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Complexity of weakly null sequences

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CONTENTS

Abstract

We introduce an ordinal index which measures the complexity of a weakly null sequence, and show that a construction due to J. Schreier can be iterated to produce for each $\alpha < \omega_1$, a weakly null sequence $(x_n^\alpha)_n$ in $C(\omega^{\omega^\alpha})$ with complexity α. As in the Schreier example each of these is a sequence of indicator functions which is a suppression-1 unconditional basic sequence. These sequences are used to construct Tsirelson-like spaces of large index. We also show that this new ordinal index is related to the Lavrent'ev index of a Baire-1 function and use the index to sharpen some results of Alspach and Odell on averaging weakly null sequences.

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0. Introduction

In this paper we investigate the oscillatory behavior of pointwise converging sequences. Our main tool is a new ordinal index which measures the oscillation of such a sequence. We show that there are weakly null sequences of indicator functions in $C(K)$ with arbitrarily large oscillation index and that the oscillation index is smaller than other similar ordinal indices. In particular, the oscillation index of a pointwise converging sequence is directly compared to the Lavrent'ev index of its limit, the ℓ^1-index defined by Bourgain, and the averaging index. Many of the results are directly related to those in [A-O] where the averaging index is used and we extend some results of that paper.

The first example of a weakly null sequence with no subsequence with averages going to zero in norm was constructed by Schreier [Sch]. His example is a sequence of indicator functions and, as observed by Pełczyński and Szlenk [P-S], can be realized on the space of ordinals less than or equal to ω^ω in the order topology. Because these are indicator functions the failure of the Banach–Saks property is solely dependent on the intersection properties of the sets. Our construction is also based on intersection properties of sets. Thus one viewpoint on the constructions in this paper is that there are families of sets with very complicated intersection properties. The purpose of the ordinal index is to measure the complexity of these intersection properties. The examples of weakly null sequences that we construct are, like Schreier's example, suppression-1 unconditional basic sequences and can be considered as generalizations of Schreier's construction.

Let us note that there is a strong relationship between this work and some unpublished results of Rosenthal on the unconditional basic sequence problem. Rosenthal showed that any weakly null sequence of indicator functions in $C(K)$ has a subsequence which is an unconditional basic sequence. In our work we found that Rosenthal's notion of weakly independent sets fits naturally into our viewpoint and we have incorporated some of Rosenthal's work into the exposition. While we believe that our work explains some difficulties with unconditionality, it is based on intersection properties which cannot provide a complete explanation. For example, in contrast to Rosenthal's result on sequences of indicator functions, it is known that given a weakly null sequence one cannot always find a subsequence which is an unconditional basic sequence ([M-R] and [O]). Thus understanding the unconditional basic sequence problem is more complicated than just understanding the oscillation properties of weakly null sequences. On the other hand, it is clear that the oscillation properties do play a fundamental role in unconditionality. It would be interesting if there were some way to incorporate these more subtle properties of weakly null sequences into an ordinal index.

The paper is organized into six sections. In the first we recall the definitions of some ordinal indices and trees. In the second we introduce the oscillation index and an essentially equivalent index which we call the spreading model index. In the third we prove that the oscillation index is essentially smaller than the ℓ^1-index and show that it is related to the Lavrent'ev index of a Baire-1 function. In the fourth section we define for each countable ordinal α a weakly null sequence of indicator functions and compute the oscillation index of the sequence and the size of the smallest $C(K)$ space which can contain the sequence. In the fifth section we use an idea of Odell to show that the construction in Section 4 can be used to construct reflexive spaces similar to Tsirelson space with large oscillation index. In the sixth section we show that the averaging index, which has a definition similar to that of the spreading model index, is not smaller than the ℓ^1-index. In particular, a space is constructed which does not contain ℓ^1 but has averaging index ω_1. We also extend some results from [A-O] in order to better characterize those sequences which can be averaged a predictable finite number of times in order to get a weakly null sequence.

We will use standard terminology and notation from Banach space theory as may be found in the books of Lindenstrauss and Tzafriri [L-T,I] and [L-T,II] and Diestel [D1] and [D2]. If α is an ordinal, we will use α rather than $\alpha+1$ to denote the space of ordinals less than or equal to α in the order topology and $C(\alpha)$ to denote the space of continuous functions on α.

1. Preliminaries

In this section we will recall the definitions of the ℓ^1-index of Bourgain, the Szlenk index, and the averaging index. In order to define the Bourgain ℓ^1-index we first need to define trees and some related notions.

Definition. Given a set S, a *tree* $\mathcal{T}$ on S is a subset of $\bigcup_{n=1}^{\infty} S^n \cup S^{\infty}$ such that if $b \in \mathcal{T}$ then any initial segment of b is also in $\mathcal{T}$, i.e., if $b \in \mathcal{T}$ and $b = (s_1, s_2, \ldots, s_n, s_{n+1})$ or $b = (s_1, s_2, \ldots, s_n, s_{n+1}, \ldots)$ then $(s_1, s_2, \ldots, s_n) \in \mathcal{T}$. We will call the elements of the tree *nodes* and say that the node b, as above, is a *successor of* or is *below* $(s_1, s_2, \ldots, s_n)$. If b and c are nodes, we will write $b > c$ to indicate that b is below c. In particular, $(s_1, s_2, \ldots, s_n, s_{n+1})$ is an *immediate successor* of $(s_1, s_2, \ldots, s_n)$. If a node $x \in S^n$ then n is the *length* or *level* of x. A *branch* of a tree is a maximal linearly ordered subset of the tree under this natural initial segment ordering. A *subtree* of a tree $\mathcal{T}$ is a subset of $\mathcal{T}$ which is a tree. A tree $\mathcal{T}$ is *finitely branching* if the number of immediate successor nodes of any node is finite. $\mathcal{T}$ is *dyadic* if this number is at most two for all nodes. Finally, a tree is *well-founded* if all of its branches are of finite cardinality.

For well-founded trees there is a standard way to define the ordinal index of the tree.

DEFINITION. Suppose that $\mathcal{T}$ is a well-founded tree on a set X. Let $\mathcal{T}^0 = \mathcal{T}$ and for each ordinal α define

$$\mathcal{T}^{\alpha+1} = \bigcup_{n=1}^{\infty} \{(x_1, x_2, \dots, x_n) \in \mathcal{T}^{\alpha} : \text{there is an } x \in X \text{ such that } (x_1, x_2, \dots, x_n, x) \in \mathcal{T}^{\alpha}\}.$$

For a limit ordinal α let $\mathcal{T}^{\alpha} = \bigcap_{\beta<\alpha} \mathcal{T}^{\beta}$. Let $o(\mathcal{T})$ be the smallest ordinal γ such that $\mathcal{T}^{\gamma} = \emptyset$. This is the *order* of the tree $\mathcal{T}$.

Note that if $\overline{x} = (x_1, x_2, \dots, x_j) \in \mathcal{T}^{\alpha} \backslash \mathcal{T}^{\alpha+1}$, the tree

$$\mathcal{T}_{\overline{x}} = \bigcup_{n=1}^{\infty} \{(y_1, y_2, \dots, y_n) : \overline{x} + (y_1, y_2, \dots, y_n)\},$$

where "+" indicates concatenation, is of order α.

Now we are ready to define the ℓ^1-index of Bourgain.

DEFINITION. Let X be a Banach space and let $\delta > 0$. Put

$$\mathcal{T}(X, \delta) = \bigcup_{n=1}^{\infty} \Big\{(x_1, x_2, \dots, x_n) \in X^n : \|x_j\| \le 1 \text{ for all } j \le n \text{ and } \Big\|\sum_j \lambda_j x_j\Big\| \ge \delta \sum |\lambda_j| \text{ for all } (\lambda_j) \in \mathbb{R}^n\Big\}.$$

The δ-ℓ^1-index of X is $\ell(X, \delta) = o(\mathcal{T}(X, \delta))$. If $\mathcal{T}(X, \delta)^{\alpha} \ne \emptyset$ for all countable α then $\ell(X, \delta) = \omega_1$.

It is easy to see that if ℓ^1 is isomorphic to a subspace of X then $\ell(X, \delta) = \omega_1$ for some $\delta > 0$ and thus, by a result of James [J], for all δ, $0 < \delta < 1$. The converse is also true but requires the fact that if $\mathcal{T}^{\alpha}(X, \delta) \ne \emptyset$ for all countable α then the tree has an infinite branch (see [Bo]). Bourgain observed that the examples employed by Szlenk [Sz] to show that there are separable reflexive spaces with Szlenk index greater than any given countable ordinal also have large ℓ^1-index. Note that this indicates that the Baire-1 functions in X^{**} provide only a very weak indication as to the ℓ^1-index of X.

Next we will define Bourgain's index for Boolean independence. This index is a technical convenience which we will use in establishing lower bounds on the ℓ^1-index.

Definition. Let K be a set and $\{(A_n,B_n)\}$ be a sequence of pairs of subsets of K. Let

$$\mathcal{T}(\{(A_n,B_n)\}) = \bigcup_{k=1}^{\infty} \Big\{(n_1,n_2,\ldots,n_k)\in\mathbb{N}^k : \text{for all } (\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k)\in\{-1,1\}^k,\ \bigcap_{i=1}^{k}\varepsilon_i A_{n_i}\neq\emptyset\Big\},$$

where $(-1)A_i = B_i$. Let $\mathcal{B}(\{(A_n,B_n)\})$ be the order of the tree $\mathcal{T}(\{(A_n,B_n)\})$.

We will usually take pairs of disjoint sets (A_n,B_n) when we employ this index. Note that by Rosenthal's theorem on ℓ^1, [R], a bounded sequence (x_n) in a Banach space X has a subsequence equivalent to the unit vector basis of ℓ^1 if (and only if) it has no weak Cauchy subsequence. Moreover, for any such sequence there is a subsequence $(x_n)_{n\in M}$ and two numbers $\delta>0$ and r such that if

$$A_n = \{x^*\in X^* : x^*(x_n)\ge r+\delta,\ \|x^*\|\le 1\},$$
$$B_n = \{x^*\in X^* : x^*(x_n)\le r,\ \|x^*\|\le 1\}$$

then $\mathcal{T}(\{(A_n,B_n): n\in M\}) = \bigcup_{k=1}^{\infty} M^k$ and thus $\mathcal{B}(\{(A_n,B_n): n\in\mathbb{N}\}) = \omega_1$. Hence if we consider such sets (A_n,B_n), the index is ω_1 for some $\delta>0$ and r if and only if (x_n) has a subsequence with no weak Cauchy subsequence. In particular, if (x_n) converges w^* sequentially to some $x^{**}\in X^{**}$, the index is countable.

Now we wish to recall some other ordinal indices which are more classical in spirit. Each of these indices is defined in terms of the oscillation of sequences of functions on the unit ball of the dual in the w^* topology. First we recall the Szlenk index. Let X be a Banach space and for each $\alpha<\omega_1$ let

$$P_{\alpha+1}(\varepsilon,B_X,B_{X^*}) = \{x^* : \text{ there exists } x_n^*\in P_\alpha(\varepsilon,B_X,B_{X^*}) \text{ and } x_n\in B_X \text{ such that } x_n^*\xrightarrow{w^*}x^*,\ x_n\xrightarrow{w}0, \text{ and } \lim x_n^*(x_n)\ge\varepsilon\}.$$

At a limit ordinal β we take the intersection, that is,

$$P_\beta(\varepsilon,B_X,B_{X^*}) = \bigcap_{\alpha<\beta} P_\alpha(\varepsilon,B_X,B_{X^*}).$$

Now fix a normalized weakly null sequence $\{e_j\}$ in X. The next two ordinal indices are defined for each such sequence. The first occurred in a paper of Zalcwasser [Z] and in a paper of Gillespie and Hurwitz [G-H] and was used to prove that a pointwise converging sequence of bounded continuous functions on a compact metric space has a sequence of convex combinations going to zero in norm. Let

$$Z_{\alpha+1}(\varepsilon,\{e_j\},B_{X^*}) = \{x^* : \text{there exists } x_n^*\in Z_\alpha(\varepsilon,\{e_j\},B_{X^*}) \text{ and } e_{j_n} \text{ such that } x_n^*\xrightarrow{w^*}x^* \text{ and } \lim|x_n^*(e_{j_{2n}})-x_n^*(e_{j_{2n-1}})|\ge\varepsilon\}.$$

As before, $Z_\beta = \bigcap_{\alpha<\beta} Z_\alpha$ for a limit ordinal β.

In [A-O] the following index was introduced in order to obtain some more precise information about the nature of the convex combinations obtained in these early papers in the context of weakly null sequences in a Banach space. Let

$$A_{\alpha+1}(\varepsilon,\{e_j\},B_{X^*}) = \{x^* : \text{for every neighborhood } \mathcal{N} \text{ of } x^* \text{ relative to } A_\alpha(\varepsilon,\{e_j\},B_{X^*}) \text{ there exists an infinite set } L\subset\mathbb{N} \text{ with } \ell^1\text{-}SP(\{e_{i|\mathcal{N}}\}_{i\in L})\ge\varepsilon\}$$

where

$$\ell^1\text{-}SP(\{e_i\}) = \lim_k \liminf_m \Big\{k^{-1}\Big\|\sum_{i=1}^k e_{n_i}\Big\| : m\le n_1<n_2<\ldots<n_k\Big\}\,.$$

As before, $A_\beta = \bigcap_{\alpha<\beta} A_\alpha$ if β is a limit ordinal. We will refer to this as the averaging index.

In each case the successor set (if non-empty) is a w^* closed nowhere dense subset of the set. (For the case of the Szlenk sets we need to assume that X^* is separable.) Thus the Baire Theorem gives in each case a largest ordinal α with the αth set non-empty and the $(\alpha+1)$th empty. Denote by $o(P,\varepsilon)$, $o(Z,\varepsilon)$, and $o(A,\varepsilon)$ the indices for the Szlenk, Zalcwasser, and average sets, respectively. Clearly $o(P,\varepsilon)\ge o(Z,2\varepsilon)$ and $o(Z,\varepsilon)\ge o(A,\varepsilon)$.

2. Weakly null sequences and the ℓ^1-index

We wish to introduce an index similar to the Szlenk index which will measure ℓ^1-ness in a different way than the ℓ^1-index of Bourgain.

Definition. Let (K,d) be a Polish space and let (f_n) be a pointwise convergent sequence of continuous functions on K. Fix $\varepsilon>0$ and let

$$A^+_{n,m} = \{k\in K : f_n(k)-f_m(k)>\varepsilon\}\,,$$
$$A^-_{n,m} = \{k\in K : f_n(k)-f_m(k)<-\varepsilon\}\,.$$

For each countable ordinal α we define inductively a subset of K by

$$\mathcal{O}^0(\varepsilon,(f_n),K) = K\,,$$

$$\mathcal{O}^{\alpha+1}(\varepsilon,(f_n),K) = \Big\{k\in\mathcal{O}^\alpha(\varepsilon,(f_n),K) : \text{ for every neighborhood } \mathcal{N} \text{ of } k \text{ there is an } N\in\mathbb{N} \text{ such that for all } n\ge N \text{ there exists an } M\in\mathbb{N} \text{ such that } \bigcap_{m\ge M} A^+_{n,m}\cap\mathcal{O}^\alpha(\varepsilon,(f_n),K)\cap\mathcal{N}\ne\emptyset \text{ or } \bigcap_{m\ge M} A^-_{n,m}\cap\mathcal{O}^\alpha(\varepsilon,(f_n),K)\cap\mathcal{N}\ne\emptyset\Big\}\,.$$

If β is a limit ordinal, $\mathcal{O}^\beta(\varepsilon,(f_n),K)=\bigcap_{\alpha<\beta}\mathcal{O}^\alpha(\varepsilon,(f_n),K)$.

Note that if (f_n) converges pointwise to 0 and $\varepsilon'<\varepsilon$, then for large enough M

$$\{k: f_n(k)\ge\varepsilon'\}\supset\bigcap_{m\ge M}A^+_{n,m}\supset\{k: f_n(k)\ge\varepsilon\}\,.$$

Thus $\mathcal{O}^\alpha(\varepsilon,(f_n),K)$ is essentially the Szlenk index set, $P_\alpha(\varepsilon,(f_n),K)$, except that we do not allow the use of subsequences. Thus $P_\alpha(\varepsilon',(f_n),K)\supset\mathcal{O}^\alpha(\varepsilon,(f_n),K)$ in this case for all $\varepsilon'<\varepsilon$. Also, it is easy to see that $\mathcal{O}^\alpha(\varepsilon,(f_n),K)$ is always closed. If the limit f is not continuous, the relationship with the Szlenk sets is less clear. However, in the definition of the Szlenk sets it is possible to use weak Cauchy sequences in place of weakly null sequences. The sets obtained in this way behave in essentially the same way as the original Szlenk sets provided the dual is separable. Thus if $[f_n]^*$ is separable, $\mathcal{O}^\alpha(\varepsilon,(f_n),K)$ will be empty for some $\alpha<\omega_1$. Actually, the index is always countable but this will be a consequence of our result relating this index to the ℓ^1-index.

This discussion (and some later results) justifies the following.

Definition. Suppose (f_n) is a pointwise converging sequence of continuous functions on a Polish space K. $\mathcal{O}(\varepsilon)=\mathcal{O}(\varepsilon,(f_n),K)$ is the largest ordinal α such that $\mathcal{O}^\alpha(\varepsilon,(f_n),K)$ is non-empty. We will refer to $\mathcal{O}(\varepsilon)$ as the *ε-oscillation index* of the sequence (f_n). (Some authors, e.g. [K-L], use the term oscillation index for an ordinal index defined in terms of the oscillation of a fixed function near a point. Here we only apply the term to sequences so no confusion should result.)

It will be convenient to use a more restrictive index at times. Let us set $\mathcal{O}^0_+(\varepsilon,(f_n),K)=K$. For each countable ordinal α define inductively a subset of K by

$$\mathcal{O}^{\alpha+1}_+(\varepsilon,(f_n),K)=\Big\{k\in\mathcal{O}^\alpha_+(\varepsilon,(f_n),K):\text{ for every neighborhood } \mathcal{N}\text{ of } k \text{ there is an } N\in\mathbb{N}\text{ such that for all } n\ge N \text{ there exists an } M\in\mathbb{N}\text{ such that } \bigcap_{m\ge M}A^+_{n,m}\cap\mathcal{O}^\alpha_+(\varepsilon,(f_n),K)\cap\mathcal{N}\ne\emptyset\Big\}\,.$$

If β is a limit ordinal then $\mathcal{O}^\beta_+(\varepsilon,(f_n),K)=\bigcap_{\alpha<\beta}\mathcal{O}^\alpha_+(\varepsilon,(f_n),K)$. As above, the positive ε-oscillation index $\mathcal{O}_+(\varepsilon)$ is the largest ordinal α such that $\mathcal{O}^\alpha_+(\varepsilon,(f_n),K)\ne\emptyset$. In a similar way we define $\mathcal{O}^\alpha_-(\varepsilon,(f_n),K)$ and $\mathcal{O}_-(\varepsilon,(f_n),K)$.

Remark 2.1. We do not need to have the sequence converging pointwise for the definition of the index to make sense. However, if (f_n) has no pointwise convergent subsequence then, by Rosenthal's ℓ^1 theorem [R], there is an infinite set M and real numbers $\delta>0$ and r such that $(\{k: f_n(k)\ge r+\delta\},\{k: f_n(k)\le r\})_{n\in M}$ is Boolean independent. From this it follows that there is a Cantor set $C\subset K$ so that relative to C and taking n and m from M, $\bigcap_{m>n}A_{n,m}$ is an

ε_n-net in C and $\varepsilon_n \to 0$ as $n \to \infty$. Hence $\mathcal{O}^1(\delta/2, (f_n)_{n\in M}, C) = C$. Thus for this subsequence the oscillation index is ω_1.

Definition. The *ε-oscillation index of a Banach space* is the supremum of the ε-oscillation index over all weak Cauchy sequences in the unit ball of the space as functions on the dual ball with the w^* topology.

The next lemma follows easily from the definitions but we state it for future reference.

Lemma 2.2. *If K is a w^* closed subset of the dual ball of X, then $\mathcal{O}^\alpha(\varepsilon, (f_n), K) \subset \mathcal{O}^\alpha(\varepsilon, (f_n), B_{X^*})$ for all α and ε. In particular, if T is a bounded operator from a Banach space X to a Banach space Y then*

$$T^*\mathcal{O}^\alpha(\varepsilon, (Tf_n), B_{Y^*}) = \mathcal{O}^\alpha(\varepsilon, (f_n), T^*B_{Y^*}) \subset \mathcal{O}^\alpha(\varepsilon/\|T\|, (f_n), B_{X^*}),$$

and consequently

$$\mathcal{O}(\varepsilon, (Tf_n), B_{Y^*}) \le \mathcal{O}(\varepsilon/\|T\|, (f_n), B_{X^*}).$$

On the other hand, we do not know if there is an essential equivalence between $\sup_M \mathcal{O}(\varepsilon, (f_n)_{n\in M}, B_{X^*})$ and $\sup_M \mathcal{O}(\varepsilon\lambda, (f_n)_{n\in M}, K)$ if K is a λ norming subset of B_{X^*}. Simple examples show that considering subsequences is necessary in order for the indices to be approximately the same size.

Next we present some examples in which we can compute some bounds on the oscillation index and compare this index to the Bourgain ℓ^1-index. The first nontrivial example we wish to present is the Schreier sequence, [Sch].

Example 1. Let $\mathcal{F}_1 = \{F \subset \mathbb{N} : \min F \ge \operatorname{card} F\} \cup \{\emptyset\}$. We will identify $\mathcal{F}_1$ with $\{1_F : F \in \mathcal{F}_1\}$ and note that the latter is a countable compact (metric) space in the topology of pointwise convergence on $\mathbb{N}$. Define $f_n(F) = 1_F(n)$. Because each F is finite (f_n) converges pointwise to 0 on $\mathcal{F}_1$. It is easy to see that each f_n is continuous on $\mathcal{F}_1$. A computation shows that

$$\mathcal{O}^j(1, (f_n), \mathcal{F}_1) = \bigcup_{m>j} \{F \in \mathcal{F}_1 : \min F \ge m \text{ and } \operatorname{card} F \le m - j\} \cup \{\emptyset\}.$$

Thus $\mathcal{O}^\omega(1, (f_n), K) = \{\emptyset\}$ and $\mathcal{O}(1) = \omega$. Because these are indicator functions the same is true for all ε, $1 \ge \varepsilon > 0$. Note that this is the maximal index because $\mathcal{F}_1$ is homeomorphic to ω^ω in the order topology and hence there are only ω non-empty topological derived sets.

It follows from the combinatorial properties of $\mathcal{F}_1$ that $[f_n]$ contains ℓ^1_n's uniformly and thus $\ell([f_n]) \ge \omega$ (see [P-S] or Section 4). Also, it is not hard to see that the ε-oscillation index of $C(\omega^\omega)$ is essentially the same as the Szlenk index, $\omega[1/\varepsilon]$.

Example 2 (Tsirelson space T; see [C-S] or Section 5). The natural unit vector basis of T is a weakly null sequence. T is reflexive and contains many ℓ^1_n's. Thus $\ell(T, \delta) \ge \omega$, for $\delta = \frac{1}{4}$, for example. The computation of $\mathcal{O}^j(\varepsilon, (e_n), B_{T^*})$ does not seem to be easy. However, note that the functionals 1_F for $F \in \mathcal{F}_1$ in Example 1 above act on T by $1_F(\sum a_n e_n) = \sum_{n\in F} a_n$ and all $\|1_F\|_{T^*} \le 2$. Moreover, if $1_{F_k} \to 1_F$ pointwise on $\mathbb{N}$, then $1_{F_k}(x) \to 1_F(x)$ for all $x \in T$. Therefore the map $S : T \to C(\mathcal{F}_1) = C(\omega^\omega)$ defined by $Sx(F) = 1_F(x)$ is well defined and bounded by 2, and $Te_n = f_n$. It follows from Example 1 and Lemma 2.2 that $\mathcal{O}^\omega(\frac{1}{2}, (e_n), B_{X^*}) \ne \emptyset$.

EXAMPLE 3 (ℓ^1). There are no non-trivial weakly null sequences in ℓ^1 and thus the oscillation index is 0. On the other hand, the ℓ^1-index is ω_1.

The anomalous behavior of the index for ℓ^1 can be corrected if we allow general sequences and modify the definition of $\mathcal{O}^{\alpha+1}$. However, this seems pointless in view of Rosenthal's characterization of ℓ^1.

Remark 2.3. Suppose that (f_n) is a pointwise converging sequence of uniformly bounded continuous functions on a compact metric space K. Then the mapping S from K into c (the space of convergent sequences under the sup norm), $S(k) = (f_n(k))$, is continuous if the range is given the $\sigma(c, \ell^1(\mathbb{N}))$ topology. If the functions (f_n) separate points on K then S is also one-to-one. From this viewpoint we are investigating $\sigma(c, \ell^1(\mathbb{N}))$ compact subsets of the ball of c. The oscillation set $\mathcal{O}^\alpha(\varepsilon, (f_n), K)$ is mapped by S to the set $\mathcal{O}^\alpha(\varepsilon, (e_n), S(K))$, where e_n denotes the functional evaluation at n, and thus the index may be computed in c. If the sequence (f_n) converges to 0 then the sets are actually in c_0 and the topology $\sigma(c_0, \ell^1(\mathbb{N}))$ is the weak topology.

3. Comparison with the ℓ^1-index

The ℓ^1-index defined by Bourgain [Bo] measures the degree to which ℓ^1 isomorphically embeds in the space. Bourgain showed that this index is related to an ordinal index of Baire-1 functions in the second dual. In the next section we will show that the presence of certain weakly null sequences can also raise the ℓ^1-index. Thus in contrast to the Baire-1 case the pointwise limit itself provides no information but rather the sequence carries the information. Our method is to use the oscillation index defined in the previous section. In this section we will show that this oscillation index is essentially bounded above by the ℓ^1-index, and prove that any sequence of continuous functions converging pointwise to a Baire-1 function of index α must have a large oscillation index. We then get Bourgain's result as a corollary. These ideas are also related to some work of Haydon, Odell and Rosenthal [H-O-R] on what they term Baire-1/2 and Baire-1/4 functions in the second dual.

Now we will prove that a large oscillation index implies a large ℓ^1-index. For an ordinal $\alpha = \omega^\gamma k + \beta$ with $\beta < \omega^\gamma$ and $k \in \mathbb{N}$, we define $\alpha/2 = \omega^\gamma[(k+1)/2]$, where $[\cdot]$ denotes the greatest integer function.

Theorem 3.1. *If* (f_n) *is a pointwise converging sequence on a compact metric space* K *and* $\mathcal{O}^\alpha(\varepsilon,(f_n),K)\neq\emptyset$ *then if* $\varepsilon'<\varepsilon$, $\ell([f_n],\varepsilon'/2)\ge\alpha/2$. *Moreover, there is an* ℓ^1*-index tree on* (f_n) *with index* $\alpha/2$.

The proof is easier if we assume that $\mathcal{O}^\alpha_+(\varepsilon,(f_n),K)\neq\emptyset$, and in this case we get α rather than $\alpha/2$ for the ℓ^1-index. The proof naturally divides into two parts: first a reduction to the case $\mathcal{O}^{\alpha/2}_+\neq\emptyset$ and then the proof of the result in this case. The reduction is proved as Lemma 3.4. The main idea in the remainder of the argument is to construct a tree of Boolean independent pairs of sets with large order where the pairs of sets are subsets of the $A^+_{n,m}$'s. Consequently, we will actually construct our ℓ^1-tree on $(f_n-f_m)_{n,m\in\mathbb{N}}$ with constant $\varepsilon/2$. A final argument is needed to get an ℓ^1-tree on (f_n).

Before we proceed to the proof we need a few lemmas which describe some sufficient conditions for a tree to be an ℓ^1-tree. The following is an unpublished lemma of Rosenthal. (We actually only use the weaker version in which r does not depend on n.)

Lemma 3.2. *Suppose* $(f_n)_{n=1}^m$ *is a finite sequence of norm one functions on* K, $\delta>0$, *and for each* n *there is a number* r_n *such that if* $A_n=\{f_n\ge r_n+\delta\}$ *and* $B_n=\{f_n\le r_n\}$, $\{(A_n,B_n)\}$ *is Boolean independent. Then* (f_n) *is* $2/\delta$*-equivalent to the unit vector basis of* ℓ^1_m.

Proof. Suppose that $a_n\in\mathbb{R}$ for all n. Let $F=\{n:a_n\ge 0\}$ and $G=\{n:a_n<0\}$. Let $t\in\bigcap_{n\in F}A_n\cap\bigcap_{n\in G}B_n$ and $t'\in\bigcap_{n\in G}A_n\cap\bigcap_{n\in F}B_n$. Then

$$\Big|\sum a_nf_n(t)-\sum a_nf_n(t')\Big|\ge\sum a_n[f_n(t)-f_n(t')]\ge\sum|a_n|\delta\,.$$

Indeed, if $n\in F$ (i.e. $a_n\ge0$) then $f_n(t)\ge r_n+\delta$ and $f_n(t')\le r_n$, and if $n\in G$ (i.e. $a_n<0$) then $f_n(t)\le r_n$ and $f_n(t')\ge r_n+\delta$. Hence either $|\sum a_nf_n(t)|\ge\sum|a_n|\delta/2$ or $|\sum a_nf_n(t')|\ge\sum|a_n|\delta/2$. ■

We would like to construct a Boolean independent tree of pairs of subsets of K where the pairs are related to the elements of the sequence (f_n). If we could, for example, choose sets of the form A_n and B_n as in the lemma above, we would immediately get an ℓ^1-tree on (f_n). Actually somewhat less is sufficient.

Lemma 3.3. *Suppose that* (x_n) *is a sequence of functions on a set* K *with values in* $[-1,1]$ *and* $\mathcal{T}$ *is a tree on* (x_n) *of order* $\alpha<\omega_1$. *Further, assume that there is a* $\delta>0$ *such that for each branch* $\mathcal{B}$ *of* $\mathcal{T}$ *there is a mapping* $\varrho_{\mathcal{B}}:\mathcal{B}\to 2^K\times 2^K$ *such that*

a) *if* $\varrho_{\mathcal{B}}(x_1,x_2,\ldots,x_n)=(A,B)$ *then* $A\subset\{x_n\ge r+\delta\}$ *and* $B\subset\{x_n\le r\}$ *for some* $r\in\mathbb{R}$ (r *may depend on* n *and* $\mathcal{B}$),

b) $\varrho_{\mathcal{B}}(\mathcal{B})$ *is a Boolean independent sequence of sets.*

Then $\mathcal{T}$ *is an* ℓ^1*-tree of order* α *with constant* $\delta/2$.

Proof. According to the previous lemma if $\mathcal{B} = \{(x_1), (x_1, x_2), \ldots\}$ then (x_i) is $2/\delta$-equivalent to the unit vector basis of ℓ^1. Thus each branch of $\mathcal{T}$ also satisfies the requirements for an ℓ^1-tree, and thus we have an ℓ^1-tree of order α. ■

Our next lemma allows us to reduce to the case of positive oscillation index. Below $p(\alpha) = \inf\{\beta+\varrho : \varrho+\beta = \alpha\}$. In particular, if $\alpha = \omega^\gamma k+\beta$, where $\beta < \omega^\gamma$, then $p(\alpha) = \omega^\gamma k$.

LEMMA 3.4. *If $\mathcal{O}^\alpha(\delta, (f_n), K) \neq \emptyset$ and $\gamma \leq \alpha/2$, then for $\varepsilon = +$ or $-$, $\mathcal{O}^\gamma_\varepsilon(\delta, (f_{n_j}), K) \neq \emptyset$, for some subsequence (f_{n_j}).*

Proof. Induction on α. We will actually prove that if $t \in \mathcal{O}^\alpha(\varepsilon, (f_n), K)$ then there is an infinite set $L \subset \mathbb{N}$ and ordinals γ and λ such that

$$\min(\gamma + \lambda, \lambda + \gamma) \geq p(\alpha)$$

and

$$t \in \mathcal{O}^\gamma_+(\delta, (f_n)_{n\in L}, K) \cap \mathcal{O}^\lambda_-(\delta, (f_n)_{n\in L}, K)\,.$$

Suppose that $p(\alpha) = \omega^v \cdot k$. Because the result depends only on $p(\alpha)$, we need only consider ordinals α of the form $\omega^v \cdot k$ in the induction.

If $\alpha = 1$, let $t \in \mathcal{O}^\alpha(\delta, (f_n), K)$ and for $\varepsilon = +$ or $-$ and $i \in \mathbb{N}$ consider the set

$$N^i_\varepsilon = \Big\{n : \text{ there exists an } M \in N \text{ such that } \bigcap_{m\geq M} A^\varepsilon_{nm} \cap \mathcal{N}_i \neq \emptyset\Big\}$$

where $\mathcal{N}_i$ is a decreasing sequence of neighborhoods with intersection $\{t\}$. For at least one choice of ε, N^i_ε is infinite for all i. For that ε let L be an infinite subset of $\mathbb{N}$ such that $L\backslash N^i_\varepsilon$ is finite for each i. Clearly $t \in \mathcal{O}^1_\varepsilon(\delta, (f_n)_{n\in L}, K)$.

Now suppose that the lemma is true for all $\beta < \alpha$ and let $t \in \mathcal{O}^\alpha(\delta, (f_n), K)$. If $\alpha = \omega^v$, let $\alpha_i \uparrow \alpha$. The inductive assumption implies that there are sequences γ_i and λ_i and $L_i \subset \mathbb{N}$ such that

$$\min(\gamma_i + \lambda_i, \lambda_i + \gamma_i) \geq p(\alpha_i)$$

and

$$t \in \mathcal{O}^{\gamma_i}_+(\delta, (f_n)_{n\in L_i}, K) \cap \mathcal{O}^{\lambda_i}_-(\delta, (f_n)_{n\in L_i}, K)\,.$$

We may assume that $L_i \subset L_{i-1}$ for all i and that the sequences (γ_i) and (λ_i) are non-decreasing. It now follows that if L is an infinite subset of $\mathbb{N}$ such that $L\backslash L_i$ is finite for all i then

$$t \in \mathcal{O}^\gamma_+(\delta, (f_n)_{n\in L}, K) \cap \mathcal{O}^\lambda_-(\delta, (f_n)_{n\in L}, K)\,,$$

where $\gamma = \lim \gamma_i$ and $\lambda = \lim \lambda_i$. We have $\min(\gamma + \lambda, \lambda + \gamma) \geq p(\alpha_i)$ for all i and $\min(\gamma + \lambda, \lambda + \gamma) \geq \lim p(\alpha_i)$. Thus if $p(\alpha_i)$ is increasing to $p(\alpha)$, we are done.

Now we may assume that $p(\alpha_i) = \omega^v \cdot k$ for all i and $p(\alpha) = \omega^v \cdot (k+1)$. (This argument will also apply to the successor ordinal case.) For each $s \in \mathcal{O}^{\omega^v\cdot k}(\delta, (f_n), K)$ there is an infinite set L_s and ordinals γ_s, λ_s such that

$$s \in \mathcal{O}^{\gamma_s}_+(\delta, (f_n)_{n\in L_s}, K) \cap \mathcal{O}^{\lambda_s}_-(\delta, (f_n)_{n\in L_s}, K)$$

and

$$\min(\gamma_s + \lambda_s, \lambda_s + \gamma_s) \geq \omega^v \cdot k\,.$$

Let $S = \{s(m) : m \in \mathbb{N}\}$ be a countable dense subset of $\mathcal{O}^{\omega^v \cdot k}(\delta, (f_n), K)$. For each $m \in \mathbb{N}$ there is an infinite set $L_m \subset L_{m-1} \subset \mathbb{N}$ and ordinals γ_m and λ_m such that

$$s(m) \in \mathcal{O}_+^{\gamma_m}(\delta, (f_n)_{n\in L_m}, K) \cap \mathcal{O}_-^{\lambda_m}(\delta, (f_n)_{n\in L_m}, K)$$

and

$$\min(\gamma_m + \lambda_m, \lambda_m + \gamma_m) \geq \omega^v \cdot k\,.$$

By a diagonalization argument we may assume that the set L does not depend on m and that $\gamma_m = \omega^v \cdot k_m$ and $\lambda_m = \omega^v \cdot j_m$. Hence

$$\mathcal{O}^{\omega^v \cdot k}(\delta, (f_n), K) \subset \bigcup_{i+j=k} \mathcal{O}_+^{\omega^v \cdot i}(\delta, (f_n)_{n\in L}, K) \cap \mathcal{O}_-^{\omega^v \cdot j}(\delta, (f_n)_{n\in L}, K).$$

Because each of these sets is closed, we need only consider those sets which contain t. Moreover, observe that if

$$t \in \mathcal{O}_+^{\omega^v \cdot i}(\delta, (f_n)_{n\in L}, K) \cap \mathcal{O}_-^{\omega^v \cdot j}(\delta, (f_n)_{n\in L}, K)$$

for two different pairs (i,j) and (i',j') then

$$t \in \mathcal{O}_+^{\omega^v \cdot i''}(\delta, (f_n)_{n\in L}, K) \cap \mathcal{O}_-^{\omega^v \cdot j''}(\delta, (f_n)_{n\in L}, K)$$

where $i'' = \max(i,i')$ and $j'' = \max(j,j')$ and $\omega^v \cdot i'' + \omega^v \cdot j'' \geq p(\alpha)$, as required. If there is only one such pair (i,j), we may assume that a neighborhood of t (relative to $\mathcal{O}^{\omega^v \cdot k}(\delta, (f_n)_{n\in L}, K)$) is contained in $\mathcal{O}_+^{\omega^v \cdot i''}(\delta, (f_n)_{n\in L}, K) \cap \mathcal{O}_-^{\omega^v \cdot j''}(\delta, (f_n)_{n\in L}, K)$ where $i'' = i$, $j'' = j$. We have

$$t \in \mathcal{O}^{\omega^v}(\delta, (f_n)_{n\in L}, \mathcal{O}_+^{\omega^v \cdot i''}(\delta, (f_n)_{n\in L}, K) \cap \mathcal{O}_-^{\omega^v \cdot j''}(\delta, (f_n)_{n\in L}, K))$$

and thus the case $\alpha = \omega^v$ gives

$$t \in \mathcal{O}_\varepsilon^{\omega^v}(\delta, (f_n)_{n\in L'}, \mathcal{O}_+^{\omega^v \cdot i''}(\delta, (f_n)_{n\in L}, K) \cap \mathcal{O}_-^{\omega^v \cdot j''}(\delta, (f_n)_{n\in L}, K))$$

for some infinite $L' \subset L$ and $\varepsilon = +$ or $-$. Hence

$$t \in \mathcal{O}_+^{\omega^v \cdot i}(\delta, (f_n)_{n\in L'}, K) \cap \mathcal{O}_-^{\omega^v \cdot j}(\delta, (f_n)_{n\in L'}, K),$$

where $i = i'' + 1$ and $j = j''$ if $\varepsilon = +$, and $i = i''$ and $j = j'' + 1$ if $\varepsilon = -$, as required. ■

Now we are ready to begin

P r o o f o f T h e o r e m 3.1. By Lemma 3.4 we need only prove it in the simpler case indicated above, i.e., suppose that $\mathcal{O}_+(\varepsilon) \geq \alpha$. Let $f(t) = \lim f_n(t)$. Fix ϱ, $\varepsilon/8 > \varrho > 0$. For any $g \in C(K)$ let $C(g, k_0) = \{k : |g(k) - f(k_0)| < \varrho\}$.

The proof is by induction on α. As usual, we must actually prove a little stronger statement to make the induction work.

INDUCTIVE HYPOTHESIS: Suppose that $k_1, k_2, \ldots, k_j$ are a finite number of points in $\mathcal{O}_+^\alpha(\varepsilon, (f_n), K)$ and $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_j$ are neighborhoods of $k_1, k_2, \ldots, k_j$, respectively; then for every $\beta < \alpha$ there is a tree $\mathcal{T}$ on $(f_n - f_m)$ of order β such that for $i = 1, 2, \ldots, j$,

$$\mathcal{T}_{\mathcal{N}_i} = \{(x_{1|\mathcal{N}_i}, x_{2|\mathcal{N}_i}, \ldots, x_{n|\mathcal{N}_i}) : (x_1, x_2, \ldots, x_n) \in \mathcal{T}\}$$

is an ℓ^1-tree of order β as in Lemma 3.3, i.e., for each i and for each branch there is a mapping into the pairs of subsets of K satisfying a) and b) with $\delta = \varepsilon/2$ and $r = \varepsilon/4$. Moreover, we may assume that for each pair of sets

$$\varrho_{\mathcal{B}}(f_{n_1} - f_{m_1}, \ldots, f_{n_j} - f_{m_j}) = (A_j, B_j)\,,$$

we have

$$A_j \subset \bigcap_{i=1}^{j-1} C(f_{n_i}, k) \cap C(f_{m_i}, k) \cap C(f_{m_j}, k) \cap \{s : |f_{n_j}(s) - f_{n_j}(k)| < \varrho\}\,,$$

$$B_j \subset \bigcap_{i=1}^{j} C(f_{n_i}, k') \cap C(f_{m_i}, k')$$

for some k and k' in K.

Assume the inductive hypothesis holds for all $\beta < \alpha$. First suppose that α is not a limit ordinal and that $\mathcal{O}_+^\alpha(\varepsilon, (f_n), K) \neq \emptyset$. Let (k_i) and $(\mathcal{N}_i)$ be as above. For each i there is an $N_i \in \mathbb{N}$ such that for each $n \geq N_i$, there is an M_n^i such that

$$\bigcap_{m \geq M_n^i} A_{n,m}^+ \cap \mathcal{O}_+^{\alpha-1}(\varepsilon, (f_n), K) \cap \mathcal{N}_i \neq \emptyset\,.$$

Let $n \geq \max\{N_i\}$ such that $|f_n(k_i) - f(k_i)| < \varrho$, let $M = \max\{M_n^i\}$ and choose a point $k_i' \in \bigcap_{m \geq M} A_{n,m}^+ \cap \mathcal{O}_+^{\alpha-1}(\varepsilon, (f_n), K) \cap \mathcal{N}_i$, for each i. We may also assume that for all $p \geq M$,

$$|f_p(k_i) - f(k_i)| < \varrho\,, \quad |f_p(k_i') - f(k_i')| < \varrho\,,$$

for $i = 1, 2, \ldots, j$. Fix $m \geq M$ and for each i let $\mathcal{N}_i'$ be a neighborhood of k_i' contained in

$$\mathcal{N}_i \cap A_{n,m}^+ \cap C(f_m, k_i') \cap \{k : |f_n(k) - f_n(k_i')| < \varepsilon/8\}$$

and let $\mathcal{N}_i''$ be a neighborhood of k_i contained in

$$\mathcal{N}_i \cap C(f_m, k_i) \cap C(f_n, k_i)\,.$$

Now by the inductive hypothesis for every $\beta < \alpha$ there is a tree $\mathcal{T}$ on $(f_p - f_q)$ such that $\mathcal{T}_{\mathcal{N}_i'}$ and $\mathcal{T}_{\mathcal{N}_i''}$ are ℓ^1-trees of order β for $i = 1, 2, \ldots, j$, and satisfy a) and b) of Lemma 3.3 with $\delta = \varepsilon/2$ and $r = \varepsilon/4$. Let

$$\mathcal{T}' = \{(f_n - f_m, x_1, x_2, \ldots, x_n) : (x_1, x_2, \ldots, x_n) \in \mathcal{T}\}\,.$$

Clearly this is a $(\beta+1)$-tree on (f_n). We need to check the hypothesis of the lemma for $\mathcal{T}'_{\mathcal{N}_i}$, $i=1,2,\ldots,j$. To define $\varrho'_{\mathcal{B}'}$ on a branch

$$\mathcal{B}'=\{(f_n-f_m),(f_n-f_m,x_1),(f_n-f_m,x_1,x_2),\ldots\}$$

of $\mathcal{T}'_{\mathcal{N}_i}$, we let

$$\varrho'_{\mathcal{B}'}(f_n-f_m,x_1,x_2,\ldots,x_n)=\varrho_{\mathcal{B}}((x_1,x_2,\ldots,x_n)),$$
$$\varrho'_{\mathcal{B}'}((f_n-f_m))=(\mathcal{N}'_i,\mathcal{N}''_i),$$

where $\varrho_{\mathcal{B}}$ denotes the mapping from the branch $\mathcal{B}=\{(x_1),(x_1,x_2),\ldots\}$ of $\mathcal{T}_{\mathcal{N}_i}$. If $r=\varepsilon/4$ and $\delta=\varepsilon/2$, the hypothesis of the lemma is satisfied and thus $\mathcal{T}'_{\mathcal{N}_i}$ is an ℓ^1-tree of order $\beta+1$. Because this is true for all $\beta<\alpha$ we get an $(\alpha+1)$-ℓ^1-tree.

For limit ordinals the conclusion is obvious. (Note we can actually get an $(\alpha+1)$-ℓ^1-tree in this case as well.)

The "moreover" assertion allows us to conclude that we can construct an ℓ^1-tree on (f_n) of the same order. Indeed, we claim that the tree obtained by replacing in each coordinate f_n-f_m by f_n is the required one. First if f is continuous we can choose a neighborhood $\mathcal{N}$ of the point $k_0\in\mathcal{O}^\alpha(\varepsilon,(f_n),K)$ so that $\mathcal{N}\subset\{k:|f(k)-c|<\varrho\}$, where $c=f(k_0)$, and restrict all of the functions to this set $\mathcal{N}$. The proof then shows that the sets $\{k:f_{n_i}\ge c+\varepsilon-\varrho\}$ and $\{k:f_{n_i}\le c+\varrho\}$ are Boolean independent for any $(f_{n_1}-f_{m_1},f_{n_2}-f_{m_2},\ldots,f_{n_j}-f_{m_j})$ in the tree constructed.

If f is not continuous we use the following lemma.

LEMMA 3.5. *If (x_n) is a uniformly bounded sequence in a Banach space X such that $\|\sum a_nx_n\|\ge\delta\sum|a_n|$ for all $(a_n)\in\mathbb{R}^{\mathbb{N}}$ and $y\in X$, $\|y\|\le1$, then*

$$\Big\|\sum a_n(x_n+y)\Big\|\ge\delta'\sum|a_n|,$$

where $\delta'=\max\{\delta d(y,[x_n])/2,\delta-\|y\|\}$.

We omit the simple proof.

Our tree was actually constructed using sequences $(f_{n_i}-f_{m_i})_{i=1}^N$ and sets (A_{n_i},B_{n_i}) where $|f_{m_j}(k)-f(k')|<\varrho$ for all $k\in\bigcap_{i=1}^N A_{n_i}^{\varepsilon_i}$, for all j, for some $k'=k(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_N)\in\bigcap_{i=1}^N A_{n_i}^{\varepsilon_i}$. Thus we can replace each f_{m_i} by g where

$$g(k)=\begin{cases} f(k(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_N)) & \text{if } k\in\bigcap_{i=1}^N A_{n_i}^{\varepsilon_i},\\ 0 & \text{otherwise}\end{cases}$$

with a loss of at most ϱ. Observe that because $\lim f_n=f$, $\liminf_N d(f,[f_n-f:n\ge N])\ge\|f\|$. Thus we get the estimate

$$\delta'\ge\max\left\{\left(\frac{\varepsilon}{2}-\varrho\right)\|f\|,\varepsilon-\|f\|-\varrho\right\}$$

from the lemma. ∎

Let us now examine the relationship between the oscillation index and ℓ^1-spreading models.

PROPOSITION 3.6. *Let (f_n) be a pointwise converging sequence on a Polish space (K,d) with limit* 0. *If $k \in \mathcal{O}^\omega(\varepsilon,(f_n),K)$ and $\mathcal{N}$ is a neighborhood of k, then for every infinite $M \subset \mathbb{N}$ there is an $L \subset M$ such that $(f_{n|\mathcal{N}})_{n\in L}$ has ℓ^1-spreading model constant at least ε.*

Proof. Fix $j \in \mathbb{N}$ and choose $L \subset M$ such that $(f_{n|\mathcal{N}})_{n\in L}$ has a spreading model. Because $k \in \mathcal{O}^{2j}((f_n),K,\delta)$, a careful examination of the proof of the previous theorem shows that there is an $\varepsilon = +$ or $-$ such that given any N there is an $\mathcal{L} \subset \{N, N+1, \ldots\}$ with cardinality j with $\mathcal{N} \cap \bigcap_{l\in\mathcal{L}} A^\varepsilon_{l,m} \neq \emptyset$, for all large m. Therefore $(f_{n|\mathcal{N}})_{n\in L}$ has ℓ^1-spreading model constant at least ε. ∎

We wish to present two more examples, but before doing so we will introduce another ordinal index which we call the *spreading model index* and show that it is closely related to the oscillation index. This index is defined in terms of the spreading model constant and should be compared with the averaging index.

Let $S^0(\varepsilon,(f_n),K) = K$ and assuming that $S^\alpha(\varepsilon,(f_n),K)$ has been defined let

$$\begin{aligned} S^{\alpha+1}(\varepsilon,(f_n),K) = \{x^* :\ & \text{for every neighborhood } \mathcal{N} \text{ of } x^* \text{ relative to} \\ & S^\alpha(\varepsilon,(f_n),K) \text{ and infinite } M \subset \mathbb{N} \text{ there is} \\ & \text{an } L \subset N \text{ with } \ell^1\text{-}SP((f_{n|\mathcal{N}}))_{n\in L} \geq \varepsilon\}\,. \end{aligned}$$

As usual, if α is a limit ordinal, then $S^\alpha(\varepsilon,(f_n),K) = \bigcap_{\beta<\alpha} S^\beta(\varepsilon,(f_n),K)$ and the ordinal index will be denoted by $o(S,\varepsilon)$.

Next we will show that the oscillation index and the spreading model index measure essentially the same thing.

PROPOSITION 3.7. *Suppose that (f_n) is a weakly null sequence in $B_{C(K)}$ for some compact metric space K. Then*

(i) *If $S^1(\varepsilon,(f_n),K) \neq \emptyset$, then $\mathcal{O}^1(\varepsilon',(f_n),K) \neq \emptyset$ for every $\varepsilon' < \varepsilon$.*

(ii) *If $\mathcal{O}^\omega(\varepsilon,(f_n),K) \neq \emptyset$, for some $\varepsilon > 0$, then $S^1(\varepsilon,(f_n),K) \neq \emptyset$.*

Proof. Suppose that $\mathcal{O}^1(\varepsilon',(f_n),K) = \emptyset$. Then for each $k \in K$ there is a neighborhood $\mathcal{N}_k$ of k and a subsequence $(f_n)_{n\in M_k}$ such that $\|f_{n|\mathcal{N}_k}\| \leq \varepsilon'$ for all $n \in M_k$. Clearly $\|\sum_{n\in F} f_{n|\mathcal{N}_k}\| \leq |F|\varepsilon'$ for every finite subset F of M_k. Hence $S^1(\varepsilon,(f_n),K) = \emptyset$ for every $\varepsilon > \varepsilon'$.

The second assertion follows from Proposition 3.6. ∎

COROLLARY 3.8. *Suppose that (f_n) is a weakly null sequence on a compact metric space K. Then*

(i) *If $o(S,\varepsilon) \geq \alpha$, then $\mathcal{O}(\varepsilon') \geq \alpha$, for every $\varepsilon' < \varepsilon$.*

(ii) *If $\mathcal{O}(\varepsilon) \geq \omega^{1+\alpha}$, then $o(S,\varepsilon) \geq \omega^\alpha$.*

EXAMPLE 4 (L^1). Of course $\ell(L^1,1)=\omega_1$. However, if (f_n) is a weakly convergent sequence in L^1, then (f_n) is uniformly integrable and hence by Dor's Theorem [Dor] for every $K<\infty$ there is an n such that if F is a set of integers of cardinality n, $(f_j)_{j\in F}$ is not K-equivalent to the unit vector basis of ℓ^1_n. Therefore ℓ^1-$SP(f_n)_{n\in L}=0$ for all L and thus $\mathcal{O}^\omega(\varepsilon,(f_n),K)=\emptyset$ for every $\varepsilon>0$.

EXAMPLE 5. The spaces Szlenk [Sz] used to show that there are reflexive spaces with arbitrarily large (countable) Szlenk index are defined inductively as $X_1=\ell^2$, $X_{\alpha+1}=(X_\alpha\oplus\ell^2)_1$, and for a limit ordinal α, $X_\alpha=(\sum_{\beta<\alpha}X_\beta)_2$. The ℓ^1-index of X_α is α for $\varepsilon=1$ and increases to $\alpha\omega$ as ε goes to 0.

Now let us consider the oscillation index. First suppose that α is a limit ordinal. If (f_n) is a weakly null sequence in X_α, then by passing to a subsequence (which can only increase the index) we may assume that $f_n=g_n+h_n$ where $g_n\in\sum_{\beta\le\lambda}X_\beta$ for all n for some $\lambda<\alpha$ and $h_n\in\sum_{\beta\in B_n}X_\beta$ where $B_n=\{\beta:\beta_n<\beta\le\beta_{n+1}\}$ for all n and $\beta_1>\lambda$. Note that $[h_n]=\ell^2$ and thus the ℓ^1-spreading model index of X_α is the supremum of the indices of X_β, $\beta<\alpha$. Now suppose that $\alpha=\beta+k$ for some integer n. Then

$$X_\alpha=\Big(X_\beta\oplus\underbrace{\ell^2\oplus\ell^2\oplus\cdots\oplus\ell^2}_{k}\Big)_1,$$

and we may write $f_n=g_n+h_n$ where $g_n\in X_\beta$ and $h_n\in\sum_1^k\oplus\ell^2$ for all n. However, we again have ℓ^1-$SP(f_n)=\ell^1$-$SP(g_n)$ and hence the spreading model index of (f_n) is the spreading model index of (g_n). Thus the spreading model index of X_α is the same, for all α, as the index of ℓ^2, namely 0.

This last family of examples illustrates the fact that the oscillation index and the spreading model index are really measuring something stronger than the existence of many ℓ^1_n's in a space.

Next we will examine the ordinal index of a Baire-1 function f on a Polish space (K,d). The index is defined by considering two real numbers c and d, $c<d$, and the disjoint G_δ sets

$$C=\{k\in K: f(k)\le c\}\quad\text{and}\quad D=\{k\in K: f(k)\ge d\}.$$

$L(f,c,d)$ is the smallest ordinal α such that there is a decreasing family of closed sets F_β, $\beta\le\alpha$, with $F_0=K$, $F_\alpha=\emptyset$, and for all $\beta<\alpha$, $F_\beta\setminus F_{\beta+1}$ disjoint from C or from D and at a limit ordinal γ, $F_\gamma=\bigcap_{\beta<\gamma}F_\beta$. (See [Bo] where the definition is given in complementary terms or [K, p. 452].)

Bourgain shows that if (f_n) is a pointwise converging sequence of continuous functions with limit f and $\varepsilon<(d-c)/2$ then $\omega^{\ell([f_n],\,\varepsilon)+1}$ is greater than $L(f,c,d)$. (Actually his result gives a slightly smaller bound.) We wish to show that in fact the oscillation index is also large. The proof of the following proposition is similar to that of Lemma 5 of [H-O-R].

PROPOSITION 3.9. *Suppose that* (f_n) *is a pointwise converging sequence of continuous functions on a compact metric space* K. *Then if* $d-c>\varepsilon$ *and if* $L(f,c,d)=\beta+m$, *where* β *is a limit ordinal and* $m<\omega$, *then* $\mathcal{O}_+(\varepsilon,(f_n),K)\geq\beta+(m-1)/2$.

Proof. Let $C=\{k: f(k)\leq c\}$ and $D=\{k: f(k)\geq d\}$. Consider the following family of closed sets: $F_0=K$, $F_1=\overline{D}$, $F_2=\overline{F_1\cap C}$, $F_3=\overline{F_2\cap D}$, and in general, $F_{\alpha+2n+1}=\overline{F_{\alpha+2n}\cap D}$ and $F_{\alpha+2n+2}=\overline{F_{\alpha+2n+1}\cap C}$ if α is even and $n\in\mathbb{N}$. (Limit ordinals are even.) If α is a limit ordinal, then $F_\alpha=\bigcap_{\beta<\alpha}F_\beta$.

It is easy to see that if α is even, then $(F_\alpha\setminus F_{\alpha+1})\cap D=\emptyset$ and $(F_{\alpha+1}\setminus F_{\alpha+2})\cap C=\emptyset$. Next we will show that $F_{\alpha+2}\cap D\subset\mathcal{O}^1_+(\varepsilon,(f_n),F_{\alpha+1})$ for all α even.

Let $d'\in F_{\alpha+2}\cap D$. This implies that $d'\in\overline{F_{\alpha+1}\cap C}\cap D$. $\overline{F_{\alpha+1}\cap C}\cap D$ contains points from D which are (non-trivial) limits of points in $F_{\alpha+1}\cap C$ and hence there exists a sequence (c_k) in $F_{\alpha+1}\cap C$ with limit d'. Moreover, we may assume that no c_k is in $\mathcal{O}^1_+(\varepsilon,(f_n),F_\alpha)$ (else d' would also be). Choose $N\in\mathbb{N}$ such that $|f_n(d')-f(d')|$ is less than $\delta/4$ for all $n\geq N$, where $0<\delta<(d-c)-\varepsilon$. For each $k\in\mathbb{N}$ there is an M_k such that $f_m(c_k)<\delta/4+f(c_k)$ for all $m\geq M_k$. Now if $\mathcal{N}$ is a neighborhood of d' and $n\geq N$ then for some $L\in\mathbb{N}$, $c_k\in\mathcal{N}$ and $f_n(c_k)>f(d')-\delta/4$, for all $k\geq L$. Because

$$f_n(c_k)-f_m(c_k)>f(d')-\delta/4-f(c_k)-\delta/4\geq d-c-\delta/2\,,$$

for all $k\geq L$ and $m\geq M_k$, we have $c_k\in\mathcal{N}\cap\bigcap_{m\geq M_k}A^+_{n,m}$. Hence $d'\in\mathcal{O}^1_+(\varepsilon,(f_n),F_{\alpha+1})$ and $F_{\alpha+3}=\overline{F_{\alpha+2}\cap D}\subset\mathcal{O}^1_+(\varepsilon,(f_n),F_{\alpha+1})$, for all α even.

Because $\mathcal{O}^{\alpha+1}_+(\varepsilon,(f_n),K)=\mathcal{O}^1_+(\varepsilon,(f_n),\mathcal{O}^\alpha_+(\varepsilon,(f_n),K))$, a simple induction argument shows that $F_{\alpha+2m+1}\subset\mathcal{O}^m_+(\varepsilon,(f_n),F_{\alpha+1})$ for any integer m and even ordinal α. It follows that if β is a limit ordinal then $F_\beta\subset\mathcal{O}^\beta_+(\varepsilon,(f_n),K)$ and for any $m\in\mathbb{N}$, $F_{\beta+2m+1}\subset\mathcal{O}^{\beta+m}_+(\varepsilon,(f_n),K)$. ∎

COROLLARY 3.10. *Suppose that* f *is a Baire*-1 *function on a compact metric space* K *and that* (f_n) *is a sequence of continuous functions on* K *with* $\|f_n\|\geq 1$ *which converge to* f *pointwise. If* $L(f,c,d)=\beta+m$, *where* $c<d$, β *is a limit ordinal and* $m\in\mathbb{N}$, *then for any* $\varepsilon<d-c$ *there is an* $\varepsilon/2$-ℓ^1-*tree of order* $\beta+m/2$ *on* (f_n-f_m), *and there is an* $\varepsilon/2$-ℓ^1-*tree on* (f_n) *of the same order.*

Proof. From the above proposition we get $\mathcal{O}^{\beta+(m-1)/2}_+(\varepsilon,(f_n),K)\neq\emptyset$. The proof of Theorem 3.1 shows that there is an ℓ^1-tree on (f_n-f_m) of order $\beta+(m-1)/2+1$ with lower estimate ε. The second assertion follows from an examination of the proofs of Proposition 3.9 and Theorem 3.1. It is easy to see that in the proof of the theorem we can replace $A^+_{n,m}$ by $\{k: f_n(k)-c>\varepsilon+\varrho\}$, where $\varrho<d-c-\varepsilon$, and always choose the points k_i and k'_i from C. We then conclude that the sets $(\{k: f_{n_s}(k)>c+\varepsilon+\varrho\},\{k: f_{n_s}(k)<c+\varrho\})^j_{s=1}$ are Boolean independent for any node $(f_{n_1},f_{n_2},\ldots,f_{n_j})$ of the tree constructed. ∎

4. Construction of weakly null sequences with large oscillation index

In this section we wish to generalize Schreier's construction of a weakly null sequence (x_n) with no subsequence having the Banach–Saks property. As observed by Pełczyński and Szlenk [P-S] the Schreier sequence is a 1-suppression unconditional basis in $C(\omega^\omega)$. Our goal is to prove

THEOREM 4.1. *For every $\alpha < \omega_1$ there is a weakly null sequence (x_n^α) in $C(\omega^{\omega^\alpha})$ with $\mathcal{O}^{\omega^\alpha}(1-\varepsilon,(x_n^\alpha),\omega^{\omega^\alpha}) \neq \emptyset$ for every $\varepsilon > 0$. Moreover, for each α, (x_n^α) is a sequence of indicator functions and (x_n^α) is a 1-suppression unconditional basic sequence.*

To construct these sequences and verify their properties it is useful to have several different viewpoints. The first viewpoint is contained in the following result.

PROPOSITION 4.2. *Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$ such that*

(i) *if $F \in \mathcal{F}$ and $G \subset F$ then $G \in \mathcal{F}$, i.e., the family is adequate,*
(ii) *$\{n\} \in \mathcal{F}$ for all $n \in \mathbb{N}$,*
(iii) *if $F_j \in \mathcal{F}$ for $j = 1,2,\ldots$ and 1_{F_j} converges pointwise to 1_F then $F \in \mathcal{F}$.*

Let $x_n = 1_{\{F\in\mathcal{F}:n\in F\}}$ for $n = 1,2,\ldots$ Then $\mathcal{F}$ is a countable compact metric space under the topology induced by identifying $\mathcal{F}$ with $\{1_F : F \in \mathcal{F}\}$ under the topology of pointwise convergence. (x_n) is a weakly null 1-suppression unconditional basic sequence in $C(\mathcal{F})$.

Proof. The first assertion is an easy consequence of (iii) and we omit the argument. Note that $x_n(F) \neq 0$ if and only if $n \in F$. Hence

$$\Big\|\sum a_n x_n\Big\| = \sup\Big\{\Big|\sum_{n\in F} a_n\Big| : F \in \mathcal{F}\Big\}.$$

If $G \subset \mathbb{N}$, then by (i)

$$\begin{aligned}\Big\|\sum_{n\in G} a_n x_n\Big\| &= \sup\Big\{\Big|\sum_{n\in G\cap F} a_n\Big| : F \in\mathcal{F}\Big\}\\ &= \sup\Big\{\Big|\sum_{n\in F} a_n\Big| : F \in \mathcal{F} \text{ and } F \subset G\Big\}.\end{aligned}$$

Clearly this is not greater than $\|\sum a_n x_n\|$. Because each $F \in \mathcal{F}$ is finite, $x_n(F) \neq 0$ for only finitely many n. Thus (x_n) is weakly null. ∎

This proposition gives us an easy way of defining and verifying the properties of the Schreier sequence. As in the previous section let

$$\mathcal{F}_1 = \{F \subset \mathbb{N} : \min F \geq \operatorname{card} F\}.$$

(We consider $\emptyset$ to be in $\mathcal{F}_1$.) If (x_n) is defined as in the proposition then it follows that (x_n) is a 1-unconditional basic sequence and is weakly null. Finally, if $L \subset \mathbb{N}$

is infinite and for each $k \in \mathbb{N}$ we let L_k be the first k elements of L then

$$\Big\| \sum_{n \in L_{2k}} x_n \Big\| \geq k$$

because $L_{2k} \setminus L_k \in \mathcal{F}_1$ and

$$\Big\| \sum_{n \in L_{2k+1}} x_n \Big\| \geq k$$

because $L_{2k+1} \setminus L_{k+1} \in \mathcal{F}_1$. Hence

$$\Big\| \sum_{n \in L_k} x_n \Big\| \Big/ k \geq (k-1)/(2k) \quad \text{for all } k\,,$$

and therefore $(x_n)_{n \in L}$ fails the Banach–Saks property (see [D1, p. 78]).

The major drawback to this representation of the Schreier sequence is that it is difficult to understand the topology of $\mathcal{F}_1$ and its relation to (x_n). In this section we will prove some results twice. First we will give proofs based on representations like that above for the Schreier sequence. The second will be given by using trees. We have found that this second viewpoint is more intuitive (it is easy to draw pictures of the trees) and we used it to establish these results originally.

Our next goal then is to use trees to describe the Schreier sequence and the underlying topological space and in particular to show that this sequence and its generalizations could be obtained by beginning with the coordinate functions on the Cantor set, and then essentially restricting them to suitable subsets of the Cantor set. More precisely, if we let $C = \{-1,1\}^{\mathbb{N}}$ and $r_n((\varepsilon_i)) = \varepsilon_n$, then $x_n = (r_n + 1)/2$ is a sequence of indicator functions. If K is a compact subset of C, then the sequence $(x_{n|K})$ is a sequence of indicator functions in $C(K)$ which will be equivalent to the Schreier sequence if K is properly chosen.

Now let us work backwards from a sequence of indicator functions to find an appropriate minimal underlying topological space. Suppose that (x_n) is a sequence of indicator functions on a set K. Define a tree $\mathcal{T}$ on $\{-1,1\}$ by

$$\mathcal{T} = \bigcup_{n=1}^{\infty} \Big\{ (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) : \bigcap_{i=1}^{n} (\operatorname{supp} x_i)^{\varepsilon_i} \neq \emptyset \Big\}$$

where

$$\operatorname{supp} x_i = (\operatorname{supp} x_i)^1 = \{k \in K : x_i(k) = 1\}$$

and

$$(\operatorname{supp} x_i)^{-1} = K \backslash (\operatorname{supp} x_i)^1.$$

Let $\overline{\mathcal{T}} = \mathcal{T} \cup \{(\varepsilon_1, \varepsilon_2, \ldots) : \bigcap_{i=1}^{n} (\operatorname{supp} x_i)^{\varepsilon_i} \neq \emptyset \text{ for all } n\}$. $\overline{\mathcal{T}}$ is also a tree and both are subtrees of the full dyadic tree

$$\mathcal{D} = \bigcup_{n=1}^{\infty} \{-1,1\}^n \cup \{-1,1\}^{\mathbb{N}}\,.$$

$\mathcal{D}$ has a natural topology given by coordinate-wise convergence. Of course in this topology $\{-1,1\}^{\mathbb{N}}$ is just the Cantor set and each finite sequence is an isolated point. Also, any infinite sequence is the limit of its restrictions to the first n coordinates, i.e., the nodes above it. Note also that this tree is closely related to the Boolean independence tree $\mathcal{T}\{((\operatorname{supp} x_n)^1, (\operatorname{supp} x_n)^{-1})\}$.

Lemma 4.3. *$\overline{\mathcal{T}}$ is the closure of $\mathcal{T}$ in $\mathcal{D}$.*

Proof. Obvious. ■

Throughout the remainder of this section we will assume that the sequence (x_n) is pointwise convergent to 0 on K, K is a compact metric space and each x_n is continuous.

Given such a sequence (x_n) the tree $\mathcal{T}((x_n))$ defined above will be called the *tree associated with* (x_n). Because K is compact and (x_n) is weakly null, $\overline{\mathcal{T}}$ contains no elements with infinitely many coordinates equal to 1 and every node is on an infinite branch. Hence $\overline{\mathcal{T}}$ is countable and therefore homeomorphic to some countable ordinal in the order topology. Unfortunately, because the tree is not well-founded, one cannot use the order of the tree $\mathcal{T}$ to determine the topological type. To get around this problem we must study the relationship between the topology of $\overline{\mathcal{T}}$ and the structure of the tree.

Before we embark on the study of the topology let us consider a property of trees analogous to property (i) of Proposition 4.2. In an unpublished paper H. P. Rosenthal introduced the following notion.

Definition. We say that a tree $\mathcal{T}$ on $\{-1,1\}$ is *weakly independent* if $(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_j)\in\mathcal{T}$ implies that for all $(\varepsilon_1',\varepsilon_2',\ldots,\varepsilon_j')$ such that $\varepsilon_i'=-1$ if $\varepsilon_i=-1$ and $\varepsilon_i'=1$ or -1 if $\varepsilon_i=1$, $(\varepsilon_1',\varepsilon_2',\ldots,\varepsilon_j')\in\mathcal{T}$. We say that a sequence of indicator functions is *weakly independent* if the associated tree is.

Rosenthal also proved the following result in a slightly different form.

Proposition 4.4. *A weakly null sequence of* (*non-zero*) *indicator functions on a compact metric space K determines a weakly independent tree if and only if it is a 1-suppression unconditional basic sequence.*

Proof. First suppose that (x_n) is a sequence of indicator functions which is a 1-suppression unconditional basic sequence and $\mathcal{T}$ is the associated tree. Then for any sequence (a_n) of real numbers we have

$$\Big\|\sum a_n x_n\Big\| = \sup\Big\{\Big|\sum a_n x_n(k)\Big| : k\in K\Big\} = \sup\Big\{\Big|\sum_{n:k\in\operatorname{supp} x_n} a_n\Big| : k\in K\Big\}.$$

In particular, suppose that $F\subset\{1,2,\ldots,j\}$ with

$$\bigcap_{n\in F}(\operatorname{supp} x_n)^{-1}\cap\bigcap_{\substack{n\notin F\\ n\le j}}\operatorname{supp} x_n\ne\emptyset,$$

i.e., the node $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j) \in \mathcal{T}$, where $\varepsilon_i = 1$ if $n \notin F$ and $n \le j$, and $\varepsilon_i = -1$ if $n \in F$. In order to check weak independence it is sufficient to check the condition on a lower node. Hence we may assume that if $\{1, 2, \ldots, j\} \supset G \supset F$, then $G \ne \{1, 2, \ldots, j\}$, and we must show that

$$\bigcap_{\substack{n \notin G \\ n \le j}} \operatorname{supp} x_n \cap \bigcap_{n \in G} (\operatorname{supp} x_n)^{-1} \ne \emptyset\,.$$

Because (x_n) is 1-suppression unconditional, we have

$$(j - \text{card } G)(1 + \text{card } G) = \Big\|(1 + \text{card } G) \sum_{\substack{n \notin G \\ n \le j}} x_n + \sum_{n \notin G} (-1) x_n\Big\|$$

and thus there exists

$$k \in \bigcap_{\substack{n \notin G \\ n \le j}} (\operatorname{supp} x_n) \cap \bigcap_{n \in G} (\operatorname{supp} x_n)^{-1}\,.$$

This implies that the node $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j) \in \mathcal{T}$ where $\varepsilon_i = 1$ if $i \notin G$ and $i \le j$, and $\varepsilon_i = -1$ if $i \in G$. Thus $\mathcal{T}$ is weakly independent.

Conversely, suppose that (x_n) is a weakly null sequence of indicator functions and that the associated tree $\mathcal{T}$ is weakly independent. Then for any sequence (a_n) of real numbers and finite subset F of $\mathbb{N}$, we claim that

$$\begin{aligned}\Big\|\sum a_n x_n\Big\| &= \sup\Big\{\Big|\sum_{n : k \in \operatorname{supp} x_n} a_n\Big| : k \in K\Big\} \\ &\ge \sup\Big\{\Big|\sum_{n \in F : k \in \operatorname{supp} x_n} a_n\Big| : k \in K\Big\} = \Big\|\sum_{n \in F} a_n x_n\Big\|\,.\end{aligned}$$

To see the inequality suppose that k is any point in K and $H = \{n : x_n(k) = 1\}$. Since $\mathcal{T}$ is weakly independent there is a point k' in K such that $\{n : x_n(k') = 1\}$ $= F \cap H$. Hence each sum on the right hand side of the inequality also occurs on the left. ■

Corollary 4.5. *Suppose that (x_n) is a weakly null sequence of non-zero indicator functions in $C(K)$ for some compact metric space K. Then the following are equivalent*:

(i) *The tree associated with (x_n) is weakly independent.*
(ii) $\mathcal{F} = \{F \subset \mathbb{N} : F = \{n : x_n(k) = 1\}$ *for some* $k \in K\}$ *is adequate.*
(iii) (x_n) *is a* 1*-suppression unconditional basic sequence.*

Proof. We have already shown that (ii)⇒(iii) and (i)⇔(iii). (i)⇒(ii) is immediate from the definition of weakly independent. ■

Remark 4.6. Rosenthal showed (unpublished) that any weakly null sequence of indicator functions in a $C(K)$ space has a subsequence which is an unconditional basic sequence by showing that there is a weakly independent subsequence.

Remark 4.7. Note that if (x_n) is weakly independent and $\mathcal{T}$ is the associated tree then

$$\bigcup_k \{(n_1, n_2, \ldots, n_k) : (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) \in \mathcal{T} \text{ where } \varepsilon_{n_i} = 1 \text{ for } i = 1, \ldots, k\}$$
$$= \mathcal{T}((\operatorname{supp} x_n, (\operatorname{supp} x_n)^{-1}))\,,$$

the Boolean independence tree.

In order to write tree elements more efficiently let us introduce the following notational conventions: If $\overline{x} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ and $\overline{y} = (\varepsilon'_1, \varepsilon'_2, \ldots \varepsilon'_j)$ are two elements in $\bigcup_{n=1}^{\infty} S^n$, for some set S, then $\overline{x}+\overline{y} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \varepsilon'_1, \varepsilon'_2, \ldots, \varepsilon'_j)$, the concatenation of $\overline{x}$ and $\overline{y}$. Let $\overline{e}_0$ be the empty tuple, $\overline{e}_1 = (-1)$, and inductively define $\overline{e}_{n+1} = \overline{e}_n + \overline{e}_1$, $n = 1, 2, \ldots$ Also, let $\overline{e}_\omega = (-1, -1, \ldots)$.

We will next introduce a derivation on a tree $\mathcal{T} \subset \mathcal{D}$. Let

$$\delta^1(\mathcal{T}) = \{\overline{t} \in \mathcal{T} : \text{there are infinitely many } \overline{s} \in \overline{\mathcal{T}} \setminus \mathcal{T} \text{ with } \overline{t} < \overline{s}\}$$

and if $\delta^\alpha(\mathcal{T})$ has been defined let $\delta^{\alpha+1}(\mathcal{T}) = \delta^1(\delta^\alpha(\mathcal{T}))$. If β is a limit ordinal let $\delta^\beta(\mathcal{T}) = \bigcap_{\alpha<\beta} \delta^\alpha(\mathcal{T})$. Finally, define $\delta(\mathcal{T}) = \inf\{\alpha : \delta^\alpha(\mathcal{T}) = \emptyset\}$.

Proposition 4.8. *Suppose that $\mathcal{T}$ is a weakly independent subtree of the dyadic tree with no infinite nodes, no infinite nodes in $\overline{\mathcal{T}}$ with infinitely many coordinates equal to* 1, *and every node of $\mathcal{T}$ is on some infinite branch. Then $\delta(\mathcal{T})$ determines the homeomorphic type of $\overline{\mathcal{T}}$ up to the number of points in the last derived set, i.e., $\delta^\alpha(\mathcal{T}) = \emptyset$ if and only if $\overline{\mathcal{T}}^{(1+\alpha)} = \emptyset$.*

Proof. First note that each element of $\overline{\mathcal{T}} \setminus \mathcal{T}$ is in the first derived set of $\overline{\mathcal{T}}$, and, in fact, $\overline{\mathcal{T}} \setminus \mathcal{T}$ is the first derived set. Now let us set up a correspondence between the derived sets of $\overline{\mathcal{T}}$ and the subtrees of $\mathcal{T}$. If C is any closed subset of $\overline{\mathcal{T}} \setminus \mathcal{T}$ then there is a tree $\mathcal{T}(C)$ with $\overline{\mathcal{T}(C)} \setminus \mathcal{T}(C) = C$. Indeed, let $\mathcal{T}(C)$ be the set of all nodes $\overline{x}$ of $\mathcal{T}$ for which there is some element c of C below $\overline{x}$.

As above let C be a closed subset of $\overline{\mathcal{T}} \setminus \mathcal{T}$. We claim that $\mathcal{T}(C^{(1)}) = \delta^1(\mathcal{T}(C))$. Suppose that $\overline{x} \in \mathcal{T}(C^{(1)})$. Then $\overline{x}$ is above some element $\overline{c}$ of $C^{(1)}$. Say $\overline{c} = \overline{x} + \overline{y} + \overline{e}_\omega$ where $\overline{y}$ is possibly the empty tuple. Hence there are distinct points $\overline{c}_k$ in C which converge to $\overline{c}$ and therefore for large k, $\overline{c}_k = \overline{y} + \overline{z}_k + \overline{e}_\omega$, where the $\overline{z}_k$'s are distinct. This implies that $\overline{x} \in \delta^1(\mathcal{T}(C))$. Conversely, suppose that $\overline{x} \in \delta^1(\mathcal{T}(C))$. Then there is a sequence of distinct points $(\overline{c}_i)$ in $\overline{\mathcal{T}(C)} \setminus \mathcal{T}(C) = C$ such that $\overline{c}_i > \overline{x}$ for all i. By passing to a subsequence we may assume that $\overline{c}_i \to \overline{c} \in C$. Clearly $\overline{c} \in C^{(1)}$ and $\overline{c} > \overline{x}$, therefore $\overline{x} \in \mathcal{T}(C^{(1)})$.

Because $\overline{\mathcal{T}} \setminus \mathcal{T} = \overline{\mathcal{T}}^{(1)}$ and $\mathcal{T}(\overline{\mathcal{T}} \setminus \mathcal{T}) = \mathcal{T}$, we have

$$\delta^1(\mathcal{T}) = \mathcal{T}((\overline{\mathcal{T}} \setminus \mathcal{T})^{(1)}) = \mathcal{T}([\overline{\mathcal{T}}^{(1)}]^{(1)}) = \mathcal{T}(\overline{\mathcal{T}}^{(2)})\,.$$

We claim that for every $\alpha < \omega_1$,

$$\delta^\alpha(\mathcal{T}) = \mathcal{T}([\overline{\mathcal{T}}^{(1)}]^{(\alpha)}) = \mathcal{T}(\overline{\mathcal{T}}^{(1+\alpha)}).$$

Indeed, if this is true for α we see by the previous claim that

$$\delta^{\alpha+1}(\mathcal{T}) = \delta^1[\mathcal{T}([\overline{\mathcal{T}}^{(1)}]^{(\alpha)})] = \mathcal{T}([[\overline{\mathcal{T}}^{(1)}]^{(\alpha)}]^{(1)}) = \mathcal{T}([\overline{\mathcal{T}}^{(1)}]^{(\alpha+1)}).$$

Also, observe that if (C_i) is a decreasing family of closed subsets of $\overline{\mathcal{T}} \setminus \mathcal{T}$ then $\mathcal{T}(\bigcap_{i=1}^\infty C_i) = \bigcap_{i=1}^\infty \mathcal{T}(C_i)$. Therefore if $\alpha_i \uparrow \alpha$ and the claim is true for each α_i then

$$\delta^\alpha(\mathcal{T}) = \bigcap \delta^{\alpha_i}(\mathcal{T}) = \bigcap \mathcal{T}([\overline{\mathcal{T}}^{(1)}]^{(\alpha_i)}) = \mathcal{T}\Big(\bigcap [[\overline{\mathcal{T}}^{(1)}]^{(\alpha_i)}]\Big) = \mathcal{T}([\overline{\mathcal{T}}^{(1)}]^{(\alpha)}),$$

establishing the claim. Hence $\delta^\alpha(\mathcal{T}) = \emptyset$ if and only if $\overline{\mathcal{T}}^{(1+\alpha)} = \emptyset$. ■

Let us now return to the Schreier sequence and examine the associated tree $\mathcal{S}_1$. Let

$$\mathcal{S}_0 = \bigcup_{n=0}^\infty \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) : \varepsilon_i = 1 \text{ for at most one } i, \quad \varepsilon_i = -1 \text{ otherwise}, 1 \le i \le n\}.$$

To define $\mathcal{S}_1$ we need introduce the extension of one tree by another tree. If $\mathcal{T}$ and $\mathcal{S}$ are trees on the same set let $\mathcal{T} \boxplus \mathcal{S}$ denote $\{\overline{x} + \overline{y} : \overline{x} \in \mathcal{T}$ and $\overline{e}_n + \overline{y} \in \mathcal{S}$ where n is the length of $\overline{x}\} \cup \mathcal{T}$. If $(\mathcal{T}_i)$ is a sequence of trees, we inductively define $\boxplus_{i=1}^n \mathcal{T}_i = [\boxplus_{i=1}^{n-1} \mathcal{T}_i] \boxplus \mathcal{T}_n$. If $n = 0$, let $\boxplus_{i=1}^n \mathcal{T}_i = \{\overline{e}_j : j = 1, 2, \ldots\}$. Also, we will use the notation $\mathcal{L}(\mathcal{T}, n)$ for the subtree which is the union of $\{\overline{e}_j : j = 0, 1, 2, \ldots, n-1\}$ and the set of nodes of $\mathcal{T}$ equal to or below $\overline{e}_{n-1} + (1)$. In particular, $\mathcal{L}(\mathcal{T}, n) \subset \{\overline{e}_j : j = 1, 2, \ldots, n-1\} \boxplus \mathcal{T}$. We need a way of forming a new tree $\mathcal{T}$ out of an infinite sequence of trees $\mathcal{T}_i$, $i = 1, 2, \ldots$, on $\{-1, 1\}$.

Define $\sum_{i=1}^\infty \mathcal{T}_i = \bigcup_{i=1}^\infty \mathcal{L}(\mathcal{T}_i, i)$. Clearly the resulting tree will depend on the order of the $\mathcal{T}_i$'s.

We claim that the tree associated to the Schreier sequence is

$$\mathcal{S}_1 = \sum_{i=1}^\infty \boxplus_{j=1}^i \mathcal{S}_0 .$$

Indeed, the Schreier sequence (x_n) is defined by the property that if $k, N \in \mathbb{N}$ and $k \le n_1 < n_2 < \ldots < n_k \le N$, then

$$\bigcap_{i=1}^k \operatorname{supp} x_{n_i} \cap \bigcap_{\substack{n \neq n_i \\ n < N}} (\operatorname{supp} x_n)^{-1} \neq \emptyset,$$

and these are the only non-empty intersections. Note that this is equivalent to saying that if $m \le k$ and $k = n_1 < n_2 < \ldots < n_m$, then

$$\bigcap_{i=1}^{m} \operatorname{supp} x_{n_i} \cap \bigcap_{\substack{n \ne n_i \\ n < N}} (\operatorname{supp} x_n)^{-1} \ne \emptyset\,.$$

Hence the tree associated with (x_n) contains for each $k \in \mathbb{N}$, exactly those nodes of the dyadic tree of the form $\overline{e}_{k-1} + \overline{x}$ where $\overline{x}$ has at most k coordinates equal to 1. It is easy to see that $\boxplus_{j=1}^{k} \mathcal{S}_0$ is exactly the nodes with at most k coordinates equal to 1. Thus $\mathcal{S}_1$ is the required tree.

To see what the topological type of $\overline{\mathcal{S}}_1$ is we need only compute the order of $\delta[\boxplus_{j=1}^{n} \mathcal{S}_0]$. Clearly $\delta(\mathcal{S}_0) = 2$. A straightforward induction argument shows that $\delta^n([\boxplus_{j=1}^{n} \mathcal{S}_0]) = \delta^n(\mathcal{L}(\boxplus_{j=1}^{n} \mathcal{S}_0, n)) = \{e_\omega\}$. It then follows easily that $\delta(\mathcal{S}_1) = \omega + 1$. Therefore $\overline{\mathcal{S}}_1$ has exactly ω non-empty derived sets. Actually, it is not hard to see that it is homeomorphic to ω^ω.

Now we are ready to generalize the Schreier example. Our goal is to build for each $\alpha < \omega_1$ a weakly null sequence of indicator functions on a compact metric space (homeomorphic to ω^{ω^α}) with oscillation index at least ω^α. The sequence will also be a 1-unconditional basic sequence.

We begin by defining the sequences in terms of subsets of $\mathbb{N}$ as we did with the Schreier sequence. $\mathcal{F}_1$ has been defined. Suppose that $\mathcal{F}_\beta$ has been defined for all $\beta < \alpha$. Let $\alpha_i = \alpha - 1$ if α is not a limit ordinal and $\alpha_i \uparrow \alpha$ if α is a limit ordinal. Define

$$\mathcal{F}_\alpha = \bigcup_{n=1}^{\infty} \Big\{ \bigcup_{i=1}^{n} F_i : \\ F_i \in \mathcal{F}_{\alpha_i},\ \text{for } i \le n,\ k \le F_1 < \ldots < F_i < F_{i+1} < \ldots < F_n \Big\}$$

where the notation $k \le F_1 < \ldots < F_i < F_{i+1} < \ldots < F_n$ means that if F_i and F_j are nonempty and $i < j$, then $k \le \min F_1$ and $\max F_i < \min F_j$.

There is some ambiguity here but it will not be of any significance as long as the choice of the sequence (α_i) is fixed for each limit ordinal. For each $\alpha < \omega_1$ let (x_n^α) denote the standard sequence of indicator functions on $\mathcal{F}_\alpha$, i.e., $x_n^\alpha(F) = 1$ if $n \in F$ and 0 otherwise.

PROPOSITION 4.9. *For each $\alpha < \omega_1$, $\mathcal{F}_\alpha$ is a countable compact metric space under the topology induced by identifying $\mathcal{F}_\alpha$ with $\{1_F : F \in \mathcal{F}_\alpha\}$ under the topology of pointwise convergence and (x_n^α) is a weakly null 1-suppression unconditional basic sequence in $C(\mathcal{F}_\alpha)$.*

Proof. It is sufficient to verify that $\mathcal{F}_\alpha$ satisfies the hypothesis of Proposition 4.2. Property (ii) is obviously inherited by each $\mathcal{F}_\alpha$ from $\mathcal{F}_1$. For (i) use induction and note that if $G \subset F_1 \cup F_2 \cup \ldots \cup F_n \in \mathcal{F}_\alpha$, as in the definition, then $G \cap F_i \in \mathcal{F}_{\alpha_i}$, for each i. Hence $G = (G \cap F_1) \cup (G \cap F_2) \cup \ldots \cup (G \cap F_n) \in \mathcal{F}_\alpha$. Finally, for (iii) suppose that (iii) holds for all $\beta < \alpha$, and that for each k, $F_k = \bigcup_{j=1}^{n_k} F_{kj}$ is an element of $\mathcal{F}_\alpha$, as in the definition. If F_k converges to a non-empty set F, let $n = \min F$. Then $n \in F_k$ for all large k and thus $n_k \le n$. We may assume by passing to subsequences that for each $j \le n$, F_{kj} converges to some F_j. Hence, by induction, because $F_{kj} \in \mathcal{F}_{\alpha j}$ for all k, we have $F_j \in \mathcal{F}_{\alpha j}$. The other properties are obvious and thus $F = \bigcup F_j \in \mathcal{F}_\alpha$, as claimed. ∎

Now we want to consider the size of the underlying topological space. First we will compute the size directly using the families $\mathcal{F}_\alpha$. For each $\alpha < \omega_1$ and $k, n \in \mathbb{N}$ with $n \le k$ let

$$\mathcal{F}_{\alpha,n,k} = \Big\{ \bigcup_{i=1}^{n} F_i : \; F_i \in \mathcal{F}_{\alpha_i},\ k \le F_1 < \ldots < F_i < F_{i+1} < \ldots < F_n,\ \text{and } F_n \in \mathcal{F}_{\alpha_n} \Big\},$$

where $\alpha_i = \alpha - 1$ if α is not a limit ordinal and $\alpha_i \uparrow \alpha$ if α is a limit ordinal.

PROPOSITION 4.10. *For each* $\alpha < \omega_1$, $\mathcal{F}_\alpha^{(\omega^\alpha)} = \{\emptyset\}$.

Proof. The result will follow from

CLAIM. *If* $\varrho \le \omega^{\alpha_n}$, *then*

$$\mathcal{F}_{\alpha,n,k}^{(\varrho)} = \Big\{ \bigcup_{i=1}^{n} F_i : \; F_i \in \mathcal{F}_{\alpha_i},\ k \le F_1 < \ldots < F_i < F_{i+1} < \ldots < F_n,\ \text{and } F_n \in \mathcal{F}_{\alpha_n}^{(\varrho)} \Big\}.$$

The proof of the claim is by induction on α, n, and ϱ.

If $\alpha = 0$, $\mathcal{F}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$. Clearly $\mathcal{F}_0^{(1)} = \{\emptyset\}$.

Let $\alpha = 1$ and $k \ge n \ge 1$. Then if $G \in \mathcal{F}_{1,n,k}^{(1)}$, there is a non-trivial sequence (G_m) in $\mathcal{F}_{1,n,k}$ which converges to F. Observe that we need only show $\operatorname{card} G < n$. However, this is obvious because $\operatorname{card} G_m \le n$ for all m and some portion of the G_m's must go to ∞. Conversely, if $G \in \mathcal{F}_{1,n,k}$ and card $G < n$, then $G \cup \{m\} \in \mathcal{F}_{1,n}$ for $m = k, k+1, \ldots$ Hence $G \in \mathcal{F}_{1,n,k}^{(1)}$. Finally, note that $\mathcal{F}_{1,n,k}$ is closed, $\mathcal{F}_{1,n,k}^{(1)} = \mathcal{F}_{1,n-1,k}$, and if $G_m \in \mathcal{F}_{1,m,m}$ then $G_m \to \emptyset$. Therefore $\mathcal{F}_1^{(\omega)} = \{\emptyset\}$.

Now assume the result holds for all $\gamma < \alpha$. Fix $\varrho < \omega^{\alpha_n}$ and $k \ge n \ge 1$ and assume that the result has been proved for ϱ. (Note that it always holds for $\varrho = 0$.) Let $\mathcal{F} = \mathcal{F}_{\alpha,n,k}^{(\varrho)}$. Suppose that (G_m) is a non-trivial sequence in $\mathcal{F}$ which converges to G. Suppose that $G_m = \bigcup_{i=1}^{n} G_{m,i}$. By passing to a subsequence if necessary we may assume that for each i, $G_{m,i} \to G_i$. If $(G_{m,n})$ is a non-trivial sequence, then $G_n \in \mathcal{F}_{\alpha_n}^{(\varrho+1)}$ and thus $G = \bigcup_{i=1}^{n} G_i$ where $G_i \in \mathcal{F}_{\alpha_i}$ for $i = 1, 2, \ldots, n$ and

$G_n \in \mathcal{F}_{\alpha_n}^{(\varrho+1)}$, that is,

$$G \in \Big\{ \bigcup_{i=1}^{n} F_i : \\ F_i \in \mathcal{F}_{\alpha_i},\ k \le F_1 < \ldots < F_i < F_{i+1} < \ldots < F_n,\ \text{and } F_n \in \mathcal{F}_{\alpha_n}^{(\varrho+1)} \Big\},$$

If $(G_{m,n})$ is trivial (eventually constant) then $(G_{i,n})$ is eventually constant for all i. However, this contradicts the non-triviality of the sequence (G_m).

Conversely, if

$$G \in \Big\{ \bigcup_{i=1}^{n} F_i : \\ F_i \in \mathcal{F}_{\alpha_i},\ k \le F_1 < \ldots < F_i < F_{i+1} < \ldots < F_n,\ \text{and } F_n \in \mathcal{F}_{\alpha_n}^{(\varrho+1)} \Big\},$$

then there is a non-trivial sequence (G_m) in $\mathcal{F}_{\alpha_n}^{(\varrho)}$ which converges to F_n. Clearly we may assume $\min G_m > \max \bigcup_{i=1}^{n-1} F_i$ for all m. Then $G'_m = \bigcup_{i=1}^{n-1} F_i \cup G_m \in \mathcal{F}_{\alpha,n,k}^{(\varrho)}$ for all m and (G'_m) converges to G. Therefore $G \in \mathcal{F}_{\alpha,n,k}^{(\varrho+1)}$.

Clearly if $\varrho_j \uparrow \varrho$, then

$$\mathcal{F}_{\alpha,n,k}^{(\varrho)} = \bigcap_{j=1}^{\infty} \Big\{ \bigcup_{i=1}^{n} F_i : \\ F_i \in \mathcal{F}_{\alpha_i},\ k \le F_1 < \ldots < F_i < F_{i+1} < \ldots < F_n,\ \text{and } F_n \in \mathcal{F}_{\alpha_n}^{(\varrho_j)} \Big\} \\ = \Big\{ \bigcup_{i=1}^{n} F_i : \\ F_i \in \mathcal{F}_{\alpha_i},\ k \le F_1 < \ldots < F_i < F_{i+1} < \ldots < F_n,\ \text{and } F_n \in \mathcal{F}_{\alpha_n}^{(\varrho)} \Big\}.$$

Thus the claim holds for all ordinals $\varrho \le \omega^{\alpha_n}$.

Finally, observe that by induction we have $\mathcal{F}_{\alpha,n,k}^{(\omega^{\alpha_n})} = \mathcal{F}_{\alpha,n-1,k}$ and hence $\mathcal{F}_{\alpha,n,n}^{(\omega^{\alpha_n}+\omega^{\alpha_{n-1}}+\ldots+\omega^{\alpha_1})} = \{\emptyset\}$. To see that $\mathcal{F}_{\alpha}^{(\omega^{\alpha})} = \{\emptyset\}$ note that $\mathcal{F}_\alpha = \bigcup_{n=1}^{\infty} \mathcal{F}_{\alpha,n,n}$ and that if $G_n \in \mathcal{F}_{\alpha,n,n}$ for all n then $G_n \to \emptyset$. ∎

Next we will prove the result again but using trees. First we will translate the construction into the tree representation.

Observe that there is a simple correspondence between

$$\Big\{ \bigcup_{i=1}^{n} F_i : F_i \in \mathcal{F}_{\alpha_i},\ k \le F_1 < \ldots < F_i < F_{i+1} < \ldots < F_n \Big\}$$

and $\boxplus_{i=1}^{n} \mathcal{S}_{\alpha_i}$, where $\mathcal{S}_{\alpha_i}$ is the tree corresponding to $\mathcal{F}_{\alpha_i}$. Indeed,

$$\bigcap_{n\in F_i} \operatorname{supp} x_n^{\alpha} \cap \bigcap_{\substack{n\notin F_i \\ n\le m}} (\operatorname{supp} x_n^{\alpha})^{-1} \ne \emptyset\,,$$

for all m, if and only if $F_i \in \mathcal{F}_\alpha$. Thus if $\overline{y}_i = (\varepsilon_{m_i+1}, \varepsilon_{m_i+2}, \ldots, \varepsilon_{m_{i+1}})$, where $m_i = \min F_i - 1$, and $\varepsilon_j = 1$ if $j \in F_i$, $\varepsilon_j = -1$ otherwise, then we have $\overline{y}_1 + \overline{y}_2 + \ldots + \overline{y}_n + \overline{e}_m$ in $\boxplus_{j=1}^n \mathcal{S}_\alpha$, for all m. Conversely, if we have $\overline{y}_1 + \overline{y}_2 + \ldots + \overline{y}_n$ in $\boxplus_{j=1}^n \mathcal{S}_\alpha$, let m_i be the sum of the lengths of the $\overline{y}_j$'s, $j = 1, 2, \ldots, i-1$, and $F_i = \{m_i + k : \varepsilon_k = 1\}$ for $\overline{y}_i = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m_{i+1}-m_i})$. Then $F_i \in \mathcal{F}_\alpha$, and $\max F_i < \min F_{i+1}$, for $i = 1, 2, \ldots, n$.

Next we need to take care of the $n \le \min F_1$ condition. Observe that we could define

$$\mathcal{F}_\alpha = \bigcup_{n=1}^{\infty} \Big\{ \bigcup_{n=1}^{\infty} F_i : F_i \in \mathcal{F}_{\alpha_i}, n \le F_1 < \ldots < F_i < F_{i+1} < \ldots < F_n \text{ and } n \in F_1 \Big\}$$

and the set would remain the same. We claim that the tree associated with $x_n^{\alpha+1}$ is

$$\mathcal{S}_{\alpha+1} = \sum_{i=1}^{\infty} \boxplus_{j=1}^i \mathcal{S}_\alpha \,.$$

We have already shown above that $\boxplus_{j=1}^i \mathcal{S}_\alpha$ corresponds to the union of i ordered sets from $\mathcal{F}_\alpha$. Now note that the nodes corresponding to $\boxplus_{j=1}^i \mathcal{S}_\alpha$ in the sum are all below or equal to $\overline{e}_{i-1} + (1)$, and thus correspond exactly to the unions of i ordered sets from $\mathcal{F}_\alpha$ with smallest element of the first set equal to i. Thus $\mathcal{S}_{\alpha+1}$ is the correct tree. A similar argument shows that for a limit ordinal α

$$\mathcal{S}_\alpha = \sum_{i=1}^{\infty} \boxplus_{j=1}^i \mathcal{S}_{\alpha_j}$$

where α_i is the defining sequence for $\mathcal{F}_\alpha$.

Now that we have the trees $\mathcal{S}_\alpha$, $\alpha < \omega_1$, we can determine the underlying topological spaces by using Proposition 4.8.

Second proof of Proposition 4.10. Inductively assume that $\delta(\mathcal{L}(\mathcal{S}_\beta, j))$ is ω^β, for all $\beta < \alpha$ and $j \in \mathbb{N}$. Let $\alpha_i = \alpha - 1$ if α is a successor and $\alpha_i \uparrow \alpha$ otherwise.

Claim. $\delta(\mathcal{L}(\boxplus_{i=1}^n \mathcal{S}_{\alpha_i}, j)) = \omega^{\alpha_n} + \omega^{\alpha_{n-1}} + \ldots + \omega^{\alpha_1} + 1$ *for all* $j \in \mathbb{N}$.

Indeed, if $\overline{x} \in \mathcal{L}(\boxplus_{i=1}^{n-1} \mathcal{S}_{\alpha_i}, j)$, $\overline{x} = \overline{y}_1 + \overline{y}_2 + \ldots + \overline{y}_{n-1}$ and the length of $\overline{x}$ is $k \ge j$, then

$$\{\overline{e}_k + \overline{y} : \overline{x} + \overline{y} \in \mathcal{L}(\boxplus_{i=1}^n \mathcal{S}_{\alpha_i}, k+1)\} \cup \{\overline{e}_i : i = 1, 2, \ldots, k\} \supset \mathcal{L}(\mathcal{S}_{\alpha_n}, k+1)$$

Thus

$$\overline{x} \in \delta^{\omega^{\alpha_n}}(\mathcal{L}(\boxplus_{i=1}^n \mathcal{S}_{\alpha_i}, j)) \,,$$

by the inductive hypothesis. Because k is arbitrary the claim follows by induction on n. (Obviously $\delta(\mathcal{L}(\boxplus_{i=1}^n \mathcal{S}_{\alpha_i}, j)) \le \omega^{\alpha_n} + \omega^{\alpha_{n-1}} + \ldots + \omega^{\alpha_1} + 1$.)

It is now easy to see that $\delta(\mathcal{L}(\mathcal{S}_\alpha, j)) = \omega^\alpha + 1$, because the order is larger than $\lambda_n = \omega^{\alpha_n} + \omega^{\alpha_{n-1}} + \ldots + \omega^{\alpha_1}$, for all n, and the elements e_i, $i \in \mathbb{N}$, are in $\delta^{\lambda_n}(\mathcal{L}(\boxplus_{i=1}^n \mathcal{S}_{\alpha_i}, j))$ for all n. ■

Our next task is to compute the oscillation index of (x_n^α). As before we will compute it in two ways, first by using the family $\mathcal{F}_\alpha$ and then by using trees.

PROPOSITION 4.11. *$\mathcal{O}^{\omega^\alpha}(\varepsilon, (x_n^\alpha), \mathcal{F}_\alpha) \neq \emptyset$, for all $\alpha < \omega_1$, $\varepsilon < 1$.*

Proof. We will show that $\mathcal{O}^\lambda(\varepsilon, (x_n^\alpha), \mathcal{F}_\alpha) = \mathcal{F}_\alpha^{(\lambda)}$ for all λ. In view of Proposition 4.10, it is sufficient to show that if $F \in \mathcal{F}_\alpha^{(\lambda+1)}$ then there is a $N \in \mathbb{N}$ such that for all $n \geq N$, $F \cup \{n\} \in \mathcal{F}_\alpha^{(\lambda)}$. We will use induction on α and λ.

If $\alpha = 1$, then for any $\lambda \in \mathbb{N}$ the result is immediate from the definition of $\mathcal{F}_1$ and Proposition 4.10.

Now assume that $\alpha > 1$ and that the result is true for all $\beta < \alpha$ and all λ. If $F \in \mathcal{F}_\alpha^{(1)}$ then either $F = \emptyset$ and the result is obvious or $F \in \mathcal{F}_{\alpha,k,k}^{(1)}$ for some $k \in \mathbb{N}$. By Proposition 4.10,

$$\mathcal{F}_{\alpha,k,k}^{(1)} = \Big\{ \bigcup_{i=1}^{k} F_i : \\ F_i \in \mathcal{F}_{\alpha_i},\ k \leq F_1 < \ldots < F_i < F_{i+1} < \ldots < F_k,\ \text{and } F_k \in \mathcal{F}_{\alpha_k}^{(1)} \Big\}.$$

Suppose that $F = \bigcup_{i=1}^k F_i$ as above. By the inductive hypothesis $F_k \cup \{n\} \in \mathcal{F}_{\alpha_k}$ for all $n \geq N$, for some $N \in \mathbb{N}$, and thus for $n > \max\{N\} \cup F_k$, $\bigcup_{i=1}^{k-1} F_i \cup F_k \cup \{n\} \in \mathcal{F}_{\alpha,k,k}^{(0)}$.

Next assume the result for all $\gamma \leq \lambda$ and let $F \in \mathcal{F}_\alpha^{(\lambda+1)}$. If $F = \emptyset$ then $\{n\} \in \mathcal{F}_\alpha^{(\lambda)}$ for all sufficiently large n. Otherwise $F \in \mathcal{F}_{\alpha,k,k}^{(\lambda+1)}$ for some k. Let $\lambda = \omega^{\alpha_k} + \omega^{\alpha_{k-1}} + \ldots + \omega^{\alpha_j} + \varrho$ for some $j \leq k+1$ and $\varrho < \omega^{\alpha_{j-1}}$. By the claim in the proof of Proposition 4.10

$$\mathcal{F}_{\alpha,k,k}^{(\lambda+1)} = \mathcal{F}_{\alpha,j-1,k}^{(\varrho+1)} = \Big\{ \bigcup_{i=1}^{j-1} F_i : \\ F_i \in \mathcal{F}_{\alpha_i},\ k \leq F_1 < \ldots < F_i < F_{i+1} < \ldots < F_{j-1},\ \text{and } F_{j-1} \in \mathcal{F}_{\alpha_{j-1}}^{(\varrho+1)} \Big\}.$$

Suppose that $F = \bigcup_{i=1}^{j-1} F_i$ as above. By the inductive hypothesis $F_{j-1} \cup \{n\} \in \mathcal{F}_{\alpha_{j-1}}$ for all $n \geq N$, for some $N \in \mathbb{N}$, and thus for $n > \max\{N\} \cup F_{j-1}$, $\bigcup_{i=1}^{j-2} F_i \cup F_{j-1} \cup \{n\} \in \mathcal{F}_{\alpha,j-1,k}^{(\varrho)}$. ■

In the proof above we established that we can always use the special sequence $(F \cup \{n\})_{n \geq N}$ to reach a set F from a smaller derived set. The next definition describes this same property for the associated tree.

Definition. Let $\mathcal{T}$ be a tree on $\{-1,1\}$ with no nodes in $\overline{\mathcal{T}}$ with infinitely many coordinates equal to 1. We will say that $\mathcal{T}$ has *property FB* (fully branching) if $\overline{x}+\overline{e}_\omega\in\overline{\mathcal{T}}$ implies that there is an $N\in\mathbb{N}$ such that either

(i) $\overline{x}+\overline{e}_j+(1)\in\mathcal{T}$ for all $j\ge N$, or
(ii) $\overline{x}+\overline{e}_j+(1)\notin\mathcal{T}$ for all $j\ge N$.

Lemma 4.12. *Suppose that $\mathcal{T}$ and $\mathcal{U}$ are weakly independent trees with property FB and all nodes of $\mathcal{T}$ and $\mathcal{U}$ are on branches with limit of the form $x+e_\omega$. Then $\mathcal{T}\boxplus\mathcal{U}$ has property FB, and for all n, $\mathcal{L}(\mathcal{T},n)$ has property FB.*

Proof. Suppose that $\overline{x}+\overline{e}_\omega\in\overline{\mathcal{T}}$. Because $\mathcal{T}\boxplus\mathcal{U}\supset\mathcal{T}$, if (i) occurs in $\mathcal{T}$, the same is true in $\mathcal{T}\boxplus\mathcal{U}$. If (ii) occurs in $\mathcal{T}$ but not in $\mathcal{T}\boxplus\mathcal{U}$, then there is a sequence of incomparable nodes of the form $\overline{e}_n+\overline{y}_j$ in $\mathcal{U}$ where n is greater than the length of $\overline{x}$ and does not depend on j, and $\overline{y}_j$ has at least one coordinate equal to one. By passing to a subsequence we may assume that $\overline{e}_n+\overline{y}_j$ converges to $\overline{e}_n+\overline{z}+\overline{e}_\omega$. The fact that the $\overline{e}_n+\overline{y}_j$'s are incomparable guarantees that $(\overline{e}_n+\overline{z})+\overline{e}_\omega$ does not satisfy (ii). Because $\mathcal{U}$ has property FB there is an N such that $\overline{e}_n+\overline{z}+\overline{e}_j+(1)\in\mathcal{U}$ for all $j>N$. Because $\mathcal{U}$ is weakly independent this implies that $\overline{e}_k+\overline{e}_j+(1)\in\mathcal{U}$ for all $j>N$, where k equals n plus the length of $\overline{z}$. Hence $\overline{x}+\overline{e}_m+(1)\in\mathcal{T}\boxplus\mathcal{U}$ for all $m>N+k$, i.e., $\overline{x}+\overline{e}_\omega$ satisfies (i).

If $\overline{x}+\overline{y}+\overline{e}_\omega\in\overline{\mathcal{T}\boxplus\mathcal{U}}\setminus\overline{\mathcal{T}}$, where $\overline{e}_n+\overline{y}\in\mathcal{U}$, n is the length of $\overline{x}$ and $\overline{x}\in\mathcal{T}$, then $\overline{x}+\overline{y}+\overline{e}_\omega$ will satisfy (i), respectively (ii), if $\overline{e}_n+\overline{y}+\overline{e}_\omega$ satisfies (i), respectively (ii).

The second assertion is obvious. ∎

Proposition 4.13. *Suppose that (x_n) is a weakly independent sequence of continuous indicator functions on a compact metric space K which converge pointwise to 0 and that the associated tree $\mathcal{T}$ has property FB. Then for any $\varepsilon<1$, and $\alpha<\omega_1$, $\mathcal{O}^\alpha(\varepsilon,(x_n),K)\ne\emptyset$ if and only if $\overline{\mathcal{T}}^{(1+\alpha)}\ne\emptyset$.*

Proof. For each n and $(\varepsilon_i)\in\overline{\mathcal{T}}\setminus\mathcal{T}$, let $\widehat{x}_n((\varepsilon_i))=1$ if $\varepsilon_n=1$, and 0 otherwise. In this way we have defined a sequence of indicator functions on $Q=\overline{\mathcal{T}}\setminus\mathcal{T}$ with span isometric to the span of (x_n). (Actually, $\overline{\mathcal{T}}\setminus\mathcal{T}$ is homeomorphic to the natural quotient of K determined by the x_n's.)

Clearly $\mathcal{O}^\alpha(\varepsilon,(x_n),K)\ne\emptyset$ if and only if $\mathcal{O}^\alpha(\varepsilon,(\widehat{x}_n),Q)\ne\emptyset$. Now observe that $(\varepsilon_i)+\overline{\varepsilon}_\omega\in\mathcal{O}^1(\varepsilon,(\widehat{x}_n),Q)$ if and only if there is an $N\in\mathbb{N}$ such that for all $j\ge N$, $(\varepsilon_j)+\overline{e}_j+(1)\in\mathcal{T}$ (use the weak independence of $(\widehat{x}_n)$). Clearly this latter condition implies that $(\varepsilon_i)+\overline{e}_\omega\in\overline{\mathcal{T}}^{(2)}=Q^{(1)}$. Conversely, if $(\varepsilon_1)\in Q^{(1)}$, then weak independence and property FB imply that $(\varepsilon_i)+\overline{e}_j+(1)\in\mathcal{T}$, for all $j\ge N$, for some $N\in\mathbb{N}$. Finally, note that weak independence and property FB are inherited by $\mathcal{T}(Q^{(1)})$ which is the tree associated with $(\widehat{x}_{n|Q^{(1)}})$. Also, if $\alpha_k\uparrow\alpha$, then $\bigcap\mathcal{T}(Q^{(\alpha_n)})=\mathcal{T}(Q^{(\alpha)})$ and it is straightforward to check that weak independence and property FB are inherited by the intersection. Thus transfinite induction may be used to complete the proof. ∎

Proposition 4.11 follows as a corollary of this result, that is, $\mathcal{O}^{\omega^\alpha}(\varepsilon,(x_n^\alpha),K) \neq \emptyset$, for all $\alpha<\omega_1$, $\varepsilon<1$.

Because the underlying space for (x_n^α) is ω^{ω^α}, ω^α is the maximal possible oscillation index. Also, note that because each (x_n^α) is an unconditional basic sequence, all of the spaces are isomorphic to complemented subspaces of the Pełczyński universal space U_1, [L-T,I, p. 92].

5. Reflexive spaces with large oscillation index

In the previous section we constructed weakly null sequences with oscillation index ω^α. Because these sequences were in $C(\mathcal{F}_\alpha)$ it follows that the span of (x_n^α) contains c_0 and thus is not reflexive. In this section we will explore an idea of E. Odell for constructing Tsirelson-like spaces with large oscillation index.

To define these spaces we begin with the space $[x_n^\alpha]$ in place of c_0 in the Tsirelson construction [C-S, p. 14]. Suppose that $x=\sum a_n t_n^\alpha$ where (t_n^α) is the unit vector basis of the space of sequences with only finitely many nonzero coordinates. Let

$$\|x\|_0=\Big\|\sum a_n x_n^\alpha\Big\|$$

and inductively define

$$\|x\|_{m+1}=\max\Big\{\|x\|_m,\ 2^{-1}\max_{\{p_i\}\in\mathcal{F}_1}\sum_{i=1}^k\Big\|\sum_{n=p_i}^{p_{i+1}-1}a_n t_n^\alpha\Big\|_m\Big\},$$

where k is the cardinality of $\{p_i\}$. Let $\|x\|=\lim\|x\|_m$. Let T_α be the completion of $\operatorname{span}\{t_n^\alpha\}$ under $\|\cdot\|$. Observe that Tsirelson space is T_0 in this construction, i.e., if we let $\mathcal{F}_0=\{\{n\}:n\in\mathbb{N}\}$, then $(x_n^0)=(1_{\{\{n\}\}})$ is equivalent to the c_0 basis. Also, observe that for each α the norm on T_α satisfies

$$(*)\qquad \|x\|=\max\Big\{\|x\|_0,2^{-1}\max_{\{p_i\}\in\mathcal{F}_1}\sum_{i=1}^k\Big\|\sum_{n=p_i}^{p_{i+1}-1}a_n t_n^\alpha\Big\|\Big\}.$$

PROPOSITION 5.1. *For each $\alpha<\omega_1$, T_α is a reflexive Banach space with unconditional basis with no subspace isomorphic to ℓ^p, $1\le p<\infty$, or c_0. Moreover, $\mathcal{O}^{\omega^\alpha}(\varepsilon,(t_n^\alpha),B_{(T_\alpha)^*})\neq\emptyset$ for all $\varepsilon<1$.*

Proof. Fix $\alpha<\omega_1$. Clearly $\|x\|_0\ge\|x\|_{c_0}$. Therefore $\|x\|\ge\|x\|_T$ for all $x\in T_\alpha$ where $\|\cdot\|_T$ denotes the norm on Tsirelson space. (This is T_0 above.) It follows from $(*)$ that if $\{u_i\}$ is a sequence of k normalized blocks of the basis in T_α with support beyond k then

$$\sum_{i=1}^k|c_i|\ge\Big\|\sum_{i=1}^k c_i u_i\Big\|_T\ge 2^{-1}\sum_{i=1}^k|c_i|,$$

for all choices of scalars (c_i). Therefore T_α does not contain c_0 or ℓ^p, for any $p>1$.

To see that ℓ^1 is not isomorphic to a subspace of T_α we need only examine the proof that T does not contain ℓ^1 as given in [C-S, p. 17]. The only properties of T that are used in the argument are that the norm satisfies equation $(*)$ and that the c_0 norm of a long average is small. The proof for T will then carry over to T_α provided we use a sequence with small $\|\cdot\|_0$. To do this let (u_i) be a normalized block basis of (t_n^α) such that

$$\sum |a_i| \ge \Big\| \sum a_i u_i \Big\| \ge \frac{8}{9} \sum |a_i| \,.$$

Because (x_n^α) is weakly null in $C(\mathcal{F}_\alpha)$, so is (u_i) and thus there is a sequence of disjoint convex combinations of (u_i),

$$y_j = \sum_{i\in E_j} \lambda_i u_i \,,$$

where $E_1 < E_2 < \ldots, \sum_{i\in E_j} \lambda_i = 1$, and $\lambda_i \ge 0$ for all i, such that $\|y_j\|_0 \le 2^{-j}$ for $j = 0, 1, \ldots$ It follows then that

$$\sum |a_i| \ge \Big\| \sum a_i y_i \Big\| \ge \frac{8}{9} \sum |a_i| \,,$$

and

$$\|y_0 + r^{-1}(y_1 + y_2 + \ldots + y_r)\|_0 \le 1 + r^{-1} \,.$$

By using these y_i's the remainder of the proof carries over without change to T_α.

Obviously T_α has an unconditional basis and thus by a result of James [L-T, I, p. 97], T_α is reflexive.

Finally, to see that the oscillation index is large we use the observation that the operator S from T_α to $[x_n^\alpha]$ defined by $S(\sum c_n t_n) = \sum c_n x_n^\alpha$ is bounded by 1 and thus $\mathcal{O}^\lambda(\varepsilon, (t_n^\alpha), B_{T_\alpha^*}) \supset S^* \mathcal{O}^\lambda(\varepsilon, (x_n^\alpha), \mathcal{F}_\alpha) \ne \emptyset$, for every $\lambda \le \omega^\alpha$, by Lemma 2.2. ∎

While these spaces T_α share important properties with T let us note that they do not possess the property that every block basis dominates a subsequence of the basis. In particular

PROPOSITION 5.2. *For every $\alpha < \omega_1$ there is a block basis (u_i) of (t_n^α) and an increasing sequence of integers (k_i) such that*

$$\Big\| \sum a_i t_i \Big\|_T \le \Big\| \sum a_i u_i \Big\| \le 2(1+\varepsilon) \Big\| \sum a_i t_{k_i} \Big\|_T$$

for any sequence of scalars (a_i).

Proof. Fix $\alpha < \omega_1$. Let (v_i) be a normalized block basic sequence of (x_n^α) which is $(1+\varepsilon)$-equivalent to the usual unit vector basis of c_0 and let $u_i = v_i/\|v_i\|$, $i = 1, 2, \ldots$ The idea is to show that for any sequence of scalars (a_i)

$$(**) \qquad \Big\| \sum a_i t_i \Big\|_{T,m} \le \Big\| \sum a_i u_i \Big\|$$

and

$$(***)\qquad \Big\|\sum a_i u_i\Big\|_m \le 2(1+\varepsilon)\Big\|\sum a_i t_{2i}\Big\|_T ,$$

$m = 0, 1, \ldots,$ where

$$\Big\|\sum a_i t_i\Big\|_{T,m+1} = \max\Big\{\Big\|\sum a_i t_i\Big\|_{T,m},\ 2^{-1}\max_{\{p_i\}\in\mathcal{F}_1}\sum_{i=1}^{k}\Big\|\sum_{n=p_i+1}^{p_{i+1}} a_n t_n\Big\|_{T,m}\Big\},$$

and

$$\Big\|\sum a_i t_i\Big\|_{T,0} = \sup|a_i| .$$

Once this is accomplished we use the fact that there is a constant K such that

$$\Big\|\sum a_i t_i\Big\|_T \le \Big\|\sum a_i t_{2i}\Big\|_T \le K\Big\|\sum a_i t_i\Big\|_T$$

and hence $[u_i]$ is isomorphic to T. (See [C-S, p. 26]. In fact, the argument given here is derived from the arguments of Casazza, Johnson and Tzafriri [C-S, p. 34–38].)

We establish $(**)$ by induction on m. For $m = 0$, $(**)$ is immediate. Now assume the inequality holds for m and we will prove it for $m+1$.

Let k_n be the first element in the support of u_n. Then

$$\begin{aligned}
&\Big\|\sum a_i t_i\Big\|_{T,m+1}\\
&\quad= \max\Big\{\Big\|\sum a_i t_i\Big\|_{T,m},\ 2^{-1}\max_{\{p_i\}\in\mathcal{F}_1}\sum_{i=1}^{k}\Big\|\sum_{n=p_i}^{p_{i+1}-1} a_n t_n\Big\|_{T,m}\Big\},\\
&\quad\le \max\Big\{\Big\|\sum a_i u_i\Big\|,\ 2^{-1}\max_{\{p_i\}\in\mathcal{F}_1}\sum_{i=1}^{k}\Big\|\sum_{n=p_i}^{p_{i+1}-1} a_n u_n\Big\|\Big\},\\
&\quad\le \max\Big\{\Big\|\sum a_i u_i\Big\|,\ 2^{-1}\max_{\{p_i\}\in\mathcal{F}_1}\sum_{i=1}^{k}\Big\|P_i\sum a_n u_n\Big\|\Big\} = \Big\|\sum a_i u_i\Big\|
\end{aligned}$$

where $P_i\sum b_n t_n^\alpha = \sum_{n=q_i}^{q_{i+1}-1} b_n t_n^\alpha$. The last inequality above holds because $\{q_i\} = \{k_{p_i}\}\in\mathcal{F}_1$ if $\{p_i\}\in\mathcal{F}_1$.

For the inequality $(***)$ we need to work a little harder. We have

$$\Big\|\sum a_n u_n\Big\|_{m+1} = \max\Big\{\Big\|\sum a_i u_i\Big\|_m,\ 2^{-1}\max_{\{q_i\}\in\mathcal{F}_1}\sum_{i=1}^{k}\Big\|P_i\sum a_n u_n\Big\|_m\Big\}$$

where P_i denotes the basis projection onto $[t_j : q_i \le j < q_{i+1}]$. Fix $\{q_i\}\in\mathcal{F}_1$ and for each i let $G_i = \{n : q_i \le k_n < k_{n+1} < q_{i+1}\}$. For each n let

$$H_n = \{i : k_n \le q_i < k_{n+1} \text{ or } k_n < q_{i+1} \le k_{n+1}, \text{ and } n \notin G_i\}\,.$$

Consider the sum corresponding to the q_i's:

$$\begin{aligned}
2^{-1}\sum_{i=1}^{k}\Big\|P_i \sum a_n u_n\Big\|_m \\
&\le 2^{-1}\sum_{i=1}^{k}\Big\|\sum_{n\in G_i} a_n u_n\Big\|_m + 2^{-1}\sum_{n=1}^{\infty}\sum_{i\in H_n}\|P_i a_n u_n\|_m \\
&\le 2^{-1}\sum_{i=1}^{k}\Big\|\sum_{n\in G_i} a_n u_n\Big\|_m + 2^{-1}\sum_{n:H_n\ne\emptyset} 2\|a_n u_n\|_{m+1} \\
&\le 2^{-1}\sum_{i=1}^{k} 2(1+\varepsilon)\Big\|\sum_{n\in G_i} a_n t_{2k_{n+1}}\Big\|_T + \sum_{n:H_n\ne\emptyset}\|a_n t_{2k_{n+1}}\|_T \\
&\le (1+\varepsilon)\Big[\sum_{i=1}^{k}\Big\|\sum_{n\in G_i} a_n t_{2k_{n+1}}\Big\|_T + \sum_{n:H_n\ne\emptyset}\|a_n t_{2k_{n+1}}\|_T\Big]\,.
\end{aligned}$$

Observe that there are at most k integers n such that $H_n \ne \emptyset$ and that $k \le q_1 < k_{m+1}$, where m is the smallest integer such that $P_1 u_m \ne 0$. Let $n_i = \min G_i$ for $i = 1, 2, \ldots, k$. Then

$$\{2k_{n_i+1} : i = 1, 2, \ldots, k\} \cup \{2k_{n+1} : H_n \ne \emptyset\}$$

is a set of at most $2k$ integers greater than $2k$. Hence this sum in brackets is at most $2\|\sum a_n t_{2k_{n+1}}\|_T$. Therefore

$$\begin{aligned}
\max\Big\{\Big\|\sum a_i u_i\Big\|_m,\ 2^{-1}\max_{\{q_i\}\in\mathcal{F}_1}\sum_{i=1}^{k}\Big\|P_i\sum a_n u_n\Big\|_m\Big\} \\
&\le \max\Big\{\Big\|\sum a_i t_{2k_{i+1}}\Big\|_T,\ 2(1+\varepsilon)\max_{\{q_i\}\in\mathcal{F}_1}\sum_{i=1}^{k}\Big\|P_i\sum a_n t_{2k_{n+1}}\Big\|_T\Big\} \\
&= 2(1+\varepsilon)\Big\|\sum a_i t_{2k_{i+1}}\Big\|_T\,,
\end{aligned}$$

as claimed. ■

Remark 5.3. Argyros [A] has modified the construction to obtain spaces X_α, $\alpha < \omega_1$, such that all of the subspaces of X_α have index at least α. To accomplish this he uses the sets $\mathcal{F}_\alpha$ in the definition of the norm instead of starting with the space $[x_n^\alpha]$.

6. Comparison with the averaging index

In [A-O] the averaging index (see Section 1 for the definition) was used to get a somewhat more constructive version of Mazur's Theorem. In this section we will show that the averaging index is much larger than the spreading model index by showing that there exists a Banach space X such that for every $\alpha < \omega_1$ there is a weakly null sequence (x_n) in X with averaging index at least α, and yet ℓ^1 is not isomorphic to a subspace of X. We will also show that the spreading model index can be used to strengthen some of the results in [A-O].

The construction of the example will be based on the infinite branching James tree construction [J] in combination with ω^ω. Let

$$\mathcal{T} = \bigcup_{n=1}^{\infty} \{(\alpha_1, \alpha_2, \ldots, \alpha_n) : \alpha_i < \omega^\omega \text{ for each } i\}.$$

Let X_0 be the linear subspace of the functions from $\mathcal{T}$ into $C_0(\omega^\omega)$ which are nonzero at only finitely many points of $\mathcal{T}$. We will use the notation $(f_{\bar{t}})$, where $f_{\bar{t}} \in C(\omega^\omega)$ for all $\bar{t}$ in $\mathcal{T}$, to denote an element of X_0 with the understanding that the index $\bar{t}$ runs over $\mathcal{T}$. We will also use $f_{\bar{t}}$ to denote the element of X_0 which is 0 except at $\bar{t}$ and $f_{\bar{t}}$ at $\bar{t}$.

Next we introduce some linear functionals on X_0. For each $\bar{s} \in \overline{\mathcal{T}}$ and $i < |\bar{s}|$ (the length of $\bar{s}$) define

$$L_{(\bar{s},i)}(f_{\bar{t}}) = \begin{cases} 0 & \text{if } \bar{s} \not> \bar{t} \text{ or } |\bar{t}| < i, \\ f_{\bar{t}}(\bar{s}(|\bar{t}|+1)) & \text{if } \bar{s} > \bar{t} \text{ and } i \le |\bar{t}| < |\bar{s}|, \end{cases}$$

where $\bar{s} > \bar{t}$ denotes that $\bar{s}$ is below $\bar{t}$, and extend linearly to X_0.

We will refer to a pair $(\bar{s}, i)$ as a *segment* and define it to be the set of nodes of length at least i which are above $\bar{s}$. We also want to have a notion of incomparable segments. Suppose that $j \le i$ and that $(\bar{s}, i)$ and $(\bar{t}, j)$ are segments; then $(\bar{t}, j)$ and $(\bar{s}, i)$ are *incomparable* if they are disjoint and $\bar{t}(m) \neq \bar{s}(m)$ for some $m \le j$, i.e., they are on branches that split by level j. Note that if $\{(\bar{s}_k, i_k)\}$ are pairwise incomparable nodes then $L_{(\bar{s}_k, i_k)}(f_{\bar{t}}) \neq 0$ for at most one k for each $\bar{t} \in \mathcal{T}$.

Now we introduce a norm on X_0. For $F \in X_0$ define

$$\|F\| = \sup\Big\{\Big[\sum_j [L_{(\bar{s}_j, i_j)}(F)]^2\Big]^{1/2} : \{(\bar{s}_j, i_j)\} \text{ pairwise incomparable}\Big\}.$$

Let X be the completion of X_0 under this norm. Clearly $C_0(\omega^\omega)$ is isometric to $X_{\bar{t}} = [f_{\bar{t}} : f \in C_0(\omega^\omega)]$ for each fixed $\bar{t}$. For each $k \in \mathbb{N}$ define a projection P_k on X by

$$P_k((f_{\bar{t}}))_{\bar{s}} = \begin{cases} f_{\bar{s}} & \text{if } |\bar{s}| \le k, \\ 0 & \text{if } |\bar{s}| > k. \end{cases}$$

It is easy to see that $\|P_k\| = 1$ for all k.

PROPOSITION 6.1. *ℓ^1 is not isomorphic to a subspace of X.*

The example is similar to an example of Odell [O] and his arguments can be modified to prove Propositon 6.1. However, we will give a slightly different proof which does not directly use the branch functionals.

Proof. Suppose that (y_n) is a normalized sequence in X which is K-equivalent to the usual unit vector basis of ℓ^1. Because ℓ^1 is not isomorphic to a subspace of $C_0(\omega^\omega)$ and $\mathrm{range}(P_k - P_{k-1})$ is isometric to $[\sum C_0(\omega^\omega)]_{\ell^2}$, it can be shown by induction that ℓ^1 is not isomorphic to a subspace of range P_k for any k. Therefore by passing to a subsequence we may assume that $(P_k(y_n))_n$ is weakly Cauchy for each k. Because for each k there are convex combinations of $(P_k(y_{2n} - y_{2n-1}))$ with small norm, we can find a sequence of disjointly supported (relative to the y_n's) convex combinations of $(y_{2n} - y_{2n-1})$, (z_j), and an increasing sequence of intergers (k_j) such that

$$\sum_{j=m}^{\infty} \|P_{k_m} z_j\| < 2^{-m},$$

for $m = 1, 2, \ldots$ Moreover, because X_0 is dense in X we may assume (by passing to a subsequence) that

$$\sum_{j=1}^{m-1} \|(I - P_{k_m}) z_j\| < 2^{-m},$$

for $m = 1, 2, \ldots$ In this way we get a sequence equivalent to the unit vector basis of ℓ^1 which is essentially supported on disjoint levels of $\mathcal{T}$. By a standard perturbation argument we may assume that $(I - P_{k_{m+1}}) z_m = 0 = P_{k_m} z_m$, and that $z_{m|\bar{t}} \neq 0$ for only finitely many $\bar{t}$ for all m.

Next note that by a theorem of James [J] we may assume that (z_j) is $(1+\varepsilon)$-equivalent to the basis of ℓ^1. For any node $\bar{s}$ let $\mathcal{W}(\bar{s}) = \{\bar{t} : \bar{t} > \bar{s}\}$, the wedge determined by $\bar{s}$. For each i there are finitely many nodes $\bar{s}(i,j)$ of length k_i such that if $\bar{t} \notin \bigcup_j \mathcal{W}(\bar{s}(i,j))$ then $z_{i|\bar{t}} = 0$. Let N be the number of nodes in the support of z_1. We claim that for each i there is a set $\mathcal{F} = \mathcal{F}(i)$ of cardinality at most N such that

$$\|z_i - (z_{i|\bigcup_{j\in\mathcal{F}} \mathcal{W}(\bar{s}(i,j))})\| < 4\varepsilon.$$

Indeed, if not let $\mathcal{S} = \{(\bar{s}, i)\}$ be a family of incomparable segments which compute the norm of $z_1 + z_i$. Let $\mathcal{S}'$ be the set of segments in $\mathcal{S}$ which intersect the support of z_1 and $\mathcal{S}'' = \mathcal{S} \setminus \mathcal{S}'$. Clearly $\mathcal{S}'$ contains at most N segments. We have

$$\begin{aligned} 4(1+\varepsilon)^{-2} \le \|z_1 + z_i\|^2 &= \sum_{(\bar{s},j)\in\mathcal{S}'} [L_{(\bar{s},j)}(z_1 + z_i)]^2 + \sum_{(\bar{s},j)\in\mathcal{S}''} [L_{(\bar{s},j)}(z_1 + z_i)]^2 \\ &\le \Big[\Big[\sum_{(\bar{s},j)\in\mathcal{S}'} [L_{(\bar{s},j)}(z_1)]^2\Big]^{1/2} + \Big[\sum_{(\bar{s},j)\in\mathcal{S}'} [L_{(\bar{s},j)}(z_i)]^2\Big]^{1/2}\Big]^2 \\ &\quad + \sum_{(\bar{s},j)\in\mathcal{S}''} [L_{(\bar{s},j)}(z_1 + z_i)]^2 \end{aligned}$$

$$\leq (1+(1-4\varepsilon))^2 + (4\varepsilon)^2 = 4(1-4\varepsilon+8\varepsilon^2)\,.$$

Clearly this is impossible for small enough ε.

It follows by another perturbation argument that we may assume that each z_i is supported in at most N wedges. As above let $\overline{s}(i,j)$, $j=1,2,\ldots,N$, be the nodes of length k_i so that z_i is supported in $\bigcup_j \mathcal{W}(\overline{s}(i,j))$. We will next refine our sequence (z_i) to get a subsequence such that there are branches $\overline{b}_1,\ldots,\overline{b}_k$, $k\leq N$, such that if $\overline{s}$ is any branch then $\overline{s}>\overline{s}(i,j)$ for at most N nodes not on some $\overline{b}_m$ and $\overline{b}_j>\overline{s}(i,j)$ for $j=1,2,\ldots,k$ for all i. Such a subsequence is easily determined by induction on N. Indeed, if $N=1$, then either there are infinitely many i and incomparable branches $(\overline{t}_i,k_i)$ such that $\overline{t}_i>\overline{s}(i,1)$ and $\overline{t}_i\not>\overline{s}(m,1)$ for any $m\neq i$, or there is a branch $\overline{b}_1$ which contains all but finitely many of the nodes $\overline{s}(i,1)$. Now suppose that $\overline{b}_1,\ldots,\overline{b}_k$ are branches such that if $\overline{s}$ is any branch then $\overline{s}>\overline{s}(i,j)$ for at most $N-1$ nodes not on some $\overline{b}_m$, and $\overline{b}_j>\overline{s}(i,j)$ for $j=1,2,\ldots,k$ for all i. As above if there is some branch which contains all but finitely many of the nodes $\overline{s}(i,N)$ then we add that branch to our list as $\overline{b}_{k+1}$ and pass to a subsequence $(z_i)_{i\in M}$ such that $\overline{b}_{k+1}>\overline{s}(i,N)$ for all $i\in M$. If this is not the case then we can find a subsequence such that there is at most one of the nodes $\overline{s}(i,N)$ on any branch. Clearly this subsequence has the required properties.

To complete the argument we need to make a few observations about the norm on X. First, observe that if $\overline{s}$ is any branch and for each node $\overline{t}$ on $\overline{s}$, $f_{\overline{t}}$ is a fixed function in $C_0(\omega^\omega)$ then $[f_{\overline{t}}]_{\overline{s}>\overline{t}}$ is isomorphic to c_0. Second, if the norm of $\sum a_i z_i$ is computed using only segments which intersect at most N of the supports of the z_i's, i.e., $\overline{s}(i,j)$ for at most N j's, then the norm is at most $(\sum a_i^2)^{1/2}N^{1/2}$. Finally, observe that for the sequence (z_i) a segment can intersect more than N of the supports of the z_i's only if it lies along one of the branches $\overline{b}_k$. A straightforward computation using these observations shows that the z_i's are not equivalent to the unit vector basis of ℓ^1. ∎

Our next goal is to show that the averaging index of X is uncountable. The basic idea is to construct for each $\alpha<\omega_1$ a weakly null sequence with averaging index α by using well-founded subtrees of $\mathcal{T}$ of order α.

PROPOSITION 6.2. *Let $\mathcal{S}$ be a well-founded subtree of $\mathcal{T}$ and for each $\overline{s}\in\mathcal{S}$ let $[f_{\overline{s}}^n]$ be a normalized weakly null sequence in $X_{\overline{s}}$. Then $[f_{\overline{s}}^n]_{n\in\mathbb{N},\overline{s}\in\mathcal{S}}$ (reordered) is a weakly null sequence in X.*

Proof. X^* is the closed linear span of the functionals $[L_{(\overline{t},i)}]_{\overline{t}\in\overline{\mathcal{T}},i\in\mathbb{N}}$. Because $\mathcal{S}$ is well-founded, for any $\overline{t}\in\overline{\mathcal{T}}$ we have $\overline{s}<\overline{t}$ for only finitely many $\overline{s}\in\mathcal{S}$. Hence for any $\varepsilon>0$, $\overline{t}\in\overline{\mathcal{T}}$, and $i\in\mathbb{N}$, we obtain $|L_{(\overline{t},i)}[f_{\overline{s}}^n]|\geq\varepsilon$ for only finitely many n and $\overline{s}$. Therefore $[f_{\overline{s}}^n]_{n\in\mathbb{N},\overline{s}\in\mathcal{S}}$ is a weakly null sequence in X. ∎

In order to estimate the averaging index we need to have some information about the w^* topology on the functionals $L_{(\overline{t},i)}$.

Lemma 6.3. *Let $\bar{t} \in \mathcal{T}$ and let (α_i) be a sequence in ω^ω which converges to α. Then*

(i) *If $\alpha < \omega^\omega$, then $L_{(\bar{t}+(\alpha_i),j)} \overset{w^*}{\to} L_{(\bar{t}+(\alpha),j)}$.*

(ii) *If $\alpha = \omega^\omega$, then $L_{(\bar{t}+(\alpha_i),j)} \overset{w^*}{\to} L_{(\bar{t},j)}$.*

Proof. We need only consider the values of the functionals at $f_{\bar{s}}$ for those $\overline{s} \in \mathcal{T}$ such that $\overline{s} \le \overline{t}$ and $|\overline{t}| - 1 \le |\overline{s}| \le |\overline{t}|$. For all others the values do not depend on (α_i). If $\overline{s} = \overline{t}$ then

$$L_{(\bar{t}+(\alpha_i),j)}[f_{\bar{s}}] = f_{\bar{s}}(\alpha_i) \to f_{\bar{s}}(\alpha) = \begin{cases} L_{(\bar{t}+(\alpha),j)}[f_{\bar{s}}] & \text{if } \alpha_i \to \alpha < \omega^\omega, \\ 0 = L_{(\bar{t},j)}[f_{\bar{s}}] & \text{if } \alpha_i \to \omega^\omega. \end{cases}$$

If $\overline{t} = \overline{s} + (\beta)$, then

$$L_{(\bar{t}+(\alpha_i),j)}[f_{\bar{s}}] = f_{\bar{s}}(\beta) \to f_{\bar{s}}(\beta) = \begin{cases} L_{(\bar{t}+(\alpha),j)}[f_{\bar{s}}] & \text{if } \alpha_i \to \alpha < \omega^\omega, \\ L_{(\bar{t},j)}[f_{\bar{s}}] & \text{if } \alpha_i \to \omega^\omega. \ \blacksquare \end{cases}$$

We will need to use well-founded subtrees of $\mathcal{T}$ of a special type, so as a technical convenience we introduce the following.

Definition. A well-founded tree $\mathcal{S} \subset \mathcal{T}$ is said to be *complete* if

(i) $\overline{s} + (\beta) \in \mathcal{S}$ for some $\beta < \omega^\omega$ and $\overline{s} \in \mathcal{T}$ then $\overline{s} + (\alpha) \in \mathcal{S}$ for all $\alpha < \omega^\omega$ and $\overline{s} \in \mathcal{S}$,

(ii) $(\alpha) \in \mathcal{S}$ for all $\alpha < \omega^\omega$.

Next we will verify that such things exist.

Proposition 6.4. *For every $\alpha < \omega_1$ there exists a complete well-founded tree $\mathcal{S} \subset \mathcal{T}$ of order at least α.*

Proof. For $\alpha = 1$ this is obvious. Suppose for all $\beta < \alpha$ there is a complete well-founded subtree $\mathcal{S}$ of $\mathcal{T}$ of order β. If $\alpha = \beta + 1$, we define $\mathcal{U} = \{(\eta) + \overline{s} : \overline{s} \in \mathcal{S} \cup \{()\}$ and $\eta < \omega^\omega\}$. Clearly $o(\mathcal{U}) = o(\mathcal{S}) + 1$ and it is obvious that $\mathcal{U}$ is complete. If α is a limit ordinal and $\alpha_i \uparrow \alpha$, for each i let $\mathcal{S}_{\alpha_i}$ be a complete well-founded subtree of $\mathcal{T}$ of order α_i. Let $\mathcal{U} = \{(\eta) + \overline{s} : \overline{s} \in \bigcup_i \mathcal{S}_{\alpha_i} \cup \{()\}$ and $\eta < \omega^\omega\}$. Clearly $\mathcal{U}$ is complete, $o(\mathcal{U}) \ge o(\mathcal{S}_{\alpha_i})$ for all i and thus $o(\mathcal{U}) \ge \alpha$. $\blacksquare$

We are now ready to show that there are weakly null sequences in X with large averaging index.

Proposition 6.5. *Suppose that $\mathcal{S}$ is a complete well-founded subtree of $\mathcal{T}$ and for each $\overline{s} \in \mathcal{S}$, $[f^n_{\bar{s}}]_{n\in\mathbb{N}}$ is the Schreier sequence. Then for every $\beta < o(\mathcal{S})$ and $\overline{s} \in \mathcal{S}^{(\beta)}$, $L_{(\bar{s},1)} \in A_\beta[\frac{1}{2}, [f^n_{\bar{t}}]_{n\in\mathbb{N},\bar{t}\in\mathcal{S}}, B_{X^*}]$.*

Proof. We proceed by induction on β. It is sufficient to prove the result for $\beta + 1$ assuming it for β. If $\overline{s} \in \mathcal{S}^{(\beta+1)}$, then because $\mathcal{S}$ is complete $\overline{s} + (\alpha) \in \mathcal{S}^{(\beta)}$ for all $\alpha < \omega^\omega$. Moreover, $L_{(\bar{s}+(\alpha),1)}[f^n_{\bar{s}}] = f^n_{\bar{s}}(\alpha)$. In Section 4 it was shown that $\omega^\omega \in \mathcal{O}^1(\frac{1}{2},(f^n),\omega^\omega) \subset A_1(\frac{1}{2},(f^n),\omega^\omega)$ where (f^n) is the Schreier sequence. Therefore by Lemma 6.3,

$$L_{(\bar{s},1)} \in A_1[\tfrac{1}{2},[f^n_{\bar{s}}]_{n\in\mathbb{N}}, A_\beta[\tfrac{1}{2},[f^n_{\bar{t}}]_{n\in\mathbb{N},\bar{t}\in\mathcal{S}}, B_{X^*}]] \,.$$

Because the definition of the averaging index only requires that a subsequence have ℓ^1-$SP \ge \frac{1}{2}$, it follows that

$$L_{(\bar{s},1)} \in A_1[\tfrac{1}{2},[f^n_{\bar{t}}]_{n\in\mathbb{N},\bar{t}\in\mathcal{S}}, A_\beta[\tfrac{1}{2},[f^n_{\bar{t}}]_{n\in\mathbb{N},\bar{t}\in\mathcal{S}}, B_{X^*}]] = A_{\beta+1}[\tfrac{1}{2},[f^n_{\bar{t}}]_{n\in\mathbb{N},\bar{t}\in\mathcal{S}}, B_{X^*}]. \ \blacksquare$$

COROLLARY 6.6. *For every $\alpha < \omega_1$ there is a weakly null sequence in X such that $o(A,\frac{1}{2}) \ge \alpha$.*

Next we want to consider the relationship between the spreading model index and the constructive version of Mazur's Theorem as presented in [A-O]. One purpose of that paper was to try to use the averaging index to determine if given a Banach space X there is an integer k such that any weakly null sequence needs to be averaged at most k times in order to get a norm null sequence. In what follows we will use the notation of [A-O] and refer the reader there for the relevant definitions.

The definition of the spreading model index makes some arguments of the type used in [A-O] difficult because of its sensitivity to passing to subsequences. On the other hand, we are going to consider all weakly null sequences in the space X so it seems natural to look for sequences which are in some sense extremal.

PROPOSITION 6.7. *Suppose that X is a subspace of $C(K)$ for some compact metric space K and that (x_n) is a weakly null sequence in the unit ball of X. Then for every $\varepsilon > 0$ there is a subsequence $(x_n)_{n\in M}$ of (x_n) such that for any infinite $L \subset M$ and $\alpha < \omega_1$,*

$$S^\alpha(\varepsilon,(x_n)_{n\in L},K) = S^\alpha(\varepsilon,(x_n)_{n\in M},K)\,.$$

Proof. The idea is to use repeatedly the following lemma [A-O].

LEMMA 6.8. *Let K be a second countable compact Hausdorff space and let (x_n) be a weakly null sequence in $C(K)$. Then there is a subsequence $(x_n)_{n\in M}$ such that for every $t \in K$ and every neighborhood $\mathcal{N}$ of t there is a neighborhood $\mathcal{N}'$ of t, $\mathcal{N}' \subset \mathcal{N}$, such that $(x_{n|\mathcal{N}'})$ has a spreading model.*

We will prove the proposition by induction on α. Note that there is an ordinal α_0 such that $S^{\alpha_0}(\varepsilon,(y_n),K) = \emptyset$ for every $\varepsilon > 0$ and weakly null sequence (y_n) in $C(K)$, and thus the induction is over a countable set. Fix $\varepsilon > 0$.

INDUCTION HYPOTHESIS. If (x_n) is a weakly null sequence in the unit ball of X, then for every $\varepsilon > 0$ there is a subsequence $(x_n)_{n\in M}$ of (x_n) such that for any infinite $L \subset M$ and $\beta \le \alpha$,

$$S^\beta(\varepsilon,(x_n)_{n\in L},K) = S^\beta(\varepsilon,(x_n)_{n\in M},K)\,.$$

Observe that the lemma proves the result for $\alpha = 1$. Now suppose that the result is true for all $\beta < \alpha$. If $\alpha = \beta + 1$, for some β, then let $(x_n)_{n\in L}$ be a subsequence such that

$$S^\lambda(\varepsilon, (x_n)_{n\in J}, K) = S^\lambda(\varepsilon, (x_n)_{n\in L}, K),$$

for all infinite $J \subset L$ and $\lambda \le \beta$. Apply the lemma to $(x_{n|S^\beta(\varepsilon,(x_n)_{n\in J},K)})_{n\in L}$ to get a subsequence $(x_n)_{n\in M}$ and note that the case $\alpha = 1$ applies to show that

$$\begin{aligned} S^{\beta+1}(\varepsilon, (x_n)_{n\in J}, K) &= S^1(\varepsilon, (x_{n|S^\beta(\varepsilon,(x_n)_{n\in L},K)})_{n\in J}, S^\beta(\varepsilon, (x_n)_{n\in L}, K)) \\ &= S^1(\varepsilon, (x_{n|S^\beta(\varepsilon,(x_n)_{n\in L},K)})_{n\in M}, S^\beta(\varepsilon, (x_n)_{n\in L}, K)) \\ &= S^{\beta+1}(\varepsilon, (x_n)_{n\in M}, K). \end{aligned}$$

for all infinite $J \subset M$.

If α is a limit ordinal, let $\alpha_i \uparrow \alpha$ and choose infinite sets $M_1 \subset M_2 \subset \ldots \subset M_i \subset \ldots$ such that

$$S^\lambda(\varepsilon, (x_n)_{n\in J}, K) = S^\lambda(\varepsilon, (x_n)_{n\in M_i}, K),$$

for all infinite $J \subset M_i$ and $\lambda \le \alpha_i$. Then if M is an infinite set such that $M\backslash M_i$ is finite for all i, it follows that

$$S^\lambda(\varepsilon, (x_n)_{n\in J}, K) = S^\lambda(\varepsilon, (x_n)_{n\in M_i}, K),$$

for all infinite $J \subset M$ and $\lambda < \alpha$. However, because $S^\alpha = \bigcap_{\lambda<\alpha} S^\lambda$, the equality holds for $\lambda = \alpha$ as well. ∎

COROLLARY 6.9. *Suppose that (x_n) is a weakly null sequence in the ball of $C(K)$ for some compact metric space K. Then there is a subsequence $(x_n)_{n\in M}$ such that*

$$A^\lambda(\varepsilon, (x_n)_{n\in M}, K) = S^\lambda(\varepsilon, (x_n)_{n\in M}, K),$$

for all $\lambda \le \omega_1$ and $\varepsilon > 0$.

Proof. The proof above gives us a subsequence so that passing to subsequences has no effect on ℓ^1-$SP(x_{n|\mathcal{N}})$ for some $\mathcal{N} \downarrow \{t\}$ for all t, therefore

$$A^1(\varepsilon, (x_n)_{n\in M}, K) = S^1(\varepsilon, (x_n)_{n\in M}, K).$$

Induction completes the proof. ∎

COROLLARY 6.10. *Let X be a subspace of $C(K)$, K compact metric, such that* $\sup\{o(S, (x_n), \varepsilon) : \varepsilon > 0$ *and* $(x_n) \subset B_X$ *is weakly null*$\} < \omega^{k+1}$, *for some* $k \in \{-1, 0, 1, 2, \ldots\}$. *Then X has property $A(k+2)$.*

Proof. To establish property $A(m)$ it is sufficient to show that every weakly null sequence has a subsequence with a norm null convex block subsequence of m-averages. According to the previous corollary every weakly null sequence has a subsequence for which the averaging index and spreading model index agree. Therefore by [A-O, Theorem 4.1], X has property $A(k+2)$. ∎

Because the spreading model index is in general smaller than the averaging index this corollary gives a real strengthening of [A-O, Corollary 4.2]. In particular, the space constructed above has averaging index ω_1 but spreading model index at most ω.

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