{l+2}(V,U),J\Omega^l(V,V)\rangle\,,$$

we have

$$(2.55)\quad \begin{aligned} V\varrho_2\|\Omega^l(V,V)\|^2 + \varrho_2\langle\Omega^{l+1}(V,V),\Omega^l(V,V)\rangle \\ + \varrho_1\langle\Omega^{l+1}(V,V),\Omega^l(V,U)\rangle - \gamma\delta \\ = U\varrho_1\|\Omega^l(V,V)\|^2 + \varrho_1\langle\nabla^{l+1}h(U,V,\dots,V),\Omega^l(V,V)\rangle \\ - \varrho_2\langle\nabla^{l+1}h(U,V,\dots,V),\Omega^l(V,U)\rangle + \gamma\delta\,. \end{aligned}$$

By comparing (2.52) and (2.55), we obtain $\gamma\delta = 0$. But, by (2.49), this means that at p

$$\langle\nabla^{l+1}h(V,\dots,V),J\nabla^{l+1}h(V,\dots,V,U)\rangle = 0\,.$$

Consequently, using also (6) for l and (2.24), we get

$$(2.56)\qquad \langle\nabla^{l+1}h(W_1,\dots,W_{l+3}),J\nabla^{l+1}h(Z_1,\dots,Z_{l+3})\rangle = 0$$

for any $W_1,\dots,W_{l+3},Z_1,\dots,Z_{l+3}\in T_pM$, $p\in M_{l+1}$. Hence, by the inductive assumption on (2), (2.56) is true on the whole of M.

Consider now the following form:

$$(2.57)\quad \begin{aligned} (\langle J\nabla^l h(X,\dots,X),\nabla^{l+1}h(X,\dots,X)\rangle \\ - i\langle J\nabla^l h(X,\dots,X),\nabla^{l+1}h(X,\dots,X,Y)\rangle)\,dz^{2l+5}\,. \end{aligned}$$

Since the tensor field defined by (2.27) is symmetric, (2.3) and Lemma 1.1 imply that the above form is well defined on M. By using (2.56), (8) for $l+1$ and the symmetry of (2.27), we get

$$\begin{aligned} &X\langle J\nabla^l h(X,\dots,X),\nabla^{l+1}h(X,\dots,X)\rangle\\ &\qquad=\langle J\nabla^l h(X,\dots,X),\nabla^{l+2}h(X,\dots,X)\rangle\\ &\qquad\quad+(2l+5)\langle J\nabla^l h(\nabla_X X,X,\dots,X),\nabla^{l+1}h(X,\dots,X)\rangle\,,\\ &-Y\langle J\nabla^l h(X,\dots,X),\nabla^{l+1}h(X,\dots,X,Y)\rangle\\ &\qquad=-\langle J\nabla^l h(X,\dots,X),\nabla^{l+2}h(Y,X,\dots,X,Y)\rangle\\ &\qquad\quad-(2l+4)\langle J\nabla^l h(\nabla_Y X,\dots,X),\nabla^{l+1}h(X,\dots,X,Y)\rangle\\ &\qquad\quad+\langle J\nabla^l h(X,\dots,X),\nabla^{l+1}h(X,\dots,X,\nabla_X X)\rangle \end{aligned}$$

and

$$\begin{aligned} &Y\langle J\nabla^l h(X,\dots,X),\nabla^{l+1}h(X,\dots,X)\rangle\\ &\qquad=\langle J\nabla^l h(X,\dots,X),\nabla^{l+2}(Y,X,\dots,X)\rangle\\ &\qquad\quad+(2l+5)\langle J\nabla^l h(\nabla_Y X,\dots,X),\nabla^{l+1}h(X,\dots,X)\rangle\,,\\ &X\langle J\nabla^l h(X,\dots,X),\nabla^{l+1}h(X,\dots,X,Y)\rangle\\ &\qquad=\langle J\nabla^l h(X,\dots,X),\nabla^{l+2}h(X,\dots,X,Y)\rangle\\ &\qquad\quad+(2l+4)\langle J\nabla^l h(\nabla_X X,X,\dots,X),\nabla^{l+1}h(X,\dots,X,Y)\rangle\\ &\qquad\quad+\langle J\nabla^l h(X,\dots,X),\nabla^{l+1}h(X,\dots,\nabla_X Y)\rangle\,. \end{aligned}$$

Now, using (6) for $l+1$, the inductive assumption on (9) and (1.13), (1.14) of Lemma 1.1, we see that the form given by (2.57) is holomorphic and consequently it is zero. It follows that

$$\langle\nabla^{l+1}h(W_1,\dots,W_{l+3}),J\nabla^l h(Z_1,\dots,Z_{l+2})\rangle=0 \tag{2.58}$$

for any $W_1,\dots,W_{l+3},Z_1,\dots,Z_{l+2}\in\mathfrak{X}(M)$ and this, together with (2.56) and (2.24), completes the proof of (9) for $l+1$.

Assume now that $\sigma_{l+3}\not\equiv 0$ and p is a point such that $\sigma_{l+3}(p)=0$. Let $\xi_1,\dots,\xi_m$ be orthonormal vector fields defined in a neighbourhood $\mathcal{U}$ of p and spanning the orthogonal complement to $N_1+\ldots+N_{l+1}$ in N. Consider the functions

$$w_\alpha=\langle\sigma_{l+3}(X,\dots,X),\xi_\alpha\rangle-i\langle\sigma_{l+3}(X,\dots,X,Y),\xi_\alpha\rangle \tag{2.59}$$

for $\alpha=1,\dots,m$. They satisfy

$$\frac{\partial}{\partial\bar z}w_\alpha=\sum_{\beta=1}^{m}c_{\alpha\beta}\left(\frac{\partial}{\partial\bar z}\right)w_\beta\,, \tag{2.60}$$

where $c_{\alpha\beta}$ are defined by (2.10). Indeed,

$$\begin{aligned}\frac{\partial}{\partial\overline{z}}w_\alpha = \frac{1}{2}\{&\langle\nabla\sigma_{l+3}(X,\ldots,X),\xi_\alpha\rangle - i\langle\nabla\sigma_{l+3}(X,\ldots,X,Y),\xi_\alpha\rangle\\
&+ i\langle\nabla\sigma_{l+3}(Y,X,\ldots,X),\xi_\alpha\rangle + \langle\nabla\sigma_{l+3}(Y,X,\ldots,X,Y),\xi_\alpha\rangle\\
&+ (l+3)\langle\sigma_{l+3}(\nabla_X X,\ldots,X),\xi_\alpha\rangle - i(l+2)\langle\sigma_{l+3}(\nabla_X X,\ldots,X,Y),\xi_\alpha\rangle\\
&- i\langle\sigma_{l+3}(X,\ldots,X,\nabla_X Y),\xi_\alpha\rangle + i(l+3)\langle\sigma_{l+3}(\nabla_Y X,\ldots,X),\xi_\alpha\rangle\\
&+ (l+2)\langle\sigma_{l+3}(\nabla_Y X,\ldots,X,Y),\xi_\alpha\rangle + \langle\sigma_{l+3}(X,\ldots,X,\nabla_Y Y),\xi_\alpha\rangle\\
&+ \langle\sigma_{l+3}(X,\ldots,X),D_X\xi_\alpha\rangle - i\langle\sigma_{l+3}(X,\ldots,X,Y),D_X\xi_\alpha\rangle\\
&+ i\langle\sigma_{l+3}(X,\ldots,X),D_Y\xi_\alpha\rangle + \langle\sigma_{l+3}(X,\ldots,X,Y),D_Y\xi_\alpha\rangle\}\end{aligned}$$

and

$$\begin{aligned}\sum_{\beta=1}^{m} c_{\alpha\beta}\Big(\frac{\partial}{\partial\overline{z}}\Big)w_\beta = \frac{1}{2}\{&\langle D_X\xi_\alpha,\sigma_{l+3}(X,\ldots,X)\rangle - i\langle D_X\xi_\alpha,\sigma_{l+3}(X,\ldots,X,Y)\rangle\\
&+ i\langle D_Y\xi_\alpha,\sigma_{l+3}(X,\ldots,X)\rangle + \langle D_Y\xi_\alpha,\sigma_{l+3}(X,\ldots,X,Y)\rangle\}\,.\end{aligned}$$

By using (7) for $l+1$ and (1.13), (1.14) of Lemma 1.1 we complete the proof of (2.60). Therefore, by Theorem 2.3, p is an isolated zero of σ_{l+3}, which gives (2) for $l+1$. Moreover, by repeating the same argument as in the case $l=0$, we find that $\widetilde{N}_{l+2}$ can be extended to a smooth 2-dimensional vector bundle over the whole of M. The proof of Theorem 2.11 is complete.

Remark 2.12. As in Remark 2.5 one can prove that if M is minimal and $\widetilde{M} = \mathbb{C}P^n$, then the bundle N_1 is real-analytic for any l.

If $\widetilde{M}$ is a Riemannian manifold of constant sectional curvature and M is a surface of genus 0 immersed in $\widetilde{M}$ with parallel mean curvature vector, then (1)–(7) of Theorem 2.11 are also true. In particular, if M is minimal, then the bundle $N' = \sum_{l=1}^{\infty} N_l$ is even-dimensional. Hence, by the "reduction theorem" of Erbacher (see [Er]), we see that if the codimension of M cannot be reduced, then $\dim\widetilde{M}$ is even. In the case where $\widetilde{M}$ is a Euclidean sphere, this is a theorem of Calabi (see $[\mathrm{C}]_3$). In this case the bundles N_l are also real-analytic.

3. Reduction theorems. Throughout this section we denote by S^m the sphere $S^m(1)$ and we use the same notations as in §§1 and 2. The standard metric tensor field on $\mathbb{C}^{n+1}$ and that induced on S^{2n+1} will be denoted by $\langle\,,\rangle$. Recall that the Hopf fibration $\pi : S^{2n+1} \to \mathbb{C}P^n$ is a principal fibre bundle with structural group S^1 and its vertical space at $v \in S^{2n+1}$ is the subspace of $T_v(S^{2n+1}) = \{w \in \mathbb{C}^{n+1} : \langle w,v\rangle = 0\}$ spanned by iv. The orthogonal complement T' to the vertical bundle is a connection in the Hopf bundle. Moreover, T'_v is a complex subspace of $\mathbb{C}^{n+1}$ and if we denote by π_* the differential of π, then

$$\pi_{*|T'_v} : T'_v \to T_{\pi(v)}\mathbb{C}P^n$$

is a $\mathbb{C}$-linear isomorphism. If we introduce on $\mathbb{C}P^n$ a metric tensor field in such a way that $\pi_{*|T'_v}$ is an isometry for every $v \in S^{2n+1}$, then $\mathbb{C}P^n$ becomes a Kähler manifold with the Fubini–Study metric of constant holomorphic sectional curvature 4. In what follows $\mathbb{C}P^n$ will be equipped with this structure. If W, Z are vector fields on $\mathbb{C}P^n$ and W', Z' are their horizontal lifts to S^{2n+1} (with respect to the connection T') then

$$\pi_*(\nabla'_{W'}Z') = \widetilde{\nabla}_W Z \tag{3.1}$$

where ∇', $\widetilde{\nabla}$ are the Levi-Civita connections on S^{2n+1} and $\mathbb{C}P^n$ respectively.

We will simultaneously consider the hyperbolic case. In this case $\mathbb{C}^{n+1}$ is endowed with the following Hermitian symmetric 2-form:

$$F(v,w) = -v_0\overline{w}_0 + \sum_{k=1}^{n} v_k\overline{w}_k\,,$$

where $v = (v_0,\ldots,v_n)$ and $w = (w_0,\ldots,w_n)$. Then $\operatorname{Re} F = \langle\,,\rangle$ is a semi-Riemannian metric tensor field of index 2. The usual differentiation $\overline{\nabla}$ on $\mathbb{R}^{2n+2}$ is its Levi-Civita connection. Consider the hyperquadric

$$H_1^{2n+1} = \{v \in \mathbb{C}^{n+1} : \operatorname{Re} F(v,v) = -1\}\,.$$

Then the tangent space $T_vH_1^{2n+1}$ can be identified with the space $\{w \in \mathbb{C}^{n+1} : \langle v,w\rangle = 0\}$, H_1^{2n+1} is totally umbilical in $\mathbb{C}^{n+1}$ and with respect to the induced metric tensor field it is a semi-Riemannian hyperbolic space of constant sectional curvature -1. It is called the *anti-de Sitter space*. Consider the natural action $v \to \lambda v$ of S^1 on H_1^{2n+1}. Then the quotient space H_1^{2n+1}/S^1 is a manifold, denoted by $\mathbb{C}H^n$, and H_1^{2n+1} is a principal circle bundle over $\mathbb{C}H^n$. Denote by π the bundle projection $\pi : H_1^{2n+1} \to \mathbb{C}H^n$. The vertical space at $v \in H_1^{2n+1}$ is the real subspace of $\mathbb{C}^{n+1}$ spanned by iv. If we denote by T' the orthogonal complement to the vertical bundle in TH_1^{2n+1} then T' is a connection in H_1^{2n+1}. Denote by π_* the differential of π. If we introduce on $\mathbb{C}H^n$ the almost complex structure J and the metric tensor field $\langle\,,\rangle$ via $\pi_{*|T'}$, then $\mathbb{C}H^n$ becomes an n-dimensional Kähler manifold of constant holomorphic curvature -4. Moreover, if we denote by ∇' and $\widetilde{\nabla}$ the Levi-Civita connections on H_1^{2n+1} and $\mathbb{C}H^n$ respectively and by W', Z' the horizontal lifts of $W, Z \in \mathfrak{X}(\mathbb{C}H^n)$, then (3.1) is also true in this case.

In what follows $\widetilde{M}$ will be a complex space form, i.e. a complete, simply connected Kähler manifold of constant holomorphic sectional curvature. By a *model space* we shall mean $\mathbb{C}^n$ with the standard Kähler structure, $\mathbb{C}P^n$ with the Fubini–Study metric of constant holomorphic sectional curvature 4 or $\mathbb{C}H^n$ with the structure described above. It is known that any complex space form is homothetically biholomorphic to one of the model spaces.

Now we collect some examples of totally real submanifolds of model spaces.

Example 3.1. Let $\widetilde{M} = \mathbb{C}^n$ and let M be a minimal submanifold of S^{m-1}, where $m \le n$. It is easy to observe that a minimal submanifold of a totally umbilical hypersurface of a real space form has parallel mean curvature vector.

Therefore M is a submanifold of $\mathbb{R}^m$ with parallel mean curvature vector. Since $\mathbb{R}^m$ can be imbedded into $\mathbb{C}^n$ as a totally geodesic totally real submanifold, M can be immersed in $\mathbb{C}^n$ as a totally real submanifold with parallel mean curvature vector.

EXAMPLE 3.2. Assume $\widetilde{M} = \mathbb{C}P^n$. If M is a submanifold minimally (or with parallel mean curvature vector) immersed in the sphere S^m for any $m \le n$, then it can be immersed as a minimal (or with parallel mean curvature vector) totally real submanifold into $\mathbb{C}P^n$. Namely, the sphere S^m is isometrically and totally geodesically immersed in the real projective space $\mathbb{R}P^m$ endowed with a metric of constant sectional curvature 1, and $\mathbb{R}P^m$ can be imbedded in $\mathbb{C}P^n$ as a totally real totally geodesic submanifold. Moreover, if $e_0, \ldots, e_n, ie_0, \ldots, ie_n$ is an orthonormal basis in $\mathbb{C}^{n+1}$, then $\pi(\mathrm{span}_{\mathbb{R}}\{e_0, \ldots, e_m\} \cap S^{2n+1})$ is a totally real totally geodesic submanifold of $\mathbb{C}P^n$, isometric to $\mathbb{R}P^n$.

EXAMPLE 3.3. Let $\widetilde{M} = \mathbb{C}H^n$. Denote by $\mathbb{R}_1^{m+1}$ the space $\mathbb{R}^{m+1}$ equipped with the Lorentz metric

$$\langle x, y\rangle = -x_0 y_0 + \sum_{k=1}^{m} x_k y_k \,.$$

Then the hyperquadric $H_0^m = \{x \in \mathbb{R}^{m+1} : \langle x, x\rangle = -1\}$ is a totally umbilical submanifold of $\mathbb{R}_1^{m+1}$ and the induced metric gives H_0^m the Riemannian structure of constant sectional curvature -1. Denote by $\mathbb{R}H^n$ the "upper" connected component of H_0^m, i.e. $\mathbb{R}H^m = \{x \in H_0^m : x_0 > 0\}$. We have the following totally real geodesic imbedding of $\mathbb{R}H^m$ into $\mathbb{C}H^n$ $(n \ge m)$:

$$\iota : \mathbb{R}H^m \ni (x_0, \ldots, x_m) \to \pi((x_0, \ldots, x_m, 0, \ldots, 0)) \in \mathbb{C}H^n \,.$$

Moreover, if $v \in H_1^{2n+1}$ and $e_1, \ldots, e_n, ie_1, \ldots, ie_n$ is an orthonormal basis of T'_v, then $\pi(H_1^{2n+1} \cap \mathrm{span}_{\mathbb{R}}\{v, e_1, \ldots, e_m\})$ is a totally real totally geodesic submanifold of $\mathbb{C}H^n$ isometric to $\mathbb{R}H^m$. The unit sphere S^{m-1} can be imbedded in $\mathbb{R}H^m$ as a totally umbilical submanifold:

$$\psi : S^{m-1} \ni (x_1, \ldots, x_m) \to (\sqrt{2}, x_1, \ldots, x_m) \in \mathbb{R}H^m \,.$$

If $\varphi : M \to S^{m-1}$ is a minimal immersion, then $\iota \circ \psi \circ \varphi$ is a totally real immersion in $\mathbb{C}H^n$ with parallel mean curvature vector.

Notice also that if $v \in H^{2n+1}$ and L is a complex subspace of T'_v, then $\pi((\mathrm{span}_{\mathbb{C}}\{v\} + L) \cap H_1^{2n+1})$ is a totally geodesic complex submanifold of $\mathbb{C}H^n$. Similarly, if $\widetilde{M} = \mathbb{C}P^n$ and L is a complex subspace of $\mathbb{C}^{n+1}$, then $\pi(S^{2n+1} \cap L)$ is a complex totally geodesic submanifold of $\mathbb{C}P^n$.

In the sequel we shall need the following reduction theorems. Other versions of reduction theorems can be found in [Er], [Ce], [ChHL].

PROPOSITION 3.1. *Let M be a totally real submanifold of a complex space form $\widetilde{M}$. If N'' is a subbundle of the normal bundle N such that*

(i) $N''_p \supset \widetilde{N}_{1p}$ *for every* $p \in M$,

(ii) $J(TM + N'') = TM + N''$,
(iii) N'' *is parallel with respect to* D,

then there is a totally geodesic complex submanifold M' *of* $\widetilde{M}$ *such that the image of* M *is contained in* M' *and* $T_pM' = T_pM + N''_p$ *for every* $p \in M$.

P r o o f. We may assume that $\widetilde{M}$ is a model space. Let $\widetilde{M} = \mathbb{C}^n$. We set $\mathcal{K}_p = T_pM + N''_p$ for $p \in M$. $\mathcal{K}_p$ is a complex subspace of $\mathbb{C}^n$. Let $p_0 \in M$ and let $Z_1, \ldots, Z_m$, $\xi_1, \ldots, \xi_q$ be vector fields defined in a neighbourhood $\mathcal{U} \subset M$ of p_0 spanning $TM_{|\mathcal{U}}$ and $N''_{|\mathcal{U}}$ respectively. Let p_1 be a point in $\mathcal{U}$ and let $p(t)$ be a curve in $\mathcal{U}$ joining p_0 and p_1. By the assumption (i),

$$\widetilde{\nabla}_{\dot{p}(t)} Z_j = \nabla_{\dot{p}(t)} Z_j + h(\dot{p}(t), Z_j) \in (TM + N'')_{p(t)} = \mathcal{K}_{p(t)}$$

for $j = 1, \ldots, m$. The assumption (iii) yields

$$\widetilde{\nabla}_{\dot{p}(t)} \xi_k = -A_{\xi_k} \dot{p}(t) + D_{\dot{p}(t)} \xi_k \in T_{p(t)}M + N''_{p(t)} = \mathcal{K}_{p(t)}$$

for $k = 1, \ldots, q$. Consequently, $\mathcal{K}_{p(t)}$ is parallel with respect to $\widetilde{\nabla}$ along $p(t)$, i.e. it is constant along $p(t)$. Since M is connected there is a complex subspace $\mathcal{K}$ of $\mathbb{C}^n$ such that $\mathcal{K}_p = \mathcal{K}$ for every $p \in M$. In particular, $T_pM \subset \mathcal{K}$ for every $p \in M$. Consequently, the image of M is contained in an affine complex subspace $M' = a + \mathcal{K}$ of $\mathbb{C}^n$. Of course, $T_pM' = T_pM + N''_p$ for $p \in M$.

Assume now that $\widetilde{M} = \mathbb{C}P^n$ or $\mathbb{C}H^n$ and let $\widehat{M}$ denote S^{2n+1} or H_1^{2n+1} respectively. Let $p \in M$ and $v \in \pi^{-1}(p)$. We define

$$\mathcal{K}_v = (\pi_{*|T'_v})^{-1}(N''_p + T_pM) + \mathrm{span}_{\mathbb{C}}\{v\}\,.$$

Then $\mathcal{K}_v$ is a complex subspace of $\mathbb{C}^{n+1}$. We set $M'_v := \pi(\mathcal{K}_v \cap \widehat{M})$. Then M'_v is a totally geodesic complex submanifold of $\widetilde{M}$ and $T_pM'_v = T_pM + N''_p$. We shall show that M'_v does not depend on $v \in \pi^{-1}(M)$. Of course, $M'_{v\lambda} = M'_v$ for $\lambda \in S^1$. Let $p_0 \in M$ and let $Z_1, \ldots, Z_m$, $\xi_1, \ldots, \xi_q$, $\mathcal{U}$, p_1 and $p(t)$ be as in the previous case. Denote by $\gamma = v(t)$ the horizontal lift of $p(t)$ to $\widehat{M}$ starting from $v_0 \in \pi^{-1}(p_0)$. Denote by $Z'_1, \ldots, Z'_m$, $\xi'_1, \ldots, \xi'_q$ the horizontal lifts of $Z_1, \ldots, Z_m$, $\xi_1, \ldots, \xi_q$ respectively. Then

$$\mathcal{K}_{v(t)} = \mathrm{span}_{\mathbb{R}}\{Z'_1, \ldots, Z'_m, \xi'_1, \ldots, \xi'_q\} + \mathrm{span}_{\mathbb{C}}\{v(t)\}\,.$$

By using (i), we get

$$\widetilde{\nabla}_{\dot{p}(t)} Z_j = \nabla_{\dot{p}(t)} Z_j + h(\dot{p}(t), Z_j) \in TM + N''\,.$$

Hence, by (3.1),

$$\nabla'_{\dot{v}(t)} Z'_j \in (\pi^{-1}_{*|T'_{v(t)}}(TM + N'') + \mathrm{span}_{\mathbb{R}}\{iv\}\,.$$

Therefore

$$\overline{\nabla}_{\dot{v}(t)} Z'_j = \nabla'_{\dot{v}(t)} Z'_j + \overline{h}(\dot{v}(t), Z'_j) \in \mathcal{K}_{v(t)} \tag{3.2}$$

where $\overline{h}$ is the second fundamental form of $\widehat{M}$ in $\mathbb{C}^{n+1}$.

The assumption (iii) gives

$$\widetilde{\nabla}_{\dot p(t)}\xi_j = -A_{\xi_j}\dot p(t) + D_{\dot p(t)}\xi_j \in TM + N''$$

and by using the same arguments as above, we get

$$\overline{\nabla}_{\dot v(t)}\xi'_j \in \mathcal{K}_{v(t)}\,. \tag{3.3}$$

Moreover, the umbilicity of $\widehat{M}$ in $\mathbb{C}^{n+1}$ yields

$$\overline{\nabla}_{\dot v(t)}\gamma = -\overline{A}_{v(t)}\dot v(t) + \overline{D}_{\dot v(t)}\gamma \in \operatorname{span}\{v(t), \dot v(t)\} \subset \mathcal{K}_{v(t)} \tag{3.4}$$

where $\overline{A}$ and $\overline{D}$ are the Weingarten endomorphism and the normal connection of the imbedding of $\widehat{M}$ into $\mathbb{C}^{n+1}$. Consequently,

$$\overline{\nabla}_{\dot v(t)} i\gamma = i\overline{\nabla}_{\dot v(t)}\gamma \in \mathcal{K}_{v(t)}\,. \tag{3.5}$$

Formulas (3.2)–(3.5) imply that the field of subspaces $\mathcal{K}_{v(t)}$ is parallel along $v(t)$ with respect to $\widetilde{\nabla}$, i.e. it is constant along $v(t)$. Since M is connected, the proof is complete.

PROPOSITION 3.2. *Let M be a totally real submanifold of a complex space form $\widetilde{M}$ and let N' be a subbundle of $\mathcal{H}$ such that*

(i) *$N'_p \supset \widetilde{N}_{1p}$ for every $p \in M$,*
(ii) *N' and JN' are orthogonal,*
(iii) *N' is parallel with respect to D.*

Then there is a totally real totally geodesic submanifold M' of $\widetilde{M}$ such that the image of M is contained in M' and $T_pM' = T_pM + N'_p$ for every $p \in M$.

Proof. We set $N'' = N' + JTM + JN'$. Let $W, Z \in \mathfrak{X}(M)$ and let the normal vector fields ξ, η belong respectively to N' and the orthogonal complement to N'' in N. By (iii) and (1.4), we obtain $\langle D_W J\xi, \eta\rangle = \langle JD_W\xi, \eta\rangle = 0$. Similarly, (1.2) together with the fact that $N' \subset \mathcal{H}$ and (i) yield $\langle D_W JZ, \eta\rangle = 0$. Hence N'' satisfies the assumptions of Proposition 3.1 and consequently we may assume that $N' + JN' = \mathcal{H}$. As in the proof of Proposition 3.1 it is sufficient to assume that $\widetilde{M}$ is a model space. We shall also use $\widehat{M}$, $\overline{\nabla}$, $\overline{h}$, $\overline{D}$, $\overline{A}$ as in the proof of Proposition 3.1. Let $\widetilde{M} = \mathbb{C}P^n$ or $\mathbb{C}H^n$. For any $p \in M$ and $v \in \pi^{-1}(p)$ we set

$$\mathcal{K}_v = (\pi_{*|T'_v})^{-1}(T_pM + N'_p) + \operatorname{span}_{\mathbb{R}}\{v\}\,,$$
$$\mathcal{L}_v = (\pi_{*|T'_v})^{-1}(JT_pM + JN'_p) + \operatorname{span}_{\mathbb{R}}\{iv\}\,.$$

Since $\pi_{*|T'_v}$ is a $\mathbb{C}$-linear isometry, we have $i\mathcal{K}_v = \mathcal{L}_v$, $\mathcal{K}_v$ is orthogonal to $\mathcal{L}_v$ and $\mathcal{L}_v + \mathcal{K}_v = \mathbb{C}^{n+1}$. Moreover, $M'_v := \pi(\mathcal{K}_v \cap \widehat{M})$ is a totally geodesic totally real submanifold of $\widetilde{M}$ and $T_pM'_v = T_pM + N'_p$. It is clear that $M'_{v\lambda} = M'_v$ for any $\lambda \in S^1$. We now show that M'_v does not depend on $p \in M$. We argue similarly to the proof of Proposition 3.1. Let $p_0 \in M$ and let $Z_1, \ldots, Z_m$, $\xi_1, \ldots, \xi_q$, $\mathcal{U}$, p_1,

$p(t)$, $v(t)=\gamma$, $Z'_1,\dots,Z'_m$, $\xi'_1,\dots,\xi'_q$ be as in the proof of Proposition 3.1 (where $\xi_1,\dots,\xi_q$ span $N'_{|\mathcal{U}}$). The vector fields

$$\eta'_1 = iZ'_1,\ \dots,\ \eta'_m = iZ'_m,\ \eta'_{m+1} = i\xi'_1,\ \dots,\ \eta'_{m+q} = i\xi'_q$$

are horizontal lifts of

$$\eta_1 = JZ_1,\ \dots,\ \eta_m = JZ_m,\ \eta_{m+1} = J\xi_1,\ \dots,\ \eta_{m+q} = J\xi_q$$

and

$$\begin{aligned}\mathcal{K}_{v(t)} &= \operatorname{span}_{\mathbb{R}}\{Z'_1,\dots,Z'_m,\xi'_1,\dots,\xi'_q,v(t)\}\,,\\ \mathcal{L}_{v(t)} &= \operatorname{span}_{\mathbb{R}}\{\eta'_1,\dots,\eta'_{m+q},iv(t)\}\,.\end{aligned}$$

We have

$$\widetilde{\nabla}_{\dot p(t)}\eta_j = -A_{\eta_j}\dot p(t) + D_{\dot p(t)}\eta_j\,.$$

By (i), $A_{\eta_j}\dot p(t)=0$, and by (iii), $D_{\dot p(t)}\eta_j \in \operatorname{span}\{\eta_1,\dots,\eta_{m+q}\}$. Therefore (3.1) yields

$$\nabla'_{\dot v(t)}\eta'_j \in \operatorname{span}_{\mathbb{R}}\{\eta'_1,\dots,\eta'_{m+q},iv(t)\} = \mathcal{L}_{v(t)}\,.$$

We also have

$$\overline{\nabla}_{\dot v(t)}\eta'_j = \nabla'_{\dot v(t)}\eta'_j + \overline{h}(\dot v(t),\eta'_j)\,.$$

Since $\dot v(t)$ is a horizontal lift of $\dot p(t)$ and $\widehat{M}$ is umbilical in $\mathbb{C}^{n+1}$, we have $\overline{h}(\dot v(t),\eta'_j) = 0$. Consequently, $\overline{\nabla}_{\dot v(t)}\eta'_j \in \mathcal{L}_{v(t)}$ for $j=1,\dots,m+q$. It follows that

$$\overline{\nabla}_{\dot v(t)}\xi'_j = -i\overline{\nabla}_{\dot v(t)}(i\xi'_j) \in i\mathcal{L}_{v(t)} = \mathcal{K}_{v(t)} \tag{3.6}$$

and for the same reasons

$$\overline{\nabla}_{\dot v(t)}Z'_j \in \mathcal{K}_{v(t)}\,. \tag{3.7}$$

We also have

$$\overline{\nabla}_{\dot v(t)}\gamma = -\overline{A}_{v(t)}\dot v(t) + \overline{D}_{\dot v(t)}\gamma = \operatorname{span}_{\mathbb{R}}\{\dot v(t),v(t)\} \subset \mathcal{K}_{v(t)}\,. \tag{3.8}$$

Formulas (3.6)–(3.8) imply that $\mathcal{K}_{v(t)}$ is constant for all t. In the case where $\widetilde{M}=\mathbb{C}^n$ the proof is exactly the same as the proof of Proposition 3.1 in this case.

Now we can prove Theorem 1.8 stated in the Introduction. It is sufficient to apply Proposition 3.2 to the bundle $N' = \sum_{l=1}^{\infty} N_l$, where N_l are defined in Theorem 2.11. By the construction of N_l and by (8) and (9) of Theorem 2.11 it is clear that N' satisfies the assumptions of Proposition 3.2.

Remark. In the case where M is minimal in $\mathbb{C}P^n$, Theorem 1.8 was also proved by J. Bolton, G. R. Jensen, M. Rigoli and L. M. Woodward (see [BJRW]). The proof given there is different from ours.

4. Surfaces of genus 0, $\mathcal{C}$-totally real immersed in Sasakian space forms with parallel mean curvature vector. Let $\widetilde{M}$ be a $(2n+1)$-dimensional manifold equipped with a Sasakian structure $(\varphi, \xi, \eta, \langle\,,\rangle)$, i.e. φ is a $(1,1)$-tensor field of rank $2n$ on M, ξ is a global vector field on M, η is a 1-form and

$$\eta(\xi) = 1\,, \quad \varphi^2 = -I + \eta \otimes \xi\,, \tag{4.1}$$

$$\langle \varphi W, \varphi Z\rangle = \langle W, Z\rangle - \eta(W)\eta(Z)\,, \tag{4.2}$$

$$(\widetilde{\nabla}_W \varphi)Z = \langle W, Z\rangle\xi - \eta(Z)W \quad \text{for any } W, Z \in \mathfrak{X}(\widetilde{M})\,. \tag{4.3}$$

The above formulas imply

$$\varphi(\xi) = 0\,, \quad \eta \circ \varphi = 0\,, \tag{4.4}$$

$$\eta(W) = \langle \xi, W\rangle\,, \tag{4.5}$$

$$\widetilde{\nabla}_W \xi = -\varphi W \tag{4.6}$$

for any $W \in \mathfrak{X}(\widetilde{M})$. By the *$\varphi$-sectional curvature* we mean the sectional curvature by φ-invariant planes contained in the contact distribution $\mathcal{D} = \operatorname{im}\varphi$. If M has constant φ-sectional curvature c, then

$$\begin{aligned}\widetilde{R}(W,S)Z = {} & \frac{c+3}{4}\{\langle S, Z\rangle W - \langle W, Z\rangle S\} \\ & + \frac{c-1}{4}\{\eta(W)\eta(Z)S - \eta(S)\eta(Z)W \\ & + \langle W, Z\rangle\eta(S)\xi - \langle S, Z\rangle\eta(W)\xi \\ & + \phi(Z,S)\varphi W - \phi(Z,W)\varphi S + 2\phi(W,S)\varphi Z\}\,,\end{aligned} \tag{4.7}$$

where ϕ is the *fundamental* 2*-form* of the structure (φ, ξ, η, g), i.e.

$$\phi(Z, W) = \langle W, \varphi Z\rangle\,.$$

Since $\widetilde{M}$ is Sasakian,

$$\phi = d\eta\,. \tag{4.8}$$

If $\mathcal{T}$ is a vector bundle over M, then the set of all C^∞ sections of $\mathcal{T}$ will be denoted by $\Gamma(\mathcal{T})$. If we introduce on $\widetilde{M}$ a structure $(\varphi^*, \xi^*, \eta^*, \langle\,,\rangle^*)$ by

$$\eta^* = \alpha\eta\,, \quad \xi^* = \frac{1}{\alpha}\xi\,, \quad \varphi^* = \varphi\,, \quad \langle\,,\rangle^* = \alpha\langle\,,\rangle + \alpha(\alpha-1)\eta\otimes\eta\,, \tag{4.9}$$

where α is a positive constant, then it is easy to check that $(\widetilde{M}, \varphi^*, \xi^*, \eta^*, \langle\,,\rangle^*)$ is also a Sasakian manifold. Such a deformation is called a *$\mathcal{D}$-homothetic deformation*. Let $W, S, Z \in \Gamma(\mathcal{D})$ and let $\widetilde{\nabla}^*$ denote the Levi-Civita connection of $\langle\,,\rangle^*$. Then

$$\begin{aligned}2\langle \widetilde{\nabla}^*_W S, Z\rangle^* &= W\langle S, Z\rangle^* + S\langle W, Z\rangle^* - Z\langle W, S\rangle^* \\ &\quad + \langle [W,S], Z\rangle^* + \langle [Z,W], S\rangle^* + \langle W, [Z,S]\rangle^* \\ &= \alpha W\langle S, Z\rangle + \alpha S\langle W, Z\rangle - \alpha Z\langle W, S\rangle \\ &\quad + \alpha\langle [W,S], Z\rangle + \alpha\langle [Z,W], S\rangle + \alpha\langle W, [Z,S]\rangle \\ &= 2\alpha\langle \widetilde{\nabla}_W S, Z\rangle\,.\end{aligned}$$

But $\langle \widetilde{\nabla}^*_W S, Z\rangle^* = \alpha \langle \widetilde{\nabla}^*_W S, Z\rangle$ and consequently

$$\langle \widetilde{\nabla}_W S, Z\rangle = \langle \widetilde{\nabla}^*_W S, Z\rangle\,. \tag{4.10}$$

We also have

$$\langle Z, \xi\rangle = \frac{1}{\alpha^2}\langle Z, \xi\rangle^*$$

for any $Z \in \mathfrak{X}(\widetilde{M})$. Therefore, if $W, S \in \Gamma(\mathcal{D})$, we get

$$\begin{aligned}\langle \widetilde{\nabla}^*_W S, \xi\rangle &= \frac{1}{\alpha^2}\langle \widetilde{\nabla}^*_W S, \xi\rangle = \frac{1}{\alpha}\langle \widetilde{\nabla}^*_W S, \xi^*\rangle^* \\ &= -\frac{1}{\alpha}\langle S, \widetilde{\nabla}^*_W \xi^*\rangle^* = \frac{1}{\alpha}\langle S, \varphi W\rangle^* = -\langle S, \widetilde{\nabla}_W \xi\rangle \\ &= \langle \widetilde{\nabla}_W S, \xi\rangle\,,\end{aligned}$$

i.e.

$$\langle \widetilde{\nabla}^*_W S, \xi\rangle = \langle \widetilde{\nabla}_W S, \xi\rangle\,. \tag{4.11}$$

Formulas (4.10) and (4.11) yield

$$\widetilde{\nabla}^*_W S = \widetilde{\nabla}_W S \tag{4.12}$$

for $W, S \in \Gamma(\mathcal{D})$. By using these formulas one can also easily verify that if $(\widetilde{M}, \varphi, \xi, \eta, \langle\,,\rangle)$ has constant φ-sectional curvature c, then $(\widetilde{M}, \varphi^*, \xi^*, \eta^*, \langle\,,\rangle^*)$ has also constant φ-sectional curvature, equal to $(c+3-3\alpha)/\alpha$. In particular, if $c > -3$ and if we change the given structure by means of a $\mathcal{D}$-homothetic deformation with $\alpha = (3+c)/4$, then $(\widetilde{M}, \langle\,,\rangle^*)$ has constant sectional curvature 1.

The standard Sasakian manifolds with constant φ-sectional curvature $c > -3$ are odd-dimensional spheres (see [B]). Consider, for instance, the unit hypersphere of $\mathbb{C}^{n+1}$ centred at 0, i.e. S^{2n+1}. Let

$$\xi = (S^{2n+1} \ni p \to ip) \in \mathfrak{X}(S^{2n+1})$$

and let φZ be the tangential part of iZ, for $Z \in T_p S^{2n+1}$. If $\langle\,,\rangle$ is the standard metric tensor field on S^{2n+1} and η is the 1-form dual to ξ with respect to $\langle\,,\rangle$, then S^{2n+1} with the given structure is Sasakian.

By a *$\mathcal{C}$-totally real submanifold* of a Sasakian manifold $(\widetilde{M}, \varphi, \xi, \eta, \langle\,,\rangle)$ we mean a submanifold M of $\widetilde{M}$ such that TM is perpendicular to ξ. For a $\mathcal{C}$-totally submanifold M the space $\varphi(T_pM)$ is orthogonal to T_pM. Indeed, if $W, Z \in \mathfrak{X}(M)$, then

$$\langle Z, \varphi W\rangle = \phi(Z, W) = d\eta(Z, W) = \tfrac{1}{2}\{Z\eta(W) - W\eta(Z) - \eta([Z, W])\} = 0\,.$$

We have the following standard example of a $\mathcal{C}$-totally real submanifold of S^{2n+1}. Let L be a totally real vector subspace of $\mathbb{C}^{n+1}$. Then $L \cap S^{2n+1}$ is a totally geodesic $\mathcal{C}$-totally real submanifold of S^{2n+1}.

Let M be a $\mathcal{C}$-totally real submanifold of $\widetilde{M}$. The Codazzi equation is given by (2.8) of Chapter I. By making use of the fact that T_pM is contained in $\mathcal{D}_p$ and is perpendicular to $\varphi(T_pM)$ and by (4.6) we get

$$\langle h(W,Z),\xi\rangle = \langle\widetilde{\nabla}_W Z,\xi\rangle = -\langle Z,\widetilde{\nabla}_W\xi\rangle = \langle Z,\varphi W\rangle = 0 \tag{4.13}$$

for $W,Z\in\mathfrak{X}(M)$. By the Gauss and Weingarten formulas and by (4.3) we obtain

$$\langle W,Z\rangle\xi = \widetilde{\nabla}_W\varphi Z - \varphi\widetilde{\nabla}_W Z = -A_{\varphi Z}W + D_W\varphi Z - \varphi h(W,Z) - \varphi\nabla_W Z$$

for $W,Z\in\mathfrak{X}(M)$. Consequently,

$$\langle h(W,Z),\varphi S\rangle = \langle h(W,S),\varphi Z\rangle \tag{4.14}$$

and

$$D_W\varphi Z = \varphi\nabla_W Z + n\varphi h(W,Z) + \langle W,Z\rangle\xi \tag{4.15}$$

for $W,Z,S\in\mathfrak{X}(M)$, where $n\varphi h(W,Z)$ is the normal part of $\varphi h(W,Z)$. Denote by $\mathcal{H}$ the orthogonal complement to $\varphi(TM)$ in $\mathcal{D}$. If $\delta\in\Gamma(\mathcal{H})$ and $W\in\mathfrak{X}(M)$, then

$$0 = \widetilde{\nabla}_W\varphi\delta - \varphi\widetilde{\nabla}_W\delta = D_W\varphi\delta - A_{\varphi\delta}W - \varphi D_W\delta + \varphi A_\delta W\,.$$

Consequently,

$$\langle D_W\varphi\delta,\varrho\rangle = \langle\varphi D_W\delta,\varrho\rangle \tag{4.16}$$

for any $\delta,\varrho\in\Gamma(\mathcal{H})$ and $W\in\mathfrak{X}(M)$. Denote by $\widetilde{M}^*$ the manifold $\widetilde{M}$ equipped with the Sasakian structure obtained by a $\mathcal{D}$-homothetic deformation with $\alpha = (c+3)/4$. Let M^* denote M with the metric tensor $\langle\,,\rangle^* = \alpha\langle\,,\rangle$. If $\iota: M\to\widetilde{M}$ is the given immersion, then $\iota: M^*\to\widetilde{M}^*$ is isometric and $\mathcal{C}$-totally real. By (4.12) we also have

$$h^* = h\,,\qquad D^*_W\varrho = D_W\varrho \tag{4.17}$$

for $W\in\mathfrak{X}(M)$ and $\varrho\in\Gamma(\mathcal{H})$, where h^* and D^* are the second fundamental form and the normal connection of the immersion $\iota: M^*\to\widetilde{M}^*$.

From now on we assume that $\widetilde{M}$ has constant φ-sectional curvature c, M is a compact orientable surface of genus 0, and M is $\mathcal{C}$-totally real and has parallel mean curvature vector $\varkappa$. If M is minimal, then the Gauss equation is

$$\frac{c+3}{4} = K + \frac{\|h\|^2}{2}\,, \tag{4.18}$$

where as usual K is the Gaussian curvature of M. Consequently, by the Gauss–Bonnet theorem, $c>-3$. The Ricci equation is

$$\begin{aligned} R^\perp(V,U)\delta = \frac{c-1}{4}\{&\langle\varphi U,\delta\rangle\varphi V - \langle\varphi V,\delta\rangle\varphi U\\ &+\langle h(V,U),\delta\rangle(h(V,V)-h(U,U))\\ &-\langle h(V,V)-h(U,U),\delta\rangle h(V,U)\}\end{aligned} \tag{4.19}$$

for any orthonormal basis V, U of T_pM and $p\in M$.

The lth normal space will have the same meaning as in §1. By using the same arguments as in §2 one can prove the following facts:

For any orthonormal basis V, U of T_pM the vectors

$$\varkappa, \quad \tfrac{1}{2}(h(V,V)-h(U,U)), \quad h(V,U)$$

are orthogonal and the last two vectors have the same length.

$M \setminus M_1$ consists of isolated points and $\widetilde{N}_1$ can be extended to a smooth bundle N_1 over the whole of M.

If $\dim N_1 = 1$, then M is totally umbilical and $\nabla h = 0$.

N_1 is a subbundle of $\mathcal{H}$ and $\varphi(N_1)$ is orthogonal to N_1.

As an immediate corollary of these facts we get

THEOREM 4.1. *Let M be a surface of genus* 0, *$\mathcal{C}$-totally real immersed with parallel mean curvature vector in a Sasakian manifold $\widetilde{M}(c)$ of constant φ-sectional curvature c.*

(1) *If* $\dim \widetilde{M}(c) = 5$, *then $c > -3$ and M is totally geodesic.*

(2) *If* $\dim \widetilde{M}(c) = 7$, *then M is totally umbilical and $\nabla h = 0$. If M is minimal, then $c > -3$ and M is totally geodesic.*

(3) *If* $\dim \widetilde{M}(c) = 9$, *then M is minimal or M is totally umbilical and $\nabla h = 0$.*

The assertions (1) and (2) are generalizations of Theorem (i) of [YKM].

The above theorem and the Gauss–Bonnet theorem imply the following pinching theorem:

THEOREM 4.2. *Let M be a compact surface minimal and $\mathcal{C}$-totally real in a Sasakian manifold of dimension* 7 *and of constant φ-sectional curvature. If the Gaussian curvature K of M satisfies $K \geq 0$ on the whole of M, then M is totally geodesic or $K \equiv 0$.*

By making small changes in the proof of Theorem 2.11 we obtain all the assertions of this theorem. In particular, we have

THEOREM 4.3. *Let M be a surface of genus* 0, *$\mathcal{C}$-totally real immersed with parallel mean curvature vector in a Sasakian manifold of constant φ-sectional curvature. Then for any $l \geq 1$*

(1) *$M \setminus M_l$ consists of isolated points and the bundle $\widetilde{N}_l$ can be extended to a smooth bundle, say $\widetilde{N}_l$, over the whole of M,*

(2) *N_l is a subbundle of $\mathcal{H}$,*

(3) *$\varphi(N_1 + \ldots + N_l)$ is orthogonal to $N_1 + \ldots + N_l$.*

In the following reduction theorem M is neither a surface nor a submanifold with parallel mean curvature vector:

PROPOSITION 4.4. *Let M be a $\mathcal{C}$-totally real submanifold of a Sasakian manifold $\widetilde{M}$ of constant φ-sectional curvature $c > -3$. Let N' be a subbundle of $\mathcal{H}$ satisfying the following conditions*:

(i) *$N'_p \supset \widetilde{N}_{1p}$ for every $p \in M$,*

(ii) *$\varphi(N')$ is orthogonal to N',*

(iii) *N' is parallel with respect to D.*

Then there is a $\mathcal{C}$-totally real, totally geodesic submanifold M' of $\widetilde{M}$ such that the image of M is contained in M' and $T_pM' = T_pM + N'_p$ for every $p \in M$.

Proof. By (4.17) and by the definition of the $\mathcal{D}$-homothetic transformation we can assume that $c = 1$. First consider the case where $\widetilde{M} = S^{n+1}$. We define

$$L_p = T_pM + N'_p + \operatorname{span}_{\mathbb{R}}\{p\}$$

for $p \in M$. By the assumptions L_p is totally real in $\mathbb{C}^{n+1}$. If we define $M'_p = S^{2n+1} \cap L_p$, then M' is a $\mathcal{C}$-totally real totally geodesic submanifold of S^{2n+1}, $p \in M'_p$ and $T_pM'_p = T_pM + N'_p$. It is sufficient to prove that L_p does not depend on p and this can be done similarly to the proofs of Propositions 3.1 and 3.2. Namely, let $p_0 \in M$ and $Z_1, \ldots, Z_m, \delta_1, \ldots, \delta_k$ be vector fields defined in a neighbourhood $\mathcal{U} \subset M$ of p_0 spanning $TM_{|\mathcal{U}}$ and $N'_{|\mathcal{U}}$ respectively. Let $p_1 \in \mathcal{U}$ and let $\gamma = p(t)$ be a curve joining p_0 and p_1. Of course

$$L_{p(t)} = \operatorname{span}\{Z_1, \ldots, Z_m, \delta_1, \ldots, \delta_k \text{ at } p(t), p(t)\}\,.$$

Denote by $\overline{\nabla}$ the standard Levi-Civita connection on $\mathbb{C}^{n+1}$, by $\overline{h}$ the second fundamental form of the standard imbedding of S^{2n+1} into $\mathbb{C}^{n+1}$, by $\overline{A}$ and $\overline{D}$ its Weingarten endomorphism and its normal connection respectively. By (i) we have

$$\widetilde{\nabla}_{\dot{p}(t)} Z_j = \nabla_{\dot{p}(t)} Z_j + h(\dot{p}(t), Z_j) \in T_{p(t)}M + N'_{p(t)}$$

for $j = 1, \ldots, m$. The assumption (iii) implies

$$\widetilde{\nabla}_{\dot{p}(t)} \delta_r = -A_{\delta_r}\dot{p}(t) + D_{\dot{p}(t)}\delta_r \in T_{p(t)}M + N'_{p(t)}\,.$$

Consequently,

$$\overline{\nabla}_{\dot{p}(t)} Z_j = \widetilde{\nabla}_{\dot{p}(t)} Z_j + \overline{h}(\dot{p}(t), Z_j) \in L_{p(t)}\,, \tag{4.20}$$

$$\overline{\nabla}_{\dot{p}(t)} \delta_r = \widetilde{\nabla}_{\dot{p}(t)} \delta_r + \overline{h}(\dot{p}(t), \delta_r) \in L_{p(t)}\,. \tag{4.21}$$

Since S^{2n+1} is totally umbilical in $\mathbb{C}^{n+1}$, we have

$$\overline{\nabla}_{\dot{p}(t)} \gamma = -\overline{A}_{p(t)}\dot{p}(t) + \overline{D}_{\dot{p}(t)}\gamma \in \operatorname{span}_{\mathbb{R}}\{\dot{p}(t)\} + \operatorname{span}_{\mathbb{R}}\{p(t)\} \subset L_{p(t)}\,. \tag{4.22}$$

Formulas (4.20)–(4.22) imply that $L_{p(t)}$ is parallel along $p(t)$ with respect to $\overline{\nabla}$, i.e. $L_{p(t)}$ is constant along $p(t)$. Since M is connected, L_p is constant on M.

In the above considerations the completeness of S^{2n+1} is not important, i.e. we can replace S^{2n+1} by its open subset. Now assume M is an arbitrary Sasakian manifold of constant φ-sectional curvature 1. It is known that M is locally isomorphic to S^{2n+1} (see [T]). Hence for any $p \in M$ there is a neighbourhood $\mathcal{U} \subset M$ of p and a totally geodesic $\mathcal{C}$-totally real submanifold M' of $\widetilde{M}$ such that $\mathcal{U}$ is contained in M' and $T_pM' = T_pM + N'_p$ for every $p \in \mathcal{U}$. Let M'_1 and M'_2 be two such submanifolds containing $\mathcal{U}_1$ and $\mathcal{U}_2$ respectively. Suppose that $\mathcal{U}_1 \cap \mathcal{U}_2 \neq \emptyset$ and that ψ is an isomorphism (of Sasakian manifolds) from $\widetilde{\mathcal{U}}$ onto an open subset of S^{2n+1}, where $\widetilde{\mathcal{U}}$ is an open subset of $\widetilde{M}$ containing $\mathcal{U}_1 \cap \mathcal{U}_2$. Then

$$\psi(M_1' \cap \widetilde{\mathcal{U}}) = L_1 \cap \psi(\widetilde{\mathcal{U}}) \quad \text{and} \quad \psi(M_2' \cap \widetilde{\mathcal{U}}) = L_2 \cap \psi(\widetilde{\mathcal{U}}),$$

where L_1 and L_2 are real vector subspaces of $\mathbb{C}^{n+1}$ and $\dim L_1 = \dim L_2$. If $L_1 \neq L_2$, then $\mathcal{U}_1 \cap \mathcal{U}_2$ is contained in $M'' = \psi^{-1}(S^{n+1} \cap L_1 \cap L_2) = M_1' \cap M_2' \cap \widetilde{\mathcal{U}}$ which is a submanifold of $\widetilde{M}$, $T_pM'' = T_pM' \cap T_pM_2'$ and $\dim M'' < \dim M_1'$. This is a contradiction, because $T_pM_1' = T_pM_2'$ for $p \in \mathcal{U}_1 \cap \mathcal{U}_2$. Consequently, $M_1' \cap \mathcal{U} = M_2' \cap \widetilde{\mathcal{U}}$. The proof is complete.

Now we state the main result of this section.

THEOREM 4.5. *Let M be a surface of genus* 0, *$\mathcal{C}$-totally real immersed with parallel mean curvature vector in a Sasakian manifold $\widetilde{M}$ of constant φ-sectional curvature $c > -3$. Then there is a totally geodesic $\mathcal{C}$-totally real submanifold M' of $\widetilde{M}$ such that the image of M is contained in M'.*

Proof. We set $N' = \sum_{l=1}^{\infty} N_l$. By Theorem 4.3, N' is a subbundle of $\mathcal{H}$ and satisfies (ii) of Proposition 4.4. Moreover, by the construction of N_l, for $l = 1, \ldots,$ it is clear that N' satisfies (i) and (iii) of Proposition 4.4. Hence by applying that proposition to N' we get the assertion.

By using Theorem 4.5 and the previously mentioned facts we obtain

COROLLARY 4.6. *Let M satisfy the assumptions of Theorem* 4.5 *and let M' be as in the assertion of that theorem. If M is minimal, then M' is even-dimensional. If* $\dim N_1 = 1$, *then* $\dim M' = 3$.

COROLLARY 4.7. *Let M be a surface of genus* 0 *with parallel mean curvature vector immersed in S^{2n+1}. If ip is perpendicular to T_pM for every $p \in M$, then there is a real vector subspace L of $\mathbb{C}^{n+1}$ such that* $\dim L < n + 1$ *and the image of M is contained in L.*

We also have

THEOREM 4.8. *Let M be a surface of constant Gaussian curvature minimally and $\mathcal{C}$-totally real immersed in S^{2n+1}. Then either M is locally flat, or there is a real vector subspace L of $\mathbb{C}^{n+1}$ such that* $\dim L < n + 1$ *and the image of M is contained in L.*

Proof. By Theorem 1.3 of Chapter I the Gaussian curvature K of M is nonnegative. Assume K is positive and let f be an isometry from an open neighbourhood $\mathcal{U} \subset M$ of p onto an open subset of the Euclidean sphere $S^2(K)$ of Gaussian curvature K. Then $\iota \circ f^{-1}$ is a minimal immersion which, by Theorem 1.2 of Chapter I, can be extended to a minimal immersion, say τ, of the whole of $S^2(K)$. In particular, τ is real-analytic. Since τ is $\mathcal{C}$-totally real on $f(\mathcal{U})$, it is $\mathcal{C}$-totally real on the whole of $S^2(K)$. In fact, let $\mathcal{U}_1$ be a domain of a chart on $S^2(K)$ and let $f(\mathcal{U}) \cap \mathcal{U}_1 \neq \emptyset$. If V, U is a frame defined on $\mathcal{U}_1$, then

$$\mathcal{U}_1 \ni p \to (\langle \tau_*(V), ip \rangle, \langle \tau_*(U), ip \rangle) \in \mathbb{R}^2,$$

being a real-analytic function which vanishes on a non-empty open subset of $\mathcal{U}_1$, is zero on the whole of $\mathcal{U}_1$. By the connectedness of $S^2(K)$, τ is globally $\mathcal{C}$-totally real. Therefore $\iota_{|\mathcal{U}}$ satisfies (1)–(3) of Theorem 4.3. Since p is arbitrary, M also has these properties. Now, to complete the proof we repeat the same argument as in the proof of Theorem 4.5.

References

[A] K. Abe, *Applications of a Riccati type differential equation to Riemannian manifolds with totally geodesic distributions*, Tôhoku Math. J. 25 (1973), 425–444.

[Ba] J. L. M. Barbosa, *An extrinsic rigidity theorem for minimal immersions from S^2 into S^n*, J. Differential Geom. 14 (1978), 355–368.

[BKSS] K. Benko, M. Kothe, K. D. Semler and U. Simon, *Eigenvalues of the Laplacian and curvature*, Colloq. Math. 42 (1979), 19–31.

[B] D. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math. 509, Springer, Berlin 1976.

[BJRW] J. Bolton, G. R. Jensen, M. Rigoli and L. M. Woodward, *On conformal minimal immersions of S^2 into $\mathbb{C}P^n$*, Math. Ann. 279 (1988), 599–620.

[B–W] J. Bolton and L. M. Woodward, *On the Simon conjecture for minimal immersions with S^1-symmetry*, Math. Z. 200 (1988), 111–121.

[Bo] O. Borůvka, *Sur les surfaces représentées par les fonctions sphériques de première espèce*, J. Math. Pures Appl. 12 (1933), 337–383.

[Br] R. Bryant, *Minimal surfaces of constant curvature in S^n*, Trans. Amer. Math. Soc. 290 (1985), 259–271.

$[C]_1$ E. Calabi, *Isometric imbedding of complex manifolds*, Ann. of Math. 58 (1953), 1–23.

$[C]_2$ —, *Construction and properties of some* 6*-dimensional almost complex manifolds*, Trans. Amer. Math. Soc. 87 (1958), 407–438.

$[C]_3$ —, *Minimal immersions of surfaces in Euclidean spheres*, J. Differential Geom. 1 (1967), 111–125.

[Ce] T. E. Cecil, *Geometric applications of critical point theory to submanifolds of complex projective space*, Nagoya Math. J. 55 (1974), 5–31.

$[Ch]_1$ B.-Y. Chen, *On the surface with parallel mean curvature vector*, Indiana Univ. Math. J. 22 (1973), 655–666.

$[Ch]_2$ —, *Geometry of Submanifolds*, Pure Appl. Math. 22, Marcel Dekker, New York 1973.

[ChHL] B.-Y. Chen, C. S. Houh and H.-S. Lue, *Totally real submanifolds*, J. Differential Geom. 12 (1977), 473–480.

[Ch–O] B.-Y. Chen and K. Ogiue, *On totally real submanifolds*, Trans. Amer. Math. Soc. 193 (1974), 257–266.

$[Che]_1$ S. S. Chern, *On the minimal immersions of the two-sphere in a space of constant curvature*, in: Problems in Analysis, Princeton Univ. Press, Princeton, N.J., 1970, 27–40.

$[Che]_2$ —, *On surfaces of constant mean curvature in a three-dimensional space of constant curvature*, in: Lecture Notes in Math. 1007, Springer, Berlin 1983, 104–108.

[D] F. Dillen, *Minimal immersions of surfaces into spheres*, Arch. Math. (Basel) 49 (1987), 94–96.

[DOVV]$_1$ F. Dillen, B. Opozda, L. Verstraelen and L. Vrancken, *On almost complex surfaces of the nearly Kaehler* 6*-sphere, I*, in: Collection of Scientific Papers, Faculty of Science, Univ. of Kragujevac, 8 (1987), 5–13.

[DOVV]$_2$ —, —, —, —, *On totally real* 3*-dimensional submanifolds of the nearly Kaehler* 6*-sphere*, Proc. Amer. Math. Soc. 99 (1987), 741–749.

[DOVV]$_3$ —, —, —, —, *On totally real surfaces of the nearly Kaehler* 6*-sphere*, Geom. Dedicata 27 (1988), 325–334.

[DVV] F. Dillen, L. Verstraelen and L. Vrancken, *On almost complex surfaces of the nearly Kaehler* 6*-sphere II*, Kodai Math. J. 10 (3) (1987), 261–271.

[Eh] Ch. Ehresmann, *Sur les variétés presque complexes*, in: Proc. of the Internat. Congress of Mathematicians, 1950, Vol. II, Amer. Math. Soc., 1952, 412–419.

[E]$_1$ N. Ejiri, *Totally real submanifolds in a* 6*-sphere*, Proc. Amer. Math. Soc. 83 (1981), 759–763.

[E]$_2$ —, *Equivariant minimal immersions of* S^2 *into* $S^{2m}(1)$, Trans. Amer. Math. Soc. 297 (1) (1986), 105–124.

[Er] J. Erbacher, *Reduction of codimension of an isometric immersion*, J. Differential Geom. 5 (1971), 333–340.

[Fe] D. Ferus, *Symmetric submanifolds of Euclidean space*, Math. Ann. 247 (1980), 81–93.

[Fr] A. Fröhlicher, *Zur Differentialgeometrie der komplexen Strukturen*, Math. Ann. 129 (1955), 50–95.

[F–I] F. Fukami and S. Ishihara, *Almost Hermitian structure on* S^6, Tôhoku Math. J. 7 (1955), 151–156.

[G] A. Gray, *Almost complex submanifolds of the six sphere*, Proc. Amer. Math. Soc. 20 (1969), 277–279.

[G–R] I. V. Guadelupe and L. Rodriguez, *Normal curvature of surfaces in space forms*, Pacific J. Math. 106 (1983), 95–103.

[H] H. Hopf, *Über Flächen mit einer Relation zwischen den Hauptkrümmungen*, Math. Nachr. 4 (1950/51), 232–249.

[I] T. Itoh, *A characterization of the generalized Veronese surfaces*, Proc. Amer. Math. Soc. 104 (2) (1988), 571–576.

[J–R] G. Jensen and M. Rigoli, *Minimal surfaces in spheres*, Rend. Mat. Fis. Politec. Torino, Special volume, (1983), 75–98.

[K–Y] M. Kon and K. Yano, *CR Submanifolds of Kaehlerian and Sasakian manifolds*, Progr. Math. 30, Birkhäuser, Boston 1983.

[K–S] M. Kozlowski and U. Simon, *Minimal immersions of* 2*-manifolds into spheres*, Math. Z. 186 (1984), 377–382.

[L]$_1$ H. B. Lawson, *Local rigidity theorem for minimal hypersurfaces*, Ann. of Math. 89 (2) (1969), 187–197.

[L]$_2$ —, *The Riemannian geometry of holomorphic curves*, in: Carolina Conference Proc. (1970), 45–62.

[M] K. Mashimo, *Homogeneous totally real submanifolds of* S^6, Tsukuba J. Math. 9 (1985), 185–202.

[MRU] S. Montiel, A. Ros and F. Urbano, *Curvature pinching and eigenvalue rigidity for minimal submanifolds*, Math. Z. 191 (1986), 537–548.

[N] H. Naitoh, *Parallel submanifolds of complex space forms I, II*, Nagoya Math. J. 90 (1983), 85–117, 91 (1983), 119–149.

[N–T] H. Naitoh and N. Takeuchi, *Totally real submanifolds and symmetric bounded domains*, Osaka J. Math. 19 (1982), 717–731.

[N–Ta] H. Nakagawa and R. Takagi, *On locally symmetric Kaehler submanifolds in a complex projective space*, J. Math. Soc. Japan 28 (1976), 638–667.

[O] T. Ogata, *Minimal surfaces in a sphere with Gaussian curvature less than* 1/6, Tôhoku Math. J. 37 (1985), 553–560.

[Og] K. Ogiue, *Differential geometry of Kaehler submanifolds*, Adv. in Math. 13 (1974), 73–114.

[Op]$_1$ B. Opozda, *Totally real submanifolds of* $S^6(1)$ *with parallel second fundamental form*, Suppl. Rend. Circ. Mat. Palermo 14 (1987), 247–253.

[Op]$_2$ —, *On totally real surfaces with parallel mean curvature vector*, Bull. Soc. Math. Belg. Sér. B 40 (2) (1988), 207–244.

[R]$_1$ A. Ros, *Positively curved Kaehler submanifolds*, Proc. Amer. Math. Soc. 93 (1985), 329–331.

[R]$_2$ —, *A characterization of seven compact Kaehler submanifolds by holomorphic pinching*, Ann. of Math. 121 (1985), 377–382.

[R]$_3$ —, *Kaehler submanifolds in the complex projective space*, preprint.

[R–V] A. Ros and L. Verstraelen, *On a conjecture of K. Ogiue*, J. Differential Geom. 19 (1984), 561–566.

[Ru] E. A. Ruh, *Minimal immersions of* 2*-spheres in* S^4, Proc. Amer. Math. Soc. 28 (1971), 219–222.

[S] K. Sekigawa, *Almost complex submanifolds of a* 6*-dimensional sphere*, Kodai Math. J. 6 (1983), 174–185.

[Si] J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. 88 (1968), 62–105.

[S–Y] Y.-T. Siu and S.-T. Yau, *Compact Kähler manifolds of positive bisectional curvature*, Invent. Math. 59 (1980), 189–204.

[T] S. Tanno, *Sasakian manifolds with constant* ϕ*-holomorphic sectional curvature*, Tôhoku Math. J. 21 (1969), 501–507.

[U]$_1$ F. Urbano, *Totally real minimal submanifolds of a complex projective space*, Proc. Amer. Math. Soc. 93 (1985), 332–334.

[U]$_2$ —, *Nonnegatively curved totally real submanifolds*, Math. Ann. 273 (1986), 345–348.

[W] N. Wallach, *Extension of locally defined minimal immersions into spheres*, Arch. Math. (Basel) 21 (1970), 210–213.

[We] H. C. Wente, *Counterexample to a conjecture of H. Hopf*, Pacific J. Math. 121 (1986), 193–243.

[YKM] S. Yamaguchi, M. Kon and Y. Miyahara, *A theorem on* C*-totally real minimal surfaces*, Proc. Amer. Math. Soc. 54 (1976), 276–280.

[Y] S.-T. Yau, *Submanifolds with constant mean curvature I*, Amer. J. Math. 96 (1974), 346–366.